

The Stabilizing Virtues of Fiscal vs. Monetary Policy on Endogenous Bubble Fluctuations

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Abstract

We explore the existence of endogenous fluctuations with a rational bubble and the stabilizing role of fiscal and monetary policies. Consumers' credit constraints, the role of collateral and a portfolio choice are the key ingredients of our analysis. We consider an overlapping generations model where households realize a portfolio choice between three assets with different returns (capital, money and bonds). Expectation-driven fluctuations and the multiplicity of steady states occur under a positive bubble on bonds, gross substitutability and large input substitution because of credit market imperfections. Focusing on the stabilizing role of policies, we show that a progressive taxation on capital income may rule out expectation-driven fluctuations and the multiplicity of steady states. In contrast, a monetary policy under a Taylor rule has a mitigated stabilizing role, depending on the reactivity of the policy rule and the concavity of the utility function. When the monetary authority decides instead to fix the nominal interest rate regardless the inflation, decreasing the level of the nominal interest rate can rule out expectation-driven fluctuations, restore the uniqueness of steady states, but can damage the welfare at the steady state.

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1 Introduction

In recent years, asset prices have experienced large fluctuations, and the financial sphere of the economy had strong effects on the real one. Some empirical contributions shed light on the excessive asset price volatility, and reveal that asset prices fluctuate more than their fundamental value (see Shiller (1981, 1989, 2000), LeRoy and Porter (1981) or Campbell (2003)). One explanation for this excessive volatility is the existence and the fluctuations of asset bubbles.

Despite the fact that many contributions deal with credit constraints at the level of entrepreneurs, some empirical studies highlight the existence of credit constraints faced by consumers underlying the role of collateral on their behavior (Campbell and Mankiw (1989), Jappelli (1990), Iacoviello (2004), Crook and Hochguertel (2005)). Such types of credit market imperfections may be a source of portfolio choices between different existing assets, like capital, money and bonds, but also gaps between their returns. We argue that such credit constraints play a crucial role to explain fluctuations of speculative bubbles, as illustrated during the recent subprime crisis. We also think that they can be a main transmission channel between the financial and the real spheres.¹ A first aim of this paper is precisely to provide such explanations focusing on endogenous fluctuations driven by the volatility of agents' expectations.

Economic fluctuations based on consumer credit constraints also open the door to new policy tools for stabilizing issues. Since our explanation of expectation-driven fluctuations relies on a trade-off between different assets namely capital, bonds and money, relevant stabilizing policies are those reducing the gaps between their returns. Monetary policy appears to be a natural policy tool, since it affects the opportunity cost of money holdings through the level of the nominal interest rate. As we focus on the interplay between the real and the financial spheres, another relevant policy could be one that alters the capital return. In this perspective, a progressive taxation of capital income could be a good candidate to dampen the gaps between the returns on physical and monetary assets along the dynamic path. Therefore, the second goal of this paper is to analyze the stabilizing role of monetary and fiscal policies. More precisely, we aim to provide new arguments for stabilizing monetary policies when contrary to most of the literature, the economy experiences bubble fluctuations in an economy with production (Bernanke and Woodford (1997), Woodford (1999), Benhabib et al. (2001), Grandmont (1985, 1986), Sorger (2005)). We also provide new insights in favor of progressive capital income taxation. While some recent contributions emphasize that nonlinear capital income taxation may be optimal (Saez (2013), Farhi et al. (2012)), we rather focus on its stabilizing virtues.

To underline the role of consumers' credit market imperfections and collateral in an economy with a rational bubble, we consider an otherwise simple

¹Only few contributions have analyzed the existence of bubble fluctuations with an interplay between the real and the financial spheres of the economy (Michel and Wigniolle (2003, 2005), Kamihigashi (2008), Bosi and Seegmuller (2010a), Wigniolle (2012)). Of course, the analysis of real effects of bubble fluctuations requires capital accumulation.

overlapping generations model with two-period lived households and production. Households save through bonds, money and capital. Bonds are sold by the monetary authority to supply money. Because of a binding cash-in-advance constraint, money is held by households to finance a share of their consumption in the second period of their life. Despite the fact that capital is used for the production, it also serves as a collateral: holding more capital increases the amount of collateral, and thus allows each household to reduce the share of consumption financed through money. It is important to note that the three assets have different returns. Bonds have larger return than capital because this latter is used as a collateral to relax the consumers' credit constraint, and also a larger return than money because we focus on equilibria with binding constraints. As a direct implication, the Fisher relationship is not satisfied.² The violation of this relationship will represent some portfolio choices that promote indeterminacy, and therefore endogenous fluctuations.

We first prove the existence of a steady state characterized by a positive rational bubble on bonds. Unlike Tirole (1985), any bubbly steady states experience over-accumulation of capital in the absence of capital taxation. In contrast to several existing papers (Farmer (1986), Benhabib and Laroque (1988), Rochon and Polemarchakis (2006)), expectation-driven fluctuations occur in the neighborhood of a steady state with a positive rational asset bubble under gross substitutability and reasonable values of input substitution, without requiring arbitrarily large increasing returns to scale (Cazzavillan and Pintus (2005), Azariadis and Reichlin (1996)). This result is obtained when the share of consumption purchased on credit weakly depends on collateral, but is sufficiently concave. Interestingly, this result is connected to a multiplicity of steady states, i.e. a form of global indeterminacy.

In a second step, we discuss the stabilizing role of fiscal and monetary policies. As discussed earlier, we introduce a progressive taxation of capital income, used to finance useless public spendings under a balanced budget. We show that increasing the degree of marginal progressivity may rule out expectation-driven fluctuations under gross substitutability by reducing the range of parameters for indeterminacy and for the multiplicity of steady states. This fiscal policy is therefore powerful to stabilize endogenous fluctuations. Contrary to previous contributions (see for instance Guo and Lansing (1998)), progressive capital income taxation is per se stabilizing through a new mechanism based on the portfolio choice between the different assets and on the gaps between their returns.

We pursue the analysis with the monetary policy, and investigate whether such a policy can have similar stabilizing virtues as fiscal policy. We first consider that the nominal interest rate is determined according to a Taylor rule on expected inflation. In this case, the results are mitigated. A weakly active or passive rule can even promote endogenous fluctuations for some relevant parameter configurations. In addition, such a policy has no effect on the multiplicity

²The Fisher relationship means that the gross real interest rate is equal to the gross nominal interest rate deflated by the gross inflation rate.

of steady states. One explanation is that such a rule does not strongly modify the nominal interest rate, and therefore does not alter so much the portfolio choice.

Therefore, we focus on an alternative monetary policy that more directly affects the nominal interest rate. A monetary policy that fixes the level of the nominal interest rate independently on expected inflation, like an interest rate pegging, is a relevant stabilization tool. In this case, under gross substitutability local indeterminacy and the multiplicity of steady states can be eliminated by implementing a sufficiently low interest rate. Hence, the monetary policy based on the direct management of the interest rate independently on expected inflation appears to be powerful to stabilize endogenous business cycles and global indeterminacy associated to the multiplicity of steady states. Note that this is in accordance with Rochon and Polemarchakis (2006), but in contrast to them we stabilize fluctuations with a positive bubble.

To summarize, in contrast to a Taylor rule, a progressive taxation of capital income and a direct management of the interest rate are powerful to stabilize the economy which experiences a positive rational bubble. We however show that capital taxation may reduce or even rule out over-accumulation of capital, while the stabilizing effect of a direct management of the interest rate implies a deterioration of the welfare at the stationary equilibrium.³

This paper is organized as follows. In the next section, we present the model. The intertemporal equilibrium is defined in Section 3. Section 4 is devoted to the steady state analysis. In Section 5, we analyze the occurrence of expectation-driven fluctuations when there is a positive bubble. In Section 6, we study the stabilizing role of fiscal vs. monetary policy. Concluding remarks are provided in Section 7, and all the proofs are gathered in a final Appendix.

2 The model

In this paper, we consider an overlapping generations model with production in discrete time ($t = 0, 1, \dots, +\infty$). This economy consists of identical two period-lived households, firms, a monetary authority, a government and four goods: a final good, productive capital, money and a bond.

2.1 Households

There is no population growth, and at each date t , a generation of unit size is born and lives for two periods.

A household derives utility from consumption when young (c_t) and old (d_{t+1}). Her preferences are represented by an additively separable life-cycle utility function:

$$u(c_t) + \beta v(d_{t+1}) = \frac{c_t^{1-\varepsilon_u}}{1-\varepsilon_u} + \beta \frac{d_{t+1}^{1-\varepsilon_v}}{1-\varepsilon_v}, \quad \beta > 0 \quad (1)$$

³In this case, we lose the Friedman rule.

c_t and d_{t+1} denote respectively the consumption of final good in the first and second period of life. $\varepsilon_u > 0$ and $\varepsilon_v > 0$ are the degrees of concavity of $u(c_t)$ and $v(d_{t+1})$. We further note that $\varepsilon_v < 1$ implies gross substitutability meaning that savings are an increasing function of the global return on portfolio.⁴

In her first period of life, the household is young and supplies one unit of labor inelastically remunerated at the wage w_t . With this wage, she can consume an amount c_t of final good at price p_t , and save through a diversified portfolio of productive capital per capita k_{t+1} (with rental factor R_{t+1}), nominal bonds B_{t+1} (with nominal interest rate i_{t+1}) and nominal balances M_{t+1} needed for a transaction motive. In her second period of life, she is old. She uses her remunerated savings and her monetary transfer Δ_{t+1} from the monetary authority to purchase an amount d_{t+1} of final good at price p_{t+1} and to pay a tax on her capital income. Defining $g(R_{t+1}k_{t+1})$ as the after-tax capital income of a household, the first and second-period budget constraints are written as follows:

$$p_t c_t + M_{t+1} + B_{t+1} + p_t k_{t+1} \leq p_t w_t \quad (2)$$

$$\begin{aligned} p_{t+1} d_{t+1} &\leq M_{t+1} + (1 + i_{t+1}) B_{t+1} \\ &\quad + p_{t+1} g(R_{t+1} k_{t+1}) + \Delta_{t+1} \end{aligned} \quad (3)$$

Considering a progressive taxation of capital income, we assume:

Assumption 1 Let $y_k \equiv Rk$. The function $g(y_k): [0, +\infty) \rightarrow [0, +\infty)$ is continuous, with $g(0) = 0$, and C^2 on $(0, +\infty)$. In addition, it satisfies $0 < g'(y_k) \leq 1$ and $g''(y_k) \leq 0$ for all $y_k > 0$. We define $\rho_1(y_k) \equiv y_k \frac{g'(y_k)}{g(y_k)} \in (0, 1)$ and $\rho_2(y_k) \equiv -y_k \frac{g''(y_k)}{g'(y_k)} \geq 0$ as the first and second order elasticities of after-tax capital income respectively.

The after-tax capital income $g(y_k) > 0$ is increasing, concave and satisfies $g(y_k) \leq y_k$. Assumption 1 implies that the tax function $\tau(y_k) \equiv y_k - g(y_k) \geq 0$ is non-decreasing and convex. In addition, the marginal tax rate $\tau_m(y_k) \equiv 1 - g'(y_k) \in (0, 1)$ is increasing in the tax base as $\rho_2(y_k) > 0$ and flat as $\rho_2 = 0$. We note that the elasticity $\rho_2(y_k)$ is a measure of marginal progressivity.⁵

⁴As we will see below, the consumer problem has the following structure:

$$\begin{aligned} \max \quad & \frac{c_t^{1-\varepsilon_u}}{1-\varepsilon_u} + \beta \frac{d_{t+1}^{1-\varepsilon_v}}{1-\varepsilon_v} \\ \text{st.} \quad & c_t + s_t = w_t \\ & d_{t+1} = \tilde{R}_{t+1} s_t + \Delta_{t+1}, \end{aligned}$$

where s_t represents global savings of a household, $\tilde{R}_{t+1}$ the global return on her portfolio, w_t , her labor income and Δ_{t+1} a monetary transfer. From this problem, we obtain:

$$\frac{ds_t}{d\tilde{R}_{t+1}} \frac{\tilde{R}_{t+1}}{s_t} = \frac{1 - \varepsilon_v \tilde{R}_{t+1} s_t / (\tilde{R}_{t+1} s_t + \Delta_{t+1})}{\varepsilon_u s_t / (w_t - s_t) + \varepsilon_v \tilde{R}_{t+1} s_t / (\tilde{R}_{t+1} s_t + \Delta_{t+1})},$$

which is positive for $\varepsilon_v < 1$.

⁵As $\rho_1 = 1$ and $\rho_2 = 0$, no progressive taxation of capital income is implemented.

Furthermore, at the second period of life, the household has to pay cash a part of the second period consumption d_{t+1} : her money demand is rationalized by a cash-in-advance constraint. We use the constraint introduced by Hahn and Solow (1995), i.e. $\gamma p_{t+1} d_{t+1} \leq M_{t+1}$, but extend it to take into account collateral:

$$\gamma(k_{t+1}) p_{t+1} d_{t+1} \leq M_{t+1} \quad (4)$$

Assumption 2 $\gamma(k) \in (0, 1)$ is a continuous function defined on $[0, +\infty)$, C^2 on $(0, +\infty)$, decreasing ($\gamma'(k) \leq 0$). In addition, we define:

$$\eta_1(k) \equiv \frac{[1 - \gamma(k)]' k}{1 - \gamma(k)} = -\frac{\gamma'(k)k}{1 - \gamma(k)} \geq 0, \quad (5)$$

$$\eta_2(k) \equiv -\frac{[1 - \gamma(k)]'' k}{[1 - \gamma(k)]'} = -\frac{\gamma''(k)k}{\gamma'(k)} \quad (6)$$

For instance, the following function satisfies these properties:

$$\gamma(k) = \frac{A}{s} \exp(-sk), \quad (7)$$

with $0 < A < s$. Using this example, $\eta_1(k)$ and $\eta_2(k)$ are given by:

$$\eta_1(k) = \frac{A \exp(-sk)}{1 - \frac{A}{s} \exp(-sk)} k \geq 0 \text{ and } \eta_2(k) = sk > 0$$

A binding cash-in-advance constraint means that the household has to pay cash, i.e. with her nominal balances M_{t+1} , a share $\gamma(k_{t+1}) \in (0, 1)$ of her second period consumption. Since the household holds $B_{t+1} + p_{t+1}k_{t+1}$ when young, she can consume on credit when old. Indeed, the household knows that in addition to the transfer from the monetary authority Δ_{t+1}/p_{t+1} , she will have $(1 + i_{t+1})B_{t+1}/p_{t+1} + g(R_{t+1}k_{t+1})$ at the next period. As a result, she can consume this amount on credit. In other words, the remaining share $1 - \gamma(k_{t+1})$ of her second period consumption can be financed by borrowing from a bank or a financial institution an amount equal to her remunerated savings from capital and bonds plus her monetary transfer, i.e. $(1 + i_{t+1})B_{t+1}/p_{t+1} + g(R_{t+1}k_{t+1}) + \Delta_{t+1}/p_{t+1}$, that she will pay back at the end of her second period of life. In the following, we refer to $1 - \gamma(k_{t+1})$ as the credit share.

