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On the Environmental Efficiency of Water Storage: The Case of a Conjunctive Use of Ground and Rainwater

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Abstract

Rainwater harvesting, consisting in collecting runoff from precipitation, has been widely developed to stop groundwater declines and even raise water tables. However this expected environmental effect is not self-evident. We showin a simple setting that the success of this conjunctive use depends on whether the runoff rate is above a threshold value. Moreover, the bigger the storage capacity, the higher the runoff rate must be to obtain an environmentally efficient system. We also extend the model to include other hydrological parameters and ecological damages, which respectively increase and decrease the environmental efficiency of rainwater harvesting.

Key words: groundwater management, rainwater harvesting, optimal control, conjunctive use
JEL classification: Q15, Q25, C61, D61

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1. Introduction

The depletion of groundwater has generated new interest in the oldest human technology developed to provide a water supply: rainwater harvesting systems. The first evidence of the collection of rainwater dates back 6,000 years in the Gangsu region of China, which has a semi-arid to arid continental climate [12]. Rain storage was also particularly important in Southern India, where dams were built by the villagers to capture rainwater from the monsoons [29], in small islands with no significant river systems [20] and in remote and arid locations [25]. Although this traditional technique was largely abandoned in favor of large-scale projects, it has enjoyed a revival in popularity since the early 20th century. For instance, Gibraltar has one of the largest rainwater collection systems in existence and rainwater is still the primary water source on many US ranches. Furthermore, the United Nations Environment Program[3] reported that in 2000, the Gansu province in China had built 2,183,000 rainwater tanks with a capacity of 73.1 million cubic meters, supplying drinking water and supplementary irrigation. More recently, the Rainwater Partnership, registered with the UN Commission on Sustainable Development, was established in 2004 to promote this practice and its integration into water resource management policies. In parallel, a lot of international cooperation encourages rainwater harvesting by financing projects all around the world. For example, Canada financed the "Hebei Dryland Project" in China between 1989 and 2000, while the UN Food and Agriculture Organization supported projects in Ghana, Kenya, Tanzania and Zambia, and more recently a cooperative project between Germany and Tunisia was carried out.[4]

This renewed popularity is at least partially due to the fact that this technology is often presented as the simplest and most affordable way to stop groundwater declines and even raise water tables [30]. However, to the best of our knowledge, no study, except Stahn and Tomini [35], examines whether this solution is actually environmentally acceptable, in terms of both reducing extraction and raising water tables.

However, because, the flow of collected rainwater is constrained by the capacity of the storage technology,. rainwater harvesting can only substitute for part of existing water sources. This capacity -constraint implies that groundwater may still be extracted as a supplemental source of water to meet water demands. We should therefore consider rainwater harvesting with some caution within the context of conjunctive use management.

Moreover, this method is dependent on local conditions like precipitation and surface runoff [3]. since rainwater harvesting involves either collecting rainfall directly where it falls or collecting the runoff originating from it. This allows to mitigate water losses and consequently increases water supplies. However, this additional water supply may lead to a decline in the value of groundwater, since the dependence on its stock decreases, and this may in turn encourage people to increase current extraction from the ground and thereby contribute to a long-term decline in the water table. At the opposite extreme, when all

[3] See http://www.unep.or.jp/ietc/publications/urban/urbanenv-2/9.asp)

[4] For more details, see Annex 1 of the study *Agricultural Technologies for Developing Countries* of the European parliament.

water runs overland, meaning that the aquifer is nonrenewable, some stock of reserve will remain in the ground when rainwater is collected at the surface, whereas if groundwater was the only water supply, the aquifer would be depleted. As local conditions may vary greatly from one area to another, the environmental success or failure of this technology is therefore far from self-evident.

The purpose of this paper is to extend the literature on conjunctive use to adequately consider the environmental potential of rainwater harvesting. Our results will differentiate between the situations in which conjunctive use has a positive long-term impact on the water table, and those in which it does not. More precisely, the discussion is formulated to compare the long-term water table level resulting from optimal conjunctive use of groundwater and rainwater storage with the level when no rainwater is collected. For optimal use, the collection of rainwater must begin before the steady state is reached, and it must be efficient to use both water sources simultaneously over the long run. Our model allows us to define the conditions under which such a conjunctive use optimally emerges. We then show that below a threshold value of runoff, rainwater harvesting is economically optimal but not environmentally efficient, because the water table is lower than it would be without any rainwater storage. To refine the analysis, we first introduce our results in a basic setting, and then extend our framework to include the hydrological influences of other parameters affecting storage and the recharge of the aquifer, as well as ecological damages.

The rest of the paper is organized as follows. In the next section, we provide some background on models of groundwater management, which provide various explanations of aquifer depletion. Sections 3 and 4 introduce the model of conjunctive use of groundwater and rainwater under a capacity constraint in a basic setting, and derive the optimal use of groundwater when there is no collection of rainwater. Section 5 presents an analysis of environmental efficiency with an implicit determination of the threshold value of runoff rates above which rainwater harvesting allows to raise the water table over the long term. Sections 6 and 7 derive some extensions to incorporate improvements regarding the water system and ecological damages. Finally, section 8 provides some concluding remarks. An appendix contains all the proofs. They work out in a completely integrated case which covers the results of the different sections[5].

2. Background of groundwater management

Groundwater management has generated a substantial literature in resource economics. Since the seminal work of Gisser and Sánchez [14], many studies have focused on the effects of the absence of central control (e.g., Allen and Gisser [1], Feirnermann and Knapp [10] or more recently Koundouri [17]). Such absence intuitively leads to the depletion of groundwater, because agents fail to take various externalities into account (Provencher and Burt [28]). Most studies compare myopic farmers, under perfect competition, with a social planner's solution focusing on the stock externality (Feirnerman

[5] The detailed proofs are available upon request.

and Knapp [10], Nieswiadomy [24], Brill and Burness [4]). Others address strategic behavior (Negri [23], Provencher and Burt [28] or Rubio and Casino [32]). However, the magnitude of the adverse consequences on resource stocks and the potential benefits from groundwater management are still controversial.

Another part of the literature deals with issues of groundwater quality; either from a pollution control perspective (Vickner et al. [38], Yadav [40] or Xepapadeas [39]) or explaining the relationship between contamination and water-use decisions (Roseta-Palma [31], Hellegers et al. [15]). This last set of studies highlights additional externalities due to the effect of pumping on aquifer contamination. Roseta-Palma [31] shows that the steady state value of the water table level is always higher in quantity-quality models than in quantity-only models.

Finally, another strand of the literature investigates the optimal management of groundwater under the conjunctive use of different water resources: surface water and groundwater (e.g., Burt [6], Provencher [27], Tsur [36]), multiple aquifers (Zeitouni and Dinar [41], Roumasset and Wada [33]), or rainwater and groundwater (Stahn and Tomini [35]). However, most of the models that analyze the conjunctive use of groundwater and surface water differ very little from typical groundwater-only models. They assume that groundwater is the single control variable used to supplement a fixed amount of surface water, as a back-up supply. There are a handful of papers that explore an additional externality when surface water is stochastic (Knapp and Olson [19], Gemma and Tsur [13], Tsur and Graham-Tomasi [37]). Ignoring the role of the resource as a buffer creates inefficiencies and therefore aquifer depletion. Moreover, few papers consider water sources of different qualities. Dinar and Xepapadeas [7] empirically show that an individual monitoring regime is superior to central monitoring in preventing degradation of the quality and depletion of the quantity of water in the aquifer when farmers use water from a central surface supply and simultaneously extract from the ground.

However, neither the early literature nor more recent studies consider the hydrologic link between groundwater and surface water. Since conjunctive use is often considered in discussing water conservation and the harmonious use of water resources to avoid undesirable effects (FAO [9]), this caveat deserves more attention. To the best of our knowledge, few papers study the interactions between several different water supplies. Zeitouni and Dinar [41] explore the contamination flows between several connected aquifers. Burness and Martin [5] consider the links between groundwater and surface water, in their modeling of the instantaneous rate of aquifer recharge caused by groundwater pumping, by river effects. They model these effects asas externalities that increase intensive pumping. Azaiez [2] develops a multi-stage model to determine how much surface water must be imported for irrigation and to artificially recharge the aquifer, which is also and simultaneously used for irrigation. Pongkijvorasin and Roumasset [26] develop an optimal control model where groundwater users enjoy positive externalities from canal conveyance loss and from return flows to the aquifer from irrigation. But, these studies still consider surface and groundwater as two separate entities, only related by the fact that some surface water is lost to deep percolation or is used for aquifer recharge, representing positive externalities on the aquifer. None of them considers the idea of both water resources coming from the same source, so that the use of one may limit the availability of the other. Such situations obviously occur when rainwater is harvested directly, because it prevents replenishment of the aquifers; but generally speaking they can also arise when water is diverted from rivers that contribute to groundwater recharge[6] or with dam constructions that stop flows towards the downstream parts of water basins, where aquifers may exist. We help to fill this gap in the literature by closely linking groundwater and rainwater harvesting. As a consequence, this model contributes to a deeper understanding of conjunctive use and complements the existing literature by analyzing two additional features: hydrological relationships and flow constraints of surface water.

