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Indeterminacy and sunspots in two-sector RBC models with generalized no-income-effect preferences*

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Abstract: *We analyze sunspot-driven fluctuations in the standard two-sector RBC model with moderate increasing returns to scale and generalized no-income-effect preferences à la Greenwood, Hercovitz and Huffman [13]. We provide a detailed theoretical analysis enabling us to derive relevant bifurcation loci and to characterize the steady-state local stability properties as a function of various structural parameters. We show that local indeterminacy occurs through flip and Hopf bifurcations for a large set of values for the elasticity of intertemporal substitution in consumption, provided that the labor supply is sufficiently inelastic. Finally, we provide a detailed quantitative analysis of the model. Computing, on a quarterly basis, a new set of empirical moments related to two broadly defined consumption and investment sectors, we are able to identify, among the set of admissible calibrations consistent with sunspot equilibria, the ones that provide the best fit of the data. The model properly calibrated solves several empirical puzzles traditionally associated with two-sector RBC models.*

Keywords: *Indeterminacy, sunspots, two-sector model, sector-specific externalities, real business cycles*

Journal of Economic Literature Classification Numbers: C62, E32, O41.

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1 Introduction

The aim of this paper is to provide a detailed theoretical and empirical assessment of the sunspot-driven two-sector Real Business Cycle model with productive externalities and increasing returns to scale, considering the Greenwood, Hercovitz and Huffman [13] (GHH) specification for individual preferences, characterized by the lack of income effect on labor choices. Compared to the previous literature in which the formal theoretical analysis of such models and their data confrontation step are largely divorced, we argue that providing a complete characterization of the local stability properties of the model as a function of various structural parameters is a crucial ingredient for a successful data confrontation strategy. Following this approach, we are able not only to derive new theoretical configurations for which the two-sector RBC model is locally indeterminate, but also to improve several well-known counterfactual predictions of this model when submitted to sunspot-driven shocks.

The recent literature suggests that by comparison to their one-sector equivalents, two-sector RBC models are able to generate local indeterminacy with much lower degrees of increasing returns to scale.[1] Yet, this result has often been obtained under relatively narrow specifications for technology and/or preferences, without much attention to robustness and domain of validity issues. Besides, in many cases, this result has been obtained through numerical simulations, without explicit consideration of the types of local bifurcations identified (and their implications for the local dynamics around the steady-state).

The first contribution of this paper is thus *theoretical*. It aims to provide a general theoretical analysis of local indeterminacy and local bifurcations in the canonical two-sector RBC model. Starting from the Benhabib and Farmer [4]'s formulation with increasing social returns, we consider the generalized specification of GHH preferences, enabling us to thoroughly analyze the interplays between increasing returns to scale, intertemporal substitution effects and labor supply elasticity in the emergence of local indeterminacy.

It is known that with GHH preferences and *constant* social returns, local indeterminacy occurs for sufficiently inelastic labor supply (Nishimura and Venditti [23]). Yet, for *increasing* social returns, this result has been extended only for the specific case of a logarithmic specification, and in fact essentially through numerical simulations (Guo

[1] While indeterminacy requires about 50% of increasing returns to scale in the one-sector RBC model of Benhabib and Farmer [3] and Farmer and Guo [8], this degree decreases to only 7% in its two-sector equivalent (see Benhabib and Farmer [4]). Indeterminacy also occurs with constant social returns to scale and decreasing private returns (Benhabib and Nishimura [5], Garnier *et al.* [9], Nishimura and Venditti [22]).

and Harrison [15]).[2]

We prove here that this result holds quite generally, in particular *for a large set of values for the elasticity of intertemporal substitution (EIS) in consumption*. We also prove the existence of an upper bound on the labor supply elasticity above which local indeterminacy never arises, and we show that this upper bound is decreasing with the degree of increasing social returns. We finally exhibit the existence of flip and Hopf bifurcations in the parameter space and we provide the analytical expressions for these bifurcations.[3] This allows us to show how a change in the EIS in consumption drastically affects the range of values for the other structural parameters for which the steady-state is locally indeterminate.