In addition, we assume that the amount of productive capital held by the household affects her cash-in-advance constraint. Capital acts as a collateral for the household: holding more capital in her portfolio allows the household to increase her opportunities to obtain credit from the bank or the financial institution, and to reduce her need of cash in her second period of life. This assumption is a simple manner to include some credit market imperfections and a collateral effect in our framework.

Notice that when collateral does not matter ($\eta_1(k_{t+1}) = 0$), and γ tends to 0, money is no longer needed and the credit market distortion disappears.

However, when $\gamma > 0$, there is a need of cash. When collateral matters ($\eta_1(k_{t+1}) > 0$), the households are aware of the credit share function. By increasing capital holdings, they are able to reduce the share of consumption financed through cash.

Using $\pi_{t+1} \equiv \frac{p_{t+1}}{p_t}$ and introducing the real variables $m_{t+1} \equiv \frac{M_{t+1}}{p_{t+1}}$, $b_{t+1} \equiv \frac{B_{t+1}}{p_{t+1}}$ and $\delta_{t+1} \equiv \frac{\Delta_{t+1}}{p_{t+1}}$, the constraints (2)-(4) can be rewritten as follows:

$$c_t + \pi_{t+1}m_{t+1} + \pi_{t+1}b_{t+1} + k_{t+1} \leq w_t \quad (8)$$

$$\begin{aligned} d_{t+1} &\leq m_{t+1} + (1 + i_{t+1})b_{t+1} \\ &\quad + g(R_{t+1}k_{t+1}) + \delta_{t+1} \end{aligned} \quad (9)$$

$$\gamma(k_{t+1})d_{t+1} \leq m_{t+1} \quad (10)$$

The representative household derives her optimal consumption choice (c_t , d_{t+1}) and her optimal portfolio choice ($k_{t+1}, m_{t+1}, b_{t+1}$) by maximizing her utility function (1) under her budget and cash-in-advance constraints (8)–(10).

Assumption 3 Let $\tilde{\varepsilon}_u \equiv c \frac{1+i}{\pi} \frac{[i\eta_1(1-\gamma)]^2}{\{\rho_1\rho_2g + [i\eta_1(1-\gamma)]\eta_2d\}(1+i\gamma)^2}$.⁶ For all $t \geq 0$, we assume $i_t > 0$, $\eta_2 > 0$ and $\varepsilon_u > \tilde{\varepsilon}_u$.⁷

Lemma 1 Under Assumptions 1-3, constraints (8)-(10) are binding and the second-order conditions are satisfied.

Proof. See Appendix A.

Lemma 1 requires that the function of the credit share $1 - \gamma(k_{t+1})$ is concave: capital holdings increase, at a decreasing rate, the part of second-period consumption purchased on credit. Moreover, the cash-in-advance constraint is binding if the nominal interest rate i_{t+1} is strictly positive ($i_{t+1} > 0$).

Under Assumptions 1-3, the optimal households' behavior is summarized by the following equations:

$$\frac{u'(c_t)}{\beta v'(d_{t+1})} = \frac{1 + i_{t+1}}{\pi_{t+1}} \frac{1}{1 + i_{t+1}\gamma(k_{t+1})} \quad (11)$$

$$R_{t+1}g'(R_{t+1}k_{t+1}) = \frac{1 + i_{t+1}}{\pi_{t+1}} - i_{t+1}\eta_1(k_{t+1}) \frac{1 - \gamma(k_{t+1})}{k_{t+1}} d_{t+1} \quad (12)$$

When collateral does not matter ($\eta_1(k) = 0$), and no progressive capital income taxation is implemented ($g'(Rk) = 1$), Eqs. (11) and (12) rewrite:

$$\frac{u'(c_t)}{\beta v'(d_{t+1})} = \frac{(1 + i_{t+1})/\pi_{t+1}}{1 + i_{t+1}\gamma} \quad (13a) \quad \text{and} \quad R_{t+1} = \frac{1 + i_{t+1}}{\pi_{t+1}} \quad (13b)$$

⁶For simplicity, the arguments of the functions and the time subscripts are omitted.

⁷This general condition will be supposed satisfied at the steady state.

We can note that as γ tends to 0, we obtain the intertemporal trade-off found in Diamond (1965) and Tirole (1985), where there are no credit market distortions in the economy (see Eq. (13a)). As $\gamma > 0$, a distortion exists: old households now have to pay cash γ in order to consume an additional unit of final good, and money entails an opportunity cost. Nevertheless, when collateral does not matter, capital and bonds are perfect substitutes (see Eq. (13b)).

When collateral matters ($\eta_1(k) > 0$), capital and bonds are no longer perfect substitutes. Indeed, households can now decrease their need of cash by holding more capital in their portfolio. As a consequence, the return on capital is lower than the return on bonds.⁸ In contrast, a positive marginal tax rate on capital income has an opposite effect by reducing capital holdings. The endogeneity of the credit share ensures the portfolio choices to be no longer constant through time. The trade-off between assets is endogenous, and depends on the amount of collateral held by the households, involving the existence of a “portfolio effect”. A change in expected inflation leads to reconsider the trade-off between asset holdings. Later, we will see that this portfolio effect is a key mechanism through which expectation-driven fluctuations may occur. As the portfolio choices are an explanation for fluctuations, we will focus on stabilizing policies that are able to modify the different returns of assets, and therefore, to counteract the portfolio effect. In this perspective, we will consider a taxation of capital income for the fiscal policy, then an interest rate rule and an interest rate pegging for the monetary policy.

2.2 Firms

A representative competitive firm produces the final good using capital and labor under a constant returns to scale technology $f(K/L)L$. Since we note $k = K/L$, the intensive production function $f(k)$ satisfies:

Assumption 4 $f(k)$ is a continuous function defined on $[0, +\infty)$ and C^2 on $(0, +\infty)$, strictly increasing ($f'(k) > 0$) and strictly concave ($f''(k) < 0$). Defining $\alpha(k) \equiv f'(k)k/f(k) \in (0, 1)$ as the capital share in total income and $\sigma(k) \equiv [f'(k)k/f(k) - 1] f'(k) / [kf''(k)] > 0$ as the elasticity of capital-labor substitution, we further assume $f'(1) < 1$, $\lim_{k \rightarrow 0^+} f'(k) > 1$ and $\sigma(k) > 1 - \alpha(k)$.

⁸For $\eta_1(k_{t+1}) < \frac{\gamma(k_{t+1})}{1 - \gamma(k_{t+1})} \frac{k_{t+1}}{\pi_{t+1}m_{t+1}}$, the following inequality is satisfied:

$$\frac{1}{\pi_{t+1}} < R_{t+1}g'(R_{t+1}k_{t+1}) < \frac{1 + i_{t+1}}{\pi_{t+1}} \quad (14)$$

In ascending order, there are the return on money, on capital and on bonds. When the role of collateral is not so large, money is a dominated asset. As capital allows households to reduce the tightness of the cash-in-advance constraint, a sufficiently large $\eta_1(k)$ can imply a return on capital lower than the return on money.

The competitive firm takes the prices as given and maximizes the profits $f(K_t/L_t)L_t - w_tL_t - R_tK_t$ ⁹:

$$R_t = f'(k_t) \equiv R(k_t) \quad (15)$$

$$w_t = f(k_t) - k_t f'(k_t) \equiv w(k_t) \quad (16)$$

Hence, the interest rate and wage elasticities are respectively equal to $\epsilon_R(k) \equiv \frac{R'(k)k}{R(k)} = -\frac{1-\alpha(k)}{\sigma(k)}$ and $\epsilon_w(k) \equiv \frac{w'(k)k}{w(k)} = \frac{\alpha(k)}{\sigma(k)}$. The inequality $\sigma(k) > 1 - \alpha(k)$ involves capital income $R_t k_t$ being increasing in k_t , which is not a restrictive assumption.

2.3 Monetary authority

For implementing monetary policy, the monetary authority (central bank) uses open market operations defined as the purchase or sale of bonds in exchange for nominal balances.¹⁰ At time t , the central bank creates nominal balances M_{t+1} , which offer liquidity at the next period $t+1$.¹¹ The money growth factor $\mu_t = M_{t+1}/M_t$ can be written as follows:

$$\mu_t = \pi_{t+1} \frac{m_{t+1}}{m_t} \quad (17)$$

In order to supply M_{t+1} in the economy at $t+1$, the central bank buys bonds from old households, and pays for them in cash through open market operations. As a consequence, a part of bonds held by the old households $\bar{B}_{t+1}$ corresponds to the counterpart of second period nominal balances M_{t+1} :

$$\bar{B}_{t+1} + M_{t+1} = 0 \quad (18)$$

At time t , the central bank sells bonds B_{t+1} and nominal balances M_{t+1} to young households, and buys bonds with interests $(1+i_t)B_t$ and nominal balances M_t from old households. The profits made by central bank Δ_t at time t are given by:

$$\Delta_t = B_{t+1} + M_{t+1} - (1+i_t)B_t - M_t \quad (19)$$

As bonds are the counterpart of money, we obtain $\Delta_t = i_t M_t$. These profits are distributed as dividends to the old households at time t . Thus, the budget constraint of the monetary authority at time t is written as follows:

$$B_{t+1} + M_{t+1} = (1+i_t)B_t + M_t + \Delta_t = (1+i_t)(B_t + M_t) \quad (20)$$

or equivalently:

$$\pi_{t+1}(b_{t+1} + m_{t+1}) = (1+i_t)(b_t + m_t) \quad (21)$$

⁹We assume a full capital depreciation within a period.

¹⁰To study the existence of expectation-driven fluctuations in an overlapping generations model without collateral, Rochon and Polemarchakis (2006) use similar open market operations for supplying money in the economy.

¹¹Placing a part of their savings in the form of nominal balances in their first period of life, young households will have the opportunity to obtain liquidity in their second period of life.

Let $\theta_t \equiv (1 + i_t)(b_t + m_t)$. When $\theta_t = 0$, all bonds are the counterpart of money. In this case, all money in the economy corresponds to inside money. When $\theta_t > 0$, a positive bubble on bonds exists. Indeed, $\theta_t > 0$ is equivalent to $b_t - \bar{b}_t > 0$, where b_t represents the market value of bonds and $\bar{b}_t$ the real counterpart of money. When $\theta_t < 0$, there is an excess of households' debt ($\theta_t < 0 \Leftrightarrow b_t - \bar{b}_t < 0$), in other words a negative bubble. Thus, Eq. (21) can be rewritten as follows:

$$\pi_{t+1}\theta_{t+1} = (1 + i_{t+1})\theta_t \quad (22)$$

In addition, the monetary authority chooses the nominal interest rate i_{t+1} as the monetary instrument, and implements the following interest rate rule:

$$1 + i_{t+1} = (1 + i^*) \left(\frac{\pi_{t+1}}{\pi^*} \right)^\phi, \quad (23)$$

where $\phi \geq 0$ is a measure of monetary policy responses to expected inflation. Furthermore, i^* and π^* are respectively the stationary values of the nominal interest rate and the inflation of an existing stationary equilibrium chosen as the targets by the monetary authority.

When $\phi = 0$, the central bank decides to fix the level of the nominal interest rate at its stationary level i^* . When $\phi > 0$, Eq. (23) depicts an interest rate rule, like a Taylor rule, which responds to expected inflation.¹² For $\phi \in (0, 1)$, the rule weakly reacts to expected inflation: an increase (decrease) in the inflation raises (depresses) the nominal interest rate less than proportionally, involving a decrease (increase) in the real interest rate. For $\phi > 1$, the rule strongly reacts to expected inflation: an increase (decrease) in the inflation raises (depresses) the nominal interest rate more than proportionally, involving an increase (decrease) in the real interest rate. Following Benhabib et al. (2001), we define a rule with $\phi \in (0, 1)$ as a passive one, and a rule with $\phi > 1$ as an active one.

2.4 Government

Taxes on capital income are used to finance wasteful public expenditures G_t . These public expenditures affect neither households' preferences nor the production function. Government budget is balanced at each period, that is:

$$G_t = \tau(R_t k_t) = R_t k_t - g(R_t k_t) \quad (24)$$

Note that we could have one institution by introducing a single budget constraint for the government and the monetary authority such that there is no deficit, i.e. $G_t + (1 + i_t)B_t + M_t + \Delta_t = R_t k_t - g(R_t k_t) + B_{t+1} + M_{t+1}$. This would be perfectly equivalent to our framework. θ_t would still be a non-predetermined variable since it would be deflated by the price.

¹²From Bernanke (2010), a rule which responds to expected inflation is more relevant to describe the US monetary policy than a rule responding to observed inflation. As a consequence, we consider a Taylor rule involving an interest rate response to inflation forecast.

3 Intertemporal equilibrium

At the intertemporal equilibrium, the budget and cash-in-advance constraints of households are given by:

$$c_t + \frac{\pi_{t+1}}{1 + i_{t+1}} \theta_{t+1} + k_{t+1} = w(k_t) \quad (25)$$

$$d_{t+1} = \theta_{t+1} + g(f'(k_{t+1})k_{t+1}) \quad (26)$$

$$\gamma(k_{t+1})d_{t+1} = m_{t+1} \quad (27)$$

The budget constraints of the monetary authority and the government are as follows:

$$\pi_{t+1} = (1 + i_{t+1}) \frac{\theta_t}{\theta_{t+1}} \quad (28)$$

$$G_{t+1} = f'(k_{t+1})k_{t+1} - g(f'(k_{t+1})k_{t+1}) \quad (29)$$

Substituting Eq. (28) into the first-period budget constraint Eq. (25), we determine:

$$c_t + \theta_t + k_{t+1} = w(k_t) \quad (30)$$

Using Eqs. (17), (27) and (28), we deduce the money growth factor:

$$\mu_t = (1 + i_{t+1}) \frac{\theta_t}{\theta_{t+1}} \frac{\gamma(k_{t+1})}{\gamma(k_t)} \frac{\theta_{t+1} + g(f'(k_{t+1})k_{t+1})}{\theta_t + g(f'(k_t)k_t)} \quad (31)$$

Substituting Eqs. (25) and (30) into Eq. (11), then Eqs. (26) and (28) into Eq. (12), the consumers' intertemporal trade-off and the no-arbitrage condition are respectively given by:

$$\begin{cases} \theta_t \frac{u'(f(k_t) - f'(k_t)k_t - \theta_t - k_{t+1})}{\beta v'(\theta_{t+1} + g(f'(k_{t+1})k_{t+1}))} = \frac{\theta_{t+1}}{1 + i_{t+1}\gamma(k_{t+1})} \\ \frac{\theta_{t+1}}{\theta_t} = \frac{1 + i_{t+1}}{\pi_{t+1}} = f'(k_{t+1})g'(f'(k_{t+1})k_{t+1})H(k_{t+1}, \theta_t), \end{cases} \quad (32)$$

$$\text{with } H(k_{t+1}, \theta_t) \equiv \frac{1 + i_{t+1}\eta_1(k_{t+1})[1 - \gamma(k_{t+1})][\rho_1(f'(k_{t+1})k_{t+1})]^{-1}}{1 - \theta_t i_{t+1}\eta_1(k_{t+1})[1 - \gamma(k_{t+1})]/k_{t+1}} \quad (33)$$

When collateral does not matter ($\eta_1(k) = 0$), we obtain $H(k_{t+1}, \theta_t) = 1$. Therefore, the Fisher equation ($(1 + i_{t+1})/\pi_{t+1} = f'(k_{t+1})$) holds at the intertemporal equilibrium as soon as there is no progressive taxation on capital income ($g'(f'(k_{t+1})k_{t+1}) = 1$). This means that the return on real asset (capital) is equal to the return on nominal asset (bonds) deflated by the inflation factor. The presence of collateral ($\eta_1(k) > 0$) implies the violation of the Fisher equation ($H(k_{t+1}, \theta_t) > 1$). As capital serves as a collateral, its return becomes lower than the real return on bonds ($f'(k_{t+1}) < (1 + i_{t+1})/\pi_{t+1}$). We will see

later that this violation of the Fisher equation represents some portfolio choices that promote indeterminacy, a source of expectation-driven fluctuations.

