Working Papers / Documents de travail

Optimal Student Loans and Graduate Tax under Moral Hazard and Adverse Selection

Robert J. Gary-Bobo
Alain Trannoy

WP 2014 - Nr 16

Optimal Student Loans and Graduate Tax under Moral Hazard and Adverse Selection*

Robert J. Gary-Bobo† and Alain Trannoy‡

2 April 2014

Abstract

We characterize the set of second-best optimal "menus" of student-loan contracts in a simple economy with risky labour-market outcomes, adverse selection, moral hazard and risk aversion. The model combines student loans with an elementary optimal income-tax problem. The second-best optima provide incomplete insurance because of moral hazard; they typically involve cross-subsidies between students. Generically, optimal loan repayments cannot be decomposed as the sum of an income tax, depending only on earnings, and a loan repayment, depending only on education. Therefore, optimal loan repayments must be income-contingent, or the income tax must comprise a *graduate tax*. The interaction of adverse selection and moral-hazard, *i.e.*, self-selection constraints and effort incentives, determines an *equal treatment property*; the expected utilities of different types of students are equalized at the *interim* stage, conditional on the event of academic success (*i.e.*, graduation). But individuals are *ex ante* unequal because of differing probabilities of success, and *ex post* unequal, because the income tax trades off incentives and insurance (redistribution).

KEYWORDS: Student Loans, Graduate Tax, Adverse Selection, Moral Hazard, Risk Aversion.

*We thank Kathryn Spier, three anonymous refereees of the *Rand Journal of Economics*, Christian Gollier and Pierre-André Chiappori for very useful remarks and comments. This manuscript is the revised version of a working paper by the same authors, circulated in May 2012 and entitled "Equal Treatment as a Second Best: Student Loans under Asymmetric Information". This research has been supported by a research grant of the ANR "programme blanc".

†CREST-ENSAE, 15, boulevard Gabriel Péri, 92245 Malakoff cedex, France, IDEP, CEPR and CES Ifo. Email: robert.gary-bobo@ensae.fr

‡ University of Aix-Marseille (Aix-Marseille School of Economics), EHESS and CNRS, France. email: alain.trannoy@univ-amu.fr.

1 Introduction

The importance of student loans for the accumulation of human capital, economic growth and welfare cannot be overestimated. In the United States, the total amount of outstanding student debt has reached \$1 trillion at the end of 2011. In Great Britain, the rise of tuition fees seems to have caused a sharp increase in average student debt[1]. Student loans pose interesting financial engineering and regulation problems. There are many discussions on the optimal design of these loans: for instance, the UK and Australia have a form a income-contingent repayment system, since loan repayments are based on the graduate's earnings, just like income tax, and interest rates are subsidized[2]. In many continental European countries, student loans play a negligible *role* but, given the severe shortage of public funds, they could go hand in hand with a substantial raise in tuition fees, and become a new source of funds for universities[3].

There is an important econometric literature on the impact of credit constraints on university or college attendance[4]. For recent quantitative studies of alternative student-loan policies in the US, see, *e.g.*, Ionescu (2009), Lochner and Monge-Naranjo (2010). These questions are hotly debated, yet, to the best of our knowledge, the micro-economic theory of student loans is still underdeveloped[5]. In the following, we propose a simple model of student loans, under the combined effects of risk aversion, moral hazard and adverse selection. We explore the structure of the set of second-best optimal (or interim incentive-efficient) allocations of credit to risk-averse students in an economy in which individual talents and efforts are not observed by the lender, future earnings are subject to risk and incomes can be taxed. The student-loan problem is compounded with an elementary form of the Mirrleesian optimal income-tax problem.

[1] The average student debt is predicted to be around 50,000 pounds, on leaving the university, for those starting in 2012 (for details, see *http://www.slc.co.uk/statistics*). The number of students experiencing difficulties to repay their loans reached 8.8% in the United States, after the economic downturn of 2008.

[2] New Zealand and Sweden are also providing income-contingent student loans. On these questions, see, *e.g.*, Barr and Johnston (2010).

[3] See, *e.g.*, Jacobs and Van der Ploeg (2006).

[4] See, for instance, Carneiro and Heckman (2002), Keane and Wolpin (2001), Stinebrickner and Stinebrickner (2008). For a survey, see Lochner and Monge-Naranjo (2011).

[5] See our discussion of the literature below.

We consider an economy with unobservable, *two-dimensional* individual *types*. Students are characterized by an *ex ante type* and an *ex post type*. The ex ante type represents cognitive (and other) skills that influence academic success. In contrast, the ex post type plays a role once the student finds a job, and can be interpreted as representing some job-market skills, or job-market opportunities that are random, but correlated with the quality of education. Students choose the length (or the quality) of their studies in a "menu"; they know their ex ante type before the choice of how much to invest in education. In contrast, the ex post type, which is private information too, is revealed later, after graduation.

Students exert two kinds of effort. While attending college, students exert some *study effort* that is not observable. This determines an *ex ante moral hazard* problem. Ex ante effort (or study effort) influences the probability of graduation. Students can be either *successful*, or *fail* to obtain a certificate. The distribution of individual earnings is also affected by unobservable effort exerted at work. Effort at work may also be a called *ex post effort*. The government does not observe ex post effort but observes education, success and individual earnings. A successful student with high ex post skills can choose to behave like a student with low ex post skills and save on effort costs. This is the source of an *ex post moral hazard* problem, with the consequence that the most productive students can decide to earn less if tax rates are too high.

The student's success, education and ex post type jointly determine her(his) potential wage. There are two sources of risk: random academic success and the drawing of ex post types. Thus, individual wages are risky. Because of risk aversion, students would like to be insured against these random shocks.

We characterize the set of second-best Pareto-optima of this model. These optima can be implemented by a combination of structured loans and income taxes.

Our main results are the following. The optimal menu of contracts exhibits *incomplete insurance*: this is mainly due to moral hazard. Students obtain the maximal amount of income insurance compatible with the exertion of high study effort among the less-talented. It follows from this that the talented ex ante types bear more risk than strictly necessary to ensure that they choose a high level of study effort. The planner cannot reduce the risk borne by the talented, because that would violate the self-selection constraints.

As a by-product, we find that *second-best optimal loan repayments are always income-contingent*, even in the presence of an income tax. More precisely, we find a *non-decomposition theorem*: the second-best cannot be implemented by means of the sum of an income tax, that depends only on observed earnings, and a loan repayment, that depends only on the observed quality of education (or observed years of education). It must be that, either the income tax depends on education, or the loan repayments depend on income. Since the budget is by construction balanced (we did not explore subsidies that would be financed by means of external sources of funds), the second-best optima typically exhibit *cross-subsidies* between types: the talented repay more and subsidize the less-talented, but the former are not fully exploited. The second-best solution can be interpreted, either as an income-contingent loan, or as a *graduate tax*, with a certain degree of progressivity.

In the standard utilitarian case, we find that the second-best optimal menu of loan contracts exhibits a form of *pooling*, called *equal treatment*: the students' *expected utilities*, net of repayments, must be equal, *conditional* on random academic success[6]. In other words, the post-study but pre-work expected utility of net earnings should be independent of the student's unobservable ex ante type. In spite of being treated equally in this particular sense, students are *ex ante* unequal, since the talented types have a greater probability of success, and they are *ex post* unequal, since the optimal income tax trades off the provision of *insurance* (and redistribution) against that of *incentives*. We show that equal treatment is a way of solving the screening problem in the presence of moral hazard. The *equal treatment property* results from the interplay of self-selection and effort incentives constraints. Without an adverse selection problem, each type of student would be subjected to the smallest amount of risk needed to guarantee high effort. This takes the form of a gap between the net income earned in case of success and that earned in case of failure. It is typically easier to obtain high effort from the talented types, so the gap between the utilities of the latter in case of success and in case of failure can be chosen to be smaller than what would be appropriate for a less-talented type of individual. But providing insurance to talented types in this way will always violate the self-selection constraints, with the possible result that the talented would not want to study longer than other students. Hence the equal treatment result: interim

[6] This property could also be defined as *interim* equalization.

equalization is the best way of providing insurance to good students, compatible with the revelation of academic talent, and given that complete insurance against failure is precluded.

Another contribution of the present article is to provide a characterization of the set of second-best optima in an economy with wage risk, risk aversion, moral hazard and adverse selection. In other words, we provide a study of the structure of second-best optima in a screening model with the added complication of moral hazard. The equal treatment result described above is not the textbook allocation of insurance under adverse selection, *à la* Rothschild-Stiglitz[7]. To obtain the familiar *separating* menu of loan contracts as a second-best optimum, we show that the weight of the talented types must exceed their natural frequency in the social planner's objective function, which is not the common assumption in public finance.

Our theory of student loans combines 4 main ingredients: risk aversion, ex ante moral hazard, a screening problem and ex post moral hazard. To be posed in a meaningful way, the student loan problem should embody these ingredients. None of these ingredients can be removed without losing an important aspect of our results. Without risk aversion, there is no demand for insurance and redistribution. Absent the ex ante moral hazard problem, students may be perfectly insured against the consequences of academic failure. It would not be reasonable to assume that study effort is ineffective (or that it can be perfectly monitored). If ex ante types are observable, we lose the equal treatment property. We model a society in which, to a certain extent, individuals choose their individual investment in education, and, to predict a person's career, it matters a great deal, for instance, if this person went to graduate school or has accomplished 2 years of college. This is why we should consider a menu of educational choices. When ex ante types are assumed observable, the self-selection problem is replaced with an education planning problem. We show that when ex ante self-selection constraints are dropped, the social planner will force all talented students to study longer. In the absence of an ex post moral hazard problem, the optimal solution would exhibit an unpalatable form of "exploitation of the talented": the marginal rate of income taxation could reach 100%.

Student loans are a very natural application for the theory of incentives or Mechanism

[7]See Rothschild and Stiglitz (1976), Wilson (1977), Miyazaki (1977), Spence (1978).

Design under hidden actions and hidden types[8]. The general theory of optimum (or equilibrium) contracts under moral hazard, adverse selection *and* risk aversion is known to be a very hard problem (see Arnott (1991) for comments and further references to unpublished essays on this question, see also the recent synthesis of Boadway and Sato (2012) on optimal taxation with uncertain earnings). Solutions can be exhibited when principal and agent are both risk-neutral[9]. An extension of Rothschild and Stiglitz's insurance market model to moral hazard is proposed in the often quoted, but unpublished manuscript of Chassagnon and Chiappori (1997). Our model is close to that of the latter contribution, but Chassagnon and Chiappori did not study the set of Pareto optima[10].

The question of student loans is certainly not new in economics[11]. A few papers have recently proposed a theoretical study of income-contingent loans, or of the graduate tax, comparing alternative ways of subsidizing higher education, and performed numerical explorations of their welfare properties[12]. A much smaller number of papers come to grips with asymmetric information[13]. Finally, some contributions were devoted to education in an optimal income-taxation model. This question has been studied in static and two-period settings, see, *e.g.*, Anderberg (2009), Bovenberg and Jacobs (2005), De Fraja (2002), Fleurbaey *et al.*

[8] It is well-known that microeconomic models of insurance and models of banking are formally close. Rothschild and Stiglitz's approach to screening in insurance markets has been applied to banking, albeit with adaptations (see, *e.g.*, Bester (1985)). Classic theories of credit contracts typically treat adverse selection and moral hazard separately (see Freixas and Rochet (1998)). The structure of second-best optima in insurance markets with pure adverse selection has been studied by Crocker and Snow (1985) and Henriet and Rochet (1990).

[9] See, *e.g.*, Picard (1987) and Caillaud, Guesnerie and Rey (1992); see also the discussion in Laffont and Martimort (2002, chapter 7). In the field of optimal regulation theory, a few contributions have dealt with special cases; see, McAfee and McMillan (1986), Baron and Besanko (1987), Laffont and Rochet (1998).

[10] Recent work on the Principal-Agent model in the case at hand required advanced mathematical optimisation techniques (see Faynzilberg and Kumar (2000)) or used stochastic calculus, as in the asset-pricing, continuous-time finance literature (see, *e.g.*, Sung Jaeyoung (2005)). These intimidating technicalities mainly explain why we study a simple textbook model here, but it conveys, we think, the essential intuitions and ideas (and yet, some of the proofs are not straightforward).

[11] See Friedman and Kuznets (1945), Friedman (1955), Shell *et al.* (1968), Nerlove (1975).

[12] See, *e.g.*, Hanushek *et al.* (2004), Garcia-Penalosa and Wälde (2000), Del Rey and Racionero (2010).

[13] De Fraja (2001) studies the interaction of unobservable skills and risk aversion in the allocation of higher education. Cigno and Luporini (2009) consider a microeconomic model of student loans with pure moral hazard. Chatterjee and Ionescu (2011) propose a quantitative analysis of a model of student loans with moral hazard but they do not rely on *Mechanism Design* techniques as we do here.

(2002). More recently, and closer to the present contribution, Findeisen and Sachs (2012) have studied the combination of an income-tax with income-contingent loans in an optimal taxation model with endogenous investment in human capital; they use a more complicated model than us, with a continuum of types, but have recourse to numerical simulations. They reach similar conclusions about the usefulness of income-contingent reimbursement.

In the following, Section 2 describes the model and studies first-best optima. Section 3 is devoted to a preliminary analysis of asymmetric information and incentive constraints. Section 4 presents the main results, characterizing the second-best optimum and discusses implementation by means of income-contingent student loans, the graduate tax, and the income tax. Section 5 studies what happens if some ingredients are removed in turn from the theory. Section 6 provides a complete characterization of the set of second-best optima. The proofs are in the appendix.

2 A Simple Model

2.1 Basic Assumptions

We consider a population of students with the same von Neumann-Morgenstern utility $u(.)$. Students have two-dimensional unobservable characteristics: an *ex ante* type, denoted i and an *ex post* type, denoted k. To fix ideas, the reader may interpret the *ex ante* type as representing the individual's *cognitive skills*, while the *ex post* type represents *job-markets skills and opportunities*. Both factors influence the distribution of a given individual's future wages, albeit in different manners. The important difference between the two type components i and k is the timing of information revelation. Both components are private information of individuals. The ex ante type i is known when the student decides how much to invest in education, while the ex post type is revealed only later, when the student enters the labor market. Thus, some important individual characteristics will be revealed in the future, when the student comes to grips with a real job. An individual is characterized by the pair (i, k). We assume that each component can take one of two values only, i, $k = 1, 2$. We assume that student types are independent draws from the same distribution. The frequency of the *ex ante* type i in the student population is denoted λ_i and of course, $\lambda_1 + \lambda_2 = 1$.[14]

The model has three building blocks: the education phase, the job-market phase and the loan contract. We describe each of these blocks in turn (see Fig. 1 for an illustration of the model's timing). There are three consecutive lotteries: type i, *success* or *failure* and type k. At the end, the individual chooses ex post effort.

