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Robert Becker
Stefano Bosi
Cuong Le Van
Thomas Seegmuller

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On existence and bubbles of Ramsey equilibrium with borrowing constraints*

Robert Becker[†], Stefano Bosi[‡], Cuong Le Van[§] and Thomas Seegmuller[¶]

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Abstract

We study the existence of equilibrium and rational bubbles in a Ramsey model with heterogeneous agents, borrowing constraints and endogenous labor.

Applying a nonstandard fixed-point theorem by Gale and Mas-Colell's (1975), we prove the existence of equilibrium in a time-truncated bounded economy. A common argument shows this solution to be an equilibrium for any unbounded economy with the same fundamentals. Taking the limit of a sequence of truncated economies, we eventually obtain the existence of equilibrium in the Ramsey model.

In the second part of the paper, we address the issue of rational bubbles and we prove that they never occur in a productive economy à la Ramsey.

Keywords: Existence of equilibrium, bubbles, Ramsey model, heterogeneous agents, borrowing constraint, endogenous labor.

JEL classification: C62, D31, D91, G10.

1 Introduction

Frank Ramsey's (1928) seminal article on optimal capital accumulation ends with a famous conjecture: "*... equilibrium would be attained by a division into*

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[†]Department of Economics, Indiana University. E-mail: becker@indiana.edu.

[‡]EPEE, University of Evry. E-mail: stefano.bosi@univ-evry.fr.

[§]IPAG Business School, CNRS, VCREME and PSE. E-mail: Cuong.Le-Van@univ-paris1.fr.

[¶]Aix-Marseille University (Aix-Marseille School of Economics), CNRS-GREQAM, EHESS. E-mail: Thomas.seegmuller@univ-amu.fr.

two classes, the thrifty enjoying bliss and the improvident at the subsistence level". In the long run, the most patient agent(s) would hold all the capital, while the others would consume at the minimum level necessary to sustain their lives. Becker (1980) demonstrated the Ramsey conjecture in the case of a stationary equilibrium when households face borrowing constraints. Without such a constraint, markets are complete and the impatient households would borrow against the future stream of their labor incomes, consume more in the present and accept their consumption converges to zero as time tends to infinity (Le Van and Vailakis (2003), and Becker (2012)).[1] In contrast, borrowing constraints result in impatient agents' positive consumption (equal to wage) at a steady state.

In the last three decades, the framework introduced by Becker (1980) has been used for different purposes. For instance, Becker and Foias (1987, 1994) and Sorger (1994, 1995) prove that persistent cycles of period two as well as chaotic solutions arise when the capital income monotonicity fails. Under additional market imperfections (strategic behavior on capital markets and progressive capital taxation), Becker and Foias (2007), Sarte (1997) or Sorger (2002, 2005, 2008) prove that impatient households may hold capital in the long run. Bosi and Seegmuller (2010) extend the Ramsey model with heterogeneous households to endogenous labor.

Our Ramsey model with heterogeneous households, endogenous labor and borrowing constraints addresses two important issues: the existence of equilibrium on the one hand and rational bubbles on the other hand.

To the best of our knowledge, the existence of bubbles has never been considered in such a context, while the existence of the equilibrium was tackled only by Becker et al. (1991) and Bosi and Seegmuller (2010). In the latter, labor supply is endogenous, but the existence of the intertemporal equilibrium is shown only in a neighborhood of the steady state. Becker et al. (1991) focus on the model with inelastic labor supply, but provide a global existence argument. Their proof rests on the introduction of a tâtonnement continuous map in which each fixed point yields an equilibrium. However, their argument no longer works when labor is elastically supplied. Our proof of existence holds in this more general and challenging case and, by the way, a proof of nonexistence for rational bubbles is also provided.[2]

[1] The introduction of a labor-leisure arbitrage implies in addition that impatient agents work less today to enjoy the leisure time but more tomorrow to repay their debt (Le Van et al. (2007)).

[2] To construct this continuous map, the authors require the intertemporal utility to be continuous for the product topology on the whole space of sequences and the productivity at the origin larger than the inverse of the time preference (β). In our paper, we only require the utility to be continuous for the product topology on the feasible set and the productivity at the origin to be larger than the capital depreciation rate.

However, even if we take the assumptions of Becker et al. (1991) on the utility function and the productivity, the proof of Becker et al. (1991) can not be carried over our model under endogenous labor supply. Indeed, their assumptions allow to have the capital per head bounded away from zero and, since the labor supply is exogenous in their paper, the paths of capital stock are bounded away from zero. In our paper, since labor supply is endogenous, labor and capitals are no longer ensured to be bounded away from zero.

We show the existence of Ramsey equilibrium in three steps. (1) We start by considering a bounded time-truncated economy and adapting a proof by Florenzano (1999) based on the Gale and Mas-Colell's (1975) fixed-point theorem.[3] (2) This solution remains an equilibrium as the uniform bounds are relaxed. (3) Taking the limit of a sequence of time-truncated economies, we eventually prove the existence of equilibrium in the infinite-horizon economy. To the best of our knowledge, there are no papers that prove the existence of equilibrium under imperfections in the financial markets (borrowing constraint) in the case of capital accumulation and endogenous labor supply.

Our setup is also suitable to address the important issue of existence of rational bubbles in a general equilibrium model. The seminal models on the existence of rational bubbles in a general equilibrium context are overlapping generations (OG) economies (Tirole (1985)). Financially constrained economies seem to share some of the same properties found in OG models. In connection with the Tirole's (1982) idea that new traders should enter the market each period, more recent contributions have shown that bubbles may exist in exchange economies with heterogeneous infinite-lived households facing some borrowing constraints (Kocherlakota (1992), Huang and Werner (2000)). Conversely, we prove that rational bubbles fail to exist when production is taken into account because of a positive interest rate. Our condition for ruling out any bubble is less demanding than the one one finds in Kocherlakota (1992) or in Huang and Werner (2000), that is an endowment growth rate lower than the interest rate.

Section 2 introduces the model. Sections 3 and 4 focus on the definition and the existence of a Ramsey equilibrium. Sections 5 proves that rational bubbles never emerge. Technical details are gathered in Appendices 1 to 3.

2 The Ramsey model

The Ramsey equilibrium model specifies the behavior of a finite collection of households and the profit motive governing production. Each household is infinitely-lived and enjoys a felicity, or reward, at each time based on its consumption and leisure time. Lifetime utility is the discounted sum of felicities and each household's discount factor is a given constant. The production technology is defined by a one-sector model with a single all purpose consumption-capital good. Households supply capital goods and labor services to the production sector at each time. Time is discrete and there is an infinite horizon. Markets are perfectly competitive and households act with perfect foresight when composing their consumption and investment decisions. Each household's budget constraint also reflects a borrowing constraint at each time.

Assumptions governing household behavior and relationships with the production sector are detailed below.

[3] Our proof is quite general and holds even if some initial individual capital endowments are zero and the capital depreciation rate equals one.

2.1 The production sector

Consumption goods and new capital goods are produced at each time. The technology is represented by a constant returns to scale production function: $F(K_t, L_t)$, where K_t and L_t denote the input demands for capital and labor at time t. Profit maximization occurs at each time. All intertemporal decisions reside with households as there are no adjustment costs in the production sector. Under competition zero profits are earned at each time in this sector. Standard assumptions are imposed on F as well as a boundary condition when the labor supply is maximal. This additional condition simplifies the existence argument.

Let $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}^2_+ = \mathbb{R}_+ \times \mathbb{R}_+$. Let m denote the maximum possible labor supply within any period. This can occur if each household provides one unit of labor at each time and there are m households. Note that all labor services provided by the households are alike.

Assumption 1 *The production function $F : \mathbb{R}^2_+ \to \mathbb{R}_+$ is C^1, homogeneous of degree one, strictly increasing, concave and strictly concave separately in $K > 0$ and $L > 0$. Inputs are essential: $F(0, L) = F(K, 0) = 0$. Limit conditions for production hold: $F(K, L) \to \infty$ either when $L > 0$ and $K \to \infty$ or when $K > 0$ and $L \to \infty$. Moreover, $(\partial F/\partial K)(0, m) > \delta$, where $\delta \in (0, 1]$ denotes the rate of capital depreciation.*

Remark 1 *Since F is homogeneous of degree one and $F(1, 0) = 0$, we obtain $(\partial F/\partial K)(\infty, m) = 0$ and $(\partial F/\partial L)(1, \infty) = 0$. Indeed,*

$$F(K, mL) \geq (\partial F/\partial K)(K, mL) K = (\partial F/\partial K)(K/L, m) K.$$

This implies $F(1, mL/K) \geq (\partial F/\partial K)(K/L, m)$. Letting K/L go to ∞ yields $0 = F(1, 0) \geq (\partial F/\partial K)(\infty, m) \geq 0$. The proof that $(\partial F/\partial L)(1, \infty) = 0$ is left to the reader.

2.2 Households

We consider an economy without population growth where m households work and consume. Each household i is endowed with k_{i0} units of capital at period 0 and one unit of time per period which may be divided between labor supply and leisure time. The leisure demand of agent i at time t is denoted by λ_{it} and that agent's labor supply is $l_{it} = 1 - \lambda_{it}$. This individual's capital supply and consumption demands at time t are denoted by k_{it} and c_{it}, respectively. Agents allocate their income at time t to consumption, c_{it}, and capital accumulation, denoted k_{it+1}.

The aggregate, or total, initial capital endowment is positive.

Assumption 2 $k_{i0} \geq 0$ *for* $i = 1, \ldots, m$, $\sum_i k_{i0} > 0$.

Each household maximizes a lifetime utility function which is separable over time: $\sum_{t=0}^{\infty} \beta_i^t u_i(c_{it}, \lambda_{it})$, where $\beta_i \in (0, 1)$ is agent i's discount factor.

Assumption 3 *$u_i : \mathbb{R}_+^2 \to \mathbb{R}$ is C^0, strictly increasing and concave. Without loss of generality, we assume $u_i(0,0) = 0$ for any i.*

For the proof of non-existence of bubbles, the following additional assumption is required.

Assumption 4 *(Inada conditions) The utility function is differentiable with*

$$\frac{\partial u_i}{\partial c}(0,\lambda) = \infty \text{ if } \lambda > 0 \text{ and } \frac{\partial u_i}{\partial \lambda}(c,0) = \infty \text{ if } c > 0$$

Household heterogeneity can arise in terms of endowments (k_{i0}), discounting (β_i) and per-period utility (u_i).

In any period, the household faces a budget constraint:

$$p_t \left[c_{it} + k_{it+1} - (1-\delta) k_{it}\right] \leq r_t k_{it} + w_t (1 - \lambda_{it})$$

It is known that, in economies with heterogenous discounting factors and no borrowing constraints, the more impatient agents borrow, consume more and work less in the short run than the most patient agents. Over the longer run those impatient agents consume less and work more in order to repay their debts to patient agents (see Le Van et al. (2007)). In our model, as in Becker (1980), agents are prevented from borrowing: $k_{it} \geq 0$ for $i = 1, \ldots, m$ and $t = 1, 2, \ldots$ This constraint implies the model has an incomplete market structure as no markets exist where agents can borrow against their future wage income in order to consume more today.