Since the portfolio choices are key ingredients for the existence of fluctuations, we will focus on policies that can alter it. As regards fiscal policy, we consider a progressive taxation on capital income $g'(f'(k_{t+1})k_{t+1}) < 1$. Indeed, such a fiscal policy can cancel out the collateral effect (see (32)). Interestingly, the level of nominal interest rate can also offset the collateral effect (see Eq. (33)). Thus, considering a usual interest rate rule, like a Taylor one, but also an alternative, like an interest rate pegging, are a priori relevant to study the stabilizing role of monetary policy.

From the budget constraint of the monetary authority Eq. (28) and the monetary rule Eq. (23), we deduce that:

$$\frac{\pi_{t+1}}{\pi^*} = \left(\frac{\theta_t}{\theta_{t+1}} \right)^{\frac{1}{1-\phi}} \quad (34)$$

Substituting Eq. (34) into Eq. (23), we obtain the nominal interest rate at the equilibrium:

$$i_{t+1} = (1 + i^*) \left(\frac{\theta_t}{\theta_{t+1}} \right)^{a_\phi} - 1, \text{ where } a_\phi \equiv \frac{\phi}{1-\phi} \in (-\infty, -1) \cup [0, +\infty) \quad (35)$$

Definition 1 *Under Assumptions 1-4, an intertemporal equilibrium with perfect foresight is a sequence $(k_t, \theta_t) \in \mathbb{R}_+^2$, $t = 0, 1, \dots, +\infty$, such that (32) is satisfied, where i_{t+1} is defined by Eq. (35) and $k(0) > 0$ is given.*

Taking into account that i_{t+1} is given by Eq. (35), we note that k_t is the only predetermined variable of this two-dimensional dynamic system (32). The intertemporal sequence of k_t and θ_t enables us to determine all the other variables, namely c_t , d_t , m_t and b_t . In particular, the dynamics of m_t and b_t are given by Eqs. (27) and (28) respectively.

4 Steady state analysis

From the system (32), we deduce that two kinds of steady state exist: $\theta = 0$ and $\theta \neq 0$. Since we are interested in fluctuations with a positive bubble, we will focus on steady states with $\theta \neq 0$. A steady state is a solution $(k, \theta) \in \mathbb{R}_{++}^2$ that satisfies the following system:

$$\begin{cases} \frac{u'(f(k) - f'(k)k - k - \theta)}{\beta v'(\theta + g(f'(k)k))} = \frac{1}{1 + i^* \gamma(k)} \\ f'(k)g'(f'(k)k)H(k, \theta) = 1 \end{cases} \quad (36)$$

$$\text{with } H(k, \theta) = \frac{1 + i^* \eta_1(k) [1 - \gamma(k)] [\rho_1(f'(k)k)]^{-1}}{1 - \theta i^* \eta_1(k) [1 - \gamma(k)] / k} \quad (37)$$

Under a constant credit share ($\eta_1(k) = 0$), we see from the system (36) that the steady state is unique, and the monetary policy does not affect the production side. Indeed, the second equation of the system (36) reduces to $f'(k)g'(f'(k)k) = 1$. When collateral matters ($\eta_1(k) > 0$), the superneutrality of money is canceled. Because of the presence of collateral, the monetary sphere affects the real one.

4.1 Existence

From Eq. (36), a steady state with $\theta \neq 0$ is a solution $k \in \mathbb{R}_+$ satisfying:

$$\begin{cases} \frac{u'(c(k))}{\beta v'(d(k))} = \frac{1}{1 + i^* \gamma(k)} \\ \theta = \frac{1 - \psi(k) \{1 + i^* \eta_1(k)[1 - \gamma(k)][\rho_1(f'(k)k)]^{-1}\}}{i^* \eta_1(k)[1 - \gamma(k)]/k} \end{cases} \quad (38)$$

$$\begin{aligned} \text{with } c(k) &= f(k) - k - \frac{k[1 - \psi(k)]}{i^* \eta_1(k)[1 - \gamma(k)]} - [kf'(k) - g(f'(k)k)], \\ d(k) &= \frac{k[1 - \psi(k)]}{i^* \eta_1(k)[1 - \gamma(k)]} \text{ and } \psi(k) = f'(k)g'(f'(k)k). \end{aligned}$$

From these equations, we deduce that $d(k) > 0$ implies $\psi(k) < 1$, and from Eqs. (28) and (31), $1 + i^* = \pi = \mu > 1$.

Proposition 1 *Let $\bar{k}$ defined by $c(\bar{k}) = 0$ and $\underline{k}$ by $\psi(\underline{k}) = 1$. Under Assumptions 1-4, there exists a steady state characterized by $-g(f'(k^*)k^*) < \theta^* < f(k^*) - k^* - f'(k^*)k^*$ and $k^* \in (\underline{k}, \bar{k})$. A positive bubble, i.e. $\theta^* > 0$, requires:*

$$[1 - \psi(k^*)] \frac{k^*}{g(f'(k^*)k^*)} > i^* \eta_1(k^*) [1 - \gamma(k^*)] \quad (39)$$

Proof. See Appendix B.

Proposition 1 indicates that a positive bubble on bonds exists at the steady state. Furthermore, the condition (39) is satisfied for small η_1 .

When there is no taxation ($g'(f'(k)k) = 1$) and collateral does not matter ($\eta_1(k) = 0$), we can see from Eq. (37) that the steady state is at the golden rule ($R(k) = 1$). As the well-known result of Tirole (1985), a positive rational asset bubble crowds out capital. When collateral matters ($\eta_1(k) > 0$), our economy experiences an over-accumulation of capital at the steady state ($R(k) < 1$). The existence of collateral incites households to hold more capital in their portfolio in order to relax the cash-in-advance constraint, and therefore, the capital return decreases. On the contrary, a taxation on capital income incites households to reduce capital accumulation. As long as there are some credit

market imperfections, implementing a tax on capital income can counteract the collateral effect, and thus reduce or even rule out over-accumulation of capital.

As regards the monetary policy, we recall that the central bank chooses the stationary values of an existing steady state for its targets. Since, the steady state k^* persists, we assume that the central bank selects this steady state, and therefore, $\pi^* = 1 + i^*$.

4.2 Normalized steady state and multiplicity

In order to facilitate the analysis of local dynamics (Sections 5 and 6), we establish the existence of a normalized steady state $k^* = 1$ (NSS). We follow the procedure introduced by Cazzavillan et al. (1998), and use the scaling parameter β to give conditions for the existence of such a steady state.

Assumption 5 *Let $\nu(\eta_1) = i^*\eta_1(1)[1 - \gamma(1)]$, we assume:*

$$f(1) - f'(1) + g(f'(1)) > 1 + \frac{1 - \psi}{\nu(\eta_1)}$$

Assumption 5 ensures that the first period consumption at the normalized steady state is positive (i.e. $c(1) > 0$), and it is satisfied when the productivity is sufficiently large.¹³

Proposition 2 *Under Assumptions 1-5, there exists a unique value $\beta^* > 0$ given by*

$$\beta^* = \frac{u' \left(f(1) - 1 - \frac{1 - \psi(1)}{\nu(\eta_1)} - [f'(1) - g(f'(1))] \right)}{v' \left(\frac{1 - \psi(1)}{\nu(\eta_1)} \right)} [1 + i^*\gamma(1)]$$

such that $k^* = 1$ is a steady state of the dynamic system (32). Assumption 5 ensures that $k^* = 1 \in (\underline{k}, \bar{k})$. Moreover, there is a positive bubble ($\theta^* > 0$) if $1 - \psi(1) - g(f'(1))\nu(\eta_1) > 0$.

Thereafter, we set $\beta = \beta^*$ so that $k^* = 1$. We further note $c^* \equiv c(1)$, $\gamma \equiv \gamma(1)$, $\eta_1 \equiv \eta_1(1)$, $\eta_2 \equiv \eta_2(1)$, $g \equiv g(f'(1))$, $\rho_1 \equiv \rho_1(1)$, $\rho_2 \equiv \rho_2(1)$, $\psi \equiv \psi(1)$, $\alpha = \alpha(1)$ and $\sigma = \sigma(1)$. At the normalized steady state, the second-order conditions are satisfied for $\varepsilon_u > \tilde{\varepsilon}_u$, with $\tilde{\varepsilon}_u \equiv c^* \frac{\nu(\eta_1)^2}{[\rho_2\psi + \eta_2(1 - \psi)](1 + i^*\gamma)^2}$ (see Assumption 3). We can now clarify the conditions for the multiplicity of steady states.¹⁴

¹³As an example, we can consider $f(k) = \tilde{A} \left(\alpha k^{\frac{\sigma-1}{\sigma}} + 1 - \alpha \right)^{\frac{\sigma}{\sigma-1}}$. Assumption 5 is satisfied for $\tilde{A} > 0$ large enough.

¹⁴We note $\varepsilon_{J_{x_i}}$ the elasticity of the function $J(x_1, \dots, x_n)$ with respect to x_i , i.e. $\varepsilon_{J_{x_i}} = \frac{\partial J(x_1, \dots, x_n)}{\partial x_i} \frac{x_i}{J(x_1, \dots, x_n)}$.

Proposition 3 *Under Assumptions 1-5, there exists a value ε_v^s such that there is a multiplicity of bubbly stationary equilibria if the following condition holds¹⁵:*

$$\varepsilon_v < \frac{\Omega}{\varepsilon_{dk}} \left(\frac{\varepsilon_u^s}{c^*} - \frac{\varepsilon_u}{c^*} \right) \equiv \varepsilon_v^s$$

Their number is generically odd.

Proof. See Appendix B.

As Clain-Chamosset-Yvrard and Seegmuller (2012), the multiplicity of steady states appears only when degrees of concavity are small enough ($\varepsilon_u < \varepsilon_u^s$ and $\varepsilon_v < \varepsilon_v^s$). Because the second-order conditions are satisfied for $\varepsilon_u > \tilde{\varepsilon}_u$, then $\varepsilon_u^s > \tilde{\varepsilon}_u$ must be satisfied. This holds true for η_1 small enough and η_2 large enough.¹⁶ This multiplicity implies the existence of several steady states with positive bubble (i.e. $\theta^* > 0$). From the second equation of (38), we can check that the size of the bubble at the steady state is increasing in the stationary capital stock. Through the collateral effect, an increase in stationary capital stock implies a decrease in stationary nominal balances, which entails a reallocation of savings towards the bubble.

We can consider the existence of multiple stationary equilibria as a form of global indeterminacy, which is a source of expectation-driven fluctuations of the bubble. We will see later that this multiplicity is connected with local indeterminacy, and that a sufficiently progressive fiscal policy or a low interest rate target can remove the multiplicity of steady states by reducing the range of parameters for the existence of this multiplicity.¹⁷

5 Expectation-driven fluctuations

Considering a framework with an interplay between the financial and the real spheres, this paper first aims to explain business fluctuations in presence of a speculative bubble. The violation of the Fisher relationship and the resulting portfolio choice between bonds, capital and money are the key ingredients to explain these fluctuations. In this section, we show that the steady state with a positive bubble on bonds can be locally indeterminate, and therefore, fluctuations driven by agents' self-fulfilling expectations can emerge. In particular, local indeterminacy occurs in the neighborhood of the normalized steady state with a positive bubble under gross substitutability and a not too weak input substitution because of credit market distortions.

¹⁵The expressions of Ω , ε_{dk} and ε_u^s are given in Appendix B.

¹⁶ $\varepsilon_u^s > \tilde{\varepsilon}_u$ is satisfied if and only if $\eta_2(1 - \psi)i^*\gamma > i^*\eta_1(1 - \gamma) \{1 - f'(1) + (1 - \frac{1-\alpha}{\sigma}) [f'(1) - \psi]\} + \psi \frac{1-\alpha}{\sigma} - \psi \rho_2 (\frac{1-\alpha}{\sigma} + i^*\gamma)$.

¹⁷Benhabib et al (2001) show however that monetary policies could have adverse effects according to the steady state of the economy.

5.1 Local dynamics: preliminaries

To derive our different results, we start by linearizing the dynamic system (32) around the normalized steady state $k^* = 1$ to obtain the characteristic polynomial. As shown in Appendix C.1, we can derive the trace $T(\varepsilon_v)$ and the determinant $D(\varepsilon_v)$ of the associated Jacobian matrix as functions of ε_v ¹⁸: $D(\varepsilon_v)$ linearly depends on $T(\varepsilon_v)$. As a result, we can apply the geometrical method developed by Grandmont et al. (1998) to discuss the local stability properties of the model.

We study the variations of the trace $T(\varepsilon_v)$ and the determinant $D(\varepsilon_v)$ in the (T, D) plane as ε_v is made to vary continuously in its admissible range $(0, +\infty)$. The locus $\Sigma \equiv \{(T(\varepsilon_v), D(\varepsilon_v)) : \varepsilon_v \geq 0\}$ describes a part of a line that we call the Σ -line.

Consider the (T, D) plane (see Figure 1). On line (AC) , one eigenvalue is equal to 1 ($D = T - 1$). On line (AB) , one eigenvalue is equal to -1 ($D = -T - 1$). Along segment $[BC]$ ($|T| < 2, D = 1$), the characteristic roots are complex conjugates with modulus equal to 1. These lines and this segment divide the space (T, D) into three different types of regions. Inside the triangle ABC , the steady state is a sink, i.e. locally indeterminate ($|T| < 1 + D$ and $D < 1$). It is a saddle point if (T, D) lies on the right or left sides of both the lines (AB) and (AC) ($|1 + D| < |T|$). It is a source otherwise.