2.1.1 Education

During the education phase, each student first learns his(her) type i, then chooses a quantity (or quality) of education q in a set $\{1, 2\}$. There are long (or highly-demanding) studies, *i.e.*, $q = 2$ and short (or less demanding) studies, *i.e.*, $q = 1$.[15] The cost of education is simply γ_q for quality q; and we assume $\gamma_2 \geq \gamma_1$. The chosen education depends on the ex ante type i only, since k is unknown when the decision is made. The educational choice q is assumed observable. Let q_i be the education of (or chosen by) type i. The efficient choice of q for type i will be $q_i = i$, due to Assumption 4, presented below in sub-section 2.2. At the end of the schooling period, each student is *successful* or *fails*. To fix ideas, success may be interpreted as *graduation*, *i.e.*, the student is granted a degree or not (but success may have slightly different interpretations). The individuals of both types have independent probabilities of success, that depend on individual effort.

Each student chooses an *ex ante effort* denoted e_i. The ex ante effort level is exerted before the realization of success. To fix ideas, we can view e_i as *study effort*. For simplicity, we assume that e_i may be equal to 0 or to 1. The *study-effort* cost is $c_i e_i$ for type i, where $c_i > 0$ is a parameter. Study effort influences the probability of success in a type-dependent way. The probability of success of a student exerting ex ante effort, e_i, is denoted $\mathbf{p}_i(e_i)$, where

$$\mathbf{p}_i(e_i) = \Pr(\text{ success} \mid i \text{ and } e_i).$$

To simplify notation, define $P_i = \mathbf{p}_i(1)$, the probability of success under high ex ante effort, and $p_i = \mathbf{p}_i(0)$, the probability of success under low ex ante effort. Type 2, the "talented

[14] λ_i is also the marginal probability of i in the distribution of $(i, k) \in \{1, 2\}^2$.

[15] If q represents a quantity, it measures, for instance, years of education, and if it's a quality, it may be a level in a hierarchy of certificates, or the quality of the educational institution.

type", is more likely to succeed given the effort level. We assume that success is observable. This is how we model *ex ante moral hazard*.

Formally, we assume,

Assumption 1. $0 < p_i < P_i < 1$, $i = 1, 2$, and $P_2 > P_1$; $p_2 > p_1$.

To sum up, types i and efforts e_i are not observable, but the public authority observes the individual's wage w, the quality of education q_i, which is recorded, and the event of success or failure.

2.1.2 Job market

We now describe the job-market phase. In case of failure, the student gets a basic job with the basic wage $\underline{w}$. Success, the second chance move, will allow the student to find a job with a salary greater than the basic wage[16].

In case of success, once the student graduated from university, a third chance move determines the student's ex post type k, or job-market skills. Depending on the realization of these skills, students will be able to occupy either a *top job* or a *middle-range job*. The probability of ex post type k, given success and education q, is defined as follows,

$$\Pr(k = q \mid q \text{ , success }) = 1 - \pi,$$

and obviously, $\Pr(k \neq q \mid q \text{ , success }) = \pi$. We assume that $\pi < 1/2$. This may be interpreted as follows: with a small probability π, an individual choosing q in $\{1, 2\}$ will be endowed with ex post characteristics $k \neq q$. In other words, a successful type $i = 2$ who chose $q = 2$ "normally" meets expectations and becomes a type $(i, k) = (2, 2)$, with high realizations of both types, and can occupy a top job, but some individuals, namely, the types $(i, k) = (2, 1)$, lack the necessary characteristics, in spite of being high quality graduates. These students will find a middle-range job. In the same fashion, a successful type $i = 1$ with education $q = 1$ would "normally" become a type $(i, k) = (1, 1)$ and obtain a middle-range

[16] The government observes success (*i.e.*, graduation). This assumption leads to some simplifications, because it allows us to reduce the number of *ex post* incentive constraints taken into consideration, again without losing anything essential in the analysis.

job, but some of these students, namely, types $(i,k) = (1,2)$, "escape their fate" because they possess the necessary skills or characteristics to occupy the top jobs.

To sum up, the quality of education increases the probability of drawing a high job-market type $k = 2$. This assumption is reasonable. The increased probability may be due to knowledge, training, to peer effects, or constitutes a return to network building during college years. It may also be due to some complementarity between job-market skills and higher education, these skills being in a sense catalyzed by education.

During the job-market phase, successful students choose *effort at work*, or *ex post effort*. The ex post effort of a type (i,k) is denoted ε_{ik}, and the ε_{ik} may be equal to 0 or to 1. For simplicity, we assume that ex post effort plays no role in case of failure. High effort at work is required from individuals occupying top jobs and low effort is the norm for middle-range jobs[17]. The wage earned by a successful individual depends on education q and effort at work ε_{ik} only. In other words, wages depend on type i through q only (academic talent is crystallized in q when $q_i = i$), and type k determines effort at work only through its impact on disutility (as indicated below).

Let $\boldsymbol{\omega}(q,\varepsilon)$ be a function measuring the wage of a graduate of quality q when ex post effort ε is exerted. We set,

$$\begin{aligned} \boldsymbol{\omega}(q,1) &= W_q, \\ \boldsymbol{\omega}(q,0) &= w_q, \end{aligned}$$

where W_q corresponds to a top job and w_q to a middle-range job, $q = 1, 2$. This means that a successful type (i,k) with education $q = i$ can earn a top salary W_q, but only at the cost of high ex post effort, $\varepsilon_{ik} = 1$; she obtains a middle-range salary w_q otherwise. The above specification is a simple way of representing the fact that, to a certain extent, a type $i = 1$ with education $q = 1$ can make up for a low expected productivity w_1 by means of high ex post effort and obtain $W_1 > w_1$. Conversely, a type $i = 2$ with education $q = 2$ can decide to exert low effort and will in that case content herself with a middle-range salary $w_2 < W_2$. To sum up, there are 5 different wage levels: the base wage $\underline{w}$, and w_1, w_2, W_1, W_2.

We now state a natural assumption on $\boldsymbol{\omega}(q,\varepsilon)$.

[17] The fact that zero ex post effort is the "norm" for middle-range jobs is just a convenient normalization.

Assumption 2.

(a) $\boldsymbol{\omega}(q,\varepsilon) \geq \underline{w}$ for all q and ε;

(b) $\boldsymbol{\omega}(2,0) > \boldsymbol{\omega}(1,0)$;

(c) $\boldsymbol{\omega}(2,1) - \boldsymbol{\omega}(2,0) \geq \boldsymbol{\omega}(1,1) - \boldsymbol{\omega}(1,0) > 0$;

(d) $\boldsymbol{\omega}(1,1) \geq \boldsymbol{\omega}(2,0)$.

Assumption 2b guarantees $w_2 > w_1$. Assumption 2c implies $W_q > w_q$. The return on ex post effort is higher when education is longer. As a consequence of Assumptions 2b and 2c, we also have,

$$W_2 - W_1 \geq w_2 - w_1 > 0, \tag{1}$$

and by 2d, the smallest W_q is larger than the highest w_q, *i.e.*, $W_1 \geq w_2$.[18]

The disutility of *effort-at-work* is equal to $\beta_{ik}\varepsilon_{ik}$ in state (i,k). Ex post types differ in this disutility. The total cost of effort is additively separable and equal to $c_i e_i + \beta_{ik}\varepsilon_{ik}$. In a nutshell, the disutility of effort at work of a type $(i,1)$ is assumed to be huge. We choose the simplest structure for the β_{ik} terms; precisely,

$$\begin{aligned} \beta_{22} &= \beta_{12} = b > 0, \\ \beta_{11} &= \beta_{21} = B > 0. \end{aligned}$$

The meaning of these assumptions is that a type $(i,2)$ incurs a disutility b to get a top job, and a type $(i,1)$ incurs disutility $B > b$ while trying to mimic a type $k = 2$. We assume that B is very large, in such a way that no individual endowed with a low ex post type will ever try to exert high effort and earn the highest wage.

The public authority does not observe k and ε_{ik}, and will not be able to infer k and ε_{ik} from the observation of wages in all cases. The tax authority will face an informational problem, because a successful person with observed education $q = q_i$ and observed wage w_q may be, either a type $(i,1)$, or a type $(i,2)$ that chose to behave as a type $(i,1)$ ex post. By assumption, the latter deviation is not observable. It follows that if the government taxes the income of types $(2,2)$ too much, these individuals will choose to occupy middle-range jobs (the same is true of types $(1,2)$ who can also occupy top jobs). This specific structure is

[18] Assumption 2d is not crucial.

a way of combining a student-loan model with an elementary form of the optimal income-tax problem in a simple and relatively tractable manner, as will be seen below.

2.1.3 Loans

We now turn to the menu of student loans and the distribution of income. Consider a public (or publicly regulated) lending system and assume that the public lending authority distributes all loans. By assumption, a student loan covers the cost of education[19]. So, the amount of a loan to an individual choosing education q is γ_q. Students initially choose in a menu of loan contracts. Reimbursement is contingent on earnings and on the quality of education, both observed by the government. Let (R_{q1}, R_{q2}, r_q) denote the repayment profile of a loan of size γ_q. A student choosing education q must repay,

(i), r_q in case of failure (and earnings are $\underline{w}$),

(ii), R_{q1} if earnings are equal to w_q, and finally,

(iii), R_{q2}, if earnings are W_q.

There are 6 possible repayment levels. This framework creates a Mirrleesian taxation problem, because the most lucky types can decide to reduce their effort *ex post* if their income is taxed too heavily through repayments R_{22} and R_{12}.

A standard bank loan would be of the form $(R_{q1}, R_{q2}, r_q) = (R_q, R_q, 0)$ with $R_{q1} = R_{q2} = R_q = (1+\rho)\gamma_q$, assuming that ρ denotes the interest rate and that the student does not reimburse the loan in case of failure: the ordinary bank loan is just a particular case in a much larger set of admissible contracts.

By definition, in this economy, an allocation is an array $\{(e_i, q_i, r_i, R_{ik}, \varepsilon_{ik})\}_{i,k=1,2}$. A menu of contracts is an array $\{(q, R_{q1}, R_{q2}, r_q)\}_{q=1,2}$. Given the above assumptions, we now write the resource constraint when (q, R_{q1}, R_{q2}, r_q) is chosen by type $i = q$ only, $i = 1, 2$.

(i) With probability $\mathbf{p}_i(e_i)(1-\pi)$, the public banker collects the amount

$$X_i = \varepsilon_{ii} R_{i2} + (1 - \varepsilon_{ii}) R_{i1}$$

[19] This is a restriction because we could have considered partial funding, but this would have required consideration of a third dimension of types: initial wealth or parental wealth. Under the simplifying assumption that all students have the same initial wealth, we would have then studied the possible impact of credit rationing on incentives.

from type i students;

(ii), with probability $\mathbf{p}_i(e_i)\pi$, the public banker collects the amount

$$Y_i = \varepsilon_{ij} R_{i2} + (1 - \varepsilon_{ij}) R_{i1},$$

with $j \neq i$, from type i students; and

(iii), with probability $1 - \mathbf{p}_i(e_i)$, the public banker collects the amount r_i from type i students.

Using the notation just defined, the resource constraint can be written,

$$\sum_i \lambda_i \{\mathbf{p}_i(e_i)[(1-\pi)X_i + \pi Y_i] + (1 - \mathbf{p}_i(e_i))r_i - \gamma_i\} \geq 0, \qquad (RC)$$

with $i = q$. We assume that the lending system is self-financed, but of course, it would be easy to add an external source of funds: that would simply add a constant in RC.[20]

Finally, we formally assume that students are risk averse.

Assumption 3. The utility function $u(.)$ is strictly increasing, strictly convave and continuously differentiable.

2.2 First-Best Optimality

Let V_i denote the *ex post* utility of a successful student with education $q_i = i$, when the ex post type is $k = i$. Let v_i denote the *ex post* utility of a student with education $q_i = i$, when $k \neq i$. The *interim* expected utility of a successful student who chose education $q_i = i$ is by definition,

$$U_i = \pi(v_i - \beta_{ij}\varepsilon_{ij}) + (1 - \pi)(V_i - \beta_{ii}\varepsilon_{ii}), \qquad (2)$$

where

$$\begin{aligned} v_i &= u[\boldsymbol{\omega}(q_i, \varepsilon_{ij}) - Y_i], \\ V_i &= u[\boldsymbol{\omega}(q_i, \varepsilon_{ii}) - X_i]. \end{aligned}$$

with $i \neq j$. The utility of an unsuccessful type i is denoted u_i, and $u_i = u(\underline{w} - r_i)$.

[20] The government may collect some taxes on tuition payments γ_i, so that the appropriate definition of γ should be net of any such taxes.

The *ex ante* expected utility of a type i student, net of expected effort costs, is simply

$$\mathbf{p}_i(e_i)U_i + (1 - \mathbf{p}_i(e_i))u_i - c_i e_i, \tag{3}$$

where $i = 1, 2$.

If we assume that welfare is higher when all types i choose education $q = i$, $i = 1, 2$, the first-best *utilitarian* optimum can be obtained as a solution of the following problem,

$$\text{Maximize} \quad \sum_i \lambda_i[\mathbf{p}_i(e_i)U_i + (1 - \mathbf{p}_i(e_i))u_i - c_i e_i] \tag{4}$$

with respect to $\{(e_i, r_i, R_{ik}, \varepsilon_{ik})\}_{i,k=1,2}$, subject to the resource constraint RC, and each effort level is chosen in the set $\{0, 1\}$. In the above formulation of the first-best problem, it is understood that type i is assigned to education $q_i = i$, for all i. This assumption is justified below.

We call the solution of this problem a *standard* utilitarian optimum because the expected utility of each type is weighted by its frequency in the population. There are of course other optima that can be generated by varying the weight of type i in the social objective. These optima are studied in Section 6 below.

To determine the first-best effort vector $(e_i^*, \varepsilon_{i1}^*, \varepsilon_{i2}^*)$, we need to compute the optimal allocation of utility in $2^6 = 64$ cases, that is, consider in turn each possible vector of efforts and compare the value of welfare for each of these combinations. But the only really interesting case is when $(e_1^*, e_2^*) = (1, 1)$, *i.e.*, high study effort is required from both types, and $(\varepsilon_{i1}^*, \varepsilon_{i2}^*) = (0, 1)$, *i.e.*, ex post effort is high if and only if job-market skills are high.