3 Definition of equilibrium

Let an infinite-horizon sequences of prices and quantities be denoted by:

$$\left(\mathbf{p}, \mathbf{r}, \mathbf{w}, (\mathbf{c}_i, \mathbf{k}_i, \boldsymbol{\lambda}_i)_{i=1}^m, \mathbf{K}, \mathbf{L}\right),$$

where

$$\begin{aligned}
(\mathbf{p}, \mathbf{r}, \mathbf{w}) &\equiv \left((p_t)_{t=0}^\infty, (r_t)_{t=0}^\infty, (w_t)_{t=0}^\infty\right) \in \mathbb{R}^\infty \times \mathbb{R}_+^\infty \times \mathbb{R}_+^\infty; \\
(\mathbf{c}_i, \mathbf{k}_i, \boldsymbol{\lambda}_i) &\equiv \left((c_{it})_{t=0}^\infty, (k_{it})_{t=1}^\infty, (\lambda_{it})_{t=0}^\infty\right) \in \mathbb{R}_+^\infty \times \mathbb{R}^\infty \times \mathbb{R}_+^\infty; \\
(\mathbf{K}, \mathbf{L}) &\equiv \left((K_t)_{t=0}^\infty, (L_t)_{t=0}^\infty\right) \in \mathbb{R}_+^\infty \times \mathbb{R}_+^\infty,
\end{aligned}$$

with $i = 1, \ldots, m$.

Definition 1 *A Walrasian equilibrium $\left(\bar{\mathbf{p}}, \bar{\mathbf{r}}, \bar{\mathbf{w}}, \left(\bar{\mathbf{c}}_i, \bar{\mathbf{k}}_i, \bar{\boldsymbol{\lambda}}_i\right)_{i=1}^m, \bar{\mathbf{K}}, \bar{\mathbf{L}}\right)$ satisfies the following conditions.*

(1) Price positivity: $\bar{p}_t, \bar{r}_t, \bar{w}_t > 0$ for $t = 0, 1, \ldots$

(2) Market clearing:

$$\begin{aligned}
goods &: \sum_{i=1}^{m}\left[\bar{c}_{it}+\bar{k}_{it+1}-(1-\delta)\,\bar{k}_{it}\right]=F\left(\bar{K}_t,\bar{L}_t\right);\\
capital &: \bar{K}_t=\sum_{i=1}^{m}\bar{k}_{it};\\
labor &: \bar{L}_t=\sum_{i=1}^{m}\bar{l}_{it},
\end{aligned}$$

for $t=0,1,\ldots$, *where* $l_{it}=1-\lambda_{it}$ *denotes the individual labor supply.*

(3) Optimal production plans: $\bar{p}_tF\left(\bar{K}_t,\bar{L}_t\right)-\bar{r}_t\bar{K}_t-\bar{w}_t\bar{L}_t$ *is the value of the program:* $\max\left[\bar{p}_tF\left(K_t,L_t\right)-\bar{r}_tK_t-\bar{w}_tL_t\right]$, *under the constraints* $K_t\geq 0$ *and* $L_t\geq 0$ *for* $t=0,1,\ldots$.

(4) Optimal consumption plans: $\sum_{t=0}^{\infty}\beta_i^tu_i\left(\bar{c}_{it},\bar{\lambda}_{it}\right)$ *is the value of the program:* $\max\sum_{t=0}^{\infty}\beta_i^tu_i\left(c_{it},\lambda_{it}\right)$, *under the following constraints:*

$$\begin{aligned}
\textit{budget constraint} &: \bar{p}_t\left[c_{it}+k_{it+1}-(1-\delta)\,k_{it}\right]\leq\bar{r}_tk_{it}+\bar{w}_t\left(1-\lambda_{it}\right)\\
\textit{borrowing constraint} &: k_{it+1}\geq 0\\
\textit{leisure endowment} &: 0\leq\lambda_{it}\leq 1\\
\textit{capital endowment} &: k_{i0}\geq 0 \textit{ given}
\end{aligned}$$

for $t=0,1,\ldots$

The following observation is a critical feature of our economic model.

Remark 2 *Under Assumption 1, individual and aggregate capital supplies, individual and aggregate consumption demands are uniformly bounded. Labor supply is uniformly bounded by* m.

Let A denote the common bound of the feasible consumption and capital stocks sequences.

Let us prove now the existence of a Walrasian equilibrium. The proof will be articulated in two parts. First, we consider an equilibrium with a finite horizon. Then, we let the horizon go to infinity to obtain a Walrasian equilibrium as limit of a sequence of finite-horizon economies.

4 Existence of equilibrium

Our main result is the existence of an equilibrium for this heterogeneous agent capital accumulation economy. The proof is given in two major steps. First, we prove the existence of equilibrium in finite-horizon economies. This demonstration has two parts: an existence theorem with an artificial bound on agents choice sets, and an extension of this theorem when those bounds are relaxed. The last step proves an equilibrium exists in the infinite-horizon economy. That economy is viewed as a limit of a sequence of truncated economies; the infinite horizon equilibrium prices and quantities are naturally shown to be limits of their finite horizon counterparts.

Consider a finite-horizon bounded economy which goes on for $T+1$ periods: $t = 0, \ldots, T$. Choose sufficiently large quantity bounds B_c, B_k, and so on, with:

$$
\begin{aligned}
X_i &\equiv \{(c_{i0}, \ldots, c_{iT}) : 0 \leq c_{it} \leq B_c\} = [0, B_c]^{T+1} \,; \\
Y_i &\equiv \{(k_{i1}, \ldots, k_{iT}) : 0 \leq k_{it} \leq B_k\} = [0, B_k]^{T} \,; \\
Z_i &\equiv \{(\lambda_{i0}, \ldots, \lambda_{iT}) : 0 \leq \lambda_{it} \leq 1\} = [0, 1]^{T+1} \,; \\
Y &\equiv \{(K_0, \ldots, K_T) : 0 \leq K_t \leq B_K\} = [0, B_K]^{T+1} \,; \\
Z &\equiv \{(L_0, \ldots, L_T) : 0 \leq L_t \leq B_L\} = [0, B_L]^{T+1} \,,
\end{aligned}
$$

where $mB_k < B_K$, $m < B_L$, $m\,(1-\delta)\,B_k + F(B_K, B_L) < B_c$. Recall that k_{i0} is given and that the borrowing constraint inequalities $k_{it} \geq 0$ model the imperfection in the credit market.[4]

Let $\mathcal{E}^T$ denote this **bounded economy** with technology and preferences as in Assumptions 1 to 3. Let X_i, Y_i and Z_i be the ith consumer-worker's bounded sets for consumption demand, capital supply and leisure demand respectively ($i = 1, \ldots, m$). The sets Y and Z are the bounded sets constraining the production sector's capital and labor demands at each time.

Theorem 3 *Under the Assumptions 1, 2 and 3, there exists an equilibrium* $\left(\bar{\mathbf{p}}, \bar{\mathbf{r}}, \bar{\mathbf{w}}, \left(\bar{\mathbf{c}}_h, \bar{\mathbf{k}}_h, \bar{\boldsymbol{\lambda}}_h\right)_{h=1}^{m}, \bar{\mathbf{K}}, \bar{\mathbf{L}}\right)$ *for the finite-horizon bounded economy* $\mathcal{E}^T$.

Proof. See Appendix 1. ■

The **unbounded economy**, with relaxed artificial bounds, is shown to possess an equilibrium price system and allocation.

Corollary 1 *Any equilibrium of* $\mathcal{E}^T$ *is an equilibrium for the finite-horizon unbounded economy.*

Proof. The standard argument applies given the existence of an equilibrium for the bounded economy. ■

Our main result is the following existence theorem for the infinite-horizon economy.

Theorem 4 *Under the Assumptions 1, 2 and 3, there exists an equilibrium in the infinite-horizon economy with endogenous labor supply and borrowing constraints.*

Proof. We consider a sequence of time-truncated economies and the associated equilibria. We prove that there exists a sequence of equilibria which converges, when the horizon T goes to infinity, to an equilibrium of the infinite-horizon economy. Appendix 2 contains the formal proof. ■

[4] The credit constraint might be generalized by requiring: $h_i \leq k_{it}$ with $h_i < 0$ given. This specification is left for another paper.

5 Non-existence of bubbles

There is considerable interest in whether or not a perfect foresight equilibrium capital asset price sequence is consistent with the notion of a rational pricing bubble. Some researchers points out that bubbles may occur with heterogeneous infinite-lived households facing borrowing constraints in an exchange economy (Kocherlakota (1992), Huang and Werner (2000)). We show this does *not* carry over to a model with productive capital accumulation.

Let $\left(\bar{\mathbf{p}}, \bar{\mathbf{r}}, \bar{\mathbf{w}}, \left(\bar{\mathbf{c}}_i, \bar{\mathbf{k}}_i, \bar{\boldsymbol{\lambda}}_i\right)_{i=1}^m, \bar{\mathbf{K}}, \bar{\mathbf{L}}\right)$ denote an equilibrium in the infinite horizon economy. We will take $\bar{p}_t = 1$ for any t. We know that, for any t, $\bar{r}_t > 0$ and $\bar{w}_t > 0$.

Claim 5 *Under Assumptions 1, 2, 3 and 4 (Inada), we have:*
(1) $\bar{c}_{it} > 0$ *if and only if* $\bar{\lambda}_{it} > 0$,
(2) for any i *and any* t, $\bar{c}_{it} > 0$ *and* $\bar{\lambda}_{it} > 0$,
(3) for any i *and any* t,

$$\frac{\partial u_i}{\partial c}\left(\bar{c}_{it}, \bar{\lambda}_{it}\right) \geq (1 - \delta + \bar{r}_{t+1})\,\beta_i \frac{\partial u_i}{\partial c}\left(\bar{c}_{it+1}, \bar{\lambda}_{it+1}\right)$$

and, if $\bar{k}_{it+1} > 0$, *then*

$$\frac{\partial u_i}{\partial c}\left(\bar{c}_{it}, \bar{\lambda}_{it}\right) = (1 - \delta + \bar{r}_{t+1})\,\beta_i \frac{\partial u_i}{\partial c}\left(\bar{c}_{it+1}, \bar{\lambda}_{it+1}\right),$$

(4) for any i *and any* t,

$$\bar{w}_t \frac{\partial u_i}{\partial c}\left(\bar{c}_{it}, \bar{\lambda}_{it}\right) \leq \frac{\partial u_i}{\partial \lambda}\left(\bar{c}_{it}, \bar{\lambda}_{it}\right) \tag{1}$$

and, if $\bar{\lambda}_{it} < 1$, *then*

$$\bar{w}_t \frac{\partial u_i}{\partial c}\left(\bar{c}_{it}, \bar{\lambda}_{it}\right) = \frac{\partial u_i}{\partial \lambda}\left(\bar{c}_{it}, \bar{\lambda}_{it}\right)$$

(5) Finally, $\bar{K}_t, \bar{L}_t > 0$.