A (local) bifurcation arises when at least one eigenvalue crosses the unit circle, that is, when the Σ -line crosses one of the loci (AB) , (AC) or $[BC]$. According to the changes of the bifurcation parameter, a pitchfork bifurcation (generically) emerges when the Σ -line crosses (AC) , as ε_v goes through ε_v^s .¹⁹ A flip bifurcation (generically) occurs when the Σ -line crosses (AB) , as ε_v goes through ε_v^f . Finally, a Hopf bifurcation (generically) arises when the Σ -line crosses the segment $[BC]$, as ε_v goes through ε_v^h .

We locate the Σ -line in the (T, D) plane by analyzing $(T(0), D(0))$, $(T(+\infty), D(+\infty))$ and its slope S (see Appendix C.2). This allows us to analyze the occurrence of local indeterminacy and endogenous cycles.

5.2 Fluctuations with a bubble

In this section, we show that expectation-driven fluctuations with a positive bubble on bonds may occur not only under large degrees of utility concavity, but also for arbitrarily small ones. Furthermore, the occurrence of fluctuations under gross substitutability is connected to a multiplicity of steady states, i.e. a form of global indeterminacy.

To highlight fluctuations with a bubble, we consider that the central bank fixes the nominal interest rate to a constant one i^* ($\phi = 0$), and no fiscal policy is implemented ($g(Rk) = Rk$). Furthermore, we limit our analysis to the case in which η_1 is arbitrarily small to have a positive bubble (see Section 4.1).

¹⁸See Lemma 3 in Appendix C.1.

¹⁹Indeed, we have (generically) an odd number of steady states (see Section 4.2).

Assumption 6 η_1 is arbitrarily small, and η_2 is arbitrarily large.

Illustration. The function $\gamma(k)$ given by Eq.(7) in Section 2.1 satisfies Assumption 6: η_1 is sufficiently small and η_2 is arbitrarily large at the normalized steady state for s large enough.

Local dynamics are studied through geometrical arguments, while technical details are relegated to Appendix C.2. To locate the Σ -line in the (T, D) plane, we first note that the value of the starting point $(T(0), D(0))$ is such that the Σ -line starts on the left-side of (AC) and inside the triangle ABC when $\varepsilon_u < \varepsilon_u^s$ and on the right-side of (AC) when $\varepsilon_u > \varepsilon_u^s$. The endpoint $(T(+\infty), D(+\infty))$ lies on the horizontal axis. More precisely, it is inside the triangle ABC with $T(+\infty) \in (-1, 0)$.

Whatever the degree of utility concavity ε_u , we can prove that under Assumptions 1 – 6, the Σ -line has a slope $S \in (0, 1)$. Moreover, the determinant $D(\varepsilon_v)$ and the trace $T(\varepsilon_v)$ are increasing in ε_v , and the bifurcation values are such that:

$$\varepsilon_v^s < \varepsilon_v^h < \varepsilon_v^f, \quad (40)$$

As a consequence, we can deduce that the Σ -line goes below the point C . Furthermore, we can easily check that $\varepsilon_v^s < 1$ and $\varepsilon_v^h > 1$ under Assumptions 1 – 6. As a result, when $\varepsilon_u < \varepsilon_u^s$, the Σ -line starts inside the triangle ABC , crosses (AC) below C , then (AB) between A and B , and ends inside the triangle ABC at $(T(+\infty), D(+\infty))$ on the horizontal axis. When $\varepsilon_u > \varepsilon_u^s$, the configuration is similar, except that the Σ -line starts outside the triangle ABC on the right-side of (AC) .

Proposition 4 *Under Assumptions 1-6, the following generically holds:*

1. When $\varepsilon_u \in (\bar{\varepsilon}_u, \varepsilon_u^s)$, the steady state is a sink for $\varepsilon_v < \varepsilon_v^s < 1$, undergoes a pitchfork bifurcation for $\varepsilon_v = \varepsilon_v^s$, is a saddle for $\varepsilon_v \in (\varepsilon_v^s, \varepsilon_v^f)$, undergoes a flip bifurcation for $\varepsilon_v = \varepsilon_v^f$, and is a sink for $\varepsilon_v > \varepsilon_v^f$.
2. When $\varepsilon_u > \varepsilon_u^s$, the steady state is a saddle for $\varepsilon_v < \varepsilon_v^f$, undergoes a flip bifurcation for $\varepsilon_v = \varepsilon_v^f$, and is a sink for $\varepsilon_v > \varepsilon_v^f$.

Proof. See Appendix C.2.

Proposition 4 shows the occurrence of persistent endogenous fluctuations around the steady state with a positive bubble under gross substitutability and a not too weak capital-labor substitution. This result extends Bosi and Seegmuller (2010a) and Clain-Chamosset-Yvrard and Seegmuller (2012) to a model with inside money.²⁰ In addition, it is important to note that the occurrence

²⁰In these two papers, the stabilizing role of policies is not addressed in the same way as here. Indeed, the fiscal policy is ignored and the monetary authority directly manages the money growth factor, while it fixes the interest rate in our framework.

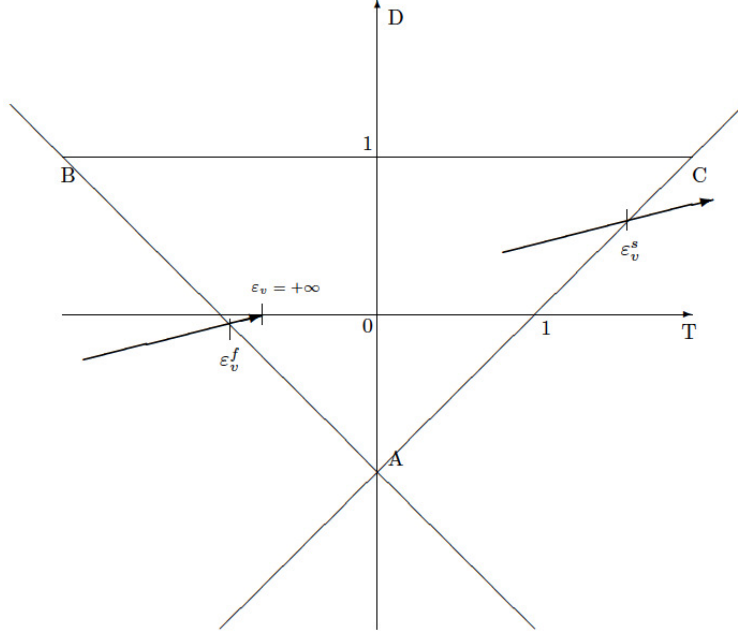


Figure 1: Fluctuations of a bubble

of fluctuations under gross substitutability is connected to the multiplicity of steady states (see Proposition 3).

When collateral does not matter ($\eta_1 = 0$), the local stability properties of the model correspond to Proposition 4.2. Our model exhibits endogenous fluctuations and two-period cycles only for large degrees of concavity on $u(c)$ and $v(d)$, i.e. for a significant income effect.²¹ More interestingly, when collateral matters ($\eta_1 > 0$), local indeterminacy also appears for small degrees of concavity on $u(c)$ and $v(d)$, in particular under gross substitutability ($\varepsilon_v < \varepsilon_v^s < 1$).

The basic mechanism for fluctuations under gross substitutability relies on a portfolio trade-off between the three assets. Because of the difference between the returns on physical and monetary assets: a reallocation between the assets takes place following a modification in agents' expectations.

Economic intuition. If households expect an increase in inflation from period t to $t + 1$, the return on bonds becomes less attractive compared to the return on capital. Because of the portfolio effect, households reallocate their savings towards capital. As a consequence, when $\varepsilon_v < \varepsilon_v^s < 1$, the portfolio effect can accelerate capital accumulation. Households consume less by cash (see Eq. (27)). The real balances m_{t+1} decrease, entailing a decrease in the return

²¹Since $\varepsilon_v > \varepsilon_v^f > 1$, income effects dominate substitution effects. Hence, global savings ($\theta_t + k_{t+1}$) are a decreasing function of their return.

on money. An effective rise in inflation takes place. The initial expectations are self-fulfilling.

Now, we will be interested in the stabilizing role of different economic policies. A policy is stabilizing in our framework as soon as it reduces the range of parameters for expectation-driven fluctuations. In the following, we will focus on empirically plausible fluctuations, namely fluctuations occurring when savings are increasing in their global return. Since savings are an increasing function of the portfolio global return when $\varepsilon_v < 1$, it appears relevant to focus on the stabilizing role of fiscal and monetary policies under gross substitutability. Hence, we will focus on the most interesting case, i.e. Proposition 4.1.

Assumption 7

$$\varepsilon_u < \varepsilon_u^s \text{ and } \varepsilon_v < 1.$$

We now investigate the second aim of this paper. We analyze the stabilizing role of fiscal and monetary policies, and compare them.

6 The stabilizing role of fiscal vs. monetary policy

In this section, we study the stabilizing role of fiscal and monetary policies on expectation-driven fluctuations with a bubble. As expectation-driven fluctuations are mainly driven by portfolio choices between capital, money and bonds, we consider policy tools that are able to modify the return on assets. We first examine the role of progressive taxation of capital income on bubble fluctuations. Second, we consider the monetary policy with an interest rate rule responding to inflation, then a direct management of the interest rate regardless the inflation. We will show that a progressive taxation of capital income and a direct management of the interest rate are most powerful to stabilize than a Taylor rule.

6.1 The stabilizing role of fiscal policy ($\phi = 0$)

To keep things as simple as possible, we consider a fixed interest rate $i_{t+1} = i^*$, and restrict our analysis to the cases where the marginal tax rate is not too high and/or an elasticity of capital-labor substitution not too small:

Assumption 8

$$\sigma \geq \frac{1 - \alpha}{g'(f'(1))} \equiv \hat{\sigma}$$

For the dynamic analysis, we focus on geometrical arguments. We show that under Assumptions 1 – 8, we obtain the same configuration as Proposition 4.1 and the same figure as in Section 5.2. The location of the Σ -line is given in Appendix C.2. When $\varepsilon_u < \varepsilon_u^s$, the Σ -line starts inside the triangle ABC , crosses

(AC) below C , then (AB) between A and B , and ends inside the triangle ABC at $(T(+\infty), D(+\infty))$ on the horizontal axis. Under Assumptions 1 – 8, local indeterminacy occurs if $\varepsilon_v < \varepsilon_v^s$.

We recall that our aim is to determine whether a fiscal policy can stabilize expectation-driven fluctuations around the bubbly steady state under gross substitutability. To do this, we examine how the critical bifurcation value ε_v^s varies as a function of ρ_2 to get a picture of the role of progressivity of capital income taxation on local indeterminacy. See also Figure 2 for a qualitative illustration.

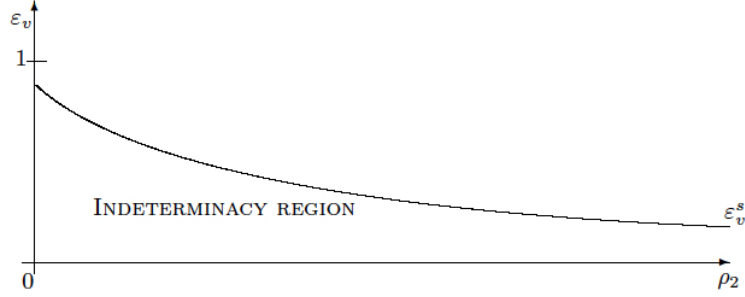


Figure 2: Stabilizing role of fiscal policy

Proposition 5 *Under Assumptions 1-8, we have $\partial \varepsilon_v^s / \partial \rho_2 < 0$. Therefore, increasing the marginal progressivity of capital income taxation reduces the range of parameters for local indeterminacy around $k^* = 1$, and the range of parameters for the multiplicity of steady states, i.e. global indeterminacy.*

Proof. See Appendix D.

Proposition 5 indicates that increasing the degree of marginal progressivity may rule out expectation-driven fluctuations under gross substitutability.

In the literature and in particular in the Ramsey models with a representative agent, the most prominent mechanism for the occurrence of endogenous fluctuations in the neighborhood of a steady state relies on the existence of the so-called wrong slopes on the labor market. In such models, the stabilizing tool is per se labor income taxation, and progressivity may rule out these wrong slopes. This stabilizing virtue of progressivity in labor income taxation has been underlined by Guo and Lansing (1998). In a Ramsey model with heterogeneous agents (see Bosi and Seegmuller (2010b)), a larger progressivity in capital income taxation promotes endogenous fluctuations because it makes the after-tax interest rate increasing in capital. In our paper, the mechanism is quite different, and relies on the portfolio choice between the different assets. A progressive capital income tax reduces the gaps between the returns on different assets along the dynamic path. More precisely, increasing the degree of

progressivity of capital income taxation counteracts the collateral effect, and thus mitigates the failure of the Fisher equation (see Eq. (12)).

Note also that even though capital income taxation is a distorting instrument, it goes in an opposite direction than the collateral effect at the steady state. Indeed, it reduces or even rules out the over-accumulation of capital at all steady states.²²

We have shown that a fiscal policy is therefore powerful to locally and globally stabilize. We can wonder now whether a monetary policy can have similar stabilizing virtues.

6.2 The stabilizing role of monetary policy ($g(Rk) = Rk$)

We examine the stabilizing virtues of different monetary rules on bubble fluctuations. To highlight the role of monetary policy, we consider the model without taxation on capital income ($g(Rk) = Rk$). First, we investigate the stabilizing virtues of a Taylor rule ($\phi > 0$). Second, we wonder whether a direct management of the nominal interest rate independently of inflation could be relevant to prevent from expectation-driven fluctuations with a speculative bubble ($\phi = 0$). Under gross substitutability such a monetary policy is powerful to prevent our economy with a speculative bubble from macroeconomic fluctuations.

6.2.1 The stabilizing role of monetary policy under a Taylor rule ($\phi > 0$)

In this section, we examine how local dynamics are modified by the implementation of an interest rate rule given by Eq. (23). To facilitate as possible this study, we make the following assumption:

Assumption 9 $f'(1) > \frac{1}{1+i^*\gamma}$

Assumption 9 ensures that money is a dominated asset, i.e. has a lower return than capital.

Enriching the model with an interest rate rule generates new configurations as regards the range of parameter values for which local indeterminacy around the normalized steady state occurs.