The second contribution of the paper is *empirical*. While two-sector RBC models are able to generate local indeterminacy and endogenous sunspot fluctuations with much lower degrees of increasing returns to scale than one-sector models, they also tend to make several inaccurate empirical predictions. In particular, the literature has identified several empirical puzzles associated with such models:[4] the *consumption-investment cyclicality puzzle* (the inability to generate simultaneous procyclical comovements of consumption and investment with output), the *consumption volatility puzzle* (the tendency to generate a volatility of consumption that exceeds that of output), the *labor comovement puzzle* (the inability to generate procyclical movements in sectoral hours worked), and the *hours worked volatility puzzle* (the inability to generate sufficiently volatile hours worked relatively to output).

We first start by computing on a quarterly basis a new set of empirical moments related to two broadly defined consumption and investment sectors, adapting a methodology initially proposed in Baxter [2] with annual data. Then, we show that, by considering the general GHH specification for individual preferences together with appropriate calibrations, all four empirical puzzles mentioned above can be resolved. Improving the model's predictions requires to find better compromises between the various economic mechanisms — EIS, income effects, wage elasticity of labor supply — identified as crucial for the local stability properties of the model. We show that the best performing calibrations are typically close to the boundary of the set of admissible calibrations consistent with indeterminacy, near the Hopf bifurcation locus identified in the theoretical analysis. This implies that appropriate calibrations must depart from the traditional

[2] In one-sector models with GHH preferences, the results are drastically different: Meng and Yip [19] and Nishimura *et al.* [21] have shown that local indeterminacy cannot arise. Jaimovich [18], using a specification that nests the GHH formulation as a special case, has proved that a minimum amount of income effect is necessary for local indeterminacy.

[3] See Grandmont [10, 11] for a simple presentation of bifurcation theory.

[4] See e.g. Benhabib and Farmer [4], Harrison [16] and Guo and Harrison [15].

logarithmic specification for the utility function extensively considered in the literature.

As a whole, we significantly generalize and improve the conclusions obtained by Guo and Harrison [15] in the particular case of GHH preferences with a logarithmic-consumption utility function. On the theoretical ground, we exhibit the existence of a Hopf bifurcation, in addition to the flip bifurcation they identify, and we show how a change in the EIS in consumption drastically affects the range of values for the other parameters for which the steady-state is locally indeterminate. In particular, for any degree of increasing returns to scale, we can get indeterminacy with *much larger labor supply elasticities*. On the empirical ground, we provide a detailed dataset enabling to derive relevant business cycle statistics to which two-sector DSGE models can be compared. We use these data to show that considering generalized GHH preferences and a calibration of structural parameters consistent with a much larger labor supply elasticity than in Guo and Harrison [15] enables to solve the four empirical puzzles traditionally associated with two-sector RBC models with indeterminacy.

The rest of this paper is organized as follows. We present the model and we characterize the intertemporal equilibrium and the steady state in Section 2. In Section 3, the complete set of conditions for indeterminacy are derived. In Section 4, we provide detailed simulations in order to discuss the ability of the model to account for the main features of observed business cycles. Some concluding remarks are provided in Section 5, whereas all the technical details are contained in an Appendix.

2 The model

We consider a standard infinite-horizon two-sector Real Business-Cycle (RBC) model with productive externalities and GHH preferences. Households are infinitely-lived, accumulate capital, and derive utility from consumption and leisure. Firms produce differentiated consumption and investment goods using capital and labor, and sell them to consumers. All markets are perfectly competitive.

2.1 Production

Firms in the consumption sector produce output Y_{ct} according to a Cobb-Douglas production function:

$$Y_{ct} = z_t K_{ct}^{\alpha} L_{ct}^{1-\alpha} \tag{1}$$

where K_{ct} and L_{ct} are capital and labor allocated to the consumption sector, and z_t is an exogenously evolving total factor productivity (TFP) level.