Proof. See Appendix 3. ■

Now define:

$$\bar{q}_{t+1} \equiv \max_i \frac{\beta_i \left(\partial u_i / \partial c\right)\left(\bar{c}_{it+1}, \bar{\lambda}_{it+1}\right)}{\left(\partial u_i / \partial c\right)\left(\bar{c}_{it}, \bar{\lambda}_{it}\right)} = \frac{1}{1 - \delta + \bar{r}_{t+1}}$$

The ratio $\bar{q}_t$ has a natural interpretation as a market discount factor. The maximum condition on the right-hand side (above) shows that this ratio reflects the marginal rate of substitution between t and $t+1$ for the highest marginal valuation consumer.

Let

$$\bar{Q}_0 \equiv 1 \tag{2}$$

$$\bar{Q}_t \equiv \prod_{s=1}^{t} \bar{q}_s \text{ for } t > 0 \tag{3}$$

Clearly, $\bar{Q}_t = \prod_{s=1}^{t} (1-\delta+\bar{r}_s)^{-1}$ for $t > 0$. $\bar{Q}_t$ is the present value of a unit of capital of period t with focal date $t = 0$. This present value is implicitly defined via the current value prices system that arises as an equilibrium configuration in our main theorem. For any t, we obtain:

$$\bar{Q}_t = \bar{Q}_{t+1} (1-\delta+\bar{r}_{t+1}) \tag{4}$$

and, by induction, $1 = \bar{Q}_0 = \bar{Q}_T (1-\delta)^T + \sum_{t=1}^{T} \bar{Q}_t \bar{r}_t (1-\delta)^{t-1}$.

We define below the fundamental value of capital considered as a long-lived asset with focal date $t = 0$. At date 1, one unit of this asset will give back $1-\delta$ unit of capital and $\bar{r}_1$ unit of consumption good as its dividend. At period 2, $1-\delta$ unit of capital will give back $(1-\delta)^2$ unit of capital and $(1-\delta)\bar{r}_2$ as its dividend. This leads to the following definition of the **Fundamental Value** of capital:

$$\mathrm{FV} \equiv \sum_{t=1}^{\infty} \bar{Q}_t (1-\delta)^{t-1} \bar{r}_t$$

Given that the price of capital at $t = 0$ is expected, with our normalizations, to be 1 if capital is priced in an efficient market (i.e. the present value of a unit of capital is its present value of future rental rates), we say there is a bubble if $\mathrm{FV} = 1 - \lim_{T\to\infty} \bar{Q}_T (1-\delta)^T < 1$. More formally:

Definition 2 *The economy is said to experience a bubble if*

$$\lim_{T\to\infty} \bar{Q}_T (1-\delta)^T > 0$$

Otherwise ($\lim_{T\to\infty} \bar{Q}_T (1-\delta)^T = 0$)*, there is no bubble.*

The crucial question concerns the existence of bubbles in a productive economy. We show that a productive economy experiences no bubbles. The proof rests on the following lemma.

Lemma 1 *If the economy experiences a bubble, then* $\bar{r}_t$ *converges to zero.*

Proof. It is equivalent to prove that, if $\bar{r}_t$ does not converge to zero, there are no bubbles. If $\bar{r}_t$ does not converge to zero, there are, equivalently, $\rho > 0$ and a strictly increasing sequence $(t_i)_{i=1}^{\infty}$ such that $\bar{r}_{t_i} \geq \rho$ for $i = 1, 2, \ldots$ For $T > t_n$, we get

$$\bar{Q}_T (1-\delta)^T = \prod_{s=1}^{T} \frac{1-\delta}{1-\delta+\bar{r}_s} \leq \prod_{i=1}^{n} \frac{1-\delta}{1-\delta+\bar{r}_{t_i}} \leq \left(\frac{1-\delta}{1-\delta+\rho}\right)^n$$

and

$$0 \leq \limsup_{T\to\infty} \bar{Q}_T (1-\delta)^T \leq \lim_{n\to\infty} \left(\frac{1-\delta}{1-\delta+\rho}\right)^n = 0$$

■

Now, we are able to prove the No-Bubble Theorem.

Theorem 6 *Under the Assumptions 1, 2, 3 and 4, our productive economy experiences no bubble.*

Proof. See Appendix 3. ■

Remark 7 *We have for any t and any i*

$$\left(\bar{Q}_t \bar{c}_{it} + \bar{Q}_t \bar{w}_t \bar{\lambda}_{it}\right) + \bar{Q}_t \bar{k}_{it+1} = \bar{Q}_t \bar{w}_t + \bar{Q}_t \left(1 - \delta + \bar{r}_t\right) \bar{k}_{it}$$

If we consider the capital as a long-lived asset, we can consider the sequence of perfectly foreseen equilibrium wages at each time, $\bar{w}_t$, as if it is agent i's endogenously determined labor income present value or endowment. Summing from $t = 0$ to $t = T$, we get:

$$\sum_{t=0}^{T} \bar{Q}_t \left(\bar{c}_{it} + \bar{w}_t \bar{\lambda}_{it}\right) + \bar{Q}_T \bar{k}_{iT+1} = \sum_{t=0}^{T} \bar{Q}_t \bar{w}_t + \bar{Q}_0 \left(1 - \delta + \bar{r}_0\right) \bar{k}_{i0}$$

Following Huang and Werner (2000) among others, we may say that, when the interest rate is high, in the sense that $\sum_{t=0}^{T} \bar{Q}_t \bar{w}_t < \infty$, then there is no bubble. Indeed, if this property holds and if $(\bar{w}_t) \in int\ l_+^\infty$, then $\sum_{t=0}^{T} \bar{Q}_t < \infty$ which implies $\lim_t \bar{Q}_t = 0$ and we have no bubble. However, here, we do not know, without additional assumptions, that $(\bar{w}_t) \in int\ l_+^\infty$ and $\sum_{t=0}^{T} \bar{Q}_t \bar{w}_t < \infty$.[5] *Nevertheless, no bubble holds in our economy.*

6 Conclusion

We have analyzed the existence of the intertemporal equilibrium and the occurrence rational bubbles in a Ramsey model with heterogeneous agents, borrowing constraints and endogenous labor.

The assumed market incompleteness from the borrowing constraint nullifies the equivalence between the planner's and the market solution characteristic of the complete markets modeling framework. For this reason, nonstandard fixed-point arguments are needed to prove the existence of a Ramsey equilibrium. Assuming borrowing constraints are the financial market imperfection in our model economy, we adopt a three-pronged proof strategy: existence in a (1) bounded and, then, (2) an unbounded truncated economy, and, last, (3) existence in an infinite-horizon economy as limit of a sequence of truncated unbounded economies. Moreover, as a by-product of our proof we demonstrate that bubbles cannot exist despite the presence of borrowing constraints because equilibrium interest-rental rates are positive.

Therefore, this paper adds to the existing Ramsey equilibrium literature on two fronts. We provide a simple general proof of existence of an equilibrium (because of the endogenous labor supply and the weaker assumptions on the fundamentals); on the other hand, our arguments also furnish a proof that bubbles are nonexistent in a productive economy.

[5] Here, we consider the interior $intl_+^\infty$ when l^∞ is endowed with the supnorm topology.

7 Appendix 1: existence of equilibrium in a finite-horizon economy

The proof of Theorem 3 requires some ingredients which are given below.

Define a bounded price set $P \equiv \triangle^{T+1}$ with the simplex

$$\triangle \equiv \{(p, r, w) : p, r, w \geq 0,\ p + r + w = 1\}$$

Focus now on the budget constraints: $p_t \left[c_{it} + k_{it+1} - (1-\delta) k_{it}\right] \leq r_t k_{it} + w_t (1 - \lambda_{it})$ for $t = 0, \ldots, T$ with $k_{iT+1} = 0$.

Consider the budget set:

$$C_i^T (\mathbf{p}, \mathbf{r}, \mathbf{w}) \equiv \left\{ \begin{array}{c} (\mathbf{c}_i, \mathbf{k}_i, \boldsymbol{\lambda}_i) \in X_i \times Y_i \times Z_i : \\ p_t \left[c_{it} + k_{it+1} - (1-\delta) k_{it}\right] \leq r_t k_{it} + w_t (1 - \lambda_{it}) \\ t = 0, \ldots, T \end{array} \right\}$$

and its interior

$$B_i^T (\mathbf{p}, \mathbf{r}, \mathbf{w}) \equiv \left\{ \begin{array}{c} (\mathbf{c}_i, \mathbf{k}_i, \boldsymbol{\lambda}_i) \in X_i \times Y_i \times Z_i : \\ p_t \left[c_{it} + k_{it+1} - (1-\delta) k_{it}\right] < r_t k_{it} + w_t (1 - \lambda_{it}) \\ t = 0, \ldots, T \end{array} \right\}$$

We denote by $\bar{B}_i^T (\mathbf{p}, \mathbf{r}, \mathbf{w})$ the closure of $B_i^T (\mathbf{p}, \mathbf{r}, \mathbf{w})$. It is obvious that, when $B_i^T (\mathbf{p}, \mathbf{r}, \mathbf{w}) \neq \varnothing$, then $C_i^T (\mathbf{p}, \mathbf{r}, \mathbf{w}) = \bar{B}_i^T (\mathbf{p}, \mathbf{r}, \mathbf{w})$. Nonemptiness of B_i^T is crucial for the existence of demands.

The following result is very useful for our proof of existence of a Walrasian equilibrium.

Lemma 2 *Under Assumptions 1, 2 and 3, if* $w_0 > 0$ *and* $r_t + w_t > 0$*, for* $t = 1, \ldots, T$*, then the set* $B_i^T (\mathbf{p}, \mathbf{r}, \mathbf{w})$ *is nonempty.*

Proof. Take $k_{i1} > 0$ and $\lambda_{i0} < 1$ such that $p_0 k_{i1} < w_0 (1 - \lambda_{i0}) \leq r_0 k_{i0} + w_0(1 - \lambda_{i0})$. Take $k_{i2} > 0$, $\lambda_{i1} < 1$ such that $p_1 k_{i2} \leq r_1 k_{i1} + w_1 (1 - \lambda_{i1})$ and so on. ■

Observe that when $\delta = 1$, the set $B_i^T (\mathbf{p}, \mathbf{r}, \mathbf{w})$ is empty if $r_t = w_t = 0$ and $p_t = 1$ for some t and, when $k_{i0} = 0$, this set is empty if $p_0 = w_0 = 0$ and $r_0 = 1$. For that reason, at the beginning, we introduce the following sets.