We conduct the analysis considering the variable $a_\phi = \phi/(1 - \phi)$ instead of ϕ . We note that $a_\phi < -1$ corresponds to an active rule ($\phi > 1$), and $a_\phi > 0$ corresponds to a passive rule ($\phi < 1$). As shown in Appendix E.1, for $a_\phi \in]-\infty, -1] \cup]0, \bar{a}_\phi]$, local indeterminacy occurs when $\varepsilon_v > \max(\varepsilon_v^f, \varepsilon_v^s)$ and when $\varepsilon_v < \min(\varepsilon_v^h, \varepsilon_v^s)$. For $a_\phi \in (\bar{a}_\phi, \bar{a}_\phi)$, local indeterminacy occurs when $\varepsilon_v < \varepsilon_v^s$. For $a_\phi \in (\bar{a}_\phi, +\infty)$, local indeterminacy occurs when $\varepsilon_v > \varepsilon_v^s$ (see Figure 3).²³ Now, we are interested in the variations of these critical values with respect to a_ϕ in order to analyze the stabilizing role of a monetary policy under a Taylor rule. See also Figure 3 for a qualitative illustration.

²²In our framework, an analysis of the welfare at the steady state appears however difficult to conduct.

²³The expressions of $\bar{a}_\phi$ and $\bar{a}_\phi$ are given in Appendix E.1.

Lemma 2 *Under Assumptions 1 – 9, the impact of the monetary policy on the bifurcation values are given by the following derivatives²⁴:*

$$\frac{\partial \varepsilon_v^f}{\partial a_\phi} > 0, \quad \frac{\partial \varepsilon_v^h}{\partial a_\phi} > 0 \quad \text{and} \quad \frac{\partial \varepsilon_v^s}{\partial a_\phi} = 0.$$

Proof. See Appendix E.1.

Appendix E.1 also allows us to construct Figure 3 that summarizes the stability properties of the economy. Grey areas correspond to indeterminacy regions. The next proposition summarizes the stabilizing role of this interest rate rule:

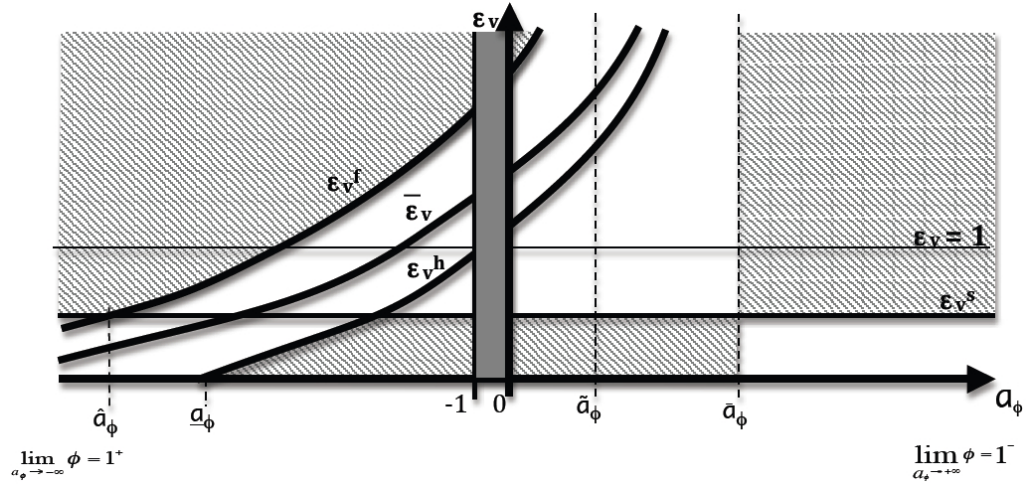


Figure 3: Stabilizing role of monetary policy under a Taylor rule

Proposition 6 *Under Assumptions 1 – 9, the following generically holds²⁵:*

1. *If $a_\phi \in]-\infty, \hat{a}_\phi]$ or if $a_\phi \in [\bar{a}_\phi, +\infty[$, local indeterminacy is ruled out for $\varepsilon_v < \varepsilon_v^s$, but occurs whatever $\varepsilon_v > \varepsilon_v^s$.*
2. *If $a_\phi \in (\hat{a}_\phi, -1)$, increasing the degree of responsiveness of the rule with respect to the expected inflation (ϕ) reduces the range of parameters for local indeterminacy when ε_v is large enough, but raises the range of parameters for local indeterminacy when ε_v is small enough.*

²⁴The functions ε_v^f and ε_v^h are homographics. ε_v^f has a vertical asymptote for $a_\phi = \bar{a}_\phi$, and ε_v^h has one for $a_\phi = \hat{a}_\phi$.

²⁵ $\hat{a}_\phi$ is given by $\varepsilon_v^f = \varepsilon_v^s$.

3. If $a_\phi \in (0, \bar{a}_\phi)$, increasing the degree of responsiveness of the rule with respect to the expected inflation (ϕ) reduces the range of parameters for local indeterminacy when ε_v is large enough, but has no impact on the range of parameters for local indeterminacy when ε_v is small enough.

Proposition 6 highlights mitigated results. In contrast to Sorger (2005) and Clain-Chamosset-Yvrard and Seegmuller (2012) where the monetary policy is however determined by the money growth factor, no clear-cut conclusion is outlined. Indeed, a weakly active or passive rule (i.e. $a_\phi \in]-\infty, \hat{a}_\phi[\cup]\bar{a}_\phi, +\infty[$) tends to destabilize promoting indeterminacy for $\varepsilon_v^s < \varepsilon_v < 1$. Recalling that the multiplicity of steady states occurs for $\varepsilon_v < \varepsilon_v^s$, Lemma 2 allows us to show:

Proposition 7 *Under Assumptions 1–9, conducting an interest rate rule given by Eq. (23) neither promotes nor rules out the multiplicity of steady states. The range of parameters for the multiplicity of steady states is not altered by such a policy.*

Our conclusion confirms the result of Clain-Chamosset-Yvrard and Seegmuller (2012): a monetary policy rule which responds only to expected inflation has no impact on the multiplicity of steady states.

A Taylor interest rate rule has no clear stabilizing virtues. One possible explanation is that it manages the level of the elasticity of the nominal interest rate with respect to the expected inflation (ϕ) and not directly the interest rate target (i^*). For instance, a weakly active or passive rule tends to destabilize for $\varepsilon_v > \varepsilon_v^s$, because such a rule has not a huge impact on the nominal interest rate, and therefore does not modify so much the portfolio choices. For that reason, we consider now an alternative monetary policy which directly manages the nominal interest rate independently on the inflation, as an interest rate pegging.

6.2.2 The stabilizing role of direct management of the interest rate ($\phi = 0$)

By such a monetary policy, we mean that the central bank fixes the nominal interest rate at a constant rate i^* . We analyze how the local stability properties of the model are modified following a change in the level of the interest rate target. We will see that decreasing the level of the target may rule out expectation-driven fluctuations with a positive rational bubble.

Under Assumption 7, we stay in Proposition 4.1.²⁶ Local indeterminacy arises for $\varepsilon_v < \varepsilon_v^s$. Therefore, we examine now how the bifurcation value ε_v^s varies with respect to i^* . See also Figure 4 for a qualitative illustration.

Proposition 8 *Under Assumptions 1-7, we have $\partial \varepsilon_v^s / \partial i^* > 0$. Therefore, a higher nominal interest rate target i^* increases the range of parameters for local indeterminacy around $k^* = 1$, and the range of parameters for the multiplicity of steady states, i.e. global indeterminacy.*

²⁶Proof of Proposition 4 given in Appendix C.2 is satisfied whatever the level of the nominal interest rate i^* satisfying Assumptions 1 – 6.

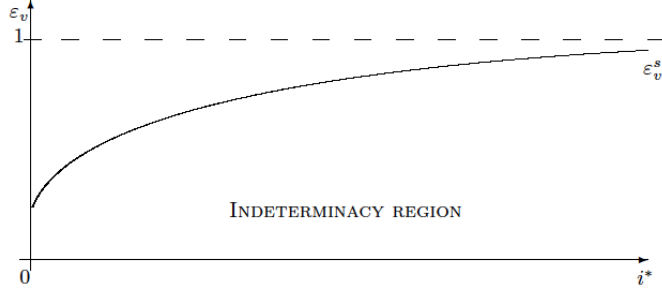


Figure 4: Stabilizing role of direct management of the interest rate

Proof. See Appendix E.2.

Proposition 8 shows that when $\varepsilon_v < 1$, a lower nominal interest rate i^* reduces the range of ε_v that guarantees indeterminacy. Under gross substitutability, lowering the level of the nominal interest rate contributes to globally and locally stabilize our economy by ruling out endogenous fluctuations and the multiplicity of steady states.

This result can be related to some previous contributions. Indeed, in a model without collateral effect, Rochon and Polemarchakis (2006) show that a weak interest rate can prevent the economy from endogenous fluctuations. However, the underlying mechanism is different since indeterminacy can only occur under a negative bubble in their framework. The monetary policy has also been investigated when the monetary authority directly manages the money growth factor. In an exchange economy, Bosi and Seegmuller (2013) show that an expansionary policy may rule out expectation-driven fluctuations that may occur however only for large income effects. In contrast, Bosi and Seegmuller (2010a) get the opposite result in an economy with capital accumulation, but without inside money.

In our framework, the monetary policy based on a direct management of the interest rate independently on expected inflation appears to be the most powerful to stabilize endogenous business cycles, and to prevent from global indeterminacy associated to the multiplicity of steady states.

However, a stabilizing policy may deteriorate the welfare at the steady state. In order to examine the trade-off between benefits and costs of the monetary policy, we evaluate the welfare at the stationary equilibrium.

Effect of monetary policy on the stationary welfare. The presence of credit constraints affected by collateral does not only play a crucial role on the existence of expectation-driven fluctuations, but allows the financial sphere to affect the real one. As a consequence, the dichotomy between monetary and real spheres is no longer relevant. Because of the role of collateral, the Fisher relationship is no more satisfied: the level of the nominal interest rate has real effects on the steady state. The superneutrality of money is canceled, and the

monetary policy alters the welfare at the steady state.²⁷

Proposition 9 *Under Assumptions 1 – 7, the effect of the interest rate target on the welfare is the following:*

1. *If $\varepsilon_v < \varepsilon_v^s(k)$, the welfare is an increasing function of the interest rate target i^* .*
2. *If $\varepsilon_v > \varepsilon_v^s(k)$, the welfare is a decreasing function of the interest rate target i^* .*

Proof. See Appendix E.3.

Proposition 9 indicates that for a sufficiently small degree of utility concavity ε_v , a decrease in the interest rate target deteriorates the welfare at the steady state. For an arbitrarily large degree of utility concavity, we reverse this result. When collateral does not matter ($\eta_1(k) = 0$), the welfare properties of the model correspond to the second case of Proposition 9. Thus, the Friedman rule, for which setting the nominal interest rate target at zero is welfare maximizing, is satisfied. By fixing the level of the nominal interest rate at zero, the monetary authority eliminates the distortion associated to money holdings, which involves the highest level of welfare.

When collateral matters ($\eta_1(k) > 0$), the Friedman rule does not hold for a small degree of concavity ε_v . One possible explanation relies on the portfolio choice. An increase in the nominal interest rate target generates a reallocation of savings towards capital. We recall that capital income is increasing in capital since we assume large input substitution. As a result, the welfare improves because households get a higher capital income and spend less through money holdings (collateral effect).

We deduce that under gross substitutability, a stabilizing goal of the monetary authority can damage the welfare as long as $\varepsilon_v < \varepsilon_v^s$. Nevertheless, decreasing the interest rate target improves the welfare as soon as there is no more room for fluctuations.

Therefore, the stabilizing role of the monetary policy has to be considered with caution since the welfare gain of stabilizing is usually small regarding the effect of the monetary policy on the steady state.

7 Concluding remarks

We develop an overlapping generations model with capital accumulation, bonds and money, where the share of consumption purchased on credit depends on the collateral. This allows us to show the existence of expectation-driven fluctuations with a positive rational bubble on bonds. This occurs when the credit share is weakly increasing, but sufficiently concave in the collateral. In addition,

²⁷When the Fisher relationship is not satisfied, we can highlight Tobin/anti-Tobin effects. Depending on values on fundamentals, the nominal interest rate enhances or dampens the capital accumulation at the steady state.

endogenous fluctuations are in accordance with gross substitutability and a not too weak substitution between inputs. The basic mechanism for fluctuations relies on a portfolio trade-off between the three assets due to the violation of the Fisher relationship.

This framework is also used to compare the stabilizing role of some fiscal and monetary policies. We show that a progressive taxation of capital income may rule out expectation-driven fluctuations, the multiplicity of steady states, and even over-accumulation of capital. When the monetary policy is fixed according to a Taylor rule on expected inflation, the results are more mitigated. One reason is that such a rule does not alter so much the portfolio choices. However, when we focus on a direct management of the level of the nominal interest rate, i.e. when the nominal interest rate is set independently of inflation, a sufficiently low interest rate may rule out expectation-driven fluctuations and the multiplicity of steady states. However, such a policy may be detrimental for welfare. To summarize, in contrast to a Taylor rule, a progressive taxation of capital income and a direct management of the interest rate are powerful to stabilize the economy which experiences a positive bubble.

8 Appendix

Appendix A

Proof of Lemma 1

We maximize the Lagrangian function:

$$\begin{aligned}\mathcal{L} = & u(c_t) + \beta v(d_{t+1}) \\ & + \lambda_{1t}(w_t - \pi_{t+1}m_{t+1} - \pi_{t+1}b_{t+1} - k_{t+1} - c_t) \\ & + \lambda_{2t}(m_{t+1} + (1 + i_{t+1})b_{t+1} + g(R_{t+1}k_{t+1}) + \delta_{t+1} - d_{t+1}) \\ & + \lambda_{3t}(m_{t+1} - \gamma(k_{t+1})d_{t+1})\end{aligned}\quad (41)$$

with respect to $(c_t, d_{t+1}, m_{t+1}, b_{t+1}, q_t, \lambda_{1t}, \lambda_{2t}, \lambda_{3t})$.

$$\frac{\partial L()}{\partial c} = u'(c_t) - \lambda_{1t} = 0 \quad (42)$$

$$\frac{\partial L()}{\partial d} = \beta v'(d_{t+1}) - \lambda_{2t} - \gamma(k_{t+1})\lambda_{3t} = 0 \quad (43)$$

$$\frac{\partial L()}{\partial m} = -\pi_{t+1}\lambda_{1t} + \lambda_{2t} + \lambda_{3t} = 0 \quad (44)$$

$$\frac{\partial L()}{\partial b} = -\pi_{t+1}\lambda_{1t} + (1 + i_{t+1})\lambda_{2t} = 0 \quad (45)$$

$$\frac{\partial L()}{\partial k} = -\lambda_{1t} + R_{t+1}g'(R_{t+1}k_{t+1})\lambda_{2t} - \lambda_{3t}\gamma'(k_{t+1})d_{t+1} = 0 \quad (46)$$

Since $\lambda_{1t} = u'(c_t) > 0$, $i_{t+1} > -1$ and from Eq. (45) $\lambda_{2t} = \lambda_{1t} \frac{\pi_{t+1}}{1+i_{t+1}} > 0$, then the constraints (8)-(9) become binding. From Eq. (44), we obtain:

$$\lambda_{3t} = \lambda_{1t} \frac{i_{t+1}\pi_{t+1}}{1+i_{t+1}} \quad (47)$$

The strict positivity of λ_{3t} requires $i_{t+1} > 0$.