In the investment sector, output Y_{It} is also produced according to a Cobb-Douglas production function but which is affected by a productive externality

$$Y_{It} = z_t A_t K_{It}^{\alpha} L_{It}^{1-\alpha} \tag{2}$$

where K_{It} and L_{It} are the numbers of capital and labor units used in the production of the investment good, and A_t is the externality parameter. Following Benhabib and Farmer [4], we assume that the externality depends on the average levels $\bar{K}_{It}$ and $\bar{L}_{It}$ of capital and labor used in the investment sector, such that:

$$A_t = \bar{K}_{It}^{\alpha\Theta} \bar{L}_{It}^{(1-\alpha)\Theta} \tag{3}$$

with $\Theta \geq 0$.[5] These economy-wide averages are taken as given by individual firms. Assuming that factor markets are perfectly competitive and that capital and labor inputs are perfectly mobile across the two sectors, the first order conditions for profit maximization of the representative firm in each sector are:

$$r_t = \frac{\alpha Y_{ct}}{K_{ct}} = p_t \frac{\alpha Y_{It}}{K_{It}}, \tag{4}$$

$$\omega_t = \frac{(1-\alpha) Y_{ct}}{L_{ct}} = p_t \frac{(1-\alpha) Y_{It}}{L_{It}} \tag{5}$$

where r_t, p_t and ω_t are respectively the rental rate of capital, the price of the investment good and the real wage rate at time t, all in terms of the price of the consumption good.

2.2 Preferences

We consider an economy populated by a continuum of unit mass of identical infinitely-lived agents. The representative agent enters each period t with a capital stock k_t inherited from the past. He then supplies elastically an amount $l_t \in [0, \ell)$ of labor (with $\ell > 0$ his exogenous time endowment), rents its capital stock k_t to the representative firms in the consumption and investment sectors, consumes c_t, and invests i_t in order to accumulate capital. Following Greenwood-Hercovitz-Huffman [13] (GHH), the instantaneous utility function is

$$U(c_t, l_t) = \frac{\left(c_t - \frac{B l_t^{1+\chi}}{1+\chi}\right)^{1-\sigma}}{1-\sigma} \tag{6}$$

with $B > 0$ a normalization constant, $\sigma > 0$ and $\chi \geq 0$. The essential feature of this specification is that the marginal rate of substitution between consumption and leisure is independent of consumption, as

$$\frac{U_2(c_t, l_t)}{U_1(c_t, l_t)} = -B l_t^{\chi} \tag{7}$$

This property illustrates the lack of income effect associated with the agent's labor supply.

[5] We do not consider externalities in the consumption good sector as they do not play any crucial role in the existence of multiple equilibria.

Denoting by y_t the GDP, the budget constraint faced by the representative household is

$$c_t + p_t i_t \equiv y_t = r_t k_t + \omega_t l_t \tag{8}$$

Assuming that capital depreciates at rate $\delta \in (0,1)$ in each period, the law of motion of the capital stock is:

$$k_{t+1} = (1-\delta)k_t + i_t \tag{9}$$

The household then maximizes its expected present discounted lifetime utility

$$\max_{\{k_{t+1}, c_t, l_t, i_t\}_{t=0}^{\infty}} \quad E_0 \sum_{t=0}^{+\infty} \beta^t \frac{\left(c_t - \frac{B l_t^{1+\chi}}{1+\chi}\right)^{1-\sigma}}{1-\sigma} \ , \tag{10}$$

with $\beta \in (0,1)$ the discount factor, subject to (8), (9) and k_0 given. The first-order conditions for an *interior* solution to this optimization problem (with $l_t < \ell$) are

$$\left(c_t - \frac{B l_t^{1+\chi}}{1+\chi}\right)^{-\sigma} = \beta E_t \left(c_{t+1} - \frac{B l_{t+1}^{1+\chi}}{1+\chi}\right)^{-\sigma} \left[\frac{r_{t+1} + (1-\delta)p_{t+1}}{p_t}\right] \tag{11}$$

$$\omega_t = B l_t^{\chi} \tag{12}$$

Equation (11) is the standard stochastic Euler equation, and (12) corresponds to the trade-off between consumption and leisure. With GHH preferences, as suggested by (7), the income elasticity of intertemporal substitution in labor is zero, and the Frisch wage elasticity of the labor supply is $1/\chi$.