3. Modeling the conjunctive use of rainwater and groundwater

At each time period t, water can be abstracted from the system by collecting rainwater, $w_r(t)$, and extracting groundwater, $w_g(t)$. Rainwater is directly captured from precipitation, R, falling within the water basin, but the storage of this water is limited by the capacity of the available technology, $w_r \leq W$. We also assume that it is not possible to collect the entire precipitation: $W < R$. Consequently, part of the remaining rain flows over the land as surface runoff, with $\rho \in (0,1)$ denoting the runoff coefficient, while the other part replenishes the aquifer. The dynamics of the groundwater elevation, h, can then be written:

$$\dot{h} = (1-\rho)\,(R - w_r(t)) - w_g(t) \tag{1}$$

The total cost of collecting rainwater is $K \cdot w_r$, where $K > 0$ is the unit cost. Following the literature, the cost of extracting w_g units of groundwater depends on the stock and is given by $C(h)w_g$. As usual, the unit pumping cost, $C(h)$, is decreasing with water elevation at a decreasing rate. This means that the dependence of the extraction cost on the stock, $C'(h)w_g$, increases with higher water head levels.

$$C'(h) < 0, \quad C''(h) \geq 0 \tag{2}$$

The capacity of the aquifer is known with certainty and the maximum elevation is denoted by $\bar{h} > 0$. At this level, the unit and marginal pumping costs are zero, while for an empty aquifer, the unit cost becomes very large.

$$C(\bar{h}) = C'(\bar{h}) = 0, \quad \lim_{h \to 0} C(h) = +\infty \tag{3}$$

Total water abstraction,$w = w_g + w_r$, is used for irrigation. As usual in the literature, irrigation demand is a positive, continuous and strictly decreasing function of price, P. We can then deduce that social benefits from water consumption are given by the area under the inverse demand curve, $P(w)$ net of water costs. We assume that the marginal social benefit is decreasing with water consumption at an increasing rate:

$$P'(w) < 0, \quad P''(w) \geq 0 \tag{4}$$

[6]For instance, water diversion in the Lower Tarim River in China induces a decline in groundwater recharge [16].

The social benefit from water consumption, net of costs, is thus given as follows:

$$\int_0^w P(x)dx - C(h(t))w_g(t) - Kw_r(t) \tag{5}$$

We can now analyze the optimal management of these interconnected water resources.

4. The optimal use of groundwater with rainwater storage

The social planner chooses the optimal consumption of both water resources in order to maximize the sum of discounted net benefits (Eq 5), given the law of groundwater dynamics (Eq 1) and subject to the constraints that both extraction (denoted GW hereafter) and rainwater collection (denoted RW hereafter) are non-negative and that RW is upper-bounded by the capacity limitation. Given a discount rate $\delta > 0$, the problem can be written as follows:

$$\begin{aligned} &\max_{w_g(t),w_r(t)} \int_0^{+\infty} \exp^{-\delta t} \left(\int_0^{w_g(t)+w_r(t)} P(x)dx - c(h(t))w_g(t) - Kw_r(t) \right) dt \\ &\text{w.r.t} \begin{cases} \dot{h} = (1-\rho)\left(R - w_r(t)\right) - w_g(t) \\ w_g(t) \geq 0,\ W \geq w_r(t) \geq 0 \\ h_0 \text{ given and } h(\infty) \text{ free} \end{cases} \end{aligned} \tag{6}$$

The current value of the Hamiltonian for this optimal management problem is:

$$\mathcal{H} = \int_0^{(w_g(t)+w_r(t))} P(x)dx - c(h(t))w_g(t) - Kw_r(t) + \lambda(t)\left((1-\rho)\left(R - w_r(t)\right) - w_g(t)\right) \tag{7}$$

where $\lambda(t) \geq 0$ is the shadow value of a unit of water held in the aquifer. The social planner seeking optimal water consumption has three options to choose from: only GW, only RW, or a combination of both. Let us add these additional constraints on controls, $w_g(t)$ and $w_r(t)$, to define the Lagrangian of the Hamiltonian (Eq 7):

$$\mathcal{L} = \mathcal{H} + \mu_g(t) \cdot w_g(t) + \mu_r(t) \cdot w_r(t) + \mu_W(t) \cdot (W - w_r(t)) \tag{8}$$

where $\mu_g(t)$ and $\mu_r(t)$ are the Lagrangian multipliers associated with the non-negativity constraints on, $w_g(t)$ and $w_r(t)$ respectively, and $\mu_W(t)$ that associated with the capacity constraint W. This formulation leads us to identify three possible regimes.[7] Optimal water uses must therefore be chosen to satisfy the following first order conditions:

$$P\left(w_g(t) + w_r(t)\right) - C\left(h(t)\right) - \lambda(t) + \mu_g(t) = 0 \tag{9}$$

$$P\left(w_g(t) + w_r(t)\right) - K - (1-\rho)\,\lambda(t) + \mu_r(t) - \mu_W(t) = 0 \tag{10}$$

$$\mu_g(t)w_g(t) = 0, \quad \mu_r(t)w_r(t) = 0, \quad \mu_W(t)\left(W - w_r(t)\right) = 0 \tag{11}$$

$$\mu_g(t), \mu_r(t), \mu_W(t) \geq 0, \quad w_g(t) \geq 0, \text{ and } w_r(t) \in [0, W] \tag{12}$$

[7] With 3 multipliers, we usually expect 2^3 cases. But *(i)* the capacity and non-negativity constraints on RW cannot both be satisfied, *(ii)* by our boundary conditions at least one source of water is used, *(iii)* if only RW is used the capacity may or may not be reached, and *(iv)* there is a boundary regime for which the use of GW or RW is indifferent.

Condition (Eq 9) represents the optimality condition, which states that the marginal benefit of GW is equal to the marginal social cost of GW extraction and its corresponding Lagrange multiplier. The social cost of GW extraction, or the full marginal cost of GW, is determined by the sum of the extraction cost and the shadow value of groundwater resource. Likewise, condition (Eq 10) represents the optimality condition for RW, taking into account the implicit price associated with the capacity constraint on RW harvesting. Conditions (Eq 11) and (Eq 12) correspond to the standard slackness conditions.

We must also take into account the dynamics of the resource and its shadow price, characterized by the two following equations:

$$\dot{h}(t) = (1-\rho)\,(R - w_r(t)) - w_g(t) \tag{13}$$

$$\dot{\lambda}(t) = \delta\lambda(t) + C'\,(h(t))\,w_g(t) \tag{14}$$

Let us first explain the structure of the stationary solution. RW can be thought of as a backstop resource for GW, since it is a perfect substitute that is developed as GW gets depleted. Remember that for full aquifers, the unit pumping cost is negligible (see Eq 3). Moreover, the concepts of backstop resource and the ordering of resource extraction are already established in the literature (Koundouri and Christou [18], Krulce et al. [21], Roumasset and Wada [33]). As a consequence, we know that optimality requires resources to be used following a *least-cost-marginal-opportunity-cost rule* first-extraction rule, where the marginal opportunity cost includes extraction and marginal user costs. As such, the optimal steady state may or may not entail groundwater use with RW harvesting. Typically, RW harvesting will not be used as long as the marginal benefit of water consumption is lower than the RW marginal opportunity cost at the steady state. However, we know that the water table declines before the steady state is reached. If the water table falls to a level such that the marginal pumping cost is higher than the harvesting cost, i.e., $c(h) > K$, but the marginal benefit of water consumption is higher than the GW marginal opportunity cost, we know for sure that we will end up with the conjunctive steady state in which RW harvesting is used at capacity.[8] This discussion is summarized in the following sufficient condition for the emergence of a conjunctive use system:

$$P(R) > K - \frac{1}{\delta}C'\left(C^{-1}(K)\right)(R-W) \tag{15}$$

Given this emergence condition (Eq 15)[9], we know that RW harvesting is economically feasible. We can now analyze equations (9) to (11) in order to characterize the steady state equilibrium of the optimization problem (6), and examine its qualitative properties. All the results are introduced below.

[8]This is clearly a sufficient condition. Conjunctive use may occur for higher water tables, but this allows us to refine the analysis.

[9]Observe that this sufficient condition is independent of the runoff. This parameter can used later without additional restrictions.