From Eqs. (42)-(43) and the expressions of λ_{2t} and λ_{3t} , we get Eq. (11). In addition, substituting the expressions of λ_{2t} and λ_{3t} into Eq. (46), we obtain Eq. (12).

Now, we can compute the Hessian matrix of the Lagrangian function (41) with respect to $(\lambda_{1t}, \lambda_{2t}, \lambda_{3t}, c_t, d_{t+1}, m_{t+1}, b_{t+1}, k_{t+1})$ ²⁸:

$$H_{88} \equiv \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & -\pi & -\pi & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1+i & Rg'(Rk) \\ 0 & 0 & 0 & 0 & -\gamma & 1 & 0 & -\gamma'd \\ -1 & 0 & 0 & u'' & 0 & 0 & 0 & 0 \\ 0 & -1 & -\gamma & 0 & \beta v'' & 0 & 0 & -\gamma'\lambda_3 \\ -\pi & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -\pi & 1+i & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & Rg'(Rk) & -\gamma'd & 0 & -\gamma'\lambda_3 & 0 & 0 & R^2g''(Rk)\lambda_2 - \lambda_3\gamma''d \end{bmatrix} \quad (48)$$

In order to get a strict local maximum, we need to check the negative definition of H_{88} over the set of points satisfying the constraints. Let p and n the numbers of constraints and variables. If the determinant of H_{88} has sign $(-1)^n$ and the last $n-p$ diagonal principal minors have alternating signs, then the optimum is a regular local maximum. In our case $n = 5$ and $p = 3$. Therefore, we need to compute the last two diagonal principal minors, that is $\det H_{88}$ and $\det H_{77}$.²⁹ Moreover, we require $\det H_{88} < 0$ and $\det H_{77} > 0$.

$$\begin{aligned} \det H_{88} &= -u''\pi^2(1+i\gamma)^2[R^2g''\lambda_2 - d\gamma''\lambda_3] - \beta v''(1+i)^2[R^2g''\lambda_2 - d\gamma''\lambda_3] \\ &\quad + (1+i)^2(\lambda_3\gamma')^2 \\ &= (\pi\lambda_1)^2 \frac{i\gamma'}{k} \left\{ \frac{\varepsilon_u}{c} \frac{(1+i\gamma)^2\pi}{1+i} \left[\frac{\rho_2\rho_1g}{i\eta_1(1-\gamma)} + \eta_2d \right] \right. \\ &\quad \left. + \frac{\varepsilon_v}{d}(1+i\gamma) \left[\frac{\rho_2\rho_1g}{i\eta_1(1-\gamma)} + \eta_2d \right] - i\eta_1(1-\gamma) \right\} \end{aligned} \quad (49)$$

As $\gamma' < 0$, $\det H_{88} < 0$ if and only if:

$$\frac{\varepsilon_u}{c} \frac{(1+i\gamma)^2\pi}{1+i} \left[\frac{\rho_2\rho_1g}{i\eta_1(1-\gamma)} + \eta_2d \right] + \frac{\varepsilon_v}{d}(1+i\gamma) \left[\frac{\rho_2\rho_1g}{i\eta_1(1-\gamma)} + \eta_2d \right] - i\eta_1(1-\gamma) > 0 \quad (50)$$

²⁸For simplicity, the arguments of the functions and the time subscripts are omitted.

²⁹ H_{88} is the Hessian matrix, while H_{77} is the Hessian matrix minus the last column and the last line.

While

$$\det H_{77} = -u''\pi^2(1+i\gamma)^2 - (1+i)^2\beta v'' > 0 \quad (51)$$

For $\eta_2 > 0$, $\frac{\varepsilon_u}{c} \frac{(1+i\gamma)^2\pi}{1+i} \left[\frac{\rho_2\rho_1g}{i\eta_1(1-\gamma)} + \eta_2d \right] - i\eta_1(1-\gamma) > 0$ is a sufficient condition for $\det H_{88} < 0$. Thus, for $\eta_2 > 0$, the second-order conditions are satisfied if³⁰:

$$\varepsilon_u > c \frac{(1+i)}{\pi} \frac{i\eta_1(1-\gamma)}{\left[\frac{\rho_2\rho_1g}{i\eta_1(1-\gamma)} + \eta_2d \right] (1+i\gamma)^2} \equiv \tilde{\varepsilon}_u \quad (52)$$

whatever the value of $\varepsilon_v \geq 0$. ■

Appendix B

Proof of Proposition 1

A steady state k is a solution of $h(k) = j(k)$, with:

$$h(k) \equiv \frac{u'(c(k))}{\beta v'(d(k))} \quad (53a) \quad j(k) \equiv \frac{1}{1+i^*\gamma(k)} \quad (53b)$$

where $c(k) \equiv f(k) - k - \frac{k[1-\psi(k)]}{i^*\eta_1(k)[1-\gamma(k)]} - [kf'(k) - g(f'(k)k)]$ and $d(k) \equiv \frac{k[1-\psi(k)]}{i^*\eta_1(k)[1-\gamma(k)]}$, with $\psi(k) \equiv f'(k)g'(f'(k)k)$.

We start by determining the admissible range of values for k . To ensure $d(k) > 0$, we get at the steady state $\psi(k) < 1$. Under Assumptions 1 and 4, $\psi(k)$ is a decreasing function of k . Hence, $k > \psi^{-1}(1) = \underline{k}$.

Now, we want to determine the range of k such that $c(k) > 0$. The decreasing returns on capital imply:

$$f(\underline{k}) > \underline{k}f'(\underline{k})$$

Since $\psi(\underline{k}) = 1$, one has $f'(\underline{k}) = \frac{1}{g'(f'(\underline{k})\underline{k})}$. Under Assumption 1, $g(y_k)$ is concave, i.e. $g(f'(k)k) > g'(f'(k)k)f'(k)k$. This involves that:

$$g(f'(\underline{k})\underline{k}) > \underline{k}$$

Hence, we deduce that:

$$c(\underline{k}) = f(\underline{k}) - \underline{k}f'(\underline{k}) + g(f'(\underline{k})\underline{k}) - \underline{k} > 0$$

³⁰For simplicity, the arguments of the functions and the time subscripts are omitted.

In addition, as $d(k) > 0$ and $\tau(f'(k)k) = kf'(k) - g(f'(k)k) > 0$, we derive the following inequality:

$$\lim_{k \rightarrow +\infty} c(k) < \lim_{k \rightarrow +\infty} f(k) - k = -\infty$$

because $f'(k) < 1$ for k large enough. As a result, there exists one value $\bar{k}$ such that $\forall k < \bar{k}$, $c(\bar{k}) > 0$. By construction, we have $\underline{k} < \bar{k}$, and therefore $(\underline{k}, \bar{k})$ is a nonempty subset.

To prove the existence of a stationary solution k , we use the continuity of $h(k)$ and $j(k)$. Using Eqs. (53a) and (53b), we determine the boundary values of $h(k)$ and $j(k)$:

$$\begin{aligned} \lim_{k \rightarrow \underline{k}} h(k) &= \frac{u'(c(\underline{k}))}{\beta v'(0)} = 0^+ & \lim_{k \rightarrow \bar{k}} h(k) &= \frac{u'(c(0))}{\beta v'(d(\bar{k}))} + \infty \\ \lim_{k \rightarrow \underline{k}} j(k) &= \frac{1}{1 + i(\underline{k})\gamma(\underline{k})} \in]0, 1] & \lim_{k \rightarrow \bar{k}} j(k) &= \frac{1}{1 + i(\bar{k})\gamma(\bar{k})} \leq 1 \end{aligned}$$

We have $\lim_{k \rightarrow \underline{k}} h(k) < \lim_{k \rightarrow \underline{k}} j(k)$ and $\lim_{k \rightarrow \bar{k}} h(k) > \lim_{k \rightarrow \bar{k}} j(k)$. Therefore, there exists at least one value $k^* \in (\underline{k}, \bar{k})$ such that $h(k^*) = j(k^*)$. ■

Proof of Proposition 3

Let

$$\varepsilon_{dk} \equiv \frac{d'(1)}{d(1)} = \frac{\psi}{1 - \psi} \left[\frac{1 - \alpha}{\sigma} + \rho_2 \left(1 - \frac{1 - \alpha}{\sigma} \right) \right] + \eta_2, \quad (54)$$

$$\nu(\eta_1) = i^* \eta_1 (1 - \gamma), \text{ and} \quad (55)$$

$$c^* = f(1) - 1 - \frac{1 - \psi}{\nu(\eta_1)} - [f'(1) - g(f'(1))] \quad (56)$$

To deal with the multiplicity of stationary solutions, we derive the following elasticities at the normalized steady state $k^* = 1$:

$$\begin{aligned} \epsilon_h \equiv \frac{h'(1)}{h(1)} &= \varepsilon_v \varepsilon_{dk} + \frac{\varepsilon_u}{c^*} \left\{ 1 - f'(1) + \frac{1 - \psi}{\nu(\eta_1)} \varepsilon_{dk} \right. \\ &\quad \left. + f'(1) \left(1 - \frac{1 - \alpha}{\sigma} \right) [1 - g'(f'(1))] \right\} \\ \epsilon_j \equiv \frac{j'(1)}{j(1)} &= \frac{\nu(\eta_1)}{1 + i^* \gamma} \end{aligned} \quad (57)$$

We can now derive the conditions for the multiplicity of steady states. Let

$$\Omega \equiv 1 - f'(1) + \frac{1 - \psi}{\nu(\eta_1)} \varepsilon_{dk} + f'(1) \left(1 - \frac{1 - \alpha}{\sigma} \right) [1 - g'(f'(1))] \quad (58)$$

$$\varepsilon_u^s \equiv c^* \Omega^{-1} \frac{\nu(\eta_1)}{1 + i^* \gamma} \quad (59)$$

A sufficient condition for the multiplicity of steady states is $\varepsilon_h < \varepsilon_j$ for $k^* = 1$. This is equivalent to the inequality written in Proposition 3, that is³¹:

$$\varepsilon_v < \frac{\Omega}{\varepsilon_{dk}} \left(\frac{\varepsilon_u^s}{c^*} - \frac{\varepsilon_u}{c^*} \right) \equiv \varepsilon_v^s \quad (60)$$

Using the notations of the proof of Proposition 1, we know that $h(\bar{k}) > j(\bar{k})$ and $h(\underline{k}) < j(\underline{k})$. Since $k^* = 1$ is a steady state, we have: $h(1) = j(1)$. If the inequality written in Proposition 3 is satisfied, we have $\varepsilon_h(1) < \varepsilon_j(1)$, then by continuity at least two other steady states exist, k_1 and k_2 such that $k_1 < 1 < k_2$. The number of steady states is generically odd. ■

Appendix C

C.1 Proofs for Section 5.1

Linearized dynamic system

We linearize the system (32) around a steady state $k^* = 1$ with respect to $(k_t, \theta_t, k_{t+1}, \theta_{t+1})$. Let $\nu(\eta_1) \equiv i^* \eta_1 (1 - \gamma) > 0$ and $a_\phi = \frac{\phi}{1-\phi}$, we obtain³²:

$$\begin{aligned} & \left[\varepsilon_v \frac{\nu(\eta_1)\psi}{1-\psi} \left(1 - \frac{1-\alpha}{\sigma} \right) + \frac{\varepsilon_u}{c^*} - \frac{\nu(\eta_1)}{1+i^*\gamma} \right] \frac{dk_{t+1}}{k} \\ & + \left[\varepsilon_v \frac{1-\psi-g\nu(\eta_1)}{1-\psi} - 1 - a_\phi \frac{\gamma(1+i^*)}{1+i^*\gamma} \right] \frac{d\theta_{t+1}}{\theta} \\ & = \frac{\varepsilon_u}{c^*} f'(1) \frac{1-\alpha}{\sigma} \frac{dk_t}{k} - \left[1 + \frac{\varepsilon_u}{c^*} \frac{1-\psi-gi^*\eta_1(1-\gamma)}{\nu(\eta_1)} + a_\phi \frac{\gamma(1+i^*)}{1+i^*\gamma} \right] \frac{d\theta_t}{\theta} \\ & - (1-\psi) \left\{ \frac{\psi}{1-\psi} \left[\frac{1-\alpha}{\sigma} + \rho_2 \left(1 - \frac{1-\alpha}{\sigma} \right) \right] + \eta_2 - \frac{\nu(\eta_1)\psi}{1-\psi} \left(1 - \frac{1-\alpha}{\sigma} \right) \right\} \frac{dk_{t+1}}{k} \\ & - \left[\psi + g\nu(\eta_1) + a_\phi (1-\psi) \frac{1+i^*}{i^*} \right] \frac{d\theta_{t+1}}{\theta} = - \left[1 + (1-\psi) a_\phi \frac{1+i^*}{i^*} \right] \frac{d\theta_t}{\theta} \end{aligned}$$

From these above dynamic equations, we derive the characteristic polynomial.

Let

$$\chi_1 = \chi_1^a a_\phi + \chi_1^b, \text{ with} \quad (61)$$

$$\chi_1^a \equiv -\psi \nu(\eta_1) \left(1 - \frac{1-\alpha}{\sigma} \right) \frac{1+i^*}{i^*}, \quad (62)$$

$$\chi_1^b \equiv [1 - \psi - g\nu(\eta_1)] \varepsilon_{dk} - \frac{\psi}{1-\psi} \nu(\eta_1) \left(1 - \frac{1-\alpha}{\sigma} \right), \quad (63)$$

³¹Under Assumptions 1-4, one has: $\varepsilon_{dk} > 0$, $\Omega > 0$ and $\varepsilon_u^s > 0$.

³²For simplicity, the arguments of the functions are omitted

$$\chi_2 = \chi_2^a a_\phi + \chi_2^b, \text{ with} \quad (64)$$

$$\chi_2^a \equiv -(1-\psi) \frac{1+i^*}{i^*}, \quad (65)$$

$$\chi_2^b \equiv -[\psi + g\nu(\eta_1)], \quad (66)$$

$$\chi_3 = \chi_3^a a_\phi + \chi_3^b, \text{ with} \quad (67)$$

$$\chi_3^a \equiv \frac{(1-\psi)(1+i^*)}{i^*(1+i^*\gamma)} \left\{ \nu(\eta_1) \left[1 + \frac{i^*\gamma\psi}{1-\psi} \left(1 - \frac{1-\alpha}{\sigma} \right) \right] - \varepsilon_{dk} i^*\gamma \right\}, \quad (68)$$

$$\chi_3^b \equiv \nu(\eta_1)^2 \frac{g}{1+i^*\gamma} + \nu(\eta_1)\psi \left(1 - \frac{1-\alpha}{\sigma} + \frac{1}{1+i^*\gamma} \right) - (1-\psi) \varepsilon_{dk}, \quad (69)$$

with ε_{dk} given by Eq. (54) in Appendix B.