It can be shown that $e_i = 1$ is optimal for all i if the effort costs c_1, c_2 are small enough and if the difference $P_i - p_i$ is large enough, that is, if effort is sufficiently effective in increasing the probabilities of success. We assume that this is indeed the case. In addition, if b is not too large, while B is very large, and if W_q is sufficiently large as compared to w_q, it will always be optimal to let the types $(i, 2)$ choose high effort ex post. In other words, under these assumptions, it is socially efficient to require that all individuals endowed with $k = 2$ occupy high-pay jobs. It follows that we have,

$$\begin{aligned} V_2 &= u(W_2 - R_{22}), \quad v_2 = u(w_2 - R_{21}), \\ V_1 &= u(w_1 - R_{11}), \quad v_1 = u(W_1 - R_{12}). \end{aligned} \tag{5}$$

Given the optimal choice of efforts, the expected wage Ew_q of a student with education q is defined as follows,

$$
\begin{aligned}
Ew_1 &= (1-\pi)w_1 + \pi W_1, \\
Ew_2 &= (1-\pi)W_2 + \pi w_2.
\end{aligned}
$$

The social surplus S_i is defined as follows:

$$S_i = P_i Ew_i + (1 - P_i)\underline{w} - \gamma_i.$$

S_i gives the expected social benefit of education for a type i student choosing $q = i$ and the "right" amounts of effort, ex ante and ex post. We assume the following,

Assumption 4.

(a) $S_i \geq \underline{w}$ for all i;

(b) $P_2[Ew_2 - Ew_1] \geq \gamma_2 - \gamma_1 \geq P_1[Ew_2 - Ew_1]$.

Thus, we assume that an education of quality $q_i = i$ is profitable, on average, for type i, as compared to the no-education alternative, with value $\underline{w}$. Assumption 4a will be satisfied, in essence, if $\boldsymbol{\omega}(q, 0)$ is high enough as compared to $\underline{w}$, and if γ_q is not too large. Assumption 4b says that assigning type i to education $q = i$ generates more surplus than choosing $q \neq i$, given the probabilities of success P_i.[21] This assumption eliminates some uninteresting corner solutions.

Now, define the inverse utility function,

$$z(x) = u^{-1}(x). \tag{6}$$

By definition, $z(u)$ is the minimal amount of resources needed to provide utility u. Using $z(.)$, we obtain

$$R_{22} = W_2 - z(V_2); \quad R_{21} = w_2 - z(v_2); \qquad R_{12} = W_1 - z(v_1); \quad R_{11} = w_1 - z(V_1); \tag{7}$$

and $r_i = \underline{w} - z(u_i)$, $i = 1, 2$.

[21] Note that $Ew_2 > Ew_1$ since under Assumption 2, $(1-\pi)W_2 + \pi w_2 > (1-\pi)W_1 + \pi w_1 > (1-\pi)w_1 + \pi W_1$.

With this interpretation in mind, it is easy to see that the expected amount of resources needed to provide expected utility $P_i U_i + (1 - P_i)u_i$ to a type i with education $q = i$ is given by the expression,

$$\mathbf{E}(z_i) = P_i[(1-\pi)z(V_i) + \pi z(v_i)] + (1 - P_i)z(u_i). \tag{8}$$

With the help of these definitions, the first-best optimality problem can be rewritten as follows. Eliminating R_{ii}, R_{ij} and r_i from the objective and the resource constraint RC, and recalling that $P_i = \mathbf{p}_i(1)$, we obtain an equivalent expression for the first-best utilitarian optimum problem,

$$\text{Maximize} \quad \sum_i \lambda_i [P_i U_i + (1 - P_i)u_i - c_i] \tag{9}$$

with respect to $(V_i, v_i, u_i)_{i=1,2}$, subject to the resource constraint,

$$\sum_i \lambda_i \{S_i - \mathbf{E}(z_i)\} \geq 0. \tag{$\overline{RC}$}$$

Given that the values of (e^*, ε^*) are now given, the cost of effort terms are just additive constants that play no role in the above maximization problem. The first-best problem becomes easy to solve.

We can state the following result.

Proposition 1. *Under Assumptions 1-4, first-best Pareto optimality implies* full insurance*, that is, for all i,*

$$V_i^* = v_i^* = u_i^*,$$

and the standard utilitarian first-best optimum exhibits full equality*, i.e.,*

$$V_1^* = V_2^*, \quad v_1^* = v_2^* \quad \text{and} \quad u_1^* = u_2^*.$$

In addition, first-best efficiency implies that the resource constraint is binding.

Proposition 1 is a standard consequence of risk aversion. Students are fully insured against labour market risk, but also against the risk of being of type 1.

This proposition describes an extremely idealized situation in which any degree of redistribution is possible, and politically acceptable.

We can also show the following.

Corollary 1. *Under Assumptions 1-4,*

(i), there does not exist an optimum with $r_i^* \geq 0$ *for all* i;

(ii), if, in a first-best Pareto optimum, individuals obtain weakly more in case of success than if they were not educated, i.e., if $W_2 - R_{22}^* \geq \underline{w}$, *and* $w_1 - R_{11}^* \geq \underline{w}$ *then, first-best efficiency implies* $r_i^* \leq 0$ *for all* i, *i.e., the students who fail receive insurance payments.*

Corollary 1 shows that the public banker should also be an insurer. If we do not permit negative repayments (*i.e.*, if the banker cannot be an insurer), it would now be easy to show that optimality implies $r_i^* = 0$: we find a *contingent reimbursement loan*, in the ordinary sense that no repayment is required in case of failure[22]. In the analysis below, negative repayments are permitted.

3 Asymmetric Information and Incentive Constraints

Let us now study the case in which types are not observed by public authorities. By definition, second-best optimal (or *interim efficient*) allocations maximize a weighted sum of the students' expected utilities, subject to resource-feasibility and incentive-compatibility constraints. Students self-select in a menu of contracts proposed by the public authorities. The allocation determines *ex post* utility values (V_i, v_i, u_i) and a quality of education q_i for each type i. Although in principle, second-best effort levels could be different from their first-best counterparts, we now assume that second-best effort levels are equal to the first-best effort levels. More precisely, we assume that the second-best optimal study effort is high for all types, that is, $e_i = 1$ for all i, and that second-best ex post effort is high ($\varepsilon_{ik} = 1$) if and only if $k = 2$. This is at the same time a reasonable assumption and the only interesting case here, given that effort variables are discrete. Other cases, in which some or all of the types exert zero effort, could be studied in a very similar way. Again, high study efforts will be optimal if the ratios $c_i/(P_i - p_i)$ and the disutility cost b of *ex post* effort are not too

[22] This case is studied in a working paper; see, Gary-Bobo and Trannoy (2012).

large, for otherwise, the social cost of providing incentives could be higher than the benefits of effort in terms of aggregate surplus. The social benefits of effort are clearly the increased probabilities of success and the increased productivity of agents on the labour market. In this model, the social cost of providing incentives is due to the addition of a dose of inequality and risk, imposed by incentive constraints, on top of the direct disutility of effort itself.

3.1 Incentives

We first consider the *ex post* incentives. Students know their ex ante types. They will reveal their ex ante types by choosing the quality of their education: type i students choose education $q_i = i$. This result will be obtained if the menu of loans $\{(q, R_{q1}, R_{q2}, r_q)_{q=1,2}\}$ is *ex ante* incentive compatible. In the ex post stage, the students with education level $q = 1$ and ex post type $k = 2$ will not choose a job with a middle-range salary *ex post* if and only if,

$$v_1 - b \geq V_1. \tag{ICX_1}$$

Similarly, a student with education level $q = 2$ and type $k = 2$ will not be tempted to behave as a type $k = 1$ *ex post* if and only if,

$$V_2 - b \geq v_2. \tag{ICX_2}$$

We will see below that these constraints must be binding at a second-best optimum. The other constraints, that is, $V_1 \geq v_1 - B$ and $v_2 \geq V_2 - B$ will always be satisfied since B is sufficiently large, as assumed above.

Given values of (V_i, v_i) satisfying ICX_1 and ICX_2, we now have a *generalized Principal-Agent problem* in the sense of Myerson (1982).[23] We apply the *extended revelation principle.* The constraints bearing on *ex ante* utilities and efforts are *revelation* and *obedience* constraints: the students should simultaneously self-select by choosing the right contract in the menu of loans *and* exert the right amount of effort ex ante. The *interim* expected utility (*i.e.*, the expected utility knowing success), can now simply be written as follows,

$$U_1 = (1 - \pi)V_1 + \pi(v_1 - b), \tag{EU_1}$$

$$U_2 = (1 - \pi)(V_2 - b) + \pi v_2. \tag{EU_2}$$

[23] On this notion, see also Laffont and Martimort (2002)

Since we assume that high ex ante effort is efficient, incentive constraints can easily be written. First, the *ex ante self-selection* constraint $\overline{IC}_i$ says that type i should not be tempted to mimic type j *ex ante* while exerting high effort,

$$P_i U_i + (1 - P_i)u_i \geq P_i U_j + (1 - P_i)u_j, \qquad (\overline{IC}_i)$$

for all $i = 1, 2$ and $j \neq i$. The cost of ex ante effort being c_i on both sides, it cancels from $\overline{IC}_i$.[24] Second, the *ex ante moral hazard* constraint MH_i says that type i should prefer to exert high study effort over low study effort *ex ante*,

$$P_i U_i + (1 - P_i)u_i - c_i \geq p_i U_i + (1 - p_i)u_i \qquad (MH_i)$$

(Recall that $p_i = \mathbf{p}_i(0)$ and $P_i = \mathbf{p}_i(1)$). In addition, the incentive constraint $\underline{IC}_i$ says that type i prefers high study effort to low study effort *and* mimicking type j *ex ante*, that is,

$$P_i U_i + (1 - P_i)u_i - c_i \geq p_i U_j + (1 - p_i)u_j, \qquad (\underline{IC}_i)$$

with $j \neq i$.

By definition, the *second-best optimality problem* is the following:

$$\text{maximize} \sum_i \lambda_i [P_i U_i + (1 - P_i)u_i - c_i]$$

subject to $\overline{RC}$, $\overline{IC}_i$, MH_i, $\underline{IC}_i$, ICX_i and EU_i.

3.2 Ex Post Moral Hazard

When the second-best optimality problem is posed as above, it is immediate that the variables (V_i, v_i) appear only in ICX_1, ICX_2 and in the resource constraint $\overline{RC}$. We can therefore decompose the problem as follows. Fix the value of U_i. Then, ex post utility levels (V_i, v_i) must be chosen in such a way that they minimize the expected cost $(1 - \pi)z(V_i) + \pi z(v_i)$ subject to ICX_i, $i = 1, 2$ and constraints EU_i. This can be proved easily, using standard techniques. Intuitively, this minimization operation is just finding the least costly way of providing incentives, given the *ex post* effort-incentive constraints. In other words, the social

[24] The cost of ex ante effort could depend on q_i. For instance, we could easily handle the following more general formulation: $C_i(q, e_i) = d(q) + c_i e_i$, but this would not change the conclusions.

planner should choose the minimal level of risk compatible with ICX_i. It follows that both ICX_i constraints must be binding at the second-best optimum. We can state this result formally.

Lemma 1. *The ex post incentive constraints ICX_i, $i = 1, 2$, must be binding at any second-best optimum, that is,*

$$v_1 = V_1 + b, \qquad \textit{and} \qquad v_2 = V_2 - b. \tag{10}$$

Given this result and constraint EU, the (V_i, v_i) variables can be completely eliminated from the welfare maximization problem. We find convenient expressions for ex ante utility, namely,

$$U_1 = V_1, \qquad \textit{and} \qquad U_2 = V_2 - b. \tag{11}$$

Eliminating (V_i, v_i) from $\overline{RC}$, we obtain a modified resource constraint, in which V_i and v_i are functions of U_i, as follows.

$$V_1 = U_1, \quad V_2 = U_2 + b, \quad v_1 = U_1 + b, \quad v_2 = U_2. \tag{12}$$

As a consequence, the resource constraint and all the incentive constraints are expressed in terms of (U_i, u_i) only.

3.3 Ex Ante Moral Hazard

We now study the role played by ex ante moral hazard or study-effort incentives. It is not difficult to see that MH_i can be rewritten as, $(P_i - p_i)(U_i - u_i) \geq c_i$, or

$$U_i - u_i \geq K_i \tag{MH_i}$$

where by definition,

$$K_i = \frac{c_i}{P_i - p_i}. \tag{13}$$

Moral hazard will thus force a gap between the reward of success and that of failure. It is natural to assume that type $i = 2$ is more efficient than type $i = 1$ while exerting effort[25]. Formally, we assume the following.

[25] $1/K_i$ measures the efficiency of effort.

Assumption 5. $K_1 \geq K_2 \geq 0$.

Can we now ignore at least one of the two moral hazard constraints? The answer is yes, under $\overline{IC}$.

Lemma 2. *Under Assumption 5, if $\overline{IC}_1$, $\overline{IC}_2$ and MH_1 hold, then MH_2 is satisfied.*

3.4 Adverse Selection

We now study the implications of the *ex ante* incentive constraints. Since $\overline{IC}_i$ can be rewritten $P_i(U_i - U_j) \geq (1 - P_i)(u_j - u_i)$, we get the string of inequalities,

$$\frac{P_2}{1 - P_2}(U_2 - U_1) \geq u_1 - u_2 \geq \frac{P_1}{1 - P_1}(U_2 - U_1). \qquad (\overline{IC})$$

Using $\overline{IC}$, we then derive the following useful result.

Lemma 3. *$\overline{IC}$ constraints imply the conditions,*

$$U_2 - u_2 \geq U_1 - u_1, \qquad (D)$$

$$U_2 \geq U_1, \qquad \text{and} \qquad u_1 \geq u_2. \qquad (D')$$

(a) If $\overline{IC}_1$ and $\overline{IC}_2$ are simultaneously binding, then $U_2 = U_1$ and $u_1 = u_2$: we get equal treatment *(but not necessarily* full insurance*).*

(b) Under $\overline{IC}_1$ and $\overline{IC}_2$, we have $u_1 = u_2$ if and only if $U_2 = U_1$.

Lemma 3 says that if equal treatment does not hold, then, either $\overline{IC}_1$ or $\overline{IC}_2$ is binding, or none of them, but not both. Assumption 5 is not needed for the proof of Lemma 3. The $\underline{IC}_i$ constraints are an added difficulty, but we can in fact ignore them, as shown by Lemma 4.

Lemma 4. *Under Assumption 5,*

(a) if $\overline{IC}_i$, $i = 1, 2$ and MH_1 hold, then $\underline{IC}_1$ is satisfied.

(b) if $\overline{IC}_2$ is satisfied, and if, in addition, $\overline{IC}_1$ and MH_1 are binding, then, $\underline{IC}_2$ is satisfied.