Let $\varepsilon > 0$ satisfy $m (1-\delta) (B_k + \varepsilon) + F(B_K, B_L) + m\varepsilon < B_c$ and $mB_k + m\varepsilon <$

B_K. We define:

$$
\begin{aligned}
& C_i^{T\varepsilon}(\mathbf{p},\mathbf{r},\mathbf{w}) \\
\equiv & \left\{ \begin{array}{c} (\mathbf{c}_i, \mathbf{k}_i, \boldsymbol{\lambda}_i) \in X_i \times Y_i \times Z_i : \\ p_t\left[c_{it} + k_{it+1} - (1-\delta)\, k_{it}\right] \leq p_t \varepsilon + p_t (1-\delta)\, \varepsilon + r_t (k_{it} + \varepsilon) + w_t (1-\lambda_{it}) \\ t = 0, \ldots, T \end{array} \right\} \\
& B_i^{T\varepsilon}(\mathbf{p},\mathbf{r},\mathbf{w}) \\
\equiv & \left\{ \begin{array}{c} (\mathbf{c}_i, \mathbf{k}_i, \boldsymbol{\lambda}_i) \in X_i \times Y_i \times Z_i : \\ p_t\left[c_{it} + k_{it+1} - (1-\delta)\, k_{it}\right] < p_t \varepsilon + p_t (1-\delta)\, \varepsilon + r_t (k_{it} + \varepsilon) + w_t (1-\lambda_{it}) \\ t = 0, \ldots, T \end{array} \right\}
\end{aligned}
$$

Remark 8 *ε represents a perturbation of the fundamental economy. In the ε-economy, the firm uses ε as an additional input. ε and k_{it} are the same capital good and experiences the same depreciation during the production process. When the process ends, they are resold at the same price p_t to earn $p_t(1-\delta)(\varepsilon + k_{it})$.*

The next lemma plays a critical role. The perturbation of the fundamental economy yields that each agent has a positive income at each time. As in standard competitive equilibrium proofs for finite exchange and/or production economies, this is required to show all agents are, in fact, finding their utility maximizing bundles subject to a budget constraint (the cheaper point property in standard equilibrium theories).

Lemma 3 *Under Assumptions 1, 2 and 3, the set $B_i^{T\varepsilon}(\mathbf{p},\mathbf{r},\mathbf{w})$ is nonempty and $C_i^{T\varepsilon}(\mathbf{p},\mathbf{r},\mathbf{w}) = \bar{B}_i^{T\varepsilon}(\mathbf{p},\mathbf{r},\mathbf{w})$. Moreover the correspondence $B_i^{T\varepsilon}$ is lower semicontinuous (lsc).*

Proof. Take $\lambda_{it} = \eta < 1$, $k_{it+1} = 0$, $c_{it} = 0$. Then, $p_t\varepsilon + p_t(1-\delta)\varepsilon + r_t(\varepsilon + k_{it}) + w_t(1-\lambda_{it}) > 0$ for any $(p_t, r_t, w_t) \in \triangle$ and, hence,

$$\alpha \equiv \min_{(p,r,w)\in\triangle} \left[p\varepsilon + p(1-\delta)\varepsilon + r\varepsilon + w(1-\eta)\right] > 0$$

We have

$$
\begin{aligned}
p_t\left[c_{it} + k_{it+1} - (1-\delta)\, k_{it}\right] &= -p_t(1-\delta)\, k_{it} \leq 0 < \alpha \\
&\leq p_t\varepsilon + p_t(1-\delta)\,\varepsilon + r_t(\varepsilon + k_{it}) + w_t(1-\lambda_{it})
\end{aligned}
$$

for any t. So, $B_i^{T\varepsilon}(\mathbf{p},\mathbf{r},\mathbf{w})$ is non empty. The proof of the remaining assertions is easy. ■

The following lemma is crucial for the proof of Theorem 3.

Lemma 4 *Under the Assumptions 1, 2 and 3, there exists*

$$\left(\bar{\mathbf{p}}(\varepsilon), \bar{\mathbf{r}}(\varepsilon), \bar{\mathbf{w}}(\varepsilon), \left(\bar{\mathbf{c}}_i(\varepsilon), \bar{\mathbf{k}}_i(\varepsilon), \bar{\boldsymbol{\lambda}}_i(\varepsilon)\right)_{i=1}^m, \bar{\mathbf{K}}(\varepsilon), \bar{\mathbf{L}}(\varepsilon)\right)$$

in the finite-horizon bounded economy $\mathcal{E}^T$ which satisfies:

(1) price positivity: $\bar{p}_t(\varepsilon), \bar{r}_t(\varepsilon), \bar{w}_t(\varepsilon) > 0$ *for* $t = 0, \ldots, T$,

(2) market clearing:

$$\begin{aligned} goods &: \sum_{i=1}^{m} \left[\bar{c}_{it}(\varepsilon) + \bar{k}_{it+1}(\varepsilon) - (1-\delta)\bar{k}_{it}(\varepsilon)\right] = F\left(\bar{K}_t(\varepsilon), \bar{L}_t(\varepsilon)\right) + m\varepsilon + m(1-\delta)\varepsilon \\ capital &: \bar{K}_t(\varepsilon) = \sum_{i=1}^{m} \bar{k}_{it}(\varepsilon) + m\varepsilon \\ labor &: \bar{L}_t(\varepsilon) = \sum_{i=1}^{m} \bar{l}_{it}(\varepsilon) \end{aligned}$$

for $t = 0, \ldots, T$, *where* $\bar{l}_{it}(\varepsilon) = 1 - \bar{\lambda}_{it}(\varepsilon)$ *denotes the individual labor supply.*

(3) Optimal production plans: $\bar{p}_t(\varepsilon)F\left(\bar{K}_t(\varepsilon), \bar{L}_t(\varepsilon)\right) - \bar{r}_t(\varepsilon)\bar{K}_t(\varepsilon) - \bar{w}_t\bar{L}_t(\varepsilon)$ *is the value of the program:* $\max\left[\bar{p}_t(\varepsilon)F(K_t, L_t) - \bar{r}_t(\varepsilon)K_t - \bar{w}_t(\varepsilon)L_t\right]$, *under the constraints* $\bar{\mathbf{K}} \in Y$ *and* $\bar{\mathbf{L}} \in Z$ *for* $t = 0, \ldots, T$. *Moreover,*

$$\bar{p}_t(\varepsilon)F\left(\bar{K}_t(\varepsilon), \bar{L}_t(\varepsilon)\right) - \bar{r}_t(\varepsilon)\bar{K}_t(\varepsilon) - \bar{w}_t\bar{L}_t(\varepsilon) = 0$$

(4) Optimal consumption plans: $\sum_{t=0}^{\infty} \beta_i^t u_i\left(\bar{c}_{it}(\varepsilon), \bar{\lambda}_{it}(\varepsilon)\right)$ *is the value of the program:* $\max \sum_{t=0}^{\infty} \beta_i^t u_i(c_{it}, \lambda_{it})$, *under the following constraints:*

$$\begin{aligned} &\bar{p}_t(\varepsilon)(c_{it} + k_{it+1}) \leq \bar{p}_t(\varepsilon)\varepsilon + \left[\bar{p}_t(\varepsilon)(1-\delta) + \bar{r}_t(\varepsilon)\right](k_{it} + \varepsilon) + \bar{w}_t(\varepsilon)(1 - \lambda_{it}) \\ &\bar{c}_i \in X_i,\ \bar{k}_i \in Y_i,\ \bar{\lambda}_{it} \in [0,1], k_{i0} \geq 0 \text{ given} \end{aligned}$$

for $t = 0, \ldots, T$.

Proof. In the spirit of Florenzano (1999), we introduce the reaction correspondences $\varphi_i\left(\mathbf{p}, \mathbf{r}, \mathbf{w}, (\mathbf{c}_h, \mathbf{k}_h, \boldsymbol{\lambda}_h)_{h=1}^m, \mathbf{K}, \mathbf{L}\right)$, $i = 0, \ldots, m+1$ defined on $P \times \left[\times_{h=1}^m (X_h \times Y_h \times Z_h)\right] \times Y \times Z$, where $i = 0$ denotes an "additional" agent, $i = 1, \ldots, m$ the consumers, and $i = m+1$ the firm. These correspondences are defined as follows.

Agent $i = 0$ (the "additional" agent):

$$\begin{aligned} &\varphi_0\left(\mathbf{p}, \mathbf{r}, \mathbf{w}, (\mathbf{c}_h, \mathbf{k}_h, \boldsymbol{\lambda}_h)_{h=1}^m, \mathbf{K}, \mathbf{L}\right) \equiv \\ &\left\{ \begin{array}{c} (\tilde{\mathbf{p}}, \tilde{\mathbf{r}}, \tilde{\mathbf{w}}) \in P : \\ \sum_{t=0}^{T} (\tilde{p}_t - p_t)\left(\sum_i \left[c_{it} + k_{it+1} - (1-\delta)k_{it}\right] - m\varepsilon - m(1-\delta)\varepsilon - F(K_t, L_t)\right) \\ + \sum_{t=0}^{T} (\tilde{r}_t - r_t)\left(K_t - m\varepsilon - \sum_{i=1}^m k_{it}\right) \\ + \sum_{t=0}^{T} (\tilde{w}_t - w_t)\left(L_t - m + \sum_{i=1}^m \lambda_{it}\right) > 0 \end{array} \right\} \end{aligned} \tag{5}$$

Agents $i = 1, \ldots, m$ (consumers-workers):

$$\begin{aligned} &\varphi_i\left(\mathbf{p}, \mathbf{r}, \mathbf{w}, (\mathbf{c}_h, \mathbf{k}_h, \boldsymbol{\lambda}_h)_{h=1}^m, \mathbf{K}, \mathbf{L}\right) \equiv \\ &\left\{ \begin{array}{c} B_i^{T\varepsilon}(\mathbf{p}, \mathbf{r}, \mathbf{w}) \text{ if } (\mathbf{c}_i, \mathbf{k}_i, \boldsymbol{\lambda}_i) \notin C_i^{T\varepsilon}(\mathbf{p}, \mathbf{r}, \mathbf{w}) \\ B_i^{T\varepsilon}(\mathbf{p}, \mathbf{r}, \mathbf{w}) \cap \left[P_i(\mathbf{c}_i, \boldsymbol{\lambda}_i) \times Y_i\right] \text{ if } (\mathbf{c}_i, \mathbf{k}_i, \boldsymbol{\lambda}_i) \in C_i^{T\varepsilon}(\mathbf{p}, \mathbf{r}, \mathbf{w}) \end{array} \right\} \end{aligned}$$

where P_i is the ith agent's set of strictly preferred allocations: $P_i(\mathbf{c}_i, \boldsymbol{\lambda}_i) \equiv \left\{ \left(\tilde{\mathbf{c}}_i, \tilde{\boldsymbol{\lambda}}_i \right) : \sum_{t=0}^{T} \beta_i^t u_i \left(\tilde{c}_{it}, \tilde{\lambda}_{it} \right) > \sum_{t=0}^{T} \beta_i^t u_i (c_{it}, \lambda_{it}) \right\}$.