2.3 Intertemporal equilibrium and steady state

We consider symmetric rational expectation equilibria which consist of prices $\{r_t, p_t, \omega_t\}_{t\geq 0}$ and quantities $\{c_t, l_t, i_t, k_t, Y_{ct}, Y_{It}, K_{ct}, K_{It}, L_{ct}, L_{It}\}_{t\geq 0}$ that satisfy the household's and the firms' first-order conditions as given by (4)–(5) and (11)–(12), the technological and budget constraints (1) to (3) and (8)–(9), and the market equilibrium conditions. All firms in the investment sector being identical, we have $\bar{K}_{It} = K_{It}$ and $\bar{L}_{It} = L_{It}$ for any t. At the equilibrium, the production function in the investment good sector is then given by

$$Y_{It} = z_t K_{It}^{\alpha(1+\Theta)} L_{It}^{(1-\alpha)(1+\Theta)} \tag{13}$$

We thus have increasing social returns which size is measured by Θ.

The market clearing conditions for the consumption and investment goods give

$$c_t = Y_{ct}, \tag{14}$$

$$i_t = Y_{It}, \tag{15}$$

while the market clearing conditions for capital and labor yield

$$K_{ct} + K_{It} = k_t \tag{16}$$

$$L_{ct} + L_{It} = l_t \tag{17}$$

For a given initial capital stock k_0, any solution that also satisfies the transversality condition

$$\lim_{t\to+\infty}\beta^t E_0\left[\left(c_t-\frac{Bl_t^{1+\chi}}{(1+\chi)}\right)^{-\sigma}k_{t+1}\right]=0$$

is called an equilibrium path.

An *interior* steady state is defined by constant equilibrium quantities and prices in the non-stochastic environment ($z_t = z$) such that $\bar{l} < \ell$. We provide the following Proposition:

Proposition 1. *Assume that $\Theta \neq (1-\alpha)/\alpha$ and consider $\underline{\chi}$ and $\hat{B}$ such that*

$$\underline{\chi} \equiv \frac{\alpha\Theta}{1-\alpha(1+\Theta)},\ \ \hat{B} \equiv (1-\alpha)\delta^{\underline{\chi}}\left(\frac{\beta\alpha}{1-\theta}\right)^{\frac{\underline{\chi}(1+\Theta)}{\Theta}}\ell^{\underline{\chi}-\chi} \tag{18}$$

with $\theta \equiv \beta(1-\delta)$. Then there exists a unique interior steady state if and only if $(\chi-\underline{\chi})(B-\hat{B})>0$.

Proof: See Appendix 6.1.

We are now able to provide a detailed local stability analysis of the steady-state, considering a family of economies parameterized by the three parameters that govern the wage elasticity of labor supply, the EIS in consumption and the degree of increasing returns to scale (IRS) in the investment sector, σ, χ and Θ. Indeed, in appendix 6.2 we show that, at the steady state and for given parameters (β, α, δ), the EIS in consumption $\epsilon_{cc} \equiv -U_1(c,l)/(U_{11}(c,l)c)$ is a function of (σ, χ), namely

$$\epsilon_{cc}(\sigma,\chi)=\frac{1}{\sigma}\left(1-\frac{1-\alpha}{(1+\chi)\left(1-\frac{\beta\delta\alpha}{1-\theta}\right)}\right) \tag{19}$$

Thus, $\epsilon_{cc}(\sigma, \chi)$ is decreasing with respect to σ but increasing with respect to χ.

3 Theory

It is easy to show that the set of equations describing an equilibrium path can be reduced to a 3-dimensional dynamic system involving c_t, k_t and z_t (the equation specifying the exogenous process for the TFP level simplifying to $z_t = z$ in a deterministic environment).

Combining (1)-(2) and firms' first-order conditions (4)-(5), we derive $p_t A_t = 1$ and that the equilibrium capital-labor ratios in the consumption and the investment sectors are identical and equal to $a_t \equiv k_t/l_t = K_{ct}/L_{ct} = K_{It}/L_{It} = \alpha\omega_t/\left((1-\alpha)r_t\right)$, with $\omega_t = (1-\alpha)z_t a_t^\alpha$ and $r_t = \alpha z_t a_t^{\alpha-1}$.