Proposition 1. *Given assumptions on water demand (4) and on cost functions (2) and (3), and the emergence condition (15), the optimal management problem (6):*
(i) *has a unique steady state which is a local saddle point;*
(ii) *leads to the conjunctive use RW and GW, with $w_r^* = W$ and $w_g^* = (1-\rho)(R-W)$, so that the total water consumption is $w^* = (1-\rho)R + \rho W$;*
(iii) *yields the long term water table $h_r(\rho, R, W)$ solving the following condition:*

$$P\left((1-\rho)R + \rho W\right) = C\left(h\right) - \frac{(1-\rho)}{\delta} C'\left(h\right)\left(R - W\right) \tag{16}$$

provided the shadow value of the resource: $\lambda_r = -\frac{(1-\rho)}{\delta} C'\left(h\right)\left(R-W\right)$

Proposition 1 states that sustainable water consumption is given by the sum of the GW that can be sustainably extracted from the aquifer and the part of runoff that can be captured. From this conjunctive use system, Eq. (16) hows that the groundwater elevation stabilizes in the long run at a level for which the water price is equal to the social cost of GW pumping. The determination of the water table level will consequently depend on the hydrologic parameters ρ and R, and the size of the RW reservoir W.

According to the literature, this solution is not far from the groundwater-only solution. It is as if the storage capacity becomes zero, $W = 0$, so we do not collect RW at the surface. More formally, the social planner maximizes the sum of discounted net benefit (5) from water consumption with respect to the single control, i.e. groundwater pumping, and subject to the state equation (1) without any possibility of collecting RW. In particular, conditions (9) and (14) look similar (given $w_r(t) = 0$), while condition (10) vanishes. And, one of the main modifications comes from water consumption which will be equivalent to water pumping. This then implies changes in condition (16). All these modifications are summarized in the following corollary.

Corollary 1. *A GW-only management problem provides a unique saddle point equilibrium defined as follows:*
(i) *In the long run we extract exactly the natural recharge, $w_g^* = (1-\rho)R$,*
(ii) *the stationary value for aquifer elevation, $h_g(\rho, R)$ solves the condition:*

$$P\left((1-\rho)R\right) = C\left(h\right) - \frac{(1-\rho)}{\delta} C'\left(h\right) R \tag{17}$$

provided the shadow value of the resource: $\lambda_g = -\frac{(1-\rho)R}{\delta} C'(h)$.

Corollary 1 gives some interesting insights into the relevant differences between solutions of a GW management problem with and without RW storage. First, water consumption is higher in the system with RW storage than in the GW-only system, because a part of surface runoff – which is lost if there is no storage - is consumed in addition to the pumped water. Consequently, the marginal benefit is lower with RW storage, which should imply a positive effect on the water table level, as in the case of increased precipitation.

Second, RW harvesting enables the social planner to reduce pumping by an equivalent amount to the storage that would have replenished the aquifer, which may alleviate the pressure on groundwater. Lower extraction rates mitigate the dependence on groundwater stocks, $c'(h)w_g$, and consequently reduce the long-term value of the aquifer. Thus, RW harvesting reduces the opportunity cost of extracting an additional unit of groundwater, which creates an incentive to extract more water, with a negative impact on the water table level.

So RW harvesting has two opposite effects on the water table elevation, which suggests that care must be taken in analyzing the trade-off between an increase in water consumption by ρW and a decrease in groundwater pumping by $(1-\rho)W$. Whether RW harvesting is a hydrologically-efficient strategy (with respect to the water table level) actually depends on the magnitude of these two effects.

5. The long-term effects of RWH on groundwater resource

Following the literature, the capture of runoff is central to the definition of this technique.[10] And the potential for application is appraised essentially according to the quantity of water that can be collected. We have underlined the fact that RW harvesting can increase the water supply by collecting water that would otherwise have been lost. However, we have also shown that this process modifies incentives with regard to extraction, so that the effects of RWH are not self-evident. The purpose of this section is therefore to analyze the extent to which runoff collection affects benefits from water consumption and decreases the burden on groundwater. This situation is considered to be environmentally efficient, rather than economically efficient, in the light of our optimal management approach.

To gain intuition of this, consider the extreme case in which there is no runoff: $\rho = 0$. IIt is straightforward that water consumption w will be the same with and without RW storage, since there is perfect equivalence between the amount of water available on the surface and in the ground. Marginal benefits from water consumption are thus identical, while the long-term marginal social costs are different. Given our assumptions on the cost function, the water table level in a GW-only management problem will be higher to satisfy simultaneously Eqs (16) and (17). Here, there is only a *cost effect*, which reduces the incentives to conserve water for later uses.

Conversely, with an extreme rate of runoff: $\rho = 1$, not one drop of rain will soak into the soil to recharge the aquifer. The problem is sharply modified since the groundwater resource is now non-renewable. In a scenario without any RW storage, the aquifer will be exhausted and water consumption will end in the long run; the long-term marginal cost becomes infinite, as does the water price. However, with RW storage, the water price is bounded by the RW supply, meaning that extraction will stop before the stock is exhausted. In this specific case, the result depends only on a *demand effect* whereby RW harvesting conserves a higher water table level.

[10] For more details, see Boers and Ben-Asher [3]

This brief analysis shows that for extremely low values of runoff, the cost effect may offset the demand effect, while for extremely high values of runoff, the demand effect may be stronger. This raises the question of the existence of a threshold value of runoff, which identifies the environmentally efficient situation. This result is summarized as follows.

Proposition 2. *There exists a unique threshold value of runoff, $\bar{\rho} \in (0, 1)$, such that the level of the water table in a system without RW harvesting is the same as the level with RW storage, $h_g(\bar{\rho}) = h_r(\bar{\rho})$. Moreover, $\forall \rho > \bar{\rho}$, RW storage is environmentally-efficient.*

Proposition 2 highlights the fact that there exists a specific runoff rate such that the decrease in the marginal benefit is perfectly balanced by a decrease in the marginal social cost of GW. In this case, the collection of RW does not preserve more water in the ground. However, for the geographic areas with higher rates of runoff, the collection of RW will enable the social planner to preserve larger groundwater stocks. For this range of values, RW harvesting is economically and environmentally- efficient. Otherwise, this technique will increase the long-term depletion of the resource.

However, this threshold value is defined for specific values of storage capacity and precipitation. In light of Proposition 1, an increase in the size of the reservoir W will increase water consumption and reduce the pumping rate. This implies a modification in the trade-off between a decrease in the marginal benefit and a decrease in the stock effect, while there is no effect for the system relying on GW alone. However, we can expect smaller and smaller demand effects, because the water price decreases at a lower rate with higher levels of consumption. This implies a larger threshold value, as summarized in the following proposition.

Proposition 3. *Since the threshold $\bar{\rho}$ is increasing with the storage capacity W, we can say that in order to be environmentally efficient::*
(i) *the implementation of large storage capacity requires a high rate of runoff;*
(ii) *for a given rate of runoff, the policy-maker should bear in mind the existence of an upper- bound to the storage capacity.*

The determination of the impact of changes in precipitation may appear easier at first sight, since water consumption and water pumping move in the same direction. In light of Proposition 1 and Corollary 1, we know that the aquifer head level will be raised separately in each water system.[11] However, it is more complicated to determine the resulting effect on the threshold. Whether the threshold value rises or falls depends on the relative strengths affecting the system with and without RW storage. Hence, it can increase, decrease or remain unchanged, as summarized in the proposition below:

[11] We can easily compute the partial derivatives: $\frac{\partial h_r}{\partial R} = -\frac{(1-\rho)\left[P'((1-\rho)R+\rho W)+\frac{C'(h)}{\delta}\right]}{-C'(h)+\frac{(1-\rho)}{\delta}C''(h)(R-W)} > 0$ and $\frac{\partial h_g}{\partial R} = -\frac{(1-\rho)\left[P'((1-\rho)R)+\frac{C'(h)}{\delta}\right]}{-C'(h)+\frac{(1-\rho)}{\delta}C''(h)R} > 0$.

Proposition 4. *The effect of higher precipitation, R, on the threshold value, $\bar{\rho}$ is ambiguous within our general setting. Nevertheless, with the following restrictions, we can state that the effect is:*
(i) *positive with a linear unit pumping cost, $C(h)$;*
(ii) *negative with a linear demand function, $P(w)$;*
(iii) *zero using the framework of Gisser and Sánchez [14], which is a combination of the two previous items.*

In contrast to the effect of the storage capacity on the threshold, the impact of changes in climatic conditions is less clear-cut. They raise the question not only of which areas experience an increase in precipitation, but also of economic specificities. In regions subject to climate change, more targeted studies are required to evaluate the environmental efficiency of an RW harvesting policy.