Lemma 3 *Let*

$$\bar{\varepsilon}_v \equiv -\chi_1^{-1} \left(\frac{\varepsilon_u}{c^*} \chi_2 + \chi_3 \right). \quad (70)$$

Under Assumptions 1-6 and $\varepsilon_v \neq \bar{\varepsilon}_v$, the characteristic polynomial, evaluated at the steady state $k^ = 1$, writes $P(X) \equiv X^2 - T(\varepsilon_v)X + D(\varepsilon_v) = 0$:*

$$D(\varepsilon_v) = -\varepsilon_u/c^* f'(1) \frac{1-\alpha}{\sigma} \frac{1 + (1-\psi) a_\phi (1+i^*)/i^*}{\chi_1 (\varepsilon_v - \bar{\varepsilon}_v)} \quad (71)$$

$$T(\varepsilon_v) = 1 + D(\varepsilon_v) - [1-\psi - g\nu(\eta_1)] \varepsilon_{dk} \frac{\varepsilon_v - \varepsilon_v^s}{\chi_1 (\varepsilon_v - \bar{\varepsilon}_v)}, \quad (72)$$

where ε_{dk} and ε_v^s are given by Eqs. (54) and (60) in Appendix B.

Characteristics of the Σ -line

The Σ -line is characterized by a starting point and a endpoint given by:

$$\begin{aligned} D(0) &= \varepsilon_u/c^* f'(1) \frac{1-\alpha}{\sigma} \frac{1 + (1-\psi) a_\phi (1+i^*)/i^*}{\chi_1 \bar{\varepsilon}_v} \\ T(0) &= 1 + D(0) - [1-\psi - g\nu(\eta_1)] \varepsilon_{dk} \frac{\varepsilon_v^s}{\chi_1 \bar{\varepsilon}_v} \\ D(+\infty) &= 0 \\ T(+\infty) &= -\frac{\psi\nu(\eta_1) \left(1 - \frac{1-\alpha}{\sigma} \right) [1/(1-\psi) + a_\phi (1+i^*)/i^*]}{\chi_1} \end{aligned}$$

We further note that the Σ -line has a slope S given by:

$$S = \frac{D'(\varepsilon_v)}{T'(\varepsilon_v)} = \frac{Z_1}{Z_1 + Z_2}, \text{ where} \quad (73)$$

$$Z_1 = (\varepsilon_u/c^*) f'(1) \frac{1-\alpha}{\sigma} \left[1 + (1-\psi) a_\phi \frac{1+i^*}{i^*} \right] \quad (74)$$

$$Z_2 = [1-\psi - g\nu(\eta_1)] \varepsilon_{dk} (\bar{\varepsilon}_v - \varepsilon_v^s) \quad (75)$$

Bifurcation values

ε_v^h is defined by $D(\varepsilon_v) = 1$:

$$\varepsilon_v^h = \frac{\Upsilon}{\chi_1} \left(\frac{\varepsilon_u}{c^*} - \frac{\chi_3}{\Upsilon} \right), \text{ where} \quad (76)$$

$$\Upsilon = \Upsilon^a a_\phi + \Upsilon^b, \text{ with} \quad (77)$$

$$\Upsilon^a \equiv -\chi_2^a - f'(1)(1-\psi) \frac{1-\alpha}{\sigma} \frac{1+i^*}{i^*}, \text{ and} \quad (78)$$

$$\Upsilon^b \equiv -\chi_2^b - f'(1) \frac{1-\alpha}{\sigma} \quad (79)$$

with χ_1 , χ_2^a , χ_2^b and χ_3 given by Eqs. (61), (65), (66) and (67) in Appendix C.1.

ε_v^s is defined by $1 - T(\varepsilon_v) + D(\varepsilon_v) = 0$:

$$\varepsilon_v^s = \frac{\Omega}{\varepsilon_{dk}} \left(\frac{\varepsilon_u^s}{c^*} - \frac{\varepsilon_u}{c^*} \right), \quad (80)$$

where Ω and ε_u^s are respectively given by Eqs. (58) and (59) in Appendix B.

ε_v^f is defined by $1 + T(\varepsilon_v) + D(\varepsilon_v) = 0$:

$$\varepsilon_v^f = \frac{\zeta_2}{\zeta_1} \left(\frac{\varepsilon_u}{c^*} - \frac{\zeta_3}{\zeta_2} \right), \text{ where} \quad (81)$$

$$\zeta_1 = \zeta_1^a a_\phi + \zeta_1^b, \text{ with} \quad (82)$$

$$\zeta_1^a \equiv 2\chi_1^a, \text{ and} \quad (83)$$

$$\zeta_1^b \equiv \chi_1^b - \frac{\psi}{1-\psi} \nu(\eta_1) \left(1 - \frac{1-\alpha}{\sigma} \right) \quad (84)$$

$$\zeta_2 = \zeta_2^a a_\phi + \zeta_2^b \text{ with} \quad (85)$$

$$\zeta_2^a \equiv 2(1-\psi) \frac{1+i^*}{i^*} \left[1 + f'(1) \frac{1-\alpha}{\sigma} \right], \text{ and} \quad (86)$$

$$\zeta_2^b \equiv 2 \left[\psi + g\nu(\eta_1) + f'(1) \frac{1-\alpha}{\sigma} \right] + \Omega [1 - \psi - g\nu(\eta_1)] \quad (87)$$

$$\zeta_3 = \zeta_3^a a_\phi + \zeta_3^b, \text{ with} \quad (88)$$

$$\zeta_3^a \equiv 2\chi_3^a, \text{ and} \quad (89)$$

$$\zeta_3^b \equiv \chi_3^b + \nu(\eta_1) \left[\psi \left(1 - \frac{1-\alpha}{\sigma} \right) + \frac{1}{1+i^*\gamma} \right] - (1-\psi) \varepsilon_{dk} \quad (90)$$

with χ_1^a , χ_1^b , χ_3^a and χ_3^b respectively given by Eqs. (62), (63), (68) and (69) in Appendix C.1 and ε_{dk} given by Eq. (54) in Appendix B.

C.2 Proofs for Proposition 4

Location of the Σ -line when $\phi = 0$ ($a_\phi = 0$)

In this section, we locate the Σ -line in the (T, D) plane, using $(T(0), D(0))$, $(T(+\infty), D(+\infty))$ and the value of its slope S when $\phi = 0$ ($a_\phi = 0$). We recall that we consider η_1 not too large (Assumption 6), since we are interested in the equilibria with a positive bubble ($\theta > 0$).

Before analyzing the location of the Σ -line, we need to determine the sign of χ_1 , χ_2 and χ_3 when $a_\phi = 0$. More precisely, we will show that for η_1 sufficiently small, $\chi_1 > 0$, $\chi_2 < 0$ and $\chi_3 < 0$. First, under Assumptions 1-5 and because $\psi < 1$ at the steady state, we have $\varepsilon_{dk} > 0$, $\Omega > 0$, and $\varepsilon_u^s > 0$.³³ There is a threshold $\eta_1^{\chi_1} \in \mathbb{R}_+$ such that $\forall \eta_1 < \eta_1^{\chi_1}$, $\chi_1 > 0$ under Assumptions 1-5. Furthermore, $\chi_2 < 0 \forall \eta_1 \in \mathbb{R}_+$ under Assumptions 1-5. Finally, there is a threshold $\eta_1^{\chi_3} \in \mathbb{R}_+$ such that $\forall \eta_1 < \eta_1^{\chi_3}$, $\chi_3 < 0$ under Assumptions 1-5. Thus, under Assumptions 1-6, one has: $\chi_1 > 0$, $\chi_2 < 0$ and $\chi_3 < 0$, and therefore, $\bar{\varepsilon}_v = -\chi_1^{-1} \left(\frac{\varepsilon_u}{c^*} \chi_2 + \chi_3 \right) > 0$. For the rest of the proof, we consider $\psi < 1$, $1 - \psi - g\nu(\eta_1) > 0$, $\varepsilon_{dk} > 0$, $\Omega > 0$, $\varepsilon_u^s > 0$, $\chi_1 > 0$, $\chi_2 < 0$, $\chi_3 < 0$ and $\bar{\varepsilon}_v > 0$.³⁴

The starting point and the endpoint of the Σ -line are such that:

$$D(0) = \varepsilon_u / c^* f'(1) \frac{1 - \alpha}{\sigma} \frac{1}{\chi_1 \bar{\varepsilon}_v} > 0 \quad (91)$$

$$1 - T(0) + D(0) = \frac{1 - \psi - g\nu(\eta_1)}{\bar{\varepsilon}_v} \varepsilon_{dk} \left(\frac{\varepsilon_u^s}{c^*} - \frac{\varepsilon_u}{c^*} \right) \quad (92)$$

$$D(+\infty) = \text{ and } T(+\infty) = -\frac{\psi}{1 - \psi} \frac{\nu(\eta_1)}{\chi_1} \left(1 - \frac{1 - \alpha}{\sigma} \right), \quad (93)$$

where ε_u^s and $\bar{\varepsilon}_v$ are respectively given by Eqs. (59) and (70).

First of all, let us see whether $D(0) < 1$. $D(0) < 1$ is equivalent to the following condition:

$$\chi_3 < \frac{\varepsilon_u}{c^*} \left\{ f'(1) \left[g'(f'(1)) - \frac{1 - \alpha}{\sigma} \right] + g\nu(\eta_1) \right\}$$

This above condition is satisfied under Assumptions 1-6 and 8. Hence, the starting point $(T(0), D(0))$ locates below the segment $[BC]$.

The sign of $1 - T(0) + D(0)$ informs us on which side of (AC) the starting point $(T(0), D(0))$ is located. When $\varepsilon_u < \varepsilon_u^s$, one has $1 - T(0) + D(0) > 0$. Nevertheless, the second-order conditions require that $\varepsilon_u^s > \bar{\varepsilon}_u$ which is equivalent to:

$$\begin{aligned} \eta_2(1 - \psi)i^*\gamma &> i^*\eta_1(1 - \gamma) \left\{ 1 - f'(1) + \left(1 - \frac{1 - \alpha}{\sigma} \right) [f'(1) - \psi] \right\} + \psi \frac{1 - \alpha}{\sigma} \\ &\quad - \psi \rho_2 \left(\frac{1 - \alpha}{\sigma} + i^*\gamma \right) \end{aligned}$$

³³The expressions of $\varepsilon_{dk} > 0$, $\Omega > 0$, and $\varepsilon_u^s > 0$ are respectively given by Eqs. (54), (58) and (59) in Appendix B.

³⁴From this proof, we deduce that $\chi_1^b > 0$, $\chi_2^b < 0$ and $\chi_3^b > 0$.

Hence, for a small degree of concavity on $u(c)$ and under Assumptions 1-6, the starting point locates on the left-side of (AC) . In addition, when $\varepsilon_u > \varepsilon_u^s$, we get $1 - T(0) + D(0) < 0$. The starting point locates on the right-side of (AC) when $\varepsilon_u > \varepsilon_u^s$.

The endpoint $(T(+\infty), D(+\infty))$ locates on the horizontal axis. Furthermore, one has $T(+\infty) < 0$ under Assumptions 1-6 (see Eq. (93)). Let us see now on which side of (AB) is $(T(+\infty), D(+\infty))$, that is if $T(+\infty)$ is lower or greater than -1 . $T(+\infty) > -1$ involves the following condition:

$$[1 - \psi - g\nu(\eta_1)] \frac{1 - \psi}{\nu(\eta_1)} \varepsilon_{dk} - 2\psi \left(1 - \frac{1 - \alpha}{\sigma} \right) > 0$$

Under Assumptions 1-5, this inequality is satisfied for a sufficiently small η_1 . Thus, the endpoint is on the right-side of (AB) under Assumptions 1-6.

Now, we study the slope S of the Σ -line and how $T(\varepsilon_v)$ and $D(\varepsilon_v)$ vary with respect to ε_v . When $\phi = 0$, the slope S of the Σ -line is given by:

$$S = \frac{D'(\varepsilon_v)}{T'(\varepsilon_v)} = \frac{Z_1}{Z_1 + Z_2} \quad (94)$$

where

$$\begin{aligned} Z_1 &= (\varepsilon_u/c^*)f'(1)\frac{1 - \alpha}{\sigma} > 0 \quad \text{and} \\ Z_2 &= Z_2^a + Z_2^b, \quad \text{with} \\ Z_2^a &= \frac{[1 - \psi - g\nu(\eta_1)] \varepsilon_{dk} \varepsilon_u}{\chi_1 c^*} \left(\chi_1 \frac{\Omega}{\varepsilon_{dk}} - \chi_2 \right), \quad \text{and} \\ Z_2^b &= -\frac{[1 - \psi - g\nu(\eta_1)] \varepsilon_{dk}}{\chi_1} \left[\frac{i^* \eta_1 (1 - \gamma)}{1 + i^* \gamma} \frac{\chi_1}{\varepsilon_{dk}} + \chi_3 \right], \end{aligned}$$

where $\chi_1 > 0$, $\chi_2 < 0$, $\chi_3 < 0$ are respectively given by Eqs. (61), (64) and (67) in Appendix C.1, and $\varepsilon_{dk} > 0$, $\Omega > 0$ and c^* by Eqs. (54), (58) and (56) in Appendix B.

Under Assumptions 1-6, we have $Z_2^a \geq 0$. Z_2^b can be rewritten as follows:

$$Z_2^b = -\frac{[1 - \psi - g\nu(\eta_1)] \varepsilon_{dk}}{\chi_1} P(\nu(\eta_1)),$$

where $P(\nu(\eta_1))$ is a quadratic polynomial defined on $\mathbb{R}_+$ such that:

$$\begin{aligned} P(\nu(\eta_1)) &= -\frac{\nu(\eta_1)^2}{1 + i^* \gamma} \varepsilon_{dk}^{-1} \frac{\psi}{1 - \psi} \left(1 - \frac{1 - \alpha}{\sigma} \right) + \nu(\eta_1) \left[\psi \left(1 - \frac{1 - \alpha}{\sigma} \right) \right. \\ &\quad \left. + \frac{1}{1 + i^* \gamma} \right] - (1 - \psi) \varepsilon_{dk} \end{aligned}$$

Under Assumptions 1-4, $P(\nu(\eta_1))$ is a concave function with $P(\nu(0)) < 0$ and reaches its maximum for $\eta_1 = \eta_{1max} > 0$. As a consequence, there is a

threshold $\hat{\eta}_1 \in \mathbb{R}_+$ such that $\forall \eta_1 < \hat{\eta}_1$, $P(\nu(\eta_1))$ is negative ($P(\nu(\eta_1)) < 0$). This implies that $Z_2^b > 0$ under Assumptions 1-6, and therefore, $Z_2 > 0$.