Combining the labor supply equation (12) with aggregate labor demand $\omega_t = (1-\alpha)z_t a_t^\alpha$ yields

$$l_t = \left(\frac{(1-\alpha)z_t}{B}\right)^{\frac{1}{\alpha+\chi}} k_t^{\frac{\alpha}{\alpha+\chi}} = l(k_t, z_t) \tag{20}$$

and thus

$$\begin{aligned} a_t &= k_t/l(k_t, z_t) = a(k_t, z_t) \\ \omega_t &= (1-\alpha)z_t(a(k_t, z_t))^\alpha = \omega(k_t, z_t), \\ r_t &= \alpha z_t(a(k_t, z_t))^{\alpha-1} = r(k_t, z_t) \end{aligned} \tag{21}$$

Using the equilibrium conditions (4), (8) and (16), we also derive :

$$\begin{aligned} y_t &= z_t k_t^\alpha (l(k_t, z_t))^{1-\alpha} = y(k_t, z_t) \\ K_{ct} &= \frac{r(k_t,z_t)c_t}{\alpha} = K_c(k_t, c_t, z_t) \\ K_{It} &= k_t - K_c(k_t, c_t, z_t) = K_I(k_t, c_t, z_t) \end{aligned}$$

Finally, combining (3) with $p_t A_t = 1$ leads to

$$p_t = (a(k_t, z_t))^{(1-\alpha)\Theta} K_I(k_t, c_t, z_t)^{-\Theta} = p(k_t, c_t, z_t) \tag{22}$$

We conclude from this analysis that, although aggregate labor l_t, the capital labor ratio a_t, and output y_t, are *control variables*, their equilibrium values at t are all pinned down by the initial aggregate capital stock k_t and the TFP level z_t, independently of households' expectations of future economic conditions.[6] This is also the case at equilibrium for the real wage ω_t and the real interest rate $r_{t.}$ Yet, the relative price of capital p_t also depends on the consumption level c_t, which adjusts at t according to households' expectations (see (11)). It follows that the economic adjustments at work in period t rest exclusively on how much consumption c_t is substituted to current investment i_t in the current period, or, equivalently, how many units of the initial capital stock $k_t = K_{ct} + K_{It}$ and of total hours worked $l_t = L_{ct} + L_{It}$ are allocated to the consumption and the investment sectors, respectively.

A complete description of the dynamics can now be obtained by referring to the households' first-order condition (11) and the budget constraint (8) rewritten as follows

$$E_t \left[\frac{r(k_{t+1}, z_{t+1}) + (1-\delta)p(k_{t+1}, c_{t+1}, z_{t+1})}{\left(c_{t+1} - \frac{B(l(k_{t+1}, z_{t+1}))^{1+\chi}}{1+\chi}\right)^\sigma} \right] = \frac{p(k_t, c_t, z_t)}{\beta} \left(c_t - \frac{B(l(k_t, z_t))^{1+\chi}}{1+\chi}\right)^{-\sigma} \tag{23}$$

$$k_{t+1} = (1-\delta)k_t + \frac{y(k_t, z_t) - c_t}{p(k_t, c_t, z_t)} \tag{24}$$

together with an exogenous process for the TFP level z_t.

When the TFP level is constant ($z_t = z$), equations (23)-(24) implicitly define a two-dimensional dynamic system in (c_t, k_t). Log-linearizing this system in a neighborhood of the interior steady-state $(\bar{c}, \bar{k})$ yields a Jacobian matrix for which the characteristic

[6] As discussed in section 3.2 below, this property results from the lack of income effect in labor supply.

polynomial is given in Appendix 6.2. In this Appendix, we also show that for a given value of χ, the Trace and Determinant of the Jacobian matrix are linear functions of σ, so that we can use the geometrical methodology described in Grandmont *et al.* [12] to study the local stability properties of the steady state. Indeed, for a given Θ, as σ is varied over $(0, +\infty)$, the Trace and Determinant move along a line denoted Δ_χ in Figures 1 and 2, and whose location depends on the value of χ.