6. Introducing additional hydrologic parameters

We now extend the previous model to incorporate some improvements regarding the water system, and we will then discuss the hydrological influence of these new parameters on the threshold value. Up to this point, we have been working with two strong assumptions: *(i)* all irrigation water is consumed, so that there is no return flow to the aquifer, and *(ii)* there is no loss of collected water.

Now, we consider that there are return flows to the aquifer from both rainwater harvesting and groundwater extraction (consequent to irrigation) at a rate $\beta \in (0,1)$, and that a part of collected rainwater is lost at the rate $\varepsilon \in (0,1)$. This loss encompasses evaporation, leakage and quality degradation. Intuitively, we state that the runoff rate is higher than the loss rate, otherwise RW storage could not capture any runoff: $\rho \geq \varepsilon$. We also assume that the proportion of rainwater naturally percolating to the aquifer is higher than the proportion of stored rainwater which is not lost and which percolates to the aquifer after use: $1-\rho \geq \beta(1-\varepsilon)$.

This slightly modifies our optimization problem (Eq 6). First, groundwater and rainwater are no longer perfect substitutes, and the water supply becomes $w = w_g + (1-\varepsilon)w_r$. Second, the dynamics of the water head becomes:

$$\dot{h} = (1-\rho)(R - w_r(t)) - w_g(t) + \beta\left(w_g(t) + (1-\varepsilon)w_r(t)\right) \tag{18}$$

Third, condition (Eq 15), which ensures the existence of a steady state with conjunctive use must be modified to take into account return flows and losses. It becomes:

$$P\left(\frac{R}{1-\beta}\right) > \frac{K}{1-\varepsilon} - \frac{1}{\delta}C'\left(C^{-1}\left(\frac{K}{1-\varepsilon}\right)\right)(R-(1-\beta)W) \tag{19}$$

The steady state solutions are obtained through a similar optimization problem taking into account all the modifications, and are characterized by the following proposition.

Proposition 5. *The optimal management problem always has a unique steady state which is a local saddle point. In this conjunctive use system, RW collection is still $w_r^* = W$, but:*
(i) *the steady state pumping rate is given by $w_g^* = \frac{(1-\rho)R - W[1-\rho-\beta(1-\varepsilon)]}{1-\beta}$, so that the total consumption is $w^* = \frac{(1-\rho)R+(\rho-\varepsilon)W}{1-\beta}$;*
(ii) *the long term water table level, $h_r(\rho, R, \beta, W, \varepsilon)$, solves the following condition:*

$$P\left(\frac{(1-\rho)R+(\rho-\varepsilon)W}{1-\beta}\right) = C(h) - \left(\frac{(1-\rho)R - W[1-\rho-\beta(1-\varepsilon)]}{\delta}\right) C'(h) \tag{20}$$

As for the GW-only system, we merely have to consider additional return flows, so the qualitative results of Corollary 1 are unchanged, apart from the first item. We state that the long-term pumping rate is $w_g^* = \frac{(1-\rho)R}{1-\beta}$ and the stationary value for the aquifer, $h_g(\rho, R, \beta)$, is obtained from the following condition:

$$P\left(\frac{(1-\rho)R}{1-\beta}\right) = C(h) - \frac{(1-\rho)R}{\delta} C'(h) \tag{21}$$

Likewise for the basic model, which is extended by the qualitative results on the existence of the threshold value in Proposition 2 extend: there is still a unique threshold value, $\bar{\rho} \in [\varepsilon, 1-\beta(1-\varepsilon)]$, above which RW storage is environmentally efficient.
In fact, there are additional restrictions on the runoff rate ρ, due to the reasonable assumptions induced by the introduction of the loss rate and the return flow rate. At the lower bound, i.e., $\rho = \varepsilon$, the water consumption is the same with or without RW harvesting (see Eqs (20) and (21)), and there is only, as previously, a cost effect which reduces the incentive to conserve water. At the upper bound, the result is slightly different. Since $1-\rho \geq \beta(1-\varepsilon)$, the proportion of rainwater which naturally percolates to the aquifer is equal to the proportion of stored rainwater which is not lost and which percolates to the aquifer after use, so that RW harvesting does not affect the dynamics of the aquifer. The long-term pumping cost is the same with or without RW harvesting, but more water is available in the first case. There is only a demand effect which increases the incentives to conserve water.

Let us now investigate the impact of the two new parameters, $\{\varepsilon; \beta\}$. Water loss in the RW harvesting technology works as runoff, reducing the available amount of water in the system. So if we assume the technology just becomes environmentally efficient and the loss rate increases, then this additional water loss must be compensated for by a higher amount of runoffs. Since at the steady state, RW harvesting is at capacity, this means that we need a higher rate of runoff: the threshold must increase.
Return flows increase the recharge of the aquifer. Consequently, as for the natural recharge R, an increase in the return flow rate modifies the system with and without RW harvesting: the same indeterminacy occurs.
To summarize:

Proposition 6. ***(i)*** *There always exists a unique threshold, which now belongs to $[\varepsilon, 1-\beta(1-\varepsilon)]$.*

(ii) *The threshold increases with a higher rate of RW losses, ε.*
(iii) *The effects of return flows on the threshold depend on economic specificities. For instance, with linear marginal pumping cost and iso-elastic demand, the result depends on the degree of iso-elasticity.*

7. A conjunctive use model with ecological damages

In this final step, we now add the values of GW-dependent ecosystems to our model, along the lines of Esteban and Albiac [8]. Depletion, $\dot{h} < 0$, creates environmental damages to associated ecosystems measured by additional costs. Otherwise, for $\dot{h} > 0$, higher water table levels improve ecosystems, generating additional benefits for society. This effect is captured by an increasing function $G(-\dot{h})$ with $G(0) = 0$. Unlike these authors, we do not introduce linearity. This is why we also require this function, as a standard cost function, to be convex: $G''(\dot{h}) > 0$. The social planner now maximizes the following net social benefit:

$$\int_0^{w_g(t)+(1-\varepsilon)w_r(t)} D(x)dx - c(h(t))w_g(t) - Kw_r(t) - G(-\dot{h}) \tag{22}$$

with respect to GW dynamics (Eq 18).

The emergence condition (Eq 19) must now be modified by adding the marginal environmental cost at the steady state mitigated by return flows. It becomes:

$$P\left(\tfrac{R}{1-\beta}\right) > \tfrac{K}{1-\varepsilon} - \tfrac{1}{\delta}C'\left(C^{-1}\left(\tfrac{K}{1-\varepsilon}\right)\right)(R-(1-\beta)W) + (1-\beta)G'(0) \tag{23}$$

The study of this maximization problem is summarized in the following proposition:

Proposition 7. *The optimal management problem always has a unique steady state which is a local saddle point. Moreover,*
(i) *Conjunctive use is not affected by ecological damages. RW collection is still $w_r^* = W$, pumping rate is given by $w_g^* = \frac{(1-\rho)R-W[1-\rho-\beta(1-\varepsilon)]}{1-\beta}$, so that the total consumption is again $w^* = \frac{(1-\rho)R+(\rho-\varepsilon)W}{1-\beta}$,*
(ii) *GW elevation solves the new condition:*

$$P\left(\tfrac{(1-\rho)R+(\rho-\varepsilon)W}{1-\beta}\right) = C\left(h\right) - \left(\tfrac{(1-\rho)R-W[1-\rho-\beta(1-\varepsilon)]}{\delta}\right)C'\left(h\right) + (1-\beta)G'(0) \tag{24}$$

The GW-only system will be modified in the same way, so that only Eq. (21) changes by adding the new term $G'(0)$. The water table level is therefore obtained from the new condition:

$$P\left(\tfrac{(1-\rho)R}{1-\beta}\right) = C\left(h\right) - \left(\tfrac{(1-\rho)R}{\delta}\right)C'\left(h\right) + (1-\beta)G'(0) \tag{25}$$

Since the term $G'(0)$ is a constant and has the same effect on Eqs. (24) and (25), the analysis of the threshold value, $\bar{\rho}$, in Proposition (6) holds, but both the threshold value and the water table will now also be dependent on ecological damages. By adding a fixed cost, the water head level is higher, whatever the system, when we take ecological damages into account.[12] But the effect on the water level with RW harvesting will be lower, because this technique reduces the dependence on the aquifer. If the runoff rate corresponds to the threshold, and environmental damages increase, we typically move to an environmentally inefficient situation. This means that the threshold is increasing with damages.