Agent $i = m+1$ (the firm):

$$\varphi_{m+1}\left(\mathbf{p}, \mathbf{r}, \mathbf{w}, (\mathbf{c}_h, \mathbf{k}_h, \boldsymbol{\lambda}_h)_{h=1}^m, \mathbf{K}, \mathbf{L}\right) \equiv \left\{ \begin{array}{c} \left(\tilde{\mathbf{K}}, \tilde{\mathbf{L}}\right) \in Y \times Z : \\ \sum_{t=0}^{T} \left[p_t F\left(\tilde{K}_t, \tilde{L}_t\right) - r_t \tilde{K}_t - w_t \tilde{L}_t \right] \\ > \sum_{t=0}^{T} \left[p_t F(K_t, L_t) - r_t K_t - w_t L_t \right] \end{array} \right\} \tag{6}$$

We observe that $\varphi_i : \Phi \to 2^{\Phi_i}$ where

$$\begin{aligned} \Phi &\equiv \Phi_0 \times \ldots \times \Phi_{m+1} \\ \Phi_0 &\equiv P \\ \Phi_i &\equiv X_i \times Y_i \times Z_i,\ i = 1, \ldots, m \\ \Phi_{m+1} &\equiv Y \times Z \end{aligned}$$

and 2^{Φ_i} denotes the set of subsets of Φ_i.

φ_i is a lower semicontinuous convex-valued correspondence for $i = 0, \ldots, m+1$.

Let us simplify the notation

$$\begin{aligned} \mathbf{v} &\equiv \left(\mathbf{p}, \mathbf{r}, \mathbf{w}, (\mathbf{c}_i, \mathbf{k}_i, \boldsymbol{\lambda}_i)_{i=1}^m, \mathbf{K}, \mathbf{L}\right) \\ \mathbf{v}_0 &\equiv (\mathbf{p}, \mathbf{r}, \mathbf{w}) \\ \mathbf{v}_i &\equiv (\mathbf{c}_i, \mathbf{k}_i, \boldsymbol{\lambda}_i) \text{ for } i = 1, \ldots, m \\ \mathbf{v}_{m+1} &\equiv (\mathbf{K}, \mathbf{L}) \end{aligned}$$

We observe the following.

(1) By definition of φ_0 (the inequality in (5) is strict): $(\mathbf{p}, \mathbf{r}, \mathbf{w}) \notin \varphi_0(\mathbf{v})$.

(2) $(\mathbf{c}_i, \mathbf{k}_i, \boldsymbol{\lambda}_i) \notin P_i(\mathbf{c}_i, \boldsymbol{\lambda}_i) \times Y_i$ implies that $(\mathbf{c}_i, \mathbf{k}_i, \boldsymbol{\lambda}_i) \notin \varphi_i(\mathbf{v})$ for $i = 1, \ldots, m$.

(3) By definition of φ_{m+1} (the inequality in (6) is also strict): $(\mathbf{K}, \mathbf{L}) \notin \varphi_{m+1}(\mathbf{v})$.

Then, for $i = 0, \ldots, m+1$, $\mathbf{v}_i \notin \varphi_i(\mathbf{v})$.

Apply Gale and Mas-Colell (1975) fixed-point theorem. There exists $\bar{\mathbf{v}} \in \Phi$ such that $\varphi_i(\bar{\mathbf{v}}) = \varnothing$ for $i = 0, \ldots, m+1$, that is, there exists $\bar{\mathbf{v}} \in \Phi$ such that the following holds.

Focus on "agent" $i = 0$. For every $(\mathbf{p}, \mathbf{r}, \mathbf{w}) \in P$,

$$\begin{aligned} & \sum_{t=0}^{T} (p_t - \bar{p}_t) \left(\sum_{i=1}^{m} \left[\bar{c}_{it} + \bar{k}_{it+1} - (1-\delta)\bar{k}_{it} \right] - m\varepsilon - m(1-\delta)\varepsilon - F\left(\bar{K}_t, \bar{L}_t\right) \right) \\ & + \sum_{t=0}^{T} (r_t - \bar{r}_t) \left(\bar{K}_t - m\varepsilon - \sum_{i=1}^{m} \bar{k}_{it} \right) + \sum_{t=0}^{T} (w_t - \bar{w}_t) \left(\bar{L}_t - m + \sum_{i=1}^{m} \bar{\lambda}_{it} \right) \\ \leq\ & 0 \end{aligned} \tag{7}$$

Consider $i=1,\ldots,m$. $\left(\bar{\mathbf{c}}_i,\bar{\mathbf{k}}_i,\bar{\boldsymbol{\lambda}}_i\right)\in C_i^{T\varepsilon}\left(\bar{\mathbf{p}},\bar{\mathbf{r}},\bar{\mathbf{w}}\right)$ and $B_i^{T\varepsilon}\left(\bar{\mathbf{p}},\bar{\mathbf{r}},\bar{\mathbf{w}}\right)\cap\left[P_i\left(\bar{\mathbf{c}}_i,\bar{\boldsymbol{\lambda}}_i\right)\times Y_i\right]=\varnothing$ for $i=1,\ldots,m$. Then, for $i=1,\ldots,m$, $\left(\mathbf{c}_i,\mathbf{k}_i,\boldsymbol{\lambda}_i\right)\in C_i^{T\varepsilon}\left(\bar{\mathbf{p}},\bar{\mathbf{r}},\bar{\mathbf{w}}\right)=\overline{B}_i^{T\varepsilon}\left(\bar{\mathbf{p}},\bar{\mathbf{r}},\bar{\mathbf{w}}\right)$ implies

$$\sum_{t=0}^{T}\beta_i^t u_i\left(c_{it},\lambda_{it}\right)\leq\sum_{t=0}^{T}\beta_i^t u_i\left(\bar{c}_{it},\bar{\lambda}_{it}\right)\tag{8}$$

Focus on the firm $i=m+1$. For $t=0,\ldots,T$ and for every $(\mathbf{K},\mathbf{L})\in Y\times Z$, we have $\sum_{t=0}^{T}\left[\bar{p}_tF\left(K_t,L_t\right)-\bar{r}_tK_t-\bar{w}_tL_t\right]\leq\sum_{t=0}^{T}\left[\bar{p}_tF\left(\bar{K}_t,\bar{L}_t\right)-\bar{r}_t\bar{K}_t-\bar{w}_t\bar{L}_t\right]$.

This is possible if and only if

$$\bar{p}_tF\left(K_t,L_t\right)-\bar{r}_tK_t-\bar{w}_tL_t\leq\bar{p}_tF\left(\bar{K}_t,\bar{L}_t\right)-\bar{r}_t\bar{K}_t-\bar{w}_t\bar{L}_t\tag{9}$$

for any t. In particular, the equilibrium profit is nonnegative.

$$\bar{p}_tF\left(\bar{K}_t,\bar{L}_t\right)-\bar{r}_t\bar{K}_t-\bar{w}_t\bar{L}_t\geq 0\tag{10}$$

Let us show that $\bar{p}_t>0$.

First, we have from the budget constraints:

$$\begin{aligned}&\bar{p}_t\sum_i\left[\bar{c}_{it}+\bar{k}_{it+1}-(1-\delta)\bar{k}_{it}\right]\\ \leq\quad& m\bar{p}_t\varepsilon+m\bar{p}_t(1-\delta)\varepsilon+m\bar{r}_t\varepsilon+\bar{r}_t\sum_i\bar{k}_{it}+\bar{w}_t\sum_i(1-\bar{\lambda}_{it})\end{aligned}$$

Combining with (10), we get

$$\begin{aligned}0\quad\geq\quad&\bar{p}_t\left(\sum_i\left[\bar{c}_{it}+\bar{k}_{it+1}-(1-\delta)\bar{k}_{it}\right]-F\left(\bar{K}_t,\bar{L}_t\right)-m(1-\delta)\varepsilon-m\varepsilon\right)\\&+\bar{r}_t\left(\bar{K}_t-m\varepsilon-\sum_i\bar{k}_{it}\right)+\bar{w}_t\left(\bar{L}_t-\sum_i\left(1-\bar{\lambda}_{it}\right)\right)\end{aligned}\tag{11}$$

Combining (7) with (11), we find

$$\begin{aligned}0\quad\geq\quad&\sum_{t=0}^{T}p_t\left(\sum_i\left[\bar{c}_{it}+\bar{k}_{it+1}-(1-\delta)\bar{k}_{it}\right]-F\left(\bar{K}_t,\bar{L}_t\right)-m(1-\delta)\varepsilon-m\varepsilon\right)\\&+\sum_{t=0}^{T}r_t\left(\bar{K}_t-m\varepsilon-\sum_i\bar{k}_{it}\right)+\sum_{t=0}^{T}w_t\left(\bar{L}_t-\sum_i\left(1-\bar{\lambda}_{it}\right)\right)\end{aligned}\tag{12}$$

and, noticing that (12) holds for any $(\mathbf{p},\mathbf{r},\mathbf{w})\in P$,

$$\bar{K}_t-m\varepsilon-\sum_i\bar{k}_{it}\quad\leq\quad 0\tag{13}$$

$$\bar{L}_t-\sum_i\left(1-\bar{\lambda}_{it}\right)\quad\leq\quad 0\tag{14}$$

$$\sum_i\left[\bar{c}_{it}+\bar{k}_{it+1}-(1-\delta)\bar{k}_{it}\right]-F\left(\bar{K}_t,\bar{L}_t\right)-m(1-\delta)\varepsilon-m\varepsilon\quad\leq\quad 0\tag{15}$$

Observe that (15) implies

$$\sum_i \bar{c}_{it} \le (1-\delta)\, mB_k + F(B_K, B_L) + m(1-\delta)\varepsilon + m\varepsilon < B_c \tag{16}$$

Suppose $\bar{p}_t = 0$. From the consumers' problem, we obtain $\bar{c}_{it} = B_c$ and $\bar{\lambda}_{it} = 1$ for any i. That is a contradiction with (16). Hence, $\bar{p}_t > 0$.

We want to prove now that, for any t,

$$\bar{p}_t F\left(\bar{K}_t, \bar{L}_t\right) - \bar{r}_t \bar{K}_t - \bar{w}_t \bar{L}_t = 0 \tag{17}$$

and $\bar{r}_t > 0$, $\bar{w}_t > 0$.

From (13) and (14), we have $\bar{K}_t \le m\varepsilon + mB_k < B_K$ and $\bar{L}_t \le m < B_L$. Suppose $\bar{p}_t F\left(\bar{K}_t, \bar{L}_t\right) - \bar{r}_t \bar{K}_t - \bar{w}_t \bar{L}_t = \pi > 0$. Choose $\mu > 1$ such that $\mu \bar{K}_t < B_K$ and $\mu \bar{L}_t < B_L$. We have

$$\bar{p}_t F\left(\mu\bar{K}_t, \mu\bar{L}_t\right) - \bar{r}_t \mu\bar{K}_t - \bar{w}_t \mu\bar{L}_t = \mu\pi > \pi = \bar{p}_t F\left(\bar{K}_t, \bar{L}_t\right) - \bar{r}_t \bar{K}_t - \bar{w}_t \bar{L}_t$$

which is a contradiction to (9).

Assume $\bar{r}_t = 0$. Then, we have $0 \ge \bar{p}_t F(K, L) - \bar{w}_t L$ for any $(K, L) \in Y \times Z$. Take $0 < K < B_K$ and $0 < L < B_L$. We obtain $0 \ge L\left[\bar{p}_t F(K/L, 1) - \bar{w}_t\right]$. Since $\bar{p}_t > 0$ and $\lim_{L\to 0} F(K/L, 1) = \infty$, we have $\left[\bar{p}_t F(K/L, 1) - \bar{w}_t\right] > 0$ when L is sufficiently close to 0, leading to a contradiction.