The proofs of all Lemmas are in the appendix. Being equipped with these preliminary results, we can now solve the optimal student-loan problem under asymmetric information.

4 The Graduate Tax Problem

The second-best optimality problem described above can be called *the graduate tax problem*, since we can easily derive the optimal loan repayment schedule from the optimal solution, and a graduate tax can be used to implement this schedule. We then inquire whether the optimal graduate tax can be decomposed as the sum of an income tax, depending only on earnings, and a loan repayment, that depends only on education. The answer to this question is no, in general. To implement the optimal graduate tax, we need a loan repayment schedule depending on education *and* income, that is, an *income-contingent repayment schedule*, or else, we need an income tax that depends also on education, *i.e.*, a *graduate tax*.

A further study of first-order conditions for optimality yields the following result, that may be called the *equal treatment* result.

Proposition 2. (Equal treatment as a second best under moral hazard and adverse selection.) *Under Assumptions 1-5, for all* $\pi < 1/2$, *the second-best optimal, standard utilitarian solution has the following properties:*

$$U_1 = U_2 = \mathbf{U}, \qquad u_1 = u_2 = \mathbf{u} \quad (\textit{equal} \quad \textit{treatment}),$$

$$\mathbf{U} = \mathbf{u} + K_1 \quad (\textit{incomplete} \quad \textit{insurance}),$$

$\overline{RC}$, MH_1, $\overline{IC}_1$, $\overline{IC}_2$, ICX_1 *and* ICX_2 *are all binding. If, in addition,* $K_1 > K_2$, *then,* MH_2, $\underline{IC}_1$ *and* $\underline{IC}_2$ *hold as strict inequalities.*

Proposition 2 shows that the second-best optimal solution exhibits *equal treatment,* in the limited sense that $U_1 = U_2$ and $u_1 = u_2$, and *incomplete insurance*, since $U_i = u_i + K_1$. The *equal treatment property* means that the post-study but pre-work expected utilities are equalized. In particular, the *interim* expected utilities U_i of both ex ante types i are the same when students are successful. They are also the same conditional on failure.

The intuition for the equal treatment result is relatively easy to convey. Suppose for a moment that the planner observes the ex ante types i. If moral hazard is the only problem in the economy, the planner should impose the minimal amount of risk compatible with high ex ante effort on each type, that is, $U_i - u_i = K_i$ for all i.[26] The talented types being more

[26] This is shown in Section 5 below.

efficient, we have $K_1 > K_2$ and therefore, $U_1 - u_1 > U_2 - u_2$. This contradicts condition D, which is a consequence of $\overline{IC}$, *i.e.*, of self-selection constraints, as shown by Lemma 3. If we now re-introduce the $\overline{IC}$ constraints, it is intuitively clear that the best is to choose the smallest admissible $U_2 - u_2$, that is, to set "equal differences" $U_2 - u_2 = U_1 - u_1 = K_1$. Under $\overline{IC}$, equal differences in turn imply $u_1 = u_2$, as can easily be checked[27]. So, we derived the equal treatment property. This shows why equal treatment is the result of the interplay of moral hazard and adverse selection.

However, inequalities will be created ex post, when the ex post types k are drawn and revealed. Indeed, the optimal solution typically involves *ex post inequality*, since $U_1 = U_2$ implies $v_1 = V_2 > V_1 = v_2$ (as explained by Proposition 3 and Corollary 3 below).

The solution also entails *ex ante inequality.* To see this, consider the ex ante expected utility of type i, namely, $\mathbf{E}u_i = P_i U_i + (1 - P_i)u_i$. The second-best optimum entails,

$$(\mathbf{E}u_2 - c_2) - (\mathbf{E}u_1 - c_1) = (P_2 - P_1)K_1 + (c_1 - c_2) > 0,$$

since $c_2 \leq c_1$. The talented types are strictly better off ex ante. Both types obtain the same expected payment in the event of success as well as in the event of failure, but *(i)*, insurance is incomplete: $U_i - u_i = K_1$, and *(ii)*, the talented types have a higher probability of success, $P_2 > P_1$.

The optimal second-best allocation is trivially not first-best efficient, since first-best efficiency requires full insurance. The solution potentially entails a limited form of *exploitation of the talented*, by means of cross-subsidies between types, since the less-talented are also producing less surplus *per capita*. In the case described by Proposition 2, this subsidy from the talented is a price paid to solve the incentive problem. Indeed, both $\overline{IC}_i$ constraints are binding and incentive compatibility holds since $U_1 = U_2$. It is only when the social welfare function sufficiently favors the talented that the incentive problem is solved by means of *screening*, imposing a higher level of risk (and return) on the most productive agents (this is shown in Section 6 below).

[27] This is because we must have $P_i(U_i - u_i) + u_i \geq P_i(U_j - u_j) + u_j$ for all $i \neq j$.

To sum up, the optimal solution involves, *(i)*, ex ante inequality of ex ante types, due to incomplete insurance; *(ii)*, equality of interim expected utilities conditional on success or failure (*i.e.*, *equal treatment*), and *(iii)*, ex post inequality, conditional on success, due to the operation of the labor market and constraints on income taxation.

We now list the *ex post* utilities associated with the solution, and derive the *optimal graduate tax schedule*. Given that $U_1 = U_2$, we have $V_1 + b = V_2$ and therefore $V_2 > V_1$ (recall that $b > 0$). Recall that ICX_i constraints impose,

$$v_1 = V_1 + b, \qquad \text{and} \qquad v_2 = V_2 - b.$$

We can therefore state the following result.

Proposition 3. *In the standard utilitarian case, the second-best optimal allocation has the following properties,*

$$\begin{aligned} u_1 &= u_2 = \mathbf{u}, \\ v_2 &= V_1 = \mathbf{u} + K_1, \\ v_1 &= V_2 = \mathbf{u} + K_1 + b, \end{aligned}$$

It follows that,

(a) $v_1 = V_2 > V_1 = v_2$;

(b) $V_1 > u_1$.

The proof of Proposition 3 is very easy, given Proposition 2, and the relationships between U_i, V_i and v_i that it implies. In practice, it follows from this result that the successful "self-made man", *i.e.*, type $(1, 2)$, earns the same after-tax income than the successful graduate from a top school with the necessary job-market skills, *i.e.*, type $(2, 2)$, since $v_1 = V_2$. A top-school student lacking the necessary job-market skills, *i.e.*, type $(2, 1)$, has the same after tax income than the plain student, *i.e.*, type $(1, 1)$, since $V_1 = v_2$. These consequences of the optimal graduate tax become explicit when we derive the implications of Propositions 2 and 3 in terms of income-contingent repayment (or education-contingent income tax). The next result shows that the equal treatment property is achieved by means of an unequal *ex post* treatment of graduates.

Corollary 3. (Optimal graduate tax, repayment schedule)

The optimal graduate tax has the following properties,

(a) $u_1 = u_2$ *implies* $r_1 = r_2$ *(equal treatment in case of failure)*;

(b) $V_2 > V_1$ *implies* $W_2 - w_1 > R_{22} - R_{11}$ *(the talented are not fully exploited)*;

(c) $v_1 = V_2$ *implies* $R_{22} > R_{12}$ *(self-made (wo)men repay less)*;

(d) $V_1 = v_2$ *implies* $R_{21} > R_{11}$ *(top-school students lacking the job-market skills repay more)*;

In the statement of Corollary 3, points *(c)* and *(d)* show that successful individuals who studied longer repay more. Thus, in case of success, past education choices matter: q_i contributes to the determination of repayments. Point *(c)* is a consequence of $v_1 = W_1 - R_{12} = W_2 - R_{22} = V_2$ implying $R_{22} - R_{12} = W_2 - W_1 > 0$. Similarly, to prove point *(d)*, we have, $V_1 = w_1 - R_{11} = w_2 - R_{21} = v_2$ implying $R_{21} - R_{11} = w_2 - w_1 > 0$.

In case of failure, point *(a)* shows that the lender "forgets" the choices of q_1 and q_2: this is mainly due to the fact that ex post effort does not matter for the students who failed. Nobody can falsely report failure, and the base wage $\underline{w}$ does not depend on effort or type. In a more general model, (r_1, r_2) would be shaped by the tradeoff between incentives and insurance, and the planner might be willing to set different values for r_1 and r_2. Point *(b)* shows that the talented cannot be fully exploited (the marginal income tax rate is smaller than 100%): this is due to the presence of an ex post moral hazard problem, for otherwise, the planner would be tempted to tax the $W_2 - w_1$ increment completely, in order to achieve greater equality.

In the first-best world, full insurance can be implemented with the help of a lump-sum tax: transfers would depend on type, success, and therefore, would also depend on education, insofar as ex ante types determine education. When negative repayments are permitted, it follows that income-contingent loans would be useless. Indeed, absent informational problems, the private lending sector could distribute standard loans efficiently, while the government would protect individuals against risk and redistribute the income. In contrast, in the second-best economy, income-transfer instruments are constrained by limited observation and incentives. The income tax is typically a function of observable income; standard loan repayments are proportional to years of education, with the addition of a bankruptcy clause that provides some form of insurance. These standard tools are not enough to achieve the second-best optimum. In particular, the loan repayments of successful individuals have no reason to be simply expressed as the product of an interest rate ρ times the cost of education, that is, $(1+\rho)\gamma_q$.[28] More flexible instruments are needed: appropriately designed income-contingent loans and (or) a graduate tax (with 100% subsidization of tuition fees) can do the required job.

In general, the optimal graduate-tax repayment schedule cannot be implemented by means of the addition of an ordinary income tax, (that would depend only on earnings) and of a standard loan repayment (that would depend only on education q_i). In other words, the optimal graduate tax cannot be expressed as the sum of two pieces: an income tax plus a standard loan repayment.

To be more precise, we would like to choose an income tax schedule $T_i = T(w_i)$ and a loan repayment schedule $L_i = L(q_i)$, with $q_i = i$, such that for any given 4 numbers $(R_{11}, R_{22}, R_{12}, R_{21})$ representing optimal repayments, we have,

$$\begin{aligned} R_{11} &= T_1 + L_1, \\ R_{22} &= T_2 + L_2, \\ R_{12} &= T_2 + L_1, \\ R_{21} &= T_1 + L_2. \end{aligned} \tag{14}$$

This implies

$$\begin{aligned} R_{22} - R_{12} &= L_2 - L_1 = R_{21} - R_{11}, \\ R_{22} - R_{21} &= T_2 - T_1 = R_{12} - R_{11}, \end{aligned}$$

which happens to be typically impossible. We can now state the non-decomposition result: the optimal graduate tax is *generically non-decomposable.*

Define the quantity,

$$\boldsymbol{\Delta} = (R_{22} - R_{21}) - (R_{12} - R_{11}). \tag{15}$$

[28] But γ_q itself may be a function of the interest rate on government bonds, and the latter rate would thus of course play a role in the resource constraint.

The linear system stated above (*i.e.*, (14)) implies that if the second-best repayments are decomposable, then $\mathbf{\Delta} = 0$. We now show that for a generic $(R_{11}, R_{22}, R_{12}, R_{21})$, these relations will not be true and $\mathbf{\Delta} \neq 0$. Using the definitions of V_i and v_i, we find

$$\mathbf{\Delta} = z(v_1) - z(V_1) + z(v_2) - z(V_2) + W_2 - W_1 + w_1 - w_2.$$

Using the properties $V_1 = v_2$ and $V_2 = v_1$ it is easy to see that,

$$\mathbf{\Delta} = (W_2 - W_1) - (w_2 - w_1). \tag{16}$$

By inequality (1), we have $\mathbf{\Delta} \geq \mathbf{0}$. We can have $\mathbf{\Delta} = \mathbf{0}$ only in a special case. Generically, we will have $\mathbf{\Delta} > \mathbf{0}$. This proves the non-decomposability property.

We have just proved the following result.

Proposition 4. (Non-decomposition Theorem: justification of income-contingent loan repayments) *Under Assumptions 1-5, it is generically impossible to decompose the second-best optimal transfers* R_{ik} and r_i *as the sum of an income tax, depending only on earnings, and student-loan repayments, depending only on education. It must be that, either the student-loan repayments are income-contingent, or the income tax is education-contingent.*

5 Application of Occam's Razor

In the above theory, we have introduced 4 important ingredients: *(i),* wage risk and risk aversion; *(ii)*, unobservable *ex ante types* leading to a *screening,* or *self-selection* problem; *(iii),* unobservable study effort generating *ex ante moral hazard*; *(iv)*, unobservable *ex post types* and effort causing an *ex post moral hazard* problem. Can we remove each of these ingredients from the model, to simplify the theory, and still obtain the key results? What do we find if we remove each of the ingredients in turn?

5.1 Without risk or risk aversion

If there is no wage risk, the problem posed above becomes rather trivial because, as soon as they would know their ex ante types, students could perfectly predict their future payoffs.

There are two sources of risk in the model: random success and ex post type draws. In a riskless world, study-effort choices would have deterministic effects and ex post types would play no role.

If, on the other hand, we impose risk neutrality, the model becomes quite degenerate, because there is no private demand for insurance and no social motive for redistribution (*i.e.*, no demand for insurance against the lack of talents, behind the veil of ignorance). Insurance is an essential element of student loan design. It follows that we need some risk and some degree of risk aversion.

5.2 Without ex ante moral hazard and ex ante screening

Before discussing what happens in the absence of moral hazard or of a screening problem ex ante, we briefly consider the situation in which ex post moral hazard is the only problem, *i.e.*, the model in which ICX_i inequalities are the only constraints. This can be called the "pure income taxation problem".

If we maximize $\Sigma_i \lambda_i (P_i U_i + (1 - P_i) u_i)$ subject to ICX_i and $\overline{RC}$, we find the following result.

Proposition 5. (Second best with observable ex ante types and observable study effort.) *When the only constraints are* $\overline{RC}$, and ICX_i, $i = 1, 2$, *then, in the standard utilitarian case, the second-best optimal solution has the following properties:*

$$u_1 = u_2 = \mathbf{u},$$

$$Ez_i' = z'(\mathbf{u}),$$

for all i, *where by definition,* $Ez_i' = \pi z'(v_i) + (1 - \pi) z'(V_i)$. The constraints $\overline{RC}$, ICX_1 *and* ICX_2 *are all binding.*

Proposition 5 is a relatively direct consequence of the first-order conditions for optimality. The solution requires *efficient interim insurance* given the remaining *ex post* moral hazard problem[29]. To achieve this goal, the marginal social cost of providing the basic

[29] Remark that when $\pi = 0$, the marginal social cost boils down to $Ez_i' = z'(U_i)$ and the condition $Ez_i' = z'(\mathbf{u})$ boils down to $z'(U_i) = z'(\mathbf{u})$, or $U_i = \mathbf{u}$.

welfare level $\mathbf{u}$, *i.e.*, $z'(\mathbf{u})$, must be equated with Ez_i', the expected marginal cost of providing the interim expected welfare level $U_i = (1-\pi)V_i + \pi v_i$. As a consequence, we have $z'(v_1) > z'(u_1) > z'(V_1)$ and $z'(v_2) < z'(u_2) < z'(V_2)$ or equivalently, $v_1 > u_1 > V_1$ and $V_2 > u_2 > v_2$. These properties are unpalatable, because a student who failed ends up being better off than an academically successful student with an unlucky draw of job market opportunities.