Proposition 8. *If we compare equations (24) and (25), we observe that:*
(i) *there again exists a unique threshold;*
(ii) *the water table increases in both cases when the damages increase, but it increases less under RW harvesting;*
(iii) *the threshold increases with damages.*

The last point may seem surprising. In ecosystems severely affected by intensive pumping, we would expect RW harvesting to alleviate the pressure on aquifers. But, here we find that the environmental efficiency of RW harvesting is reduced. This could result from the way we consider damages: they are linked to changes in the water table. It would be interesting to include the level of the water height in the formulation of damages proposed by Esteban and Albiac [8].

8. Conclusion

This paper analyzes the extent to which rainwater harvesting is environmentally efficient. More precisely, we investigate whether this technique, used in conjunction with groundwater extraction, is more effective in limiting aquifer depletion than a purely groundwater-based system. We emphasize the importance of considering the hydrological cycle in its entirety, so that ground and surface water are interdependent and must be managed as such. In a basic setting, we highlight the fact that rainwater harvesting should be seen not only as a means to collect surface runoff in order to alleviate pressure on conventional water sources, but also as an economically optimal solution, which will only be environmentally efficient when the surface runoff rate is high enough.

Moreover, the runoff rate needs to be even higher when the storage capacity is bigger. This draws attention to the fact that water policies may have an adverse impact if they encourage larger storage capacities with no consideration for specific parameters. We then examine how this result is modified by the inclusion of additional parameters like evaporation, return flows and ecological damages due to aquifer depletion. Variations in these parameters affect the likelihood that rainwater harvesting will be environmentally efficient. This technique could therefore be a long-term solution for groundwater depletion under specific conditions, which must be taken into account in water policies.

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[12] Note first that the LHS of equality (20) (respectively (21)) and (24) (respectively (25)) are unchanged, while the same positive constant is added to the RHS of (24) (respectively (25)). If we remember that $C' < 0$ and $C'' > 0$, the result follows.

Our approach remains focused on certain specific aspects, and could be extended in a number of directions. Firstly, we could enrich the ecological part of the model. For instance, we could link the environmental damages to the level of the water table. This could even be introduced in a nonlinear way, since the impact on biodiversity is stronger when the aquifer is depleted. Other characteristics of the aquifer could be included, such as stochastic recharge, leakage and GW quality.

Secondly, in this paper we assume a fixed capacity. However, the RW harvesting technology can be improved over time. This requires investment strategy and therefore the inclusion of additional dynamics in the model. This problem was explored by Stahn and Tomini [35], but they do not explore the question of the threshold in runoff rates.

Finally, it is well-known that strategic behavior modifies the aquifer level, and its effect on the threshold should therefore be taken into account in the model. This analysis quickly becomes very complex, especially when feedback strategies are introduced. A first exploration is provided by Soubeyran et al. [34].

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APPENDIX

A. The solution to the general problem

Let us introduce some notations used below:

$$\begin{array}{lll} \rho \in [\varepsilon, 1-\beta(1-\varepsilon)] & \beta \in (0,1) & \varepsilon \in (0,1) \\ R_\rho \equiv (1-\rho)R & a \equiv 1-\beta > 0 & b \equiv 1-\rho-\beta(1-\varepsilon) > 0 \\ K_\varepsilon \equiv \frac{K}{1-\varepsilon} & w(t) \equiv w_g(t) + (1-\varepsilon)w_r(t) & \\ \mu_r(t) \equiv \frac{\mu'_r(t)}{1-\varepsilon} \geq 0 & \mu_W(t) \equiv \frac{\mu'_W(t)}{1-\varepsilon} \geq 0 & \end{array}$$

A.1. The problem

Let us consider the following optimal control problem:

$$\begin{array}{rl} \max\limits_{w_g(t), w_r(t)} & \int_0^{+\infty} exp^{-\delta t} \left[\int_0^{w(t)} P(x)dx - c(h(t))w_g(t) - Kw_r(t) - G\left(-\dot{h}(t)\right) \right] dt \\ \text{w.r.t} & \begin{cases} \dot{h}(t) = (1-\rho)\left(R - w_r(t)\right) - w_g(t) + \beta\left(w_g(t) + (1-\varepsilon)w_r(t)\right) \\ w_g(t) \geq 0,\ 0 \leq w_r(t) \leq W \\ h_0 \text{ given and } h(\infty) \text{ free} \end{cases} \end{array} \tag{26}$$

The current value of the Hamiltonian for this problem is:

$$\mathcal{H} = \int_0^{w(t)} P(x)dx - c(h(t))w_g(t) - Kw_r(t) - G\left(-\dot{h}(t)\right) + \lambda(t)\left[R_\rho - aw_g(t) - bw_r(t)\right]$$

where $\lambda(t) \geq 0$ denotes the shadow value of the groundwater resource. Moreover, if we introduce the Lagrange multipliers $\mu_g(t)$ and $\mu'_r(t)$ associated with the non-negativity constraints on $w_g(t)$ and $w_r(t)$, respectively, and the implicit price $\mu'_W(t)$ of W, the capacity constraint, the Lagrangian function associated with this program is:

$$\mathcal{L} = \mathcal{H} + \mu_g(t) \cdot w_g(t) + \mu'_r(t) \cdot w_r(t) + \mu'_W(t) \cdot (W - w_r(t)) \tag{27}$$

The Hamiltonian conditions are given by systems (CTR1) and (DYN1):

$$(CTR1)\quad \begin{cases} P\left(w(t)\right) - C\left(h(t)\right) - a\left(\lambda(t) + G'\left(-\dot{h}(t)\right)\right) + \mu_g(t) = 0 & (GW_1) \\ P\left(w(t)\right) - K_\varepsilon - \frac{b}{1-\varepsilon}\left(\lambda(t) + G'\left(-\dot{h}(t)\right)\right) + \mu_r(t) - \mu_W(t) = 0 & (RW_1) \\ \mu_g(t)w_g(t) = 0, \quad \mu_r(t)w_r(t) = 0, \quad \mu_W(t)\left(W - w_r(t)\right) = 0 & (S1_1) \\ \mu_g(t), \mu_r(t), \mu_W(t) \geq 0, \quad w_g(t) \geq 0, \text{ and } w_r(t) \in [0, W] & (S2_1) \end{cases}$$

$$(DYN1)\quad \begin{cases} \dot{h}(t) = R_\rho - aw_g(t) - bw_r(t) \\ \dot{\lambda}(t) = \delta\lambda(t) + C'\left(h(t)\right)w_g(t) \end{cases}$$

A.2. The construction of the steady state

Let us consider a steady state satisfying (CTR1) and (DYN1) with $\dot{h}(t) = \dot{\lambda}(t) = 0$. There are eight possible cases,since each of the three Lagrange multipliers may or may not be strictly positive. We nevertheless immediately observe that:

Remark 1. *At a steady state: (i) both constrains on w_g cannot be simultaneously binding since $W > 0$, (ii) there must be some GW consumption (i.e , $\mu_g = 0$) otherwise $\dot{h}(t) = 0$ implies that $w_r = \frac{R_\rho}{b} = \frac{(1-\rho)}{(1-\rho)-\beta(1-\varepsilon)}R > R$, which is impossible since $w_r \leq W < R$.*

Now let us observe:

Lemma 1. *If the capacity constraint on RW harvesting is not binding (i.e., $\mu_W = 0$) at the steady state, then $K_\varepsilon > C(h)$.*

Proof of Lemma 1

Since $\mu_g = \mu_W = \dot{h} = 0$, the difference between Eqs. (GW_1) and (RW_1) is:

$$K_\varepsilon - C\left(h\right) + \left(\frac{b}{1-\varepsilon} - a\right)\left[\lambda + G'\left(0\right)\right] - \mu_r = 0 \tag{28}$$

Using (i) $\mu_r \geq 0$ and (ii) $\lambda = -\frac{1}{\delta}C'\left(h\right)w_g$ since $\dot{\lambda} = 0$, it follows that:

$$K_\varepsilon - C\left(h\right) \geq \left(\frac{b}{1-\varepsilon} - a\right)\left(\frac{1}{\delta}C'\left(h\right)w_g - G'\left(0\right)\right) \tag{29}$$

(iii) Notice that $\frac{b}{1-\varepsilon} - a = -\frac{\rho-\varepsilon}{1-\varepsilon} < 0$ since $\rho > \varepsilon$, and (iv) by assumption $C' < 0, G' > 0$. The lhs of Eq (29) is therefore positive and we can conclude that $K_\varepsilon > C(h)$. ♦

From Lemma 1, we can assert :

Lemma 2. *If we assume that*

$$P\left(\frac{R}{a}\right) > K_\varepsilon - \frac{1}{\delta}C'\left(C^{-1}\left(K_\varepsilon\right)\right)\left(R - aW\right) + aG'\left(0\right) \tag{30}$$

the RW storage capacity is always binding at a steady state.