As $Z_1 > 0$ and $Z_2 > 0$, we can conclude that the slope S of the Σ -line belongs to $(0, 1)$ under Assumptions 1-6. Moreover, as $D'(\varepsilon_v) = Z_1 / [\chi_1(\varepsilon_v - \bar{\varepsilon}_v)^2]$ and $T'(\varepsilon_v) = D'(\varepsilon_v) + Z_2 / [\chi_1(\varepsilon_v - \bar{\varepsilon}_v)^2]$, we also have $T'(\varepsilon_v) \geq 0$ and $D'(\varepsilon_v) \geq 0$.

To further analyze the slope S , we show now that the Σ -line goes below C . We need to prove that under Assumptions 1-6 (i.e. η_1 not too large), we have $\varepsilon_v^s < \varepsilon_v^h$. This is equivalent to the following inequality:

$$\begin{aligned} 0 &> \kappa_1 + \kappa_2, \quad \text{with} \\ \kappa_1 &= -\frac{\varepsilon_u}{c^*} \left\{ \frac{\Omega}{\varepsilon_{dk}} + \frac{g\nu(\eta_1) + f'(1) [g'(f'(1)) - \frac{1-\alpha}{\sigma}]}{\chi_1} \right\}, \quad \text{and} \\ \kappa_2 &= \frac{\nu(\eta_1)}{1 + i^*\gamma} \varepsilon_{dk}^{-1} + \frac{\chi_3}{\chi_1} \end{aligned}$$

We can deduce that under Assumptions 1-6 and 8, $\kappa_1 \leq 0$. As regards κ_2 , we can rewrite it as follows:

$$\kappa_2 = \frac{P(\nu(\eta_1))}{\chi_1}$$

From the previous analysis about the value of the slope S , we know that $P(\nu(\eta_1)) < 0$ under Assumptions 1-6. Therefore, $\kappa_2 < 0$ under these assumptions. As $\kappa_1 < 0$ and $\kappa_2 < 0$, we can conclude that under Assumptions 1-6 $\varepsilon_v^s < \varepsilon_v^h$. As a consequence, the Σ -line goes below C .

Note that the proof also holds true when no fiscal policies are implemented (i.e. $g(Rk) = Rk$), and in particular for $\rho_1 = 1$ and $\rho_2 = 0$.

As regards the size of ε_v^s , we can show that when no progressive fiscal policies are implemented (i.e. $\rho_1 = 1$ and $\rho_2 = 0$), $\varepsilon_v^s < 1$. Indeed, the condition $\varepsilon_u^s > \bar{\varepsilon}_u$ rewrites:

$$\nu(\eta_1) < \eta_2 i^* \gamma - \frac{f'(1)}{1 - f'(1)} \frac{1 - \alpha}{\sigma}$$

Hence, we have $\nu(\eta_1) / (1 + i^*\gamma) \varepsilon_{dk}^{-1} < 1$.³⁵

Since $\varepsilon_v^s = \nu(\eta_1) / (1 + i^*\gamma) \varepsilon_{dk}^{-1} - \frac{\varepsilon_u}{c^*} \frac{1 - f'(1)}{\nu(\eta_1)} [\nu(\eta_1) + \varepsilon_{dk}]$, we can conclude that $\varepsilon_v^s < 1$.

Furthermore, one has $\varepsilon_v^h > 1$ for a sufficiently large η_2 . Therefore, under Assumptions 1-6, we get: $\varepsilon_v^s < 1$ and $\varepsilon_v^h > 1$. ■

³⁵When $\rho_1 = 1$ and $\rho_2 = 0$, $\varepsilon_{dk} = \frac{f'(1)}{1 - f'(1)} \frac{1 - \alpha}{\sigma} + \eta_2 > 0$.

Appendix D

Proof of Proposition 5

We determine how the bifurcation value ε_v^s varies with respect to ρ_2 when $a_\phi = 0$. The derivative of ε_v^s with respect to ρ_2 is given by:

$$\frac{\partial \varepsilon_v^s}{\partial \rho_2} = \frac{1}{\varepsilon_{dk}^2} \left[-\frac{\psi}{1-\psi} \left(1 - \frac{1-\alpha}{\sigma} \right) \left\{ \frac{\nu(\eta_1)}{1+i^*\gamma} - \frac{\varepsilon_u}{c^*} \left[1 - f'(1) + \left(1 - \frac{1-\alpha}{\sigma} \right) (f'(1) - \psi) \right] \right\} \right]$$

Assumption 7 is equivalent to:

$$\frac{\nu(\eta_1)}{1+i^*\gamma} > \frac{\varepsilon_u}{c^*} \left[1 - f'(1) + \frac{1-\psi}{i^*\eta_1(1-\gamma)} \varepsilon_{dk} + \left(1 - \frac{1-\alpha}{\sigma} \right) (f'(1) - \psi) \right],$$

As $\varepsilon_{dk} > 0$ under Assumptions 1 – 4 (see Eq. (54) in Appendix B), we deduce that ε_v^s is a decreasing function of ρ_2 ($\frac{\partial \varepsilon_v^s}{\partial \rho_2} < 0$). Since indeterminacy occurs for $\varepsilon_v < \varepsilon_v^s$, this proves Proposition 5. ■

Appendix E

E.1 Variations of critical values (ε_v^f , ε_v^h , ε_v^s and $\bar{\varepsilon}_v$) with respect to $a_\phi \neq 0$ when $g(Rk) = Rk$

The expressions of $1 - T(\varepsilon_v) + D(\varepsilon_v)$, $1 + T(\varepsilon_v) + D(\varepsilon_v)$ and $D(\varepsilon_v)$ can be written as follows:

$$\begin{aligned} 1 - T(\varepsilon_v) + D(\varepsilon_v) &= \{1 - f'(1)[1 + \nu(\eta_1)]\} \varepsilon_{dk} \frac{\varepsilon_v - \varepsilon_v^s}{\chi_1 (\varepsilon_v - \bar{\varepsilon}_v)} \\ 1 + T(\varepsilon_v) + D(\varepsilon_v) &= \frac{\zeta_1 (\varepsilon_v - \varepsilon_v^f)}{\chi_1 (\varepsilon_v - \bar{\varepsilon}_v)} \\ D(\varepsilon_v) - 1 &= \frac{\varepsilon_v^h - \varepsilon_v}{\varepsilon_v - \bar{\varepsilon}_v}, \end{aligned}$$

with $\bar{\varepsilon}_v$, ε_v^s , ε_v^f , ε_v^h , χ_1 and ζ_1 are respectively given by Eqs. (70), (80), (81), (76), (61) and (82) in Appendix C.1, and ε_{dk} by Eq. (54) in Appendix B.

In this section, we graphically represent the variations of the bifurcation values (ε_v^s , ε_v^f , ε_v^h) and the critical value $\bar{\varepsilon}_v$ in the (a_ϕ, ε_v) plane, and identify the areas in which the normalized steady state is a sink.

We can see from Eq. (60) that ε_v^s does not depend on a_ϕ . On the other hand, the different bifurcation and critical values (ε_v^f , ε_v^h and $\bar{\varepsilon}_v$) are homographic functions of a_ϕ (see Eqs. (70), (81) and (76)) in Appendix C.1 with $g'(f'(1)) = 1$ and $\psi = f'(1)$. Since $\phi > 0$, a_ϕ is defined on $] -\infty, -1[\cup]0, +\infty[$.

Under Assumptions 1 – 6, $\chi_1^a < 0$, $\chi_1^b > 0$, $\chi_2^a < 0$, $\chi_2^b < 0$, $\chi_3^a < 0$, $\chi_3^b < 0$, $\zeta_1^a < 0$, $\zeta_1^b > 0$, $\zeta_2^a > 0$, $\zeta_2^b > 0$, $\zeta_3^a < 0$, $\zeta_3^b < 0$, $\Upsilon^a > 0$ and $\Upsilon^b > 0$.

We can note that ε_v^f has a vertical asymptote at $a_\phi = -\zeta_1/(2\chi_1^a) \equiv \tilde{a}_\phi > 0$ under Assumptions 1–6, and that ε_v^h and $\bar{\varepsilon}_v$ have the same vertical asymptote at $a_\phi = -\chi_1^b/\chi_1^a \equiv \bar{a}_\phi > -\zeta_1/(2\chi_1^a)$ under Assumptions 1-6. Note that we consider only the positive values of ε_v^f , ε_v^h and $\bar{\varepsilon}_v$. We can check that:

$$\begin{aligned}\varepsilon_v^f = 0 &\Leftrightarrow a_\phi^{\varepsilon_v^f} = -\frac{\zeta_2^b \varepsilon_u / c^* - \zeta_3^b}{\zeta_2^a \varepsilon_u / c^* - \zeta_3^a} \quad (< 0 \text{ under Assumptions 1–6}) \\ \varepsilon_v^h = 0 &\Leftrightarrow a_\phi^{\varepsilon_v^h} = -\frac{\Upsilon^b \varepsilon_u / c^* - \chi_3^b}{\Upsilon^a \varepsilon_u / c^* - \chi_3^a} \quad (< 0 \text{ under Assumptions 1–6}) \\ \bar{\varepsilon}_v = 0 &\Leftrightarrow a_\phi^{\bar{\varepsilon}_v} = -\frac{\chi_2^b \varepsilon_u / c^* + \chi_3^b}{\chi_2^a \varepsilon_u / c^* + \chi_3^a} \quad (< 0 \text{ under Assumption 1–6})\end{aligned}$$

Under Assumptions 1–6 and 9, we have $a_\phi^{\bar{\varepsilon}_v} < a_\phi^{\varepsilon_v^h}$, and we can clarify that $\lim_{\eta_2 \rightarrow +\infty} a_\phi^{\varepsilon_v^f} < \lim_{\eta_2 \rightarrow +\infty} a_\phi^{\bar{\varepsilon}_v}$.

Furthermore, we can derive the first derivatives with respect to a_ϕ . Under Assumptions 1–6, we obtain:

$$\begin{aligned}\frac{\partial \varepsilon_v^s}{\partial a_\phi} &= 0, \\ \frac{\partial \varepsilon_v^f}{\partial a_\phi} &= \frac{(\zeta_2^a \zeta_1^b - \zeta_2^b \zeta_1^a) \frac{\varepsilon_u}{c^*} - (\zeta_3^a \zeta_1^b - \zeta_3^b \zeta_1^a)}{(\tilde{\zeta}_1^a a_\phi + \tilde{\zeta}_1^b)^2} > 0 \\ \frac{\partial \varepsilon_v^h}{\partial a_\phi} &= \frac{(\Upsilon^a \chi_1^b - \Upsilon^b \chi_1^a) \frac{\varepsilon_u}{c^*} - (\chi_3^a \chi_1^b - \chi_3^b \chi_1^a)}{(\chi_1^a a_\phi + \chi_1^b)^2} > 0 \\ \frac{\partial \bar{\varepsilon}_v}{\partial a_\phi} &= -\frac{(\chi_2^a \chi_1^b - \chi_2^b \chi_1^a) \frac{\varepsilon_u}{c^*} + (\chi_3^a \chi_1^b - \chi_3^b \chi_1^a)}{(\chi_1^a a_\phi + \chi_1^b)^2} > 0\end{aligned}$$

We can now draw Figure 3, and deduce the different regions in which the steady state is a sink, in other words the indeterminacy regions (grey areas in Figure 3). ■

E.2 Proof of Proposition 8

We determine the variations of the bifurcation value ε_v^s with respect to i^* when $a_\phi = 0$ and $g(y_k) = y_k$. As regards the derivative of ε_v^s with respect to i^* , we get:

$$\begin{aligned}\frac{\partial \varepsilon_v^s}{\partial i^*} &= (\varepsilon_{dk} i^*)^{-1} \left[\frac{\nu(\eta_1)}{(1 + i^* \gamma)^2} + \frac{\varepsilon_u}{c^{*2}} \frac{1 - f'(1)}{\nu(\eta_1)} (\varepsilon_{dk} c^* + \Omega) \right] > 0, \\ \text{with } \Omega &= \frac{1 - f'(1)}{\nu(\eta_1)} [\nu(\eta_1) + \varepsilon_{dk}] > 0\end{aligned}$$

Since indeterminacy occurs for $\varepsilon_v < \varepsilon_v^s$, this proves Proposition 8. ■

E.3 Proof of Propostion 9

We want to determine the effect of the interest rate i^* on the household welfare. At the steady state, the household welfare level is given by Eqs. $W = U(c, d) = u(c) + \beta v(d)$, with c and d given in the proof of Proposition of 1 in Appendix B.³⁶ The elasticity of the welfare level W with respect to the interest rate i^* is written as follows:

$$\begin{aligned}\varepsilon_{Wi} &= \varepsilon_{Ud}\varepsilon_{di} \left(1 + \frac{U_c}{U_d} \frac{dc/di}{dd/di}\right), \quad \text{where} \\ \varepsilon_{di}(k) &= \varepsilon_{ki}(k)\varepsilon_{dk}(k) - 1, \quad \text{with} \\ \varepsilon_{ki}(k) &= \frac{\varepsilon_v + (\varepsilon_u/c^*(k)) \frac{k[1-f'(k)]}{i^*\eta_1(k)[1-\gamma(k)]} - \frac{i^*\gamma(k)}{1+i^*\gamma(k)}}{(\varepsilon_v - \varepsilon_v^s(k))\varepsilon_{dk}(k)}, \quad \text{and} \\ \varepsilon_{dk}(k) &= \frac{f'(k)}{1-f'(k)} \frac{1-\alpha(k)}{\sigma(k)} + \eta_2(k)\end{aligned}$$

As $dc/di = -\frac{d}{i} [\varepsilon_{ki}i^*\eta_1(1-\gamma) + (\varepsilon_{ki}\varepsilon_{dk} - 1)]$ and $dd/di = \frac{d}{i} [\varepsilon_{ki}\varepsilon_{dk} - 1]$, we obtain:

$$\varepsilon_{Wi} = \frac{\varepsilon_v}{1+i^*\gamma} \frac{\nu(\eta_1) \left(\varepsilon_v - 2 \frac{i^*\gamma}{1+i^*\gamma} \right) + \frac{\varepsilon_u}{c^*} [1-f'(k)] k (1+i^*\gamma) + \varepsilon_{dk} \frac{(i^*\gamma)^2}{1+i^*\gamma}}{\varepsilon_{dk}(\varepsilon_v^s - \varepsilon_v)}$$

Because $f'(k) < 1$, $\varepsilon_u > 0$ and $\varepsilon_{dk} > 0$ under Assumptions 1 – 4, the numerator of ε_{Wi} is positive for a sufficiently large η_2 because ε_{dk} is increasing in η_2 (see Eq. (54)). Hence, ε_{Wi} has the same sign as $\varepsilon_v^s - \varepsilon_v$ under Assumptions 1 – 6. ■

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³⁶For simplicity, the arguments of the functions are omitted.

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