5.3 Without a screening problem

If there are no ex ante types, there is no ex ante screening problem; students do not self-select by choosing their level (or quality) of education. In a simple version of the model, there would be only one level of education and all students would face the same probability of success, conditional on study effort. Student loans would then be designed so as to induce effort, that is, to solve the moral hazard problems, both ex ante and ex post, but the graduate tax decomposition problem would be vacuous, because we need at least two types (and two levels of education) to pose this problem in a meaningful way.

Yet, there is another possibility, which is to study a model with two fully observable ex ante types: each type of student would then be assigned to an education level by the authoritarian social planner. To study this case, it is enough to compute second-best optima while ignoring all self-selection constraints, that is, both $\overline{IC}_i$ and $\underline{IC}_i$ constraints. The only constraints taken into consideration are $\overline{RC}$, MH_i and ICX_i. The results are quite different. When ex ante types are public information, we find the following result.

Proposition 6. (Second best with observable ex ante types and moral hazard.) *When the constraints are* $\overline{RC}$, MH_i and ICX_i, $i = 1, 2$, *then, in the standard utilitarian case, the second-best optimal solution has the following properties:*

$$U_i = u_i + K_i, \quad \text{for all } i = 1, 2.$$

$\overline{RC}$, MH_1, MH_2, ICX_1 *and* ICX_2 *are all binding, and*

$$\mathbf{E}z_1' = \mathbf{E}z_2',$$

where by definition, $\mathbf{E}z_i' = P_i[\pi z'(v_i) + (1-\pi)z'(V_i)] + (1-P_i)z'(u_i)$.

We skip the proof of Proposition 6 here. This proof relies on a discussion of the Kuhn and Tucker conditions for optimality.

In the above statement, it is striking that the *equal treatment* property is no longer required. To see this, note that MH_1 and MH_2 are simultaneously binding. It follows that, if $K_1 \neq K_2$, we cannot have $U_1 = U_2$ and $u_1 = u_2$. In the "normal" case, that is, if $K_1 > K_2$, we find that $\overline{IC}$ constraints must be violated. Indeed, condition D (in Lemma 3 above) implies $U_2 - u_2 \geq U_1 - u_1$, or equivalently, $K_2 \geq K_1$, a contradiction.

If $K_1 > K_2$, since the MH_i are binding, we must have $u_1 - u_2 < U_1 - U_2$. It follows that $U_1 > U_2$ is now a possibility, in particular if u_1 is close to u_2. Thus, $\overline{IC}_2$ may be violated. In other words, better insurance against a lack of academic talents, in case of failure, would imply that high quality studies are not chosen, because type $i = 2$ students would all prefer q_1. As a result, if ex ante types are assumed observable, the social planner would have to force the talented students to study longer.

The solution is indeed strikingly different from that of Proposition 2 above. Proposition 6 shows that *equal treatment* is a consequence of $\overline{IC}$ constraints. Interim equalization is not simply implied by the redistribution motive (risk aversion), it is a way of solving the self-selection problem. Absent self-selection constraints, Pareto-optimality does not impose equal treatment or full equality[30].

We conclude that with observable ex ante types, we would lose an important and realistic feature of our model, namely, the fact that education is an object of individual choice, which is also used to screen individuals[31]. In addition, the ex ante selection problem has a nontrivial influence on the shape of optimal repayment schedules.

[30] This point is also shown by the proof of Proposition 1, in the Appendix: full equality is not a consequence of first-best optimality, whereas *full insurance* is implied by first-best efficiency. The two things are different here. In the next subsection, Proposition 2' shows that equal treatment is a second-best device: it is imposed to take care of ex ante self-selection constraints, even when it is not a requirement of first-best optimality.

[31] This has been discussed by Spence (1973), although in the present paper, we model screening by means of self-selection in a menu of contracts as in Rothschild and Stiglitz (1976).

5.4 With an ex ante screening problem but without ex ante moral hazard

If we remove study effort from our theory, and if ex ante types are not publicly observable, the only constraints are $\overline{RC}$, ICX_i *and* $\overline{IC}_i$, $i = 1, 2$. In this case, we know that the best we could possibly achieve is *equal treatment in case of failure* with *efficient interim insurance*, that is, $u_1 = u_2 = \mathbf{u}$ and $Ez'_i = z'(\mathbf{u})$ for all i (as in Proposition 5). But the self-selection constraints $\overline{IC}_i$ would now impose $U_1 = U_2 = \mathbf{U}$ in addition (by Lemma 3c). Given $\overline{RC}$, we would then have three equations to determine two variables, namely $(\mathbf{U}, \mathbf{u})$. This is typically impossible. It follows that the solution would not simply be the allocation of Proposition 5, with the addition of $\mathbf{U} = \mathbf{u}$: we would find $Ez'_i \neq z'(\mathbf{u})$ for at least one i.

Another way of looking at the same difficulty is to take the equal treatment solution of Proposition 2 and let the cost of study effort go to zero, *i.e.*, $c_i \to 0$ for all i, so that, in the limit, we obtain $K_1 = 0$, and ex ante moral hazard disappears as an economic problem. By continuity, the solution is now such that $\mathbf{U} = \mathbf{u}$, but we have imposed the constraint $U_1 - u_1 \geq 0$, and this constraint is not a consequence of $\overline{IC}$ alone. In fact, we need a specific treatment of this case. The results are disappointing: absent MH_i constraints, it can be shown that the structure of the optimal solution depends on the shape of inverse marginal utility $z'(.)$.

Proposition 7. (Second best with unobservable ex ante types and observable study effort.) *When the only constraints are* $\overline{RC}$, $\overline{IC}_i$ and ICX_i, $i = 1, 2$, *then, in the standard utilitarian case, the second-best optimal solution may be an* equal treatment *or a* separating *allocation. If* π *is sufficiently small,*

(a) if $z'(.)$ *is concave, equal treatment is optimal:*

$$u_1 = u_2 = \mathbf{u}, \qquad U_1 = U_2 = \mathbf{U},$$

$$Ez'_2 \geq z'(\mathbf{u}) \geq Ez'_1,$$

$z'(\mathbf{u})$ *is a weighted average of* Ez'_2 *and* Ez'_1,

$$z'(\mathbf{u}) = \frac{\Sigma_i \lambda_i P_i E z'_i}{\Sigma_i \lambda_i P_i},$$

where, by definition, $Ez'_i = \pi z'(v_i) + (1-\pi) z'(V_i)$;

(b) if $z'(.)$ *is strictly convex,* $\overline{IC}_1$ *is binding and* $\overline{IC}_2$ *is slack. We have* $U_2 > U_1$, $u_1 > u_2$, *and*

$$z'(u_1) = Ez'_1 > Ez'_2 > z'(u_2).$$

(c) There is no solution such that $\overline{IC}_2$ *is binding and* $\overline{IC}_1$ *is slack. The constraints* $\overline{RC}$ *and* ICX_i *are always binding.*

We skip the proof of this Proposition[32]. The result shows that the second-best optimal allocation depends on specific properties of the inverse marginal utility function $z' = 1/u'$. The solution is less clear-cut than when ex ante moral hazard is taken into account. In fact, adding the MH_i constraints simplifies the solution and the proof of the optimum characterization result: the solution is easier to find under moral hazard, combined with adverse selection, than under adverse selection alone. The equal treatment solution described by Proposition 2 is also more robust in many ways. It is robust to changes in the values of key parameters and in the shape of the utility function, provided that the latter is strictly concave; it is also robust to changes in the weights of types in the social welfare objective, as shown by Proposition 2' below. Finally, in the student loan problem, it is not reasonable to assume that study effort can be perfectly monitored, or that study effort does not affect the probability of success. We conclude that dropping ex ante moral hazard does not yield an equivalent, but more parsimonious theory.

5.5 Without ex post moral hazard

This case is easy to study; the solution can be obtained by letting $\pi \to 0$: the ex post type $k \neq i$ is drawn with probability zero. Proposition 2 is still valid when ex post moral hazard does not play a role: the optimum exhibits equal treatment and incomplete insurance. The important difference with the general case is now that $V_1 = V_2$. These relations indicate that the *talented are completely exploited*. We have $R_{22} - R_{11} = W_2 - w_1$. In other words, the marginal income-tax rate is 100% for earnings above w_1. The student loan problem is now combined with a first-best taxation problem. This solution is clearly unrealistic. The

[32] The proof is available upon request.

general conclusion of this discussion is that none of the "ingredients" of our model can be dropped without losing an important feature of the solution.

5.6 Non-decomposition results

If we look at the proof of the non-decomposition theorem (*i.e.*, Proposition 4), we see that the proof does not depend on the fine details of the optimal allocation. In particular, if equal treatment does not hold, *i.e.*, if $U_1 \neq U_2$, the non-decomposition theorem is still valid. The non-decomposition property is still true in the simplified versions of the model studied above. Under the assumptions of Proposition 5, for instance, when ex post moral hazard is the only problem, the non-decomposition property holds.

Proposition 8. (Non-decomposition result when ex post moral hazard is the only problem.) *When the only constraints are* $\overline{RC}$, and ICX_i, $i = 1, 2$, *then, in the standard utilitarian case, the second-best optimal transfers* R_{ij} *cannot be decomposed as the sum of an income tax, depending only on earnings, and student-loan repayments, depending only on education. It must be that, either the student-loan repayments are income-contingent, or the income tax is education-contingent.*

6 Complete Characterization of Second-best Optima

Our goal is now to characterize all second-best Pareto optima in our model. To this end, we will consider weights for ex ante types that may differ from their frequencies in the population. Let α_1 and α_2 be the weights of type $i = 1$ and type $i = 2$ in the welfare function. We assume $\alpha_1 + \alpha_2 = 1$ without any loss of generality, and $\alpha_i > 0$ for all i. A first-best optimum can be obtained as a solution of the following problem,

$$\text{Maximize} \quad \sum_i \alpha_i[\mathbf{p}_i(e_i)U_i + (1 - \mathbf{p}_i(e_i))u_i - c_i] \tag{17}$$

with respect to $\{(R_i, R_i', r_i, e_i, \varepsilon_{ik})\}_{i,k=1,2}$, subject to the resource constraint RC, and e_i, ε_{i1}, ε_{i2} are equal to 0 or 1.

Under the assumptions made above, using the inverse utility function $z(x) = u^{-1}(x)$ and eliminating R_{ik} and r_i from the objective and the resource constraint RC, the second-best optimality problem can be rewritten as follows:

$$\text{Maximize} \quad \sum_i \alpha_i[P_i U_i + (1 - P_i)u_i - c_i] \tag{18}$$

with respect to $(V_i, v_i, u_i)_{i=1,2}$, subject to the resource constraint $\overline{RC}$ and all incentive and moral-hazard constraints, exactly as defined in Section 3 above. The statement of Proposition 2 is in fact a particular case of a more general result, called Proposition 2', that we now state. The proof of this proposition is provided in the appendix.

Proposition 2'. (Equal treatment as a second best under adverse selection, ex ante and ex post moral hazard.) *Under Assumptions 1-5, for all $\pi < 1/2$, there exists an open interval $(\underline{\lambda}_2, \overline{\lambda}_2)$, including λ_2, such that if $\alpha_2 \in (\underline{\lambda}_2, \overline{\lambda}_2)$, then, the second-best optimal solution has the following properties:*

$$U_1 = U_2 = \mathbf{U}, \qquad u_1 = u_2 = \mathbf{u} \quad \text{(equal \quad treatment)},$$

$$\mathbf{U} = \mathbf{u} + K_1 \quad \text{(incomplete \quad insurance)},$$

where $\mathbf{U}$, $\mathbf{u}$ are functions of π. Constraints $\overline{RC}$, MH_1, $\overline{IC}_1$, $\overline{IC}_2$, ICX_1 and ICX_2 are all binding. If, in addition, $K_1 > K_2$, then, MH_2, $\underline{IC}_1$ and $\underline{IC}_2$ hold as strict inequalities.

Proposition 2' shows that the result of Proposition 2 can be extended: the equal treatment optima are obtained for all weights α in a neighborhood of type frequencies λ.

At this point, several remarks can be made. We first find a local invariance property: when α is close to λ, the optimal solution does not depend on α.

Corollary 2. *The second-best optimal solution of Proposition 2' is independent of α, when α_2 is small enough, i.e., if $\alpha_2 < \overline{L}_2$, we have,*

$$\frac{\partial}{\partial \alpha_2}(U_2, u_2, U_1, u_1) = 0.$$

Proof of Corollary 2: The second-best allocation is the solution of a system of four equations with four unknowns: *(i)*, $U_1 = U_2$; *(ii)*, $u_1 = u_2$; *(iii)*, $U_1 = u_1 + K_1$; and *(iv)*, given these constraints, $\overline{RC}$ pins down $u_1 = \mathbf{u}$. None of these equations involve α. *Q.E.D.*

For the sake of completeness, the remaining question is to find the second-best optimal solutions when $\alpha_2 > \overline{\lambda}_2$. We look for a second-best allocation in which a single $\overline{IC}$ constraint is binding. By Lemma 5, proved in the appendix, we know that $\overline{IC}_1$ must be the binding constraint, and we can show that this happens *only if* $\alpha_2 > \lambda_2$.

Proposition 9. (Separating Second-best Optima) *Under Assumptions 1-5, for all $\pi < 1/2$, if a second-best optimum has only one binding $\overline{IC}$ constraint, then,*

(a) $\overline{IC}_1$ is the binding constraint; $\overline{IC}_2$ is slack;

(b) constraints MH_1, $\overline{RC}$, ICX_1 and ICX_2 are binding;

(c) we have $U_2 > U_1 > u_1 > u_2$ and, necessarily, $\alpha_2 > \lambda_2$.