Proof of Lemma 2

Assume the contrary, $w_r < W$, so that $\mu_W = 0$. Since $\lambda = -\frac{1}{\delta}C'\left(h\right)w_g$, Eq. (RW_1) at the steady state can be written as:

$$\mu_r = -P\left(w\right) + K_\varepsilon - \frac{b}{1-\varepsilon}\left[\frac{1}{\delta}C'\left(h\right)w_g - G'\left(0\right)\right] \tag{31}$$

Using Lemma 1, i.e., $K_\varepsilon > C(h) \Leftrightarrow C^{-1}\left(K_\varepsilon\right) < h$, and since $C" \geq 0$, we deduce that:

$$\mu_r \leq -P\left(w\right) + K_\varepsilon - \frac{b}{\delta(1-\varepsilon)}C'\left(C^{-1}\left(K_\varepsilon\right)\right)w_g + \frac{b}{1-\varepsilon}G'\left(0\right) \tag{32}$$

Moreover from system (DYN1) at steady state we know that:

(i) $R_\rho - aw_g - bw_r = 0 \Leftrightarrow aw_g = R_\rho - bw_r < R_\rho - bW$ since we havee assumed that $w_r < W$. Moreover $b = 1-\rho-\beta(1-\varepsilon)$ and $1 > \rho > \varepsilon$, we can therefore say:

$$aw_g < (1-\rho)(R-W) + \beta(1-\varepsilon)W < R-(1-\beta)W = R-aW \tag{33}$$

(ii) The total consumption of water is $w = \frac{R_\rho - bw_r}{a} + (1-\varepsilon)w_r = \frac{1}{a}\left(R_\rho - (b-a(1-\varepsilon))\,w_r\right)$. Since $b - a(1-\varepsilon) = \varepsilon - \rho$, we have $w = \frac{1}{a}\left[R_\rho + (\rho-\varepsilon)\,w_r\right]$. If we now remember that $w_r < W < R$ and $\varepsilon < \rho < 1$, we can observe that:

$$w < \frac{1}{a}\left((1-\rho)+(\rho-\varepsilon)\right)R < \frac{R}{1-\beta} \tag{34}$$

Under assumptions $P' < 0$ and $C' < 0$ and items (i) and (ii), we can rewrite condition (32) as:

$$\mu_r < -P\left(\frac{R}{1-\beta}\right) + K_\varepsilon - \frac{b}{a\delta(1-\varepsilon)}C'\left(C^{-1}(K_\varepsilon)\right)(R-aW) + \frac{b}{1-\varepsilon}G'(0) \tag{35}$$

Finally, since $0 < \frac{b}{(1-\varepsilon)a} < 1$ (see (iii) of the proof of lemma 1), we can state that:

$$\mu_r < -P\left(\frac{R}{a}\right) + K_\varepsilon - \frac{1}{\delta}C'\left(C^{-1}(K_\varepsilon)\right)(R-aW) + aG'(0) \tag{36}$$

Given the assumption introduced in Eq (30), this implies that $\mu_r < 0$ which is a contradiction. ♦

If we now combine Remark 1 and Lemmas 1 and 2, we can say that a steady state, if it exists, verifies:

$$\begin{cases} P(w) - C(h) - a(\lambda + G'(0)) = 0, & w_g = \frac{1}{a}(R_\rho - bW) \\ \mu_W = P(w) - K_\varepsilon - \frac{b}{1-\varepsilon}(\lambda + G'(0)), & \lambda = -\frac{1}{\delta a}C'(h)(R_\rho - bW) \\ \mu_W \geq 0 \text{ and } w_r = W, & w = \frac{1}{a}(R_\rho + (\rho-\varepsilon)W) \end{cases} \tag{37}$$

In other words, a unique state exists, if there exists a unique solution $h^* \in [0, \bar{h}]$ to

$$\phi_1(h^*) = P\left(\frac{1}{a}(R_\rho + (\rho-\varepsilon)W)\right) - C(h^*) + \frac{1}{\delta}C'(h^*)(R_\rho - bW) - aG'(0) = 0 \tag{38}$$

which verifies:

$$\mu_W = P\left(\frac{1}{a}(R_\rho + (\rho-\varepsilon)W)\right) - K_\varepsilon + \frac{b}{a\delta(1-\varepsilon)}C'(h^*)(R_\rho - bW) - \frac{b}{1-\varepsilon}G'(0) \geq 0 \tag{39}$$

So let us first observe:

Lemma 3. *If condition (30) is satisfied, any solution (it is exists) to $\phi_1(h^*) = 0$ verifies $\mu_W > 0$.*

Proof of Lemma 3
Using Eq.(37) and $W < R$, we get $w < \frac{1}{a}(R_\rho + \rho W) < \frac{R}{a}$ and by Eq. (33) we know that $R_\rho - bW < R - aW$. If we now bear in mind that $P' < 0$ and $C^{'} < 0$, we can deduce from condition (30) that:

$$\begin{aligned} 0 &< P\left(\frac{R}{a}\right) - K_\varepsilon + \frac{1}{\delta}C'\left(C^{-1}(K_\varepsilon)\right)(R-aW) - aG'(0) \\ &< P(w) - K_\varepsilon + \frac{1}{\delta}C'\left(C^{-1}(K_\varepsilon)\right)(R_\rho - bW) - aG'(0) \equiv \phi_1(C^{-1}(K_\varepsilon)) \end{aligned} \tag{40}$$

But $\phi_1(h)$ is increasing in h since $C' < 0$ and $C" \geq 0$. We deduce that the steady state $h^* < C^{-1}(K_\varepsilon)$ or $C(h^*) > K_\varepsilon$. We can therefore say from equation (38) that:

$$P\left(\frac{1}{a}(R_\rho + (\rho-\varepsilon)W)\right) - K_\varepsilon + \frac{1}{\delta}C'(h^*)(R_\rho - bW) - aG'(0) > 0 \tag{41}$$

Since $\frac{b}{a(1-\varepsilon)} \in (0,1)$, $C' < 0$ and $G'(0) > 0$, we can even state:

$$P\left(\frac{1}{a}(R_\rho + (\rho-\varepsilon)W)\right) - K_\varepsilon + \frac{b}{a\delta(1-\varepsilon)}C'(h^*)(R_\rho - bW) - \frac{b}{1-\varepsilon}G'(0) > 0 \tag{42}$$

or, in other words, that $\mu_W > 0$. ♦

We can finally affirm that:

Proposition 9. *Under condition (30), the optimization problem (26) admits a unique steady state given by equations (37).*

Proof of Proposition 9:
It simply remains to show that $\phi_1(h) = 0$ admits a unique solution $h^* \in \left[0, \bar{h}\right]$. This is immediate since (i) $\phi_1' > 0$ because $C' < 0$ and $C" \geq 0$, (ii) $\lim_{h\to 0} \phi_1(h) = -\infty$ because $\lim_{h\to 0} C(h) = \infty$ and (iii) $\lim_{h\to \bar{h}} \phi_1(h) > 0$. However this last point is less obvious. So let us first observe that $\lim_{h\to \bar{h}} \phi_1(h) = P\left(\frac{1}{a}\left(R_\rho + (\rho - \varepsilon)W\right)\right) - aG'(0)$ since we have assumed that $C(\bar{h}) = C'(\bar{h}) = 0$. Now remark that $\frac{1}{a}\left(R_\rho + (\rho-\varepsilon)W\right) < \frac{R}{1-\beta}$ since $W < R$ and use condition (30) in order to obtain:

$$\lim_{h\to 0} \phi_1(\bar{h}) > P\left(\tfrac{R}{a}\right) - aG'(0) > K_\varepsilon - \tfrac{1}{\delta}C'\left(C^{-1}\left(K_\varepsilon\right)\right)(R - aW) > 0 \quad (43)$$

(remember $C' < 0$ and $R - aW > 0$). ♦

A.3. The local saddle point property

Let us return to the systems (CTR1) and (DYN1) and set $\mu_g = 0$, and $\mu_r = 0$. Since $\mu_W > 0$ at the steady state we can choose a neighborhood around the steady state such that $\mu_W(t) > 0$ and therefore set $w_r(t) = W$. In this case equation (GW_1) becomes:

$$\phi_2\left(w_g(t), h(t), \lambda(t)\right) = P\left(w_g(t) + (1-\varepsilon)W\right) - C\left(h(t)\right) - a\left(\lambda(t) + G'\left(-R_\rho + aw_g(t) + bW\right)\right) = 0 \quad (44)$$

Moreover $\partial_{w_g}\phi_2 = P' - a^2G" < 0$ by our assumptions. We can therefore apply the implicit function theorem around the steady state and build $w_g(t) = \phi_3\left(h(t), \lambda(t)\right) > 0$. We even know that

$$\partial_h\phi_3 = \tfrac{C'}{P' - a^2G"} > 0 \text{ and } \partial_\lambda\phi_3 = \tfrac{a}{P' - a^2G"} < 0 \quad (45)$$