We introduce the following parameter restrictions:

Assumption 1. *$\alpha < 1/2$, $\Theta \in (\underline{\Theta}, \bar{\Theta})$ and $\delta \in (0, \bar{\delta})$, with $\underline{\Theta} = \delta/(1-\delta)$, $\bar{\Theta} = (1-\alpha)/\alpha$, and*

$$\bar{\delta} = \frac{\sqrt{[(1-\beta)(1+\Theta)(1-\alpha(1+\Theta))+4\beta\Theta^2]^2+4\beta^2\Theta^3(1-\alpha(1+\Theta))}-\left[(1-\beta)(1+\Theta)(1-\alpha(1+\Theta))+4\beta\Theta^2\right]}{2\beta\Theta(1-\alpha(1+\Theta))}$$

These restrictions cover all empirically plausible configurations since capital shares are typically less than 50% of GDP in industrialized economies. Moreover, using a standard calibration consistent with quarterly US data, $(\alpha, \beta) = (0.3, 0.99)$, Assumption 1 implies $\underline{\Theta} \approx 0.0256$, $\bar{\Theta} \approx 2.33$ and $\bar{\delta} > 0.06$. These bounds define intervals for Θ and δ which largely cover the range of available empirical estimates for these parameters.[7] Note also that the bound $\underline{\chi}$ on the inverse of the labor supply elasticity, as given by (18), cannot be arbitrarily close to zero since the amount of externalities satisfies $\Theta > \underline{\Theta}$.

When $\Theta \in (\underline{\Theta}, \bar{\Theta})$, depending on the value of χ, the line Δ_χ has two possible locations. We have indeed the following geometrical configurations that provide a full picture of the local stability properties of the steady state:

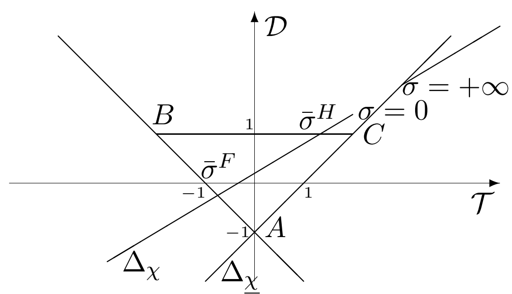

Figure 1: Local indeterminacy with GHH preferences when $\chi > \underline{\chi}$.

When $\chi = \underline{\chi}$, the line $\Delta_{\underline{\chi}}$ merges with the line generated by AC along which $\mathcal{D} = \mathcal{T} - 1$ and one characteristic root is equal to 1. If $\chi > \underline{\chi}$, Δ_χ crosses the triangle ABC in which both characteristic roots have a modulus less than 1 and local indeterminacy

[7] For example, Basu and Fernald [4] obtain a point estimates for the degree of IRS in the durable manufacturing sector in the US economy of 0.33, with standard deviation 0.11 (see also Harrison [17]) for other estimates in a similar range). A typical estimate for the annual depreciation rate in the US is 10%, implying $\delta = 0.025$.

arises (see Figure 1). Indeed, when σ increases from 0, the value of the pair $(\mathcal{T}, \mathcal{D})$ varies along Δ_χ. The steady state is first unstable for $\sigma \in [0, \bar{\sigma}^H)$, then becomes locally indeterminate when $\sigma \in (\bar{\sigma}^H, \bar{\sigma}^F)$ and is finally saddle-point stable for $\sigma > \bar{\sigma}^F$. When σ crosses $\bar{\sigma}^H$, one pair of complex characteristic roots crosses the unit circle and a Hopf bifurcation occurs generating quasi-periodic endogenous fluctuations. When σ crosses $\bar{\sigma}^F$ one negative characteristic root crosses the value -1 and a flip bifurcation occurs generating period-two cycles.

On the contrary, if $\chi < \underline{\chi}$, the line $\Delta_{\underline{\chi}}$ is located on the right of the triangle ABC and local indeterminacy is ruled out (see Figure 2). Depending on the value of σ the steady state is either saddle-point stable or unstable with the possible existence of a flip bifurcation and period-two cycles.

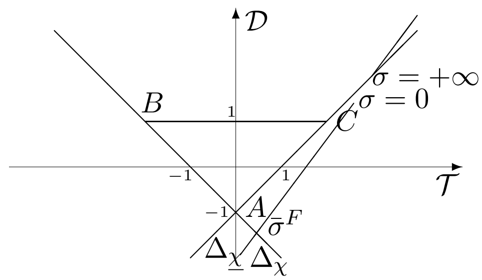

Figure 2: Local determinacy with GHH preferences when $\chi \in [0, \underline{\chi})$.