(d) The second-best solution is fully determined by the following 4 equations: $\overline{IC}_1$, MH_1 and $\overline{RC}$, expressed as equalities, and the condition,

$$\frac{\lambda_2}{\lambda_1 \alpha_2} \frac{[P_2(1-P_\alpha)Ez_2' - P_\alpha(1-P_2)z'(u_2)]}{(P_2 - P_1)} = P_1 Ez_1' + (1-P_1)z'(u_1), \tag{K}$$

where $P_\alpha = \alpha_1 P_1 + \alpha_2 P_2$ and $Ez_i' = (1-\pi)z'(V_i) + \pi z'(v_i)$.

We have completely characterized the second-best optima under adverse selection, *ex ante* and *ex post* moral hazard, in the optimal graduate tax problem. Again, since the Kuhn-Tucker conditions are necessary and sufficient, we have shown that when $\alpha_2 > \overline{\lambda}_2$, the solution is that described by Proposition 9.

Proposition 9 shows that the second-best optimum is a *separating* allocation à la Rothschild-Stiglitz, *i.e.*, $U_2 > U_1 > u_1 > u_2$, when the weight of the talented types is sufficiently higher than their frequency in the population. In other words, to get a *separating optimum*, the social planner must be willing to markedly favor the talented ex ante types. It is still true that types are *ex ante* unequal since, using $\overline{IC}_i$ and MH_i constraints, we find that

$$Eu_2 = u_2 + P_2(U_2 - u_2) > u_2 + P_1(U_2 - u_2) = u_1 + P_1(U_1 - u_1) = Eu_1.$$

But the talented types are now less well insured in case of failure, since $u_1 > u_2$.

Let us now study the possible decomposition of transfers in separating optima. To this end, compute,

$$\mathbf{\Delta} = (R_{22} - R_{21}) - (R_{12} - R_{11}).$$

If the repayment schedule can be decomposed, equations (16) above must hold, and we must have $\boldsymbol{\Delta} = 0$. Using (7) and (11)-(13), *i.e.*, the definitions of R_{ij} and the consequences of Lemma 1, we find

$$\boldsymbol{\Delta} = [z(U_1 + b) - z(U_1)] - [z(U_2 + b) - z(U_2)] + (W_2 - W_1) - (w_2 - w_1).$$

By Assumption 2, we have $(W_2 - W_1) > (w_2 - w_1)$. Since $z(.)$ is increasing and strictly convex, and since by $\overline{IC}$, we must have $U_2 > U_1$, it must be that

$$z(U_2 + b) - z(U_2) > z(U_1 + b) - z(U_1).$$

Therefore, the sign of $\boldsymbol{\Delta}$ is ambiguous, but $\boldsymbol{\Delta}$ has no reason to be exactly zero: *non-decomposability is generic*. If we draw the model's basic elements from a suitably dispersed distribution, that is, if we choose at random a strictly concave utility function, $u(.)$, productivities w_q and W_q, and a vector of parameters $(\underline{w}, \pi, b, \gamma_i, \lambda_i, p_i, P_i, c_i, \alpha_i)_{i=1,2}$ satisfying the assumptions stated above, then, quantity $\boldsymbol{\Delta}$ will almost surely be nonzero. Indeed, the difference $(W_2 - W_1) - (w_2 - w_1)$ does not vary with α_2, while U_2 and U_1 do depend on α_2 through condition K. This shows intuitively that $\boldsymbol{\Delta}$ varies with α_2 and can be zero only "by chance" on a negligible subset of the admissible models.

7 Conclusion

We have studied optimal student-loan contracts in a simple private information economy with unobservable student types. Types differ in the probability distributions of individual labour-market outcomes (adverse selection). Future earnings are risky. Students are risk-averse and choose an *ex ante* effort variable, that is not observed by the lender (moral hazard) but affects the probabilities of academic success. This poses an optimal insurance problem. Students can also reduce their effort *ex post* and thus reduce their earnings below potential. This poses an additional income taxation problem à la Mirrlees. We completely describe the set of second-best optimal (or interim efficient) incentive-compatible menus of loan contracts. There are two types of optima: the *separating* and *equal treatment* allocations. Equal treatment arises when the social weights of types are in the neighborhood of their frequencies in the student population. In this case, the *interim expected utility* of students of different types are equalized, conditional on the student's observable success or failure. However, students are *ex ante* unequal since they differ in their probability of success. In addition, students are *ex post* unequal since the second-best allocation trades off incentives and insurance-redistribution motives. This type of allocation is different from the familiar menus of separating contracts in screening models à la Rothschild-Stiglitz. The separating menus, in which the talented students bear more risk than the less-talented ones, appear only if the weight of talented types is sufficiently greater than the latter type's frequency. In both cases, the optimal menus of contracts exhibit incomplete insurance, as a consequence of moral hazard; they typically involve cross-subsidies in favor of the less-talented. The less-talented obtain the maximal amount of insurance, compatible with effort incentives. Optimal student loans are always income-contingent, even in the presence of an income tax. In other words, the second-best transfers cannot be decomposed as the sum of an income tax, depending only on earnings, and a loan repayment, depending only on education. It must be that the optimal loan-repayments are income contingent, or that the income tax is itself education-contingent. The student-loan contracts can be interpreted as a form of *graduate tax*. Future work on this topic should try to solve the second-best optimality problem with more than two types, study the case of continuous effort and also the case in which repayments are constrained to be positive.

8 References

Anderberg, Dan (2009), "Optimal Policy and the Risk Properties of Human Capital Reconsidered", *Journal of Public Economics,* **93**, 1017-1026.

Arnott, Richard J. (1991), "Moral Hazard and Competitive Insurance Markets", in Georges Dionne. ed.*, Contributions to Insurance Economics*, Kluwer Academic Publishers, Dordrecht, the Netherlands.

Baron, David P. and David Besanko (1987), "Monitoring, Moral Hazard, Asymmetric Information, and Risk Sharing in Procurement Contracting", *Rand Journal of Economics*, **18**, 509-532.

Barr, Nicholas and Alison Johnston (2010), "Interest Subsidies on Student Loans: A Better Class of Drain", CEE DP 114, *London School of Economics*, London, UK.

Bester, Helmut (1985), "Screening vs. Rationing in Credit Markets with Imperfect Information", *American Economic Review*, **75**, 850-855.

Boadway, Robin, and Motohiro Sato (2012), "Optimal Income Taxation with Uncertain Earnings: a Synthesis", *manuscript*, Queen's University, Canada.

Bovenberg, A. Lans, and Bas Jacobs (2005), "Redistribution and Education Subsidies are Siamese Twins", *Journal of Public Economics*, 89, 2005-2035.

Caillaud, Bernard, Guesnerie, Roger and Patrick Rey (1992), "Noisy Observations in Adverse Selection Models", *Review of Economic Studies*, **59**, 596-615.

Carneiro, Pedro and James J. Heckman (2002), "The Evidence on Credit Constraints in Post-secondary Schooling", *Economic Journal*, **112**, 705-734.

Chassagnon, Arnold and Pierre-André Chiappori (1997), "Insurance under Moral Hazard and Adverse Selection: The Case of Pure Competition", *Paris School of Economics, manuscript*, http://umrpse.ens.fr/users/chassagnon/pub.html.

Chatterjee, Satyajit and Felicia Ionescu (2011),"Insuring Student Loans against the Financial Risk of College Failure", *manuscript*, *Colgate University*, Hamilton, New York.

Cigno, Alessandro and Annalisa Luporini (2009), "Scholarships or Student Loans? Subsidizing Higher Education in the Presence of Moral Hazard", *Journal of Public Economic Theory*, **11(1)**, 55-87.

Crocker, Keith J. and Arthur Snow (1985), "The Efficiency of Competitive Equilibria in Insurance Markets with Asymmetric Information", *Journal of Public Economics*, **26**, 207-219.

De Fraja, Gianni (2001), "Education Policies: Equity, Efficiency and Voting Equilibrium", *Economic Journal*, Conference Papers, **111**, C104-C119.

De Fraja, Gianni (2002), "The Design of Optimal Education Policies", *Review of Economic Studies*, **69**, 437-466.

Del Rey, Elena, and Maria Racionero (2010), "Financing Schemes for Higher Education", *European Journal of Political Economy*, **26**, 104-113.

Dionne, Georges, Doherty, Neil and Nathalie Fombaron (2000), "Adverse Selection in Insurance Markets", Chapter 7 in Georges Dionne. ed., *Handbook of Insurance*, Kluwer Academic Publishers, Dordrecht, Netherlands.

Faynzilberg, Peter S. and Praveen Kumar (2000), "On the Generalized Principal-Agent Problem: Decomposition and Existence Results", *Review of Economic Design*, **5**, 23-58.

Findeisen, Sebastian, and Dominik Sachs (2012), "Education and Optimal Dynamic Taxation: The Role of Income-Contingent Student Loans", *manuscript*, University of Zurich, Switzerland.

Fleurbaey, Marc, Gary-Bobo, Robert J., and Denis Maguain (2002), "Education, Distributive Justice and Adverse Selection", *Journal of Public Economics*, **84**, 113-150.

Freixas, Xavier and Jean-Charles Rochet (1998), *Microeconomics of Banking*, MIT Press, Cambridge, Massachusetts.

Friedman, Milton and Simon Kuznets (1945), *Income from Individual Professional Practice*, National Bureau of Economic Research, New York.

Friedman, Milton (1955), "The Role of Government in Public Education", in R. A. Solo, ed., *Economics and the Public Interest*, Rutgers University Press, New Brunswick, New Jersey.

Garcia-Penalosa, Cecilia and Klaus Wälde (2000), "Efficiency and Equity Effects of Subsidies to Higher Education", *Oxford Economic Papers*, **52(4)**, 702-722.

Hanushek, Eric A., Charles Ka Yui Leung, and Kuzey Yilmaz (2004), "Borrowing Constraints, College Aid, and Intergenerational Mobility", *NBER WP 10711*, *National Bureau of Economic Research*, Cambridge, Massachusetts.

Hellwig, Martin F. (1987), "Some Recent Developments in the Theory of Competition in Markets with Adverse Selection", *European Economic Review*, **31**, 319-325.

Henriet, Dominique and Jean-Charles Rochet (1990), "Efficiency of Market Equilibria with Adverse Selection: The Case of Insurance", Chapter 9 in Paul Champsaur *et al.* eds., *Essays in Honor of Edmond Malinvaud*, *volume 1*, MIT Press, Cambridge, Massachusetts.

Ionescu, Felicia (2009), "The Federal Student Loan Program: Quantitative Implications for College Enrollment and Default Rates", *Review of Economic Dynamics*, **12**, 205-231.

Jacobs, Bas and Frederick Van der Ploeg (2006), "Guide to Reform of Higher Education: A European Perspective", *Economic Policy*, **21(47)**, 535-592.

Keane, Michael and Kenneth Wolpin (2001), "The Effect of Parental Transfers and Borrowing Constraints on Educational Attainment", *International Economic Review*, **42**, 1051-1103.

Laffont Jean-Jacques and David Martimort (2002), *The Theory of Incentives: The Principal-Agent Model*, Princeton University Press, Princeton, New Jersey.

Laffont Jean-Jacques and Jean-Charles Rochet (1998), "Regulation of a Risk-Averse Firm", *Games and Economic Behavior*, **25**, 149-173.

Lochner Lance, J. and Alexander Monge-Naranjo (2010), "The Nature of Credit Constraints and Human Capital", *manuscript, Pennsylvania State University*, forthcoming, AER.

Lochner Lance, J. and Alexander Monge-Naranjo (2011), "Credit Constraints in Education", *NBER Working Paper 17435,* NBER*,* Cambridge, Massachusetts.

McAfee, R. Preston J. and John McMillan (1986), "Bidding for Contracts: A Principal-Agent Analysis", *Rand Journal of Economics*, **17**, 326-338.

Miyazaki, Hajime (1977), "The Rat Race and Internal Labor Markets", *Bell Journal of Economics*, **8(2)**, 394-418.

Myerson, Roger B. (1982), "Optimal Coordination Mechanisms in Generalized Principal-Agent Models", *Journal of Mathematical Economics*, **10**, 67-81.

Nerlove, Marc (1975), "Some Problems in the Use of Income-contingent Loans for the Finance of Higher Education", *Journal of Political Economy*, **83(1)**, 157-184.

Picard, Pierre (1987), "On the Design of Incentive Schemes under Moral Hazard and Adverse Selection", *Journal of Public Economics*, **33**, 305-331.

Rothschild, Michael and Joseph E. Stiglitz (1976), "Equilibrium in Competitive Insurance Markets", *Quarterly Journal of Economics*, **90**, 629-650.

Shell, Karl, Fisher, Franklin M., Foley, Duncan K., Friedlander, Ann F. *et al.* (1968) "The Educational Opportunity Bank: An Economic Analysis of Contingent Repayment Loan Programs for Higher Education", *National Tax Journal*, **21**, 2-45.

Spence, Michael (1973), "Job Market Signaling", *Quarterly Journal of Economics*, **87(3)**, 355-374.

Spence, Michael (1978), "Product Differentiation and Performance in Insurance Markets", *Journal of Public Economics*, **10**, 427-447.

Stinebrickner, Todd and Ralph Stinebrickner (2008), "The Effect of Credit Constraints on the College Drop-out Decision: A Direct Approach Using a New Panel Study", *American Economic Review*, **98**, 2163-2184.

Sung Jaeyoung (2005), "Optimal Contracts under Adverse Selection and Moral Hazard: A Continuous-time Approach", *Review of Financial Studies* , **18(3)**, 1021-1073.

Wilson, Charles (1977), "A Model of Insurance Markets with Incomplete Information", *Journal of Economic Theory*, **16**, 167-207.

9 Appendix: Proofs

Maximizing a weighted sum of expected utilities.

Our goal is to characterize all first-best and second-best Pareto optima in our model. To this end we consider weights for ex ante types that may differ from their frequencies in the population. Let α_1 and α_2 be the weights of type $i = 1$ and type $i = 2$ in the welfare function. We assume $\alpha_1 + \alpha_2 = 1$ without loss of generality, and $\alpha_i > 0$ for all i.

Under the assumptions made above, the first-best optimality problem can be rewritten as follows: maximize $\Sigma_i \alpha_i [P_i U_i + (1 - P_i) u_i]$ with respect to $(V_i, v_i, u_i)_{i=1,2}$, subject to the resource constraint $\overline{RC}$, exactly as defined in the text above. Second-best optima are obtained while maximizing the same objective function, subject to the resource constraint, plus all incentives and moral hazard constraints.

9.1 First-best efficiency

Proof of Proposition 1.