It follows that the dynamics of the system is locally given by:

$$\begin{cases} \dot{h}(t) = R_\rho - a\phi_3\left(h(t), \lambda(t)\right) - bW \\ \dot{\lambda}(t) = \delta\lambda(t) + C'\left(h(t)\right)\phi_3\left(h(t), \lambda(t)\right) \end{cases} \quad (46)$$

It remain to verify that the Jacobian J of this two dimensional system has a positive trace $Tr(J)$ and a negative determinant $\det(J)$. By computation, we obtain:

$$J = \begin{bmatrix} -\frac{aC'}{P' - a^2G"} & -\frac{a^2}{P' - a^2G"} \\ C" \cdot \phi_3 + \frac{\left(C'\right)^2}{P' - a^2G"} & \delta + \frac{aC'}{P' - a^2G"} \end{bmatrix} \quad (47)$$

We can therefore say, under our assumptions, that :

$$Tr(J) = \delta > 0 \text{ and } \det(J) = \frac{-\delta aC' + a^2C"\phi_3}{P' - a^2G"} < 0 \quad (48)$$

B. The study of the threshold

Let us introduce the system:

$$\begin{cases} \phi_4(\rho, h; R, \beta, \Gamma) = 0 \\ \phi_5(\rho, h; W, R, \varepsilon, \beta, \Gamma) = 0 \end{cases} = \begin{cases} P\left(\frac{1}{a}R_\rho\right) - C(h) + \frac{1}{\delta}C'(h)R_\rho - a\Gamma = 0 \\ P\left(\frac{1}{a}\left(R_\rho + (\rho-\varepsilon)W\right)\right) - C(h) + \frac{1}{\delta}C'(h)\left(R_\rho - bW\right) - a\Gamma = 0 \end{cases} \quad (49)$$

with $\Gamma = G'(0)$. The first equation gives us the stationary water table without RWH while the second includes this option. So if we solves this system in h and ρ we obtain the threshold value, $\bar{\rho}$ for which the stationary water table h is the same without or with rainwater harvesting. Moreover this threshold can then be linked to the different parameters $(W, R, \varepsilon, \beta, \Gamma)$.

B.1. Existence of a threshold

Let us observe that for any admissible $(W, R, \varepsilon, \beta, \Gamma)$, $\forall\{i = 4, 5\}$, (i) $\partial_h \phi_i > 0$ since $C' < 0$ and $C" \geq 0$, (ii) $\lim_{h \to 0} \phi_i = -\infty$ since $\lim_{h \to 0} C(h) = -\infty$ and (iii) $\lim_{h \to \bar{h}} \phi_i > 0$ (apply the same argument as in the proof of Proposition 9). We can therefore, by the implicit function theorem, define two continuous functions $h_g(\rho)$ and $h_r(\rho)$ which respectively solve in h, $\phi_4(\rho, h; R, \beta, \Gamma) = 0$ and $\phi_5(\rho, h; W, R, \varepsilon, \beta, \Gamma) = 0$. It remains now to study $(h_g(\rho) - h_r(\rho))$ since the threshold solves $h_g(\bar{\rho}) = h_r(\bar{\rho})$.

Since $\rho \in [\varepsilon, 1 - \beta(1-\varepsilon)]$, let us first observe that $(h_g(\rho) - h_r(\rho))_{\rho=\varepsilon} > 0$. To verify this point, let us observe that, for $\rho = \varepsilon$, h_g and h_r respectively solve:

$$\begin{cases} P\left(\frac{1-\varepsilon}{1-\beta}R\right) - C(h_g) + \frac{1-\varepsilon}{\delta}C'(h_g)R - (1-\beta)\Gamma = 0 \\ P\left(\frac{1-\varepsilon}{1-\beta}R\right) - C(h_r) + \frac{1-\varepsilon}{\delta}C'(h_r)(R - (1-\beta)W) - (1-\beta)\Gamma = 0 \end{cases} \tag{50}$$

This implies:

$$\begin{aligned} & -C(h_g) + \frac{(1-\varepsilon)R}{\delta}C'(h_g) = -C(h_r) + \frac{1-\varepsilon}{\delta}C'(h_r)(R - (1-\beta)W) \\ \Leftrightarrow & \left[-C(h_g) + \frac{(1-\varepsilon)R}{\delta}C'(h_g)\right] - \left[-C(h_r) + \frac{(1-\varepsilon)R}{\delta}C'(h_r)\right] = -\frac{(1-\varepsilon)(1-\beta)W}{\delta}C'(h_r) \end{aligned} \tag{51}$$

It is straightforward that the right-hand side of Eq (51) is positive, so that:

$$\left[-C(h_g) + \frac{(1-\varepsilon)R}{\delta}C'(h_g)\right] > \left[-C(h_r) + \frac{(1-\varepsilon)R}{\delta}C'(h_r)\right] \tag{52}$$

Finally since $C' < 0$ and $C" \geq 0$, we deduce that $h_g(\varepsilon) > h_r(\varepsilon)$.

Second, observe that $(h_g(\rho) - h_r(\rho))_{\rho=1-\beta(1-\varepsilon)} < 0$. In this case, the quantities h_g and h_r solve:

$$\begin{cases} P\left(\frac{\beta(1-\varepsilon)}{1-\beta}R\right) - C(h_g) + \frac{\beta(1-\varepsilon)}{\delta}C'(h_g)R - (1-\beta)\Gamma = 0 \\ P\left(\frac{\beta(1-\varepsilon)}{1-\beta}R + (1-\varepsilon)W\right) - C(h_r) + \frac{\beta(1-\varepsilon)}{\delta}C'(h_r)R - (1-\beta)\Gamma = 0 \end{cases} \tag{53}$$

Since $P\left(\frac{\beta(1-\varepsilon)}{1-\beta}R\right) > P\left(\frac{\beta(1-\varepsilon)}{1-\beta}R + (1-\varepsilon)W\right)$ (remember that $P' < 0$) we can write:

$$C(h_g) - \frac{\beta(1-\varepsilon)}{\delta}C'(h_g)R > C(h_r) - \frac{\beta(1-\varepsilon)}{\delta}C'(h_r)R \tag{54}$$

Since $C(h) - \frac{\beta(1-\varepsilon)}{\delta}C'(h)R$ is decreasing in h, we conclude that $h_g(1-\beta(1-\varepsilon)) < h_r(1-\beta(1-\varepsilon))$.

Finally by continuity, we can affirm that there exists at least one $\bar{\rho} \in [\varepsilon, 1-\beta(1-\varepsilon)]$ such that $h_g(\bar{\rho}) - h_r(\bar{\rho}) = 0$.

B.2. Uniqueness of the threshold

Since this threshold and the associated water table are also obtained as a solution to system (49) for a given set $(W, R, \varepsilon, \beta, \Gamma)$ of parameters, we can use the Gale-Nikaido theorem (see Gale-Nikaido [11] or Mas-Colell [22]) which states that if every principal minor of $\partial_{\rho,h}(\phi_{i=4,5})$ is positive and the domain $K = [0, \bar{h}] \times [0, 1]$ of the function is a rectangle, the solution, if it exists, is unique. Let us verify this point.

$$\partial_{\rho,h}(\phi_{i=4,5}) = \begin{bmatrix} -R\left(\frac{1}{a}P'_g + \frac{1}{\delta}C'\right) & -C' + \frac{R_\rho}{\delta}C" \\ -(R-W)\left(\frac{1}{a}P'_r + \frac{1}{\delta}C'\right) & -C' + \frac{R_\rho - bW}{\delta}C" \end{bmatrix} \text{ with } \begin{pmatrix} P'_g \\ P'_r \end{pmatrix} = \begin{pmatrix} P'\left(\frac{1}{a}R_\rho\right) \\ P'\left(\frac{1}{a}(R_\rho + (\rho-\varepsilon)W)\right) \end{pmatrix}$$

Concerning the principal minors of order 1, we notice that:

(i) $-R\left(\frac{1}{a}P'_g + \frac{1}{\delta}C'\right) > 0$ since $C' < 0$ and $P' < 0$

(ii) $-C' + \frac{R_\rho - bW}{\delta}C" > 0$ since $C' < 0$, $C" \geq 0$ and $R_\rho - bW = (1-\rho)(R-W) + \beta(1-\varepsilon)W > 0$