We then get the following Proposition:

Proposition 2. *Under Assumption 1, let* $\underline{\chi} \equiv \alpha\Theta/[1 - \alpha(1 + \Theta)]$. *Then the following results hold:*

i) If $\chi > \underline{\chi}$*, the steady state is saddle-point stable when* $\sigma > \bar{\sigma}^F$*, undergoes a flip bifurcation at* $\sigma = \bar{\sigma}^F$*, becomes locally indeterminate when* $\sigma \in (\bar{\sigma}^H, \bar{\sigma}^F)$*, undergoes a Hopf bifurcation when* $\sigma = \bar{\sigma}^H$ *and becomes locally unstable when* $\sigma \in [0, \bar{\sigma}^H)$*.*

ii) If $\chi \in [0, \underline{\chi})$*, the steady state is locally unstable when* $\sigma > \bar{\sigma}^F$*, undergoes a flip bifurcation at* $\sigma = \bar{\sigma}^F$ *and becomes saddle-point stable when* $\sigma \in [0, \bar{\sigma}^F)$*.*

The Hopf and flip bifurcation values are respectively defined as:

$$\bar{\sigma}^H = \frac{\Theta(1-\delta)(1-\beta)(1-\theta)\left[\left(1-\frac{\beta\delta\alpha}{1-\theta}\right)-\frac{1-\alpha}{1+\chi}\right]}{\delta\alpha[1-\beta+(1-\theta)\Theta]}$$

and

$$\bar{\sigma}^F = \frac{(1-\theta)\left[(\chi+\alpha)\Theta[2(1-\delta)(1+\beta)+(1-\theta)\delta]-(1-\theta)\delta\chi(1-\alpha)(1+\Theta)\right]\left[\left(1-\frac{\beta\delta\alpha}{1-\theta}\right)-\frac{1-\alpha}{1+\chi}\right]}{2(\chi+\alpha)\delta\alpha[1+\beta+(1-\theta)\Theta]}$$

with $\bar{\sigma}^F > \bar{\sigma}^H$*.*

Proof: See Appendix 6.2.

Interpretation. Proposition 2 shows that local indeterminacy in a two-sector model with GHH preferences can only occur if the aggregate labor supply curve is sufficiently *inelastic* (i.e., if $\chi > \underline{\chi}$) and the EIS in consumption $\epsilon_{cc}(\sigma, \chi)$ is in an *intermediate range* (i.e., such that $\sigma \in (\bar{\sigma}^H, \bar{\sigma}^F)$). While a similar conclusion has been obtained in models with constant social returns (see e.g. Benhabib and Nishimura [5]),[8] this is a new conclusion in models with increasing social returns. Moreover, as $\partial\underline{\chi}/\partial\Theta > 0$, larger externalities requires *less* elastic labor supply curves for the existence of indeterminacy. Finally, since with $\alpha < 1/2$, we have $\underline{\Theta} < \alpha/(1-\alpha) < \bar{\Theta}$, indeterminacy is compatible with standard *negatively sloped* capital and labor equilibrium demand functions.

Proposition 2 also strongly generalizes the conclusions obtained on a numerical basis by Guo and Harrison [15] in the particular case of GHH preferences with a logarithmic-consumption utility function ($\sigma = 1$). First, we provide an explicit analytical expression for the threshold $1/\underline{\chi}$ on the labor supply elasticity above which local indeterminacy is ruled out. Second, we exhibit the existence of a Hopf bifurcation, in addition to the flip bifurcation identified by Guo and Harrison [15]. Third, we show how a change in the EIS in consumption drastically affects the range of values for the other parameters for which the steady-state is locally indeterminate. In particular, since $\partial\bar{\sigma}^H/\partial\chi > 0$ and $\partial\bar{\sigma}^F/\partial\chi > 0$, for any degree of increasing returns to scale Θ, considering smaller values for σ enables to obtain indeterminacy with *much larger labor supply elasticities* (i.e. smaller $\chi's$). This property will be fundamental in our data confrontation analysis undertaken in the next section.