The first-best optimality problem is a convex programming problem, since $z(.)$ is a convex function and the objective is a linear function of utility levels V_i, v_i, and u_i. To write the first-order necessary conditions for optimality, let κ denote the Lagrange multiplier of the resource constraint. We find, for $i = 1, 2$,

$$\alpha_i = \kappa \lambda_i z'(V_i) = \kappa \lambda_i z'(v_i) = \kappa \lambda_i z'(u_i)$$

for all i. This immediately yields,

$$\frac{\alpha_i}{\kappa^* \lambda_i} = z'(V_i^*) = z'(v_i^*) = z'(u_i^*), \tag{19}$$

$$\frac{z'(V_1^*)}{z'(V_2^*)} = \frac{\alpha_1/\lambda_1}{\alpha_2/\lambda_2} = \frac{z'(v_1^*)}{z'(v_2^*)} = \frac{z'(u_1^*)}{z'(u_2^*)}, \tag{20}$$

$$\kappa^* = \frac{1}{\Sigma_i \lambda_i z'(u_i^*)} > 0. \tag{21}$$

It follows that first-best optimality implies *full insurance*, that is, for all i,

$$V_i^* = v_i^* = u_i^*, \tag{22}$$

and the resource constraint must be binding. If, in addition, $\alpha_i = \lambda_i$, we get *full equality*, *i.e.*, $V_1^* = V_2^*$, $v_1^* = v_2^*$ and $u_1^* = u_2^*$. *Q.E.D.*

Proof of Corollary 1

(i) There does not exist an unconstrained optimum with $r_i^* \geq 0$ for all i. If such an optimum did exist, then, because of full insurance, we would have $\boldsymbol{\omega}(q_i, \varepsilon_{ii}^*) - R_{ii}^* = \boldsymbol{\omega}(q_i, \varepsilon_{ij}^*) - R_{ij}^* = \underline{w} - r_i^* \leq \underline{w}$ and therefore, $\Sigma_i \lambda_i \mathbf{E} z_i^* \leq \underline{w} < \Sigma_i \lambda_i S_i$, a contradiction, since resources would then be wasted.

(ii) First-best efficiency implies full insurance and full insurance in turn implies $\boldsymbol{\omega}(q_i, \varepsilon_{ii}^*) - R_{ii}^* = \underline{w} - r_i^*$. This yields $\boldsymbol{\omega}(q_i, \varepsilon_{ii}^*) - R_{ii}^* - \underline{w} = -r_i^*$. So, if we require $\boldsymbol{\omega}(q_i, \varepsilon_{ii}^*) - R_{ii}^* \geq \underline{w}$, we must have $r_i^* \leq 0$. *Q.E.D.*

9.2 Incentives. Proofs of Lemmas

Proof of Lemma 2:

Adding up the $\overline{IC}_i$ constraints immediately yields

$$(P_2 - P_1)(U_2 - U_1) \geq (P_2 - P_1)(u_2 - u_1)$$

and $P_2 > P_1$ in turn implies $U_2 - U_1 \geq u_2 - u_1$. With MH_1, we then obtain the following string of inequalities:

$$U_2 - u_2 \geq U_1 - u_1 \geq K_1 \geq K_2.$$

It follows that MH_2 is satisfied.

Q.E.D.

Proof of Lemma 3:

As in the proof of Lemma 2, adding up the $\overline{IC}_i$ constraints and $P_2 > P_1$ immediately yields $U_2 - U_1 \geq u_2 - u_1$: this is property D. Given $\overline{IC}$, since $P_2 > P_1$, if $U_1 > U_2$ we find a contradiction. $\overline{IC}$ above shows that $U_2 \geq U_1$ implies $u_1 - u_2 \geq 0$. This yields D'.

The proof of Lemma 3a is trivial, since $P_2 > P_1$. Lemma 3b follows from the fact that $u_1 = u_2$ and $\overline{IC}$ imply $U_2 - U_1 \geq 0 \geq U_2 - U_1$ and therefore $U_2 = U_1$. But under $\overline{IC}$, we also have that $U_2 = U_1$ implies $u_1 = u_2$. *Q.E.D.*

Proof of Lemma 4:

(a) If $\overline{IC}_1$ holds, then,

$$(1-P_1)(u_1-u_2) \geq P_1(U_2-U_1),$$

and since under IC, $U_2 - U_1 \geq 0$, and we assumed $P_1 > p_1$, we also have $(1-p_1)(u_1-u_2) \geq p_1(U_2-U_1)$. But MH_1 implies $c_1 - (P_1-p_1)(U_1-u_1) \leq 0$. This trivially implies

$$(1-p_1)(u_1-u_2) \geq p_1(U_2-U_1) + c_1 - (P_1-p_1)(U_1-u_1), \tag{23}$$

and rearranging terms we get the equivalent inequality,

$$P_1U_1 + (1-P_1)u_1 - c_1 \geq p_1U_2 + (1-p_1)u_2,$$

but this is exactly $\underline{IC}_1$.

(b) Given that MH_1 is binding, $\underline{IC}_2$ can be expressed as follows,

$$P_2U_2 + (1-P_2)u_2 - c_2 \geq p_2(u_1+K_1) + (1-p_2)u_1 = u_1 + p_2K_1. \tag{$\underline{IC}_2 + MH_1$}$$

Combining $\overline{IC}_1$ and MH_1, holding as equalities, we easily obtain,

$$u_1 + P_1K_1 = P_1U_2 + (1-P_1)u_2. \tag{$\overline{IC}_1 + MH_1$}$$

Substituting the value of u_1 derived from $(\overline{IC}_1 + MH_1)$ in $(\underline{IC}_2 + MH_1)$ yields, after some rearrangement of terms,

$$(P_2-P_1)(U_2-u_2) \geq c_2 + (p_2-P_1)K_1.$$

Dividing both sides by $(P_2-p_2) > 0$ and rearranging terms, we obtain,

$$(U_2-u_2)\left[\frac{P_2-P_1}{P_2-p_2}\right] \geq K_2 + \frac{p_2-P_1}{P_2-p_2}K_1. \tag{Z}$$

From condition D and MH_1 we know that $U_2 - u_2 \geq U_1 - u_1 = K_1$. In addition, $(P_2 - P_1)/(P_2-p_2) = 1 + (p_2-P_1)/(P_2-p_2) > 0$. Hence, the following string of inequalities:

$$(U_2-u_2)\left[\frac{P_2-P_1}{P_2-p_2}\right] \geq K_1\left[1+\frac{p_2-P_1}{P_2-p_2}\right] \geq K_2 + \frac{p_2-P_1}{P_2-p_2}K_1,$$

since, by Assumption 5, $K_1 \geq K_2$. This shows that condition Z is satisfied, hence, $\underline{IC}_2$ is satisfied when $\overline{IC}_2$ holds and when $\overline{IC}_1$ and MH_1 are binding.

Q.E.D.

9.3 Optimal graduate tax

Before the proof of Propositions 2 an 2', we state a useful preliminary result. Lemma 5 gives indications on the constraints that must be binding at a second-best optimum. There is an asymmetry between the two $\overline{IC}_i$ constraints. Lemma 5 shows that, if a single $\overline{IC}$ constraint is binding at the optimum, this constraint must be $\overline{IC}_1$.

Lemma 5. *At a second-best optimum,*

(a) $\overline{RC}$ *is binding;*

(b) if $\overline{IC}_1$ *and* $\overline{IC}_2$ *are binding, then,* MH_1 *must be binding;*

(c) if $\overline{IC}_2$ *is binding, then,* $\overline{IC}_1$ *must be binding too.*

Proof of Lemma 5:

Using Lemmata 1-4, the second-best optimality problem can be further simplified. The benevolent public banker should maximize $\Sigma_i \alpha_i (P_i U_i + (1 - P_i) u_i)$ with respect to (U_i, u_i), subject to $\overline{IC}_i$, MH_1, $\underline{IC}_2$ and $\overline{RC}$, $i = 1, 2$. To study this problem, we will also temporarily ignore (*i.e.*, relax) constraint $\underline{IC}_2$ and check at the end that it is indeed satisfied. Let κ, δ, μ_1 and μ_2 be the nonnegative Lagrange multipliers of, respectively, constraints $\overline{RC}$, MH_1, $\overline{IC}_1$ and $\overline{IC}_2$. The first-order conditions (*i.e.*, Kuhn-Tucker conditions) for the second-best optimality problem are the following. For $i, j = 1, 2$,

$$\begin{aligned}
\alpha_1 P_1 + \mu_1 P_1 - \mu_2 P_2 + \delta &= \kappa \lambda_1 P_1 E z_1'; && (FOC1)\\
\alpha_2 P_2 + \mu_2 P_2 - \mu_1 P_1 &= \kappa \lambda_2 P_2 E z_2'; && (FOC2)\\
\alpha_1 (1 - P_1) + \mu_1 (1 - P_1) - \mu_2 (1 - P_2) - \delta &= \kappa \lambda_1 (1 - P_1) z'(u_1); && (FOC3)\\
\alpha_2 (1 - P_2) + \mu_2 (1 - P_2) - \mu_1 (1 - P_1) &= \kappa \lambda_2 (1 - P_2) z'(u_2); && (FOC4)
\end{aligned}$$

with, by definition,

$$\begin{aligned}
E z_1' &= (1 - \pi) z'[U_1] + \pi z'[U_1 + b], && (24)\\
E z_2' &= (1 - \pi) z'[U_2 + b] + \pi z'[U_2],
\end{aligned}$$

and with the complementary slackness conditions, *i.e.*,

$$\begin{aligned} \delta(U_1 - u_1 - K_1) &= 0, & (CS_1) \\ \kappa\{\Sigma_i \lambda_i [S_i - \mathbf{E} z_i]\} &= 0, & (CS_2) \\ \mu_i\{P_i U_i + (1-P_i)u_i - P_i U_j - (1-P_i)u_j\} &= 0, & (CS_{3i}) \end{aligned}$$

where $j \neq i$. These conditions are necessary and sufficient for an optimum, because as noted above, the problem is convex. It follows from this that, if we find a solution in which all multipliers are nonnegative, we have found the solution.

Proof of Lemma 5(a) Adding equations FOC1 to FOC4, we easily find,

$$1/\kappa = \Sigma_i \lambda_i \{P_i E z_i' + (1-P_i) z'(u_i)\} > 0,$$

so that $\kappa > 0$, since $z'(.) > 0$. Hence, by CS_2, it follows that $\overline{RC}$ is binding.

Proof of Lemma 5(b) If $\overline{IC}_1$ and $\overline{IC}_2$ are binding, then $U_1 = U_2 = U$ and $u_1 = u_2 = u$. Suppose that MH_1 is slack, *i.e.*, $U_1 > u_1 + K_1$, at the second-best optimum, then by CS_1, we have $\delta = 0$. With $\delta = 0$, $FOC1$ and $FOC3$ form a linear system in (μ_1, μ_2); that is,

$$\begin{aligned} \mu_1 P_1 - \mu_2 P_2 &= P_1[\kappa\lambda_1 E z_1' - \alpha_1]; \\ \mu_1(1-P_1) - \mu_2(1-P_2) &= (1-P_1)[\kappa\lambda_1 z'(u) - \alpha_1]; \end{aligned}$$

This system has a nonzero determinant, equal to $P_2 - P_1 > 0$, and a unique solution (μ_1^*, μ_2^*). It is easy to check that,

$$\mu_2^* = \frac{P_1(1-P_1)}{P_2 - P_1} \kappa\lambda_1 \left(z'(u) - E z_1'\right).$$

But,

$$E z_1' = (1-\pi) z'[U] + \pi z'[U+b].$$

Since $b > 0$ and since MH_1 implies $U > u + K_1$, it follows that $E z_1' > z'(U)$ and thus, $z'(u) - E z_1' < 0$. We conclude that $\mu_2^* < 0$. This is a violation of Kuhn-Tucker conditions, since all multipliers must be non-negative. We have found a contradiction. This proves 5b.

Proof of Lemma 5(c): If $\overline{IC}_2$ is binding, and $\overline{IC}_1$ is slack, then, by CS_{31}, we have $\mu_1 = 0$. Using $FOC2$ and $FOC4$, we easily obtain, $\kappa\lambda_2 z'(u_2) = \alpha_2 + \mu_2 = \kappa\lambda_2 Ez'_2$ and therefore, $z'(u_2) = Ez'_2$. Using the definition of Ez'_i given above, $Ez'_2 = (1-\pi)z'[U_2 + b] + \pi z'[U_2]$, we find $Ez'_2 > z'(U_2)$. But $\overline{IC}$ and MH_1 together imply $U_2 \geq U_1 \geq u_1 + K_1 > u_1 \geq u_2$, so that $U_2 - u_2 \geq K_1 > 0$. It follows that we have a contradiction since $U_2 > u_2$ and hence $Ez'_2 > z'(u_2)$.

Q.E.D.

Proof of Propositions 2 and 2':

The FOCs can be rewritten,

$$\begin{aligned}
\mu_1 P_1 - \mu_2 P_2 + \delta &= P_1[\kappa\lambda_1 Ez'_1 - \alpha_1]; && (FOC1b)\\
\mu_2 P_2 - \mu_1 P_1 &= P_2[\kappa\lambda_2 Ez'_2 - \alpha_2]; && (FOC2b)\\
\mu_1(1-P_1) - \mu_2(1-P_2) - \delta &= (1-P_1)[\kappa\lambda_1 z'(u_1) - \alpha_1]; && (FOC3b)\\
\mu_2(1-P_2) - \mu_1(1-P_1) &= (1-P_2)[\kappa\lambda_2 z'(u_2) - \alpha_2]. && (FOC4b)
\end{aligned}$$

Our optimum candidate exhibits equal treatment, $U_1 = U_2 = U(\pi)$, $u_1 = u_2 = u(\pi)$, and both $\overline{IC}$ constraints are binding. By Lemma 5 above, MH_1 must then be binding too. This imposes $U(\pi) = u(\pi) + K_1$. Adding the four $FOCb$ equations easily yields, $\kappa = \kappa(\pi)$, satisfying the relation,

$$\kappa(\pi)[\Sigma_i P_i \lambda_i Ez'_i + (1 - P_\lambda)z'(u(\pi))] = 1, \qquad (\Sigma FOC)$$

where, by definition, $P_\lambda = P_1\lambda_1 + P_2\lambda_2$, and where,

$$Ez'_i = (1-\pi)z'[U(\pi) + (i-1)b] + \pi z'[U(\pi) + (2-i)b], \qquad (25)$$

for $i = 1, 2$. We know that $\overline{RC}$ is binding and it follows that $(U(\pi), u(\pi))$ is fully determined by the intersection of $\overline{RC}$ and MH_1. It is easy to check that the solution $(U(\pi), u(\pi))$ is a continuous function of π for $\pi \geq 0$, and this solution does not depend on α_2.