It therefore remains to compute the determinant of $\partial_{\rho,h}\left(\phi_{i=4,5}\right)$. This quantity is given by:

$$\begin{aligned}
\det\left(\partial_{\rho,h}\left(\phi_{i=4,5}\right)\right) &= \left(\tfrac{1}{a}P'_g+\tfrac{1}{\delta}C'\right)\left(RC'-\tfrac{(1-\rho)(R-W)R}{\delta}C\text{"}-\tfrac{\beta(1-\varepsilon)RW}{\delta}C\text{"}\right)\\
&+\left(\tfrac{1}{a}P'_r+\tfrac{1}{\delta}C'\right)\left(-RC'+\tfrac{(1-\rho)(R-W)R}{\delta}C\text{"}+WC'\right)\\
&=\underbrace{\tfrac{R}{a}\left(C'-\tfrac{(1-\rho)(R-W)}{\delta}C\text{"}\right)\left(P'_g-P'_r\right)}_{>0}-\underbrace{\tfrac{\beta(1-\varepsilon)RW}{\delta}C\text{"}\left(\tfrac{1}{a}P'_g+\tfrac{1}{\delta}C'\right)}_{<0}+\underbrace{WC'\left(\tfrac{1}{a}P'_r+\tfrac{1}{\delta}C'\right)}_{>0}
\end{aligned}$$

$$\text{hence } \det\left(\partial_{\rho,h}\left(\phi_{i=4,5}\right)\right)>0 \tag{55}$$

since $C', P'_g, P'_r<0$,$C\text{"}\geq 0$ and because $P\text{"}>0$, $P'\left(\frac{1}{a}R_\rho\right)<P'\left(\frac{1}{a}\left(R_\rho+(\rho-\varepsilon)W\right)\right)$, i.e. $P'_g-P'_r<0$.

B.3. Comparative statics

Let us denote by θ one of the following parameters $\{W,R,\beta,\varepsilon,\Gamma\}$ and let us compute $\frac{\partial\rho}{\partial\theta}$. It we differentiate system (49) with respect to (h,ρ,θ) and solve the system, we know by the Cramer rule that:

$$\frac{\partial\rho}{\partial\theta}=\det\begin{pmatrix}-\partial_\theta\phi_4 & \partial_h\phi_4\\ -\partial_\theta\phi_5 & \partial_h\phi_5\end{pmatrix}\Big/\det\begin{pmatrix}\partial_\rho\phi_4 & \partial_h\phi_4\\ \partial_\rho\phi_5 & \partial_h\phi_5\end{pmatrix}=\frac{N}{D} \tag{56}$$

From Eq. (55), we know that the denominator D is positive. It follows that the sign of $\frac{\partial\rho}{\partial\theta}$ is simply given by its numerator N. Moreover a simple exercice of computation shows that :

$$\partial_h\phi_4=-C'(h)+\frac{1}{\delta}C''(h)R_\rho>0 \text{ and } \partial_h\phi_5=-C'(h)+\frac{1}{\delta}C''(h)\left(R_\rho-bW\right)>0 \tag{57}$$

and

θ	$\partial_\theta\phi_4$	$\partial_\theta\phi_5$
W	0	$\frac{(\rho-\varepsilon)}{a}P'_r-\frac{b}{\delta}C'(h)$
R	$\frac{(1-\rho)}{a}\left(P'_g+\frac{a}{\delta}C'(h)\right)$	$\frac{(1-\rho)}{a}\left(P'_r+\frac{a}{\delta}C'(h)\right)$
ε	0	$-W\left(\dfrac{P'_r}{a}+\frac{\beta}{\delta}C'(h)\right)$
β	$\dfrac{R_\rho P'_g}{a^2}+\Gamma$	$\dfrac{\left(R_\rho+(\rho-\varepsilon)W\right)P'_r}{a^2}+\frac{1}{\delta}C'(h)(1-\varepsilon)W+\Gamma$
Γ	$-a$	$-a$

(58)

We can therefore identify, after computation, the following effects:

(i) The threshold and the storage capacity, $\frac{\partial\rho}{\partial W}$

$$sign\left(\frac{\partial\rho}{\partial W}\right)=sign\left(\left(\tfrac{\rho-\varepsilon}{a}P'_r-\tfrac{b}{\delta}C'\right)\underbrace{\left(-C'+\tfrac{R_\rho}{\delta}C\text{"}\right)}_{>0 \text{ since } C'<0,\ C\text{"}>0}\right)=sign\left(\tfrac{\rho-\varepsilon}{a}P'_r-\tfrac{b}{\delta}C'\right) \tag{59}$$

Let us now observe that the computation of the difference of the two equations (see Eq (49)) which define the threshold gives:

$$P\left(\tfrac{1}{a}R_\rho\right)-P\left(\tfrac{1}{a}\left(R_\rho+(\rho-\varepsilon)W\right)\right)+\tfrac{bW}{\delta}C'=0 \tag{60}$$

and the convexity of the inverse demand gives:

$$P\left(\tfrac{1}{a}R_\rho\right)-P\left(\tfrac{1}{a}\left(R_\rho+(\rho-\varepsilon)W\right)\right)>-\tfrac{(\rho-\varepsilon)W}{a}P'_r \tag{61}$$

These two last equations immediately show that $\left(\frac{(\rho-\varepsilon)}{a}P'_r-\frac{b}{\delta}C'\right)>0$ hence that $\frac{\partial\rho}{\partial W}>0$.

(ii) The threshold and the evaporation rate, $\frac{\partial\rho}{\partial\varepsilon}$

$$sign\left(\frac{\partial\rho}{\partial\varepsilon}\right) = sign\left(-W\left(\frac{1}{a}P_r' + \frac{\beta}{\delta}C'\right)\left(-C' + \frac{R_\rho}{\delta}C"\right)\right) > 0$$

since $P_r' < 0$, $C' < 0$ and $C" > 0$

(iii) The threshold and the marginal ecosystem damage, $\frac{\partial\rho}{\partial\Gamma}$

$$sign\left(\frac{\partial\rho}{\partial\Gamma}\right) = sign\left(-a\frac{bW}{\delta}C"\right) \leq 0$$

(iv) The threshold and the recharge, $\frac{\partial\rho}{\partial R}$

Since $\partial_h\phi_5 = \partial_h\phi_4 + \frac{bW}{\delta}C''(h)$ and $\left(\frac{1-\rho}{a}\right) > 0$, a routine computation gives:

$$sign\left(\frac{\partial\rho}{\partial R}\right) = sign\left(\underbrace{\partial_h\phi_4}_{>0}\ \underbrace{\left(-P_g' + P_r'\right)}_{\geq 0\ (\text{since } P"\geq 0)} + \underbrace{\frac{bW}{\delta}C''(h)}_{\geq 0}\underbrace{\left(P_g' + \frac{a}{\delta}C'(h)\right)}_{<0}\right)$$

This quantity is clearly unsigned. For instance if the demand is linear, we have $P_r' - P_g' = 0$ and $\frac{\partial\rho}{\partial R} < 0$. But if the unit pumping cost is linear in h so that $C" = 0$, we observe $\frac{\partial\rho}{\partial R} > 0$. Finally in the Gisser-Sanchez case (i.e. $C" = 0$ and P linear) it is immediate that $\frac{\partial\rho}{\partial R} = 0$.

(v) The threshold and the return flow, $\frac{\partial\rho}{\partial\beta}$

$$sign\left(\frac{\partial\rho}{\partial\beta}\right) = sign\left(\partial_h\phi_4\left[\frac{R_\rho}{a^2}\left(-P_g' + P_r'\right) + \frac{(\rho-\varepsilon)W}{a^2}P_r' + \frac{(1-\varepsilon)W}{\delta}C'(h)\right] + \frac{bW}{\delta}C''(h)\left(\frac{R_\rho}{a^2}P_g' + \Gamma\right)\right)$$

This quantity is again unsigned. To illustrate this point let us assume that $C" = 0$ so that $C' = -c_0$ a constant with $c_0 > 0$. If we add that $P(w)$ is linear (Gisser-Sanchez case) then $sign\left(\frac{\partial\rho}{\partial\beta}\right) = sign\left(\frac{(\rho-\varepsilon)W}{a^2}P_r' - \frac{(1-\varepsilon)W}{\delta}c_0\right) < 0$. If we now introduce an isoelastic demand $P(w) = w^{-\alpha}$ with $a > 1$, then

$$sign\left(\frac{\partial\rho}{\partial\beta}\right) = sign\left(\alpha\underbrace{\left(\left(\frac{R_\rho}{a}\right)^{-\alpha} - \left(\frac{R_\rho+(\rho-\varepsilon)W}{a}\right)^{-\alpha}\right)}_{\equiv d>0} - \underbrace{a\frac{(1-\varepsilon)W}{\delta}c_0}_{\equiv n>0}\right)$$

so if $\alpha > \max\left\{\frac{n}{d}, 1\right\}$ then $\frac{\partial\rho}{\partial\beta} > 0$.