The proof that $\bar{w}_t > 0$ is similar.

Let us show now that

$$\begin{aligned} \bar{K}_t - m\varepsilon - \sum_i \bar{k}_{it} &= 0 \\ \bar{L}_t - \sum_i \left(1 - \bar{\lambda}_{it}\right) &= 0 \\ \sum_i \left[\bar{c}_{it} + \bar{k}_{it+1} - (1-\delta)\,\bar{k}_{it}\right] - F\left(\bar{K}_t, \bar{L}_t\right) - m(1-\delta)\varepsilon - m\varepsilon &= 0 \end{aligned}$$

Since $\bar{p}_t > 0$ the budget constraints bind. Combining with (13), (14), (15) and (17), we obtain

$$\begin{aligned} & m\bar{p}_t\varepsilon + m\bar{p}_t(1-\delta)\varepsilon + m\bar{r}_t\varepsilon + \bar{r}_t \sum_i \bar{k}_{it} + \bar{w}_t \sum_i \left(1 - \bar{\lambda}_{it}\right) \\ = \;& \bar{p}_t \sum_i \left[\bar{c}_{it} + \bar{k}_{it+1} - (1-\delta)\,\bar{k}_{it}\right] \\ \le \;& \bar{p}_t \left[F\left(\bar{K}_t, \bar{L}_t\right) + m(1-\delta)\varepsilon + m\varepsilon\right] \\ = \;& \bar{r}_t \bar{K}_t + \bar{w}_t \bar{L}_t + \bar{p}_t m(1-\delta)\varepsilon + \bar{p}_t m\varepsilon \\ \le \;& m\bar{p}_t\varepsilon + m\bar{p}_t(1-\delta)\varepsilon + m\bar{r}_t\varepsilon + \bar{r}_t \sum_i \bar{k}_{it} + \bar{w}_t \sum_i \left(1 - \bar{\lambda}_{it}\right) \end{aligned}$$

Hence

$$\bar{p}_t \sum_i \left[\bar{c}_{it} + \bar{k}_{it+1} - (1-\delta)\,\bar{k}_{it}\right] = \bar{p}_t \left[F\left(\bar{K}_t, \bar{L}_t\right) + m(1-\delta)\varepsilon + m\varepsilon\right]$$

and

$$\bar{r}_t\left(\bar{K}_t - m\varepsilon - \sum_i \bar{k}_{it}\right) + \bar{w}_t\left[\bar{L}_t - \sum_i\left(1-\bar{\lambda}_{it}\right)\right] = 0 \tag{18}$$

Since $\bar{p}_t > 0$, we have

$$\sum_i\left[\bar{c}_{it} + \bar{k}_{it+1} - (1-\delta)\,\bar{k}_{it}\right] - F\left(\bar{K}_t, \bar{L}_t\right) - m\,(1-\delta)\,\varepsilon - m\varepsilon = 0$$

Since $\bar{r}_t > 0$, $\bar{w}_t > 0$ and (18) holds, inequalities (13) and (14) become

$$\bar{K}_t - m\varepsilon - \sum_i \bar{k}_{it} = 0 \text{ and } \bar{L}_t - \sum_i\left(1-\bar{\lambda}_{it}\right) = 0$$

The proof of Lemma 4 is now complete. ■

Proof of Theorem 3

Proof. Keeping in mind these results, we now prove Theorem 3. We let ε converge to 0. We denote the allocations and the prices obtained in Lemma 4 by

$$\left(\bar{\mathbf{p}}(\varepsilon), \bar{\mathbf{r}}(\varepsilon), \bar{\mathbf{w}}(\varepsilon), \left(\bar{\mathbf{c}}_i(\varepsilon), \bar{\mathbf{k}}_i(\varepsilon), \bar{\boldsymbol{\lambda}}_i(\varepsilon)\right)_{i=1}^m, \bar{\mathbf{K}}(\varepsilon), \bar{\mathbf{L}}(\varepsilon)\right)$$

We recall that, for any t, $\bar{p}_t(\varepsilon) + \bar{r}_t(\varepsilon) + \bar{w}_t(\varepsilon) = 1$. Denote

$$\begin{aligned} &\left(\bar{\mathbf{p}}, \bar{\mathbf{r}}, \bar{\mathbf{w}}, \left(\bar{\mathbf{c}}_i, \bar{\mathbf{k}}_i, \bar{\boldsymbol{\lambda}}_i\right)_{i=1}^m, \bar{\mathbf{K}}, \bar{\mathbf{L}}\right) \\ \equiv\ &\lim_{\varepsilon\to 0}\left(\bar{\mathbf{p}}(\varepsilon), \bar{\mathbf{r}}(\varepsilon), \bar{\mathbf{w}}(\varepsilon), \left(\bar{\mathbf{c}}_i(\varepsilon), \bar{\mathbf{k}}_i(\varepsilon), \bar{\boldsymbol{\lambda}}_i(\varepsilon)\right)_{i=1}^m, \bar{\mathbf{K}}(\varepsilon), \bar{\mathbf{L}}(\varepsilon)\right)\end{aligned}$$

For any t and any $(K_t, L_t) \in Y \times Z$, we have

$$0 = \bar{p}_t F\left(\bar{K}_t, \bar{L}_t\right) - \bar{r}_t\bar{K}_t - \bar{w}_t\bar{L}_t \geq \bar{p}_t F(K_t, L_t) - \bar{r}_t K_t - \bar{w}_t L_t$$

$\bar{K}_0 = \sum_i k_{i0} > 0$ and $\bar{L}_0 \leq m < B_L$.

Let us show that $\bar{w}_0 > 0$ and $\bar{r}_t + \bar{w}_t > 0$, for $t = 1, \ldots, T$.

If $\bar{w}_0 = 0$ and $\bar{p}_0 > 0$ we have

$$0 = \bar{p}_0 F\left(\bar{K}_0, \bar{L}_0\right) - \bar{r}_0\bar{K}_0 \geq \bar{p}_0 F(K_0, L_0) - \bar{r}_0 K_0 \tag{19}$$

for any $(K_0, L_0) \in Y \times Z$. Take $K_0 = \epsilon > 0$ and $L_0 = L \in (0, B_L)$. We obtain a contradiction:

$$0 \geq \bar{p}_0 F(\epsilon, L) - \bar{r}_0\epsilon = \left[\bar{p}_0 F(1, L/\epsilon) - \bar{r}_0\right]\epsilon > 0$$

when ϵ goes to zero since $F(1, \infty) = \infty$. Hence $\bar{w}_0 = 0$ implies $\bar{p}_0 = 0$ and $\bar{r}_0 = 1$ (because of the unit simplex). However, from (19), $\bar{K}_0 = 0$ which is impossible.

Assume that $\bar{w}_t = 0$ for some $t \geq 1$. The same argument previously used implies $\bar{p}_t = 0$ and $\bar{r}_t = 1$.

Assume $\bar{r}_t = 0$ for some $t \geq 1$ and $\bar{p}_t > 0$. Then,

$$0 \geq \bar{p}_t F(K_t, L_t) - \bar{w}_t L_t$$

for any $(K_t, L_t) \in Y \times Z$. Take $K_t = K \in (0, B_K)$ and $L_t = \epsilon > 0$. We obtain a contradiction

$$0 \geq \bar{p}_t F(K, \epsilon) - \bar{w}_t \epsilon = [\bar{p}_t F(K/\epsilon, 1) - \bar{w}_t] \epsilon > 0$$

when ϵ becomes sufficiently close to zero, since $F(\infty, 1) = \infty$. Hence $\bar{r}_t = 0$ implies $\bar{p}_t = 0$ and $\bar{w}_t = 1$.

From Lemma 2, the set $B_i^T(\bar{\mathbf{p}}, \bar{\mathbf{r}}, \bar{\mathbf{w}})$ is nonempty. Taking $(\mathbf{c}_i, \mathbf{k}_i, \boldsymbol{\lambda}_i) \in B_i^T(\bar{\mathbf{p}}, \bar{\mathbf{r}}, \bar{\mathbf{w}})$, we have

$$\bar{p}_t \left[\bar{c}_{it} + \bar{k}_{it+1} - (1-\delta)\bar{k}_{it}\right] < \bar{r}_t \bar{k}_{it} + \bar{w}_t \left(1 - \bar{\lambda}_{it}\right)$$

for any t. There exists $\bar{\varepsilon} > 0$ such that for any $\varepsilon < \bar{\varepsilon}$, we have for any t,

$$\begin{aligned} \bar{p}_t(\varepsilon)\left[\bar{c}_{it}(\varepsilon) + \bar{k}_{it+1}(\varepsilon) - (1-\delta)\bar{k}_{it}(\varepsilon)\right] &< \bar{r}_t(\varepsilon)\bar{k}_{it}(\varepsilon) + \bar{w}_t(\varepsilon)\left[1 - \bar{\lambda}_{it}(\varepsilon)\right] \\ &< \bar{p}_t(\varepsilon)\varepsilon + \bar{p}_t(\varepsilon)(1-\delta)\varepsilon + \bar{r}_t(\varepsilon)\varepsilon \\ &\quad + \bar{r}_t(\varepsilon)\bar{k}_{it}(\varepsilon) + \bar{w}_t(\varepsilon)\left[1 - \bar{\lambda}_{it}(\varepsilon)\right] \end{aligned}$$

Therefore, $\sum_{t=0}^T \beta_i^t u_i(c_{it}, \lambda_{it}) \leq \sum_{t=0}^T \beta_i^t u_i\left(\bar{c}_{it}(\varepsilon), \bar{\lambda}_{it}(\varepsilon)\right)$. Let ε go to 0. Then $\sum_{t=0}^T \beta_i^t u_i(c_{it}, \lambda_{it}) \leq \sum_{t=0}^T \beta_i^t u_i\left(\bar{c}_{it}, \bar{\lambda}_{it}\right)$. Let $(\mathbf{c}_i, \mathbf{k}_i, \boldsymbol{\lambda}_i) \in C_i^T(\bar{\mathbf{p}}, \bar{\mathbf{r}}, \bar{\mathbf{w}})$. There exists a sequence $(\mathbf{c}_i^n, \mathbf{k}_i^n, \boldsymbol{\lambda}_i^n)_{n=1}^\infty \subset B_i^T(\bar{\mathbf{p}}, \bar{\mathbf{r}}, \bar{\mathbf{w}})$ which converges to $(\mathbf{c}_i, \mathbf{k}_i, \boldsymbol{\lambda}_i)$. For any n, we have

$$\sum_{t=0}^T \beta_i^t u_i(c_{it}^n, \lambda_{it}^n) \leq \sum_{t=0}^T \beta_i^t u_i\left(\bar{c}_{it}, \bar{\lambda}_{it}\right)$$

Let n go to ∞. Then $\sum_{t=0}^T \beta_i^t u_i(c_{it}, \lambda_{it}) \leq \sum_{t=0}^T \beta_i^t u_i\left(\bar{c}_{it}, \bar{\lambda}_{it}\right)$. The prices $(\bar{p}_t, \bar{w}_t)$ are strictly positive since the utility functions u_i are strictly increasing. The price $\bar{r}_t$ is strictly positive since we have proved above that $\bar{r}_t = 0$ implies $\bar{p}_t = 0$.