We must check that the associated multipliers $\mu_i(\pi)$ are nonnegative. $FOC2b$ and $FOC4b$ provide us with a linear system of equations for (μ_1, μ_2). The determinant of this system is $P_2 - P_1 > 0$, so that there is a unique solution $(\mu_1(\pi), \mu_2(\pi))$. We easily derive,

$$\mu_1(\pi) = \frac{P_2(1-P_2)\kappa(\pi)\lambda_2}{(P_2-P_1)}[Ez'_2 - z'(u(\pi))], \tag{26}$$

$$\mu_2(\pi) = \frac{\lambda_2 P_2(1-P_1)}{(P_2-P_1)}\left[\kappa(\pi)Ez'_2 - \frac{\alpha_2}{\lambda_2}\right] + \frac{\lambda_2 P_1(1-P_2)}{(P_2-P_1)}\left[\frac{\alpha_2}{\lambda_2} - \kappa(\pi)z'(u(\pi))\right]. \tag{27}$$

Remark first that $Ez'_2 > Ez'_1 > z'(U(\pi)) > z'(u(\pi))$, since $U(\pi) > u(\pi)$ and $\pi < 1/2$. Using these inequalities, expression ΣFOC above, and the fact that $z' > 0$, we derive

$$1 > \kappa(\pi)z'(u(\pi)), \qquad \text{and} \qquad 1 < \kappa(\pi)Ez'_2. \tag{28}$$

From these remarks, we immediately obtain $\mu_1(\pi) > 0$. Note that this property is true for all values of α_2 in $(0,1)$ since $\mu_1(\pi)$ does not depend on α_2. For the same reasons, $\kappa(\pi)$, given by equation ΣFOC, does not depend on α_2, and thus, we always have $Ez'_2 > z'(u(\pi))$ and $\kappa(\pi) > 0$. Given the inequalities derived above, for any $\alpha_2 \leq \lambda_2$, we also have $\kappa(\pi)Ez'_2 - (\alpha_2/\lambda_2) > 0$, and therefore, using $P_2 > P_1$, we find

$$\begin{aligned} \mu_2(\pi) &> \frac{\lambda_2 P_1(1-P_2)}{(P_2-P_1)}\left[\kappa(\pi)Ez'_2 - \frac{\alpha_2}{\lambda_2} + \frac{\alpha_2}{\lambda_2} - \kappa(\pi)z'(u(\pi))\right] \\ &= \frac{\lambda_2\kappa(\pi)P_1(1-P_2)}{(P_2-P_1)}[Ez'_2 - z'(u(\pi))] > 0. \end{aligned} \tag{29}$$

In particular, if $\lambda_2 = \alpha_2$, then $\mu_2(\pi) > 0$. By continuity, there exists an interval of values of α_2, including λ_2, such that $\mu_2(\pi)$ is positive. By Lemma 4, we know that both $\underline{IC}_i$ constraints are satisfied. By Lemma 2, we know that MH_2 is also satisfied.

Finally, we must check that the multiplier $\delta = \delta(\pi) > 0$. Adding $FOC3b$ and $FOC4b$, we derive an expression for $\delta(\pi)$, that is,

$$\delta(\pi) = 1 - P_\alpha - \kappa(\pi)(1-P_\lambda)z'(u(\pi)), \tag{30}$$

where by definition, $P_\alpha = \alpha_1 P_1 + \alpha_2 P_2$. This can be rewritten

$$\frac{\delta(\pi)}{(1-P_\lambda)} = \frac{(1-P_\alpha)}{(1-P_\lambda)} - \kappa(\pi)z'(u(\pi)). \tag{31}$$

Since we have $1 > \kappa(\pi)z'(u(\pi))$, this means that there is an interval of values of α_2, around λ_2, such that $\delta(\pi) > 0$. In addition, $\delta(\pi) > 0$ for all values of $\alpha_2 \leq \lambda_2$ since $\alpha_2 \to 0$ implies $P_\alpha \to P_1 < P_\lambda$. We conclude that there exists an interval $(\underline{\lambda}_2, \overline{\lambda}_2)$ of values of α_2, including

λ_2, such that all Lagrange multiplers are positive. Therefore, we have found the optimal solution for these values of α_2 and π.

Q.E.D.

9.4 Generic non-decomposability of the graduate tax

Proof of Proposition 8.

If the ICX_i are the only constraints, Proposition 6 tells us that we must have $Ez_2' = Ez_1'$ at the optimum, or equivalently,

$$\pi z'(V_2 - b) + (1-\pi)z'(V_2) = \pi z'(V_1 + b) + (1-\pi)z'(V_1). \tag{G8}$$

We can easily prove by contradiction that the latter condition implies

$$v_1 = V_1 + b > V_2 > V_1 > V_2 - b = v_2. \tag{H8}$$

To prove condition $H8$, suppose first that $V_1 \geq V_2$. Then, from equation $G8$, we have,

$$0 \leq (1-\pi)[z'(V_1) - z'(V_2)] = \pi[z'(V_2 - b) - z'(V_1 + b)] < 0,$$

but this is impossible since our assumption also implies $V_2 - b < V_1 + b$, and it must be that the left hand side is non negative, while the right-hand side is strictly negative. We conclude that $V_2 > V_1$.

Suppose now that $V_1 \leq V_2 - b$. Then, given condition $G8$, we derive,

$$\pi z'(V_1) + (1-\pi)z'(V_2) \leq \pi z'(V_2) + (1-\pi)z'(V_1),$$

but this is impossible, since $V_2 > V_1$ and $\pi < 1/2$. We conclude that $V_1 + b > V_2$. This proves condition $H8$.

Now, define,

$$\mathbf{\Delta} = (R_{22} - R_{21}) - (R_{12} - R_{11}).$$

If the repayment schedule can be decomposed, equations (16) above must hold, and we must have $\mathbf{\Delta} = 0$. Using (7), the definitions of R_{ij} and the fact that ICX_i constraints are binding, we find,

$$\mathbf{\Delta} = [z(V_1 + b) - z(V_2)] - [z(V_1) - z(V_2 - b)] + (W_2 - W_1) - (w_2 - w_1).$$

By Assumption 2, $(W_2 - W_1) > (w_2 - w_1)$. Since $z(.)$ is increasing and strictly convex, given condition $H8$ above, it must be that

$$z(V_1 + b) - z(V_2) > z(V_1) - z(V_2 - b).$$

Therefore, $\boldsymbol{\Delta} > 0$: we conclude that the repayment schedule is not decomposable. *Q.E.D.*

9.5 Separating optima

Proof of Proposition 9:

If a second-best optimum has only one binding $\overline{IC}$ constraint, then, by Lemma 5, $\overline{IC}_1$ must be binding and $\overline{IC}_2$ is slack. Hence, $\mu_2 = 0$. By Lemmas 2 and 4, we can neglect MH_2 and $\underline{IC}_i$ constraints. By Lemma 1, ICX_i constraints are binding. Thus, using (13), define

$$Ez'_i = (1 - \pi)z'[U_i + (i - 1)b] + \pi z'[U_i + (2 - i)b].$$

Adding equations $FOC1$ to $FOC4$, we easily obtain, $1/\kappa = \Sigma_i \lambda_i (P_i E z'_i + (1 - P_i) z'(u_i)) > 0$. Hence, $\overline{RC}$ is binding. Suppose now that MH_1 is slack. Then, $\delta = 0$. From $FOC1b$ and $FOC3b$, we easily derive,

$$\kappa\lambda_1 z'(u_1) = \alpha_1 + \mu_1 = \kappa\lambda_1 E z'_1.$$

Hence, $z'(u_1) = Ez'_1$. But this leads to a contradiction, since MH_1 imposes $u_1 + K_1 < U_1$ and therefore,

$$Ez'_1 > z'(U_1) > z'(u_1 + K_1) > z'(u_1).$$

We conclude that MH_1 must be binding and $U_1 = u_1 + K_1$. From $FOC2b$, and the requirement that $\mu_1 \geq 0$, we derive

$$\mu_1 = \frac{P_2}{P_1}\left[\alpha_2 - \kappa\lambda_2 E z'_2\right] \geq 0. \qquad (A9)$$

From $FOC1b$, we derive,

$$\mu_1 + \frac{\delta}{P_1} = \kappa\lambda_1 E z'_1 - \alpha_1 \geq 0. \qquad (B9)$$

Combining $A9$ and $B9$, we obtain,

$$\frac{\alpha_2}{\lambda_2} \geq \kappa E z'_2 \qquad \text{and} \qquad \kappa E z'_1 \geq \frac{\alpha_1}{\lambda_1}. \qquad (C9)$$

We must compare Ez_2' and Ez_1'. It is easy to see that $Ez_2' > Ez_1'$ if and only if,

$$(1-\pi)z'(U_2+b)+\pi z'(U_2) > (1-\pi)z'(U_1)+\pi z'(U_1+b). \tag{D9}$$

To prove that $D9$ is true, assume that $Ez_1' \geq Ez_2'$. Then, using $U_2 > U_1$, we must have,

$$(1-\pi)z'(U_1)+\pi z'(U_1+b) \geq (1-\pi)z'(U_2+b)+\pi z'(U_2) > (1-\pi)z'(U_1+b)+\pi z'(U_1),$$

but this leads to a contradiction since, obviously, $U_1+b > U_1$ and $\pi < 1/2$. This proves $D9$. Now, it follows from $C9$ and $D9$ that we must have $\alpha_2(1-\lambda_2) > (1-\alpha_2)\lambda_2$, or $\alpha_2 > \lambda_2$.

Combining $A9$ and $B9$, and assuming that $\delta > 0$, we obtain

$$\delta = \lambda_1 P_1[\kappa E z_1' - (\alpha_1/\lambda_1)] + P_2\lambda_2[\kappa E z_2' - (\alpha_2/\lambda_2)] > 0,$$

which is equivalent to,

$$\Sigma_i \lambda_i P_i E z_i' > \frac{P_\alpha}{\kappa}. \tag{E9}$$

The allocation U_1, U_2, u_1, u_2 is determined by a system of four equations, the first three are obviously $\overline{IC}_1$, MH_1 and $\overline{RC}$, expressed as equalities. To find the fourth equation, we eliminate Lagrange multipliers from $FOC1b$, $FOC3b$ and $FOC4b$. More precisely, adding $FOC1b$ and $FOC3b$ yields

$$\mu_1 = \lambda_1\kappa[P_1 E z_1' + (1-P_1)z'(u_1)] - 1 + \alpha_2. \tag{F9}$$

On the other hand, substituting $A9$ yields,

$$\frac{P_2}{P_1}[\alpha_2 - \kappa\lambda_2 E z_2'] = \lambda_1\kappa[P_1 E z_1' + (1-P_1)z'(u_1)] - 1 + \alpha_2,$$

or equivalently,

$$\frac{P_\alpha}{\kappa} = \lambda_2 P_2 E z_2' + \lambda_1 P_1[P_1 E z_1' + (1-P_1)z'(u_1)]. \tag{G9}$$

Note that if $G9$ holds, since $Ez_1' > z'(u_1)$, then, necessarily, $E9$ holds: this confirms that $\delta > 0$. Substituting the expression for κ, derived above, in $G9$, and rearranging terms, yields the fourth equation that we need to solve the problem,

$$\frac{\lambda_2}{\lambda_1\alpha_2}\frac{[P_2(1-P_\alpha)Ez_2' - P_\alpha(1-P_2)z'(u_2)]}{(P_2-P_1)} = P_1 E z_1' + (1-P_1)z'(u_1) \tag{K}$$

The second-best optimum (U_1, u_1, U_2, u_2) is fully determined by $\overline{RC}$, $MH_1, \overline{IC}_1$ expressed as equalities and condition K. The condition $\mu_1 \geq 0$ yields a lower bound on the values of α_2 that can be derived from $A9$, that is, equivalently, from $C9$, we have,

$$\frac{\alpha_2}{\lambda_2} \geq \kappa E z_2'. \tag{$H9$}$$

Summing the $FOCs$, we found $1/\kappa = \Sigma_i \lambda_i (P_i E z_i' + (1 - P_i) z'(u_i))$. Since MH_i implies $Ez_i' > z'(u_i)$, and since by $D9$, $Ez_2' > Ez_1'$, we must have $1/\kappa < Ez_2'$ or $\kappa E z_2' > 1$. We conclude from $H9$ that $\alpha_2 > \lambda_2$ is required for this type of solution to be optimal. *Q.E.D.*

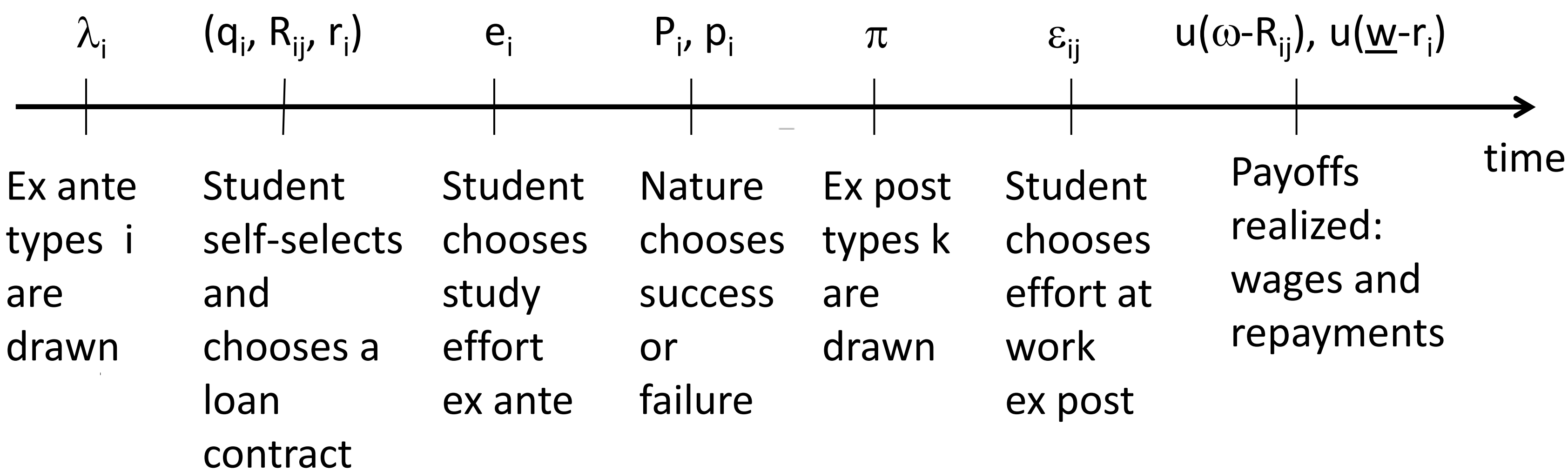

Figure 1. Timing of the model