It is easy to check that the list $\left(\bar{\mathbf{p}}, \bar{\mathbf{r}}, \bar{\mathbf{w}}, \left(\bar{\mathbf{c}}_i, \bar{\mathbf{k}}_i, \bar{\boldsymbol{\lambda}}_i\right)_{i=1}^m, \bar{\mathbf{K}}, \bar{\mathbf{L}}\right)$ is an equilibrium for the $T+1$-horizon economy. ■

8 Appendix 2: existence of equilibrium in an infinite-horizon economy

Proof of Theorem 4

Proof. We will denote by

$$\left(\bar{\mathbf{p}}(T), \bar{\mathbf{r}}(T), \bar{\mathbf{w}}(T), \left(\bar{\mathbf{c}}_i(T), \bar{\mathbf{k}}_i(T), \bar{\boldsymbol{\lambda}}_i(T)\right)_{i=1}^m, \bar{\mathbf{K}}(T), \bar{\mathbf{L}}(T)\right)$$

an equilibrium for the $T+1$-horizon economy and

$$\begin{aligned} &\left(\hat{\mathbf{p}}, \hat{\mathbf{r}}, \hat{\mathbf{w}}, \left(\hat{\mathbf{c}}_i, \hat{\mathbf{k}}_i, \hat{\boldsymbol{\lambda}}_i\right)_{i=1}^m, \hat{\mathbf{K}}, \hat{\mathbf{L}}\right) \\ \equiv\ &\lim_{T\to\infty}\left(\bar{\mathbf{p}}(T), \bar{\mathbf{r}}(T), \bar{\mathbf{w}}(T), \left(\bar{\mathbf{c}}_i(T), \bar{\mathbf{k}}_i(T), \bar{\boldsymbol{\lambda}}_i(T)\right)_{i=1}^m, \bar{\mathbf{K}}(T), \bar{\mathbf{L}}(T)\right) \end{aligned}$$

for the product topology.

We claim that $\hat{w}_0 > 0, \hat{w}_t + \hat{r}_t > 0$ for any $t \geq 1$. Indeed, we always have

$$0 = \hat{p}_0 F\left(\hat{K}_0, \hat{L}_0\right) - \hat{w}_0 \hat{L}_0 - \hat{r}_0 \hat{K}_0 \geq \hat{p}_0 F(K, L) - \hat{w}_0 L - \hat{r}_0 K$$

for any $(K, L) \in \mathbb{R}_+^2$.

If $\hat{w}_0 = 0$ and $\hat{p}_0 > 0$, then $0 \geq \hat{p}_0 F(K, L) - \hat{r}_0 K$ for any $(K, L) \in \mathbb{R}_+^2$. Take $K > 0$ and let L go to infinity to get a contradiction. Hence $\hat{w}_0 = 0$ implies $\hat{p}_0 = 0$ and $\hat{r}_0 = 1$. In this case we will have $\hat{K}_0 = 0$ which is impossible since $\hat{K}_0 = \sum_i k_{i0} > 0$. We conclude that $\hat{w}_0 > 0$.

Assume $\hat{w}_t = 0$ and $\hat{p}_t > 0$ for some $t \geq 1$. Then $0 \geq \hat{p}_t F(K, L) - \hat{r}_t K$ for any $(K, L) \in \mathbb{R}_+^2$. Take $K > 0$ and let L go to infinity to have a contradiction. Now assume $\hat{r}_t = 0$ and $\hat{p}_t > 0$ for some $t \geq 1$. Then $0 \geq \hat{p}_t F(K, L) - \hat{w}_t L$ for any $(K, L) \in \mathbb{R}_+^2$. Take $L > 0$ and let K go to infinity: a contradiction arises. Then, $\hat{r}_t + \hat{w}_t > 0$ for any t. From Lemma 2, for any $\tau \geq 1$, the set $B_i^\tau(\hat{\mathbf{p}}, \hat{\mathbf{r}}, \hat{\mathbf{w}})$ is non empty. Fix some $\tau \geq 1$. Take $(c_{it}, k_{it+1}, \lambda_{it})_{t=0}^\tau \in B_i^\tau(\hat{\mathbf{p}}, \hat{\mathbf{r}}, \hat{\mathbf{w}})$. We have

$$\hat{p}_t [c_{it} + k_{it+1} - (1-\delta) k_{it}] < \hat{r}_t k_{it} + \hat{w}_t (1 - \lambda_{it})$$

for $t = 0, \ldots, \tau$. There exists $N > \tau$ such that, for any $T \geq N$,

$$\bar{p}_t(T) [c_{it} + k_{it+1} - (1-\delta) k_{it}] < \bar{r}_t(T) k_{it} + \bar{w}_t(T) (1 - \lambda_{it})$$

for $t = 0, \ldots, \tau$. Take $T \geq N$. Define $\left(\tilde{c}_{it}(T), \tilde{k}_{it+1}(T), \tilde{\lambda}_{it}(T)\right)_{t=0}^T$ by $\tilde{c}_{it}(T) = c_{it}$, $\tilde{k}_{it+1}(T) = k_{it+1}$ and $\tilde{\lambda}_{it}(T) = \lambda_{it}$ for $t = 0, \ldots, \tau$, and $\tilde{c}_{it}(T) = \tilde{k}_{it+1}(T) = \tilde{\lambda}_{it}(T) = 0$ for $t = \tau+1, \ldots, T$. Obviously, $(\tilde{c}_{it}(T), \tilde{k}_{it+1}(T), \tilde{\lambda}_{it}(T))_{t=0}^T \in C_i^T(\bar{\mathbf{p}}(\mathbf{T}), \bar{\mathbf{r}}(\mathbf{T}), \bar{\mathbf{w}}(\mathbf{T}))$. Hence

$$\sum_{t=0}^{\tau} \beta_i^t u_i(c_{it}, \lambda_{it}) = \sum_{t=0}^{T} \beta_i^t u_i(\tilde{c}_{it}, \tilde{\lambda}_{it}) \leq \sum_{t=0}^{T} \beta_i^t u_i\left(\bar{c}_{it}(T), \bar{\lambda}_{it}(T)\right)$$

This implies

$$\sum_{t=0}^{\tau} \beta_i^t u_i(c_{it}, \lambda_{it}) \leq \lim_{T \to \infty} \sum_{t=0}^{T} \beta_i^t u_i\left(\bar{c}_{it}(T), \bar{\lambda}_{it}(T)\right) = \sum_{t=0}^{\infty} \beta_i^t u_i\left(\hat{c}_{it}, \hat{\lambda}_{it}\right) \quad (20)$$

Now let $(c_{it}, k_{it+1}, \lambda_{it})_{t=0}^\infty \in \mathbb{R}_+^\infty \times \mathbb{R}_+^\infty \times [0,1]^\infty$ satisfy:

$$\hat{p}_t [c_{it} + k_{it+1} - (1-\delta) k_{it}] \leq \hat{r}_t k_{it} + \hat{w}_t (1 - \lambda_{it})$$

for $t = 0, \ldots, \infty$. In this case, $(c_{it}, k_{it+1}, \lambda_{it})_{t=0}^\tau \in C_i^\tau(\hat{\mathbf{p}}, \hat{\mathbf{r}}, \hat{\mathbf{w}})$. There exists a sequence $\left((c_{it}^n, k_{it+1}^n, \lambda_{it}^n)_{t=0}^\tau\right)_n \subset B_i^\tau(\hat{\mathbf{p}}, \hat{\mathbf{r}}, \hat{\mathbf{w}})$ converging to $(c_{it}, k_{it+1}, \lambda_{it})_{t=0}^\tau$. We then have, from (20):

$$\sum_{t=0}^{\tau} \beta_i^t u_i(c_{it}^n, \lambda_{it}^n) \leq \sum_{t=0}^{\infty} \beta_i^t u_i\left(\hat{c}_{it}, \hat{\lambda}_{it}\right)$$

Let n go to ∞: $\sum_{t=0}^{\tau}\beta_i^t u_i(c_{it},\lambda_{it}) \leq \sum_{t=0}^{\infty}\beta_i^t u_i\left(\hat{c}_{it},\hat{\lambda}_{it}\right)$. Let τ go to ∞: $\sum_{t=0}^{\infty}\beta_i^t u_i(c_{it},\lambda_{it}) \leq \sum_{t=0}^{\infty}\beta_i^t u_i\left(\hat{c}_{it},\hat{\lambda}_{it}\right)$. We have proved that $\left(\hat{\mathbf{c}},\hat{\boldsymbol{\lambda}}\right)$ solves the consumer's problem in the infinite-horizon economy. The prices $(\hat{p}_t,\hat{w}_t)$ are strictly positive thanks to the strict increasingness of the utility functions. The price $\hat{r}_t > 0$ since we have proved $\hat{r}_t = 0$ implies $\hat{p}_t = 0$.

It is now easy to check that the list $\left(\hat{\mathbf{p}},\hat{\mathbf{r}},\hat{\mathbf{w}},\left(\hat{\mathbf{c}}_i,\hat{\mathbf{k}}_i,\hat{\boldsymbol{\lambda}}_i\right)_{i=1}^{m},\hat{\mathbf{K}},\hat{\mathbf{L}}\right)$ is an equilibrium for the infinite-horizon economy. ■

9 Appendix 3: non-existence of bubbles

Proof of Claim 5

Proof. (1) $\bar{c}_{it} + \bar{w}_t\bar{\lambda}_{it} + \bar{k}_{it+1} = (1-\delta+\bar{r}_t)\bar{k}_{it} + \bar{w}_t$. Suppose $\bar{c}_{it} > 0$ and $\bar{\lambda}_{it} = 0$. By Assumption 4, we can decrease $\bar{c}_{it}$ and increase $\bar{\lambda}_{it}$ to have a higher utility for period t. Hence $\bar{\lambda}_{it} > 0$. The converse is proved by the same argument.

(2) We first prove that $\bar{c}_{it} = \bar{\lambda}_{it} = 0$ for any t is excluded. Suppose it is not true. Then $\sum_{t=0}^{\infty}\beta_i^t u_i\left(\bar{c}_{it},\bar{\lambda}_{it}\right) = 0$. Define $k_{it} = c_{it} = \lambda_{it} = 0$ for any $t \geq 1$ and $c_{i0} + \bar{w}_0\lambda_{i0} = (1-\delta+\bar{r}_0)k_{i0} + \bar{w}_0$ with $c_{i0} > 0$ and $\lambda_{i0} \in (0,1)$. Then $\sum_{t=0}^{\infty}\beta_i^t u_i(c_{it},\lambda_{it}) = u_i(c_{i0},\lambda_{i0}) > 0$, that is a contradiction.

Without loss of generality, we can assume that $t = 1$ is the first period where the consumption and leisure are positive, i.e. $\bar{c}_{i1} > 0$ and $\bar{\lambda}_{i1} > 0$ (because of point (1)). Hence, $\bar{c}_{i0} = \bar{\lambda}_{i0} = 0$. Define

$c_{i0} + \bar{w}_0\lambda_{i0} = \varepsilon > 0$, $\lambda_{i0} \in (0,1)$, $c_{i0} > 0$, $k_{i1} = \bar{k}_{i1} - \varepsilon > 0$,

$c_{i1} = \bar{c}_{i1} - (1-\delta+\bar{r}_1)\varepsilon > 0$, $\lambda_{i1} = \bar{\lambda}_{i1}$, $k_{i2} = \bar{k}_{i2}$,

$c_{it} = \bar{c}_{it}$, $\lambda_{it} = \bar{\lambda}_{it}$, $k_{it+1} = \bar{k}_{it+1}$ for any $t \geq 2$.

The sequence $(\mathbf{c}_i,\mathbf{k}_i,\boldsymbol{\lambda}_i)$ belongs to the budget set of agent i. And we have, by Assumption 4 (Inada), $\sum_{t=0}^{\infty}\beta_i^t u_i(c_{it},\lambda_{it}) > \sum_{t=0}^{\infty}\beta_i^t u_i\left(\bar{c}_{it},\bar{\lambda}_{it}\right)$ for ε sufficiently close to 0. This leads to a contradiction. Hence $\bar{c}_{i0} > 0$ and $\bar{\lambda}_{i0} > 0$. By induction, we obtain also $\bar{c}_{it} > 0$ and $\bar{\lambda}_{it} > 0$ for any i and any t.

(3) We have

$$\begin{aligned}
\bar{c}_{it} + \bar{k}_{it+1} &= \bar{k}_{it}(1-\delta+\bar{r}_t) + \bar{w}_t(1-\bar{\lambda}_{it}) \\
(\bar{c}_{it}-\varepsilon) + \left(\bar{k}_{it+1}+\varepsilon\right) &= \bar{k}_{it}(1-\delta+\bar{r}_t) + \bar{w}_t\left(1-\bar{\lambda}_{it}\right) \\
[\bar{c}_{it+1} + \varepsilon(1-\delta+\bar{r}_{t+1})] + \bar{k}_{it+2} &= \left(\bar{k}_{it+1}+\varepsilon\right)(1-\delta+\bar{r}_{t+1}) + \bar{w}_{t+1}\left(1-\bar{\lambda}_{it+1}\right)
\end{aligned}$$

Then

$$
\begin{aligned}
0 &\geq u_i\left(\bar{c}_{it}-\varepsilon,\bar{\lambda}_{it}\right)-u_i\left(\bar{c}_{it},\bar{\lambda}_{it}\right) \\
&\quad +\beta_i\left[u_i\left(\bar{c}_{it+1}+\varepsilon\left(1-\delta+\bar{r}_{t+1}\right),\bar{\lambda}_{it+1}\right)-u_i\left(\bar{c}_{it+1},\bar{\lambda}_{it+1}\right)\right] \\
&\geq \frac{\partial u_i}{\partial c}\left(\bar{c}_{it}-\varepsilon,\bar{\lambda}_{it}\right)(-\varepsilon) \\
&\quad +\beta_i\frac{\partial u_i}{\partial c}\left(\bar{c}_{it+1}+\varepsilon\left(1-\delta+\bar{r}_{t+1}\right),\bar{\lambda}_{it+1}\right)\varepsilon\left(1-\delta+\bar{r}_{t+1}\right) \\
0 &\geq -\frac{\partial u_i}{\partial c}\left(\bar{c}_{it}-\varepsilon,\bar{\lambda}_{it}\right) \\
&\quad +\beta_i\frac{\partial u_i}{\partial c}\left(\bar{c}_{it+1}+\varepsilon\left(1-\delta+\bar{r}_{t+1}\right),\bar{\lambda}_{it+1}\right)\left(1-\delta+\bar{r}_{t+1}\right)
\end{aligned}
$$

if $\varepsilon>0$ and small enough.

Let ε go to zero. Then,

$$0\geq-\frac{\partial u_i}{\partial c}\left(\bar{c}_{it},\bar{\lambda}_{it}\right)+\beta_i\frac{\partial u_i}{\partial c}\left(\bar{c}_{it+1},\bar{\lambda}_{it+1}\right)\left(1-\delta+\bar{r}_{t+1}\right)$$

If $\bar{k}_{it+1}>0$, then we can take also $\varepsilon<0$ small enough in absolute value and let it go to zero to obtain the reverse inequality.

(4) Since $\bar{\lambda}_{it}>0$, we can choose $0<\varepsilon<\bar{\lambda}_{it}$. Define $c_{it}=\bar{c}_{it}+\bar{w}_t\varepsilon$ and $\lambda_{it}=\bar{\lambda}_{it}-\varepsilon$. The budget constraint is satisfied. In addition, we have for $\varepsilon\in\left(0,\bar{\lambda}_{it}\right)$

$$
\begin{aligned}
0 &\geq u_i\left(c_{it},\lambda_{it}\right)-u_i\left(\bar{c}_{it},\bar{\lambda}_{it}\right)\geq\frac{\partial u_i}{\partial c}\left(c_{it},\lambda_{it}\right)\bar{w}_t\varepsilon+\frac{\partial u_i}{\partial\lambda}\left(c_{it},\lambda_{it}\right)(-\varepsilon) \\
0 &\geq \frac{\partial u_i}{\partial c}(c_{it},\lambda_{it})\bar{w}_t-\frac{\partial u_i}{\partial\lambda}(c_{it},\lambda_{it})
\end{aligned}
$$

Let ε go to zero. Then

$$0\geq\bar{w}_t\frac{\partial u_i}{\partial c}\left(\bar{c}_{it},\bar{\lambda}_{it}\right)-\frac{\partial u_i}{\partial\lambda}\left(\bar{c}_{it},\bar{\lambda}_{it}\right)$$

Now, if $\bar{\lambda}_{it}<1$, then we can take $\varepsilon>0$ such that $\bar{\lambda}_{it}+\varepsilon<1$ and let ε go to zero to get the reverse inequality.

(5) We have $\bar{C}_t+\bar{K}_{t+1}=F\left(\bar{K}_t,\bar{L}_t\right)+(1-\delta)\bar{K}_t$. If $\bar{K}_t=0$, then $\bar{C}_t=0$ and $\bar{c}_{it}=0$ for any i contradicting point (2).

If $\bar{L}_t=0$, then we have

$$0=F(\bar{K}_t,\bar{L}_t)=\bar{r}_t\bar{K}_t+\bar{w}_t\bar{L}_t=\bar{r}_t\bar{K}_t$$

Hence $\bar{K}_t=0$, since $\bar{r}_t>0$. As above, this contradicts the point (2) of the claim. ■

Proof of Theorem 6

Proof. First, observe that the production function F satisfies Assumption 1 and

$$\lim_{b\to0^+}\left(\partial F/\partial L\right)(1,b)>0 \tag{21}$$

Since F is homogeneous of degree one, we have, for $K > 0$ and $L > 0$, $(\partial F/\partial K)(K,L) = (\partial F/\partial K)(K/L,1)$ and $(\partial F/\partial L)(K,L) = (\partial F/\partial L)(1,L/K)$. Let $(\bar{K}_t, \bar{L}_t)$ be an equilibrium sequence of aggregate capital stocks and labors. Observe that $\bar{r}_t = (\partial F/\partial K)(\bar{K}_t/\bar{L}_t, 1)$. Since F is differentiable and concave, we have for any t

$$\bar{r}_t \geq \lim_{a\to\infty} (\partial F/\partial K)(a,1) \tag{22}$$

Suppose the economy has a bubble in prices. Then, from Lemma 1, $\bar{r}_t$ converges to zero. But from (22), $\bar{K}_t/\bar{L}_t$ tends to infinity, or equivalently, $\bar{L}_t/\bar{K}_t$ goes to 0. Since $\bar{K}_t$ is positive and bounded above, we obtain $\bar{L}_t \to 0$. Recall that

$$\bar{C}_t + \bar{K}_{t+1} = F(\bar{K}_t, \bar{L}_t) + (1-\delta)\bar{K}_t = \bar{K}_t\left[F(1, \bar{L}_t/\bar{K}_t) + 1 - \delta\right]$$

and choose $\varepsilon > 0$ such that $F(1,\varepsilon) + 1 - \delta < 1$. There exists T such that for any $t > T$, $K_{t+1} \leq \bar{K}_t\left[F(1, \bar{L}_t/\bar{K}_t) + 1 - \delta\right] < [F(1,\varepsilon) + 1 - \delta]\bar{K}_t$. This implies $\bar{K}_t \to 0$ when t tends to infinity, and $\bar{C}_t \to 0$ too.

Reconsider the first-order conditions of point (4) of Claim 5:

$$\bar{w}_t \frac{\partial u_i}{\partial c}(\bar{c}_{it}, \bar{\lambda}_{it}) \leq \frac{\partial u_i}{\partial \lambda}(\bar{c}_{it}, \bar{\lambda}_{it}) \tag{23}$$

It is easy to see that $\bar{w}_t = (\partial F/\partial L)(\bar{K}_t, \bar{L}_t)$. Since $\bar{L}_t/\bar{K}_t$ converges to 0, according to (21), we have $\lim_{t\to\infty} \bar{w}_t = \lim_{t\to\infty} (\partial F/\partial L)(1, \bar{L}_t/\bar{K}_t) > 0$. We claim that, for any i, $\lim_{t\to\infty} \bar{\lambda}_{it} = 0$. Assume the contrary: $\lim_{t\to\infty} \bar{\lambda}_{jt} = \bar{\lambda} > 0$ for some j. Since $\lim_{t\to\infty} \bar{C}_t = 0$, we have $\lim_{t\to\infty} \bar{c}_{it} = 0$ for any i. Thus,

$$\lim_{t\to\infty}\left[\bar{w}_t \frac{\partial u_j}{\partial c}(\bar{c}_{jt}, \bar{\lambda}_{jt})\right] = \infty$$

Since

$$\frac{\partial u_j}{\partial \lambda}(\bar{c}_{jt}, \bar{\lambda}_{jt})\bar{\lambda}_{jt} \leq u_j(\bar{c}_{jt}, \bar{\lambda}_{jt}) - u_j(\bar{c}_{jt}, 0) \leq u_j(\bar{c}_{jt}, \bar{\lambda}_{jt}) \leq u_j(A, 1)$$

(23) implies a contradiction:

$$\infty = \lim_{t\to\infty} \bar{w}_t \frac{\partial u_j}{\partial c}(\bar{c}_{jt}, \bar{\lambda}_{jt}) \leq \limsup_t \frac{\partial u_j}{\partial \lambda}(\bar{c}_{jt}, \bar{\lambda}_{jt}) \leq \frac{u_j(A,1)}{\bar{\lambda}} < \infty$$

Hence, for any i, $\lim_{t\to\infty} \bar{\lambda}_{it} = 0$ and $\lim_{t\to\infty} \bar{L}_t = m$, that is a contradiction.

■

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