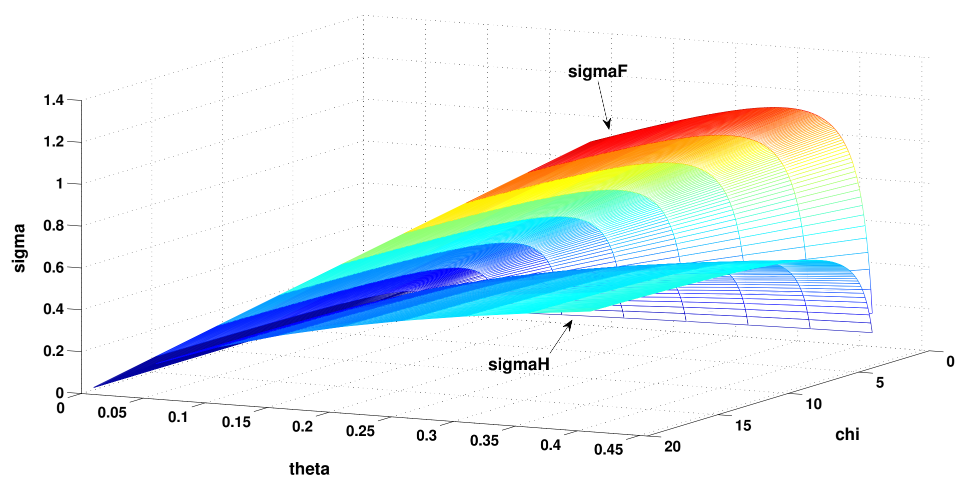

Figure 3: *Local stability properties in the* (Θ, χ, σ) *space*

Note: The indeterminacy zone is the interior area delimited by the flip (upper curve) and the Hopf (lower curve) bifurcation loci.

[8] In two-sector models with constant social returns *à la* Benhabib and Nishimura [5] and GHH preferences, local indeterminacy remains compatible with an infinitely elastic labor supply provided σ is close enough to 0 (see Nishimura and Venditti [23]).

As an illustration to Proposition 2, Figure 3 displays the determinacy-indeterminacy areas in the (Θ, χ, σ) space, once a fixed calibration $(\alpha, \delta, \beta) = (0.3, 0.025, 0.99)$ for the other three structural parameters is chosen. The indeterminacy zone is the interior area delimited by the flip (upper curve) and the Hopf (lower curve) bifurcation loci. The figure confirms, numerically, that a large set of values for (Θ, χ, σ) are consistent with indeterminacy – including low values for the degree of increasing returns to scale –, and that indeterminacy with a sufficiently elastic labor supply (χ sufficiently close to 0) requires *low values for* σ (typically less than 0.5). The configuration considered by Guo and Harrison [15], obtained as the cross section of the three-dimensional plane at $\sigma = 1$, clearly strongly restricts the range of admissible configurations regarding the various economic mechanisms — EIS, income effects, wage elasticity of labor supply — important in the business cycle properties of the model, in particular as concerns the labor supply elasticity.[9] Our aim is now to explore if considering generalized GHH preferences enables to provide better empirical performance when the model is submitted to sunspot shocks, in particular as regards the four empirical puzzles mentioned in the introduction.

4 Quantitative analysis

We now turn to the quantitative analysis of the model. Our aim is to investigate if a two-sector model with GHH preferences is able to reproduce the main features of observed business cycles when subject to indeterminacy and sunspot disturbances. Previous papers in the literature have already performed this data confrontation step, using different types of preferences,[10] and have identified several empirical regularities that are hardly accounted for by two-sector models: the consumption/investment cyclicality puzzle (the inability to generate procyclical comovements of consumption and investment with output), the consumption volatility puzzle (the propensity to generate a volatility of consumption larger than that of output), the labor comovement puzzle (the inability to generate procyclical comovements in sectoral hours worked), and the hours worked volatility puzzle (the inability to generate sufficiently volatile hours worked relatively to output).

The main contribution of this section is to show that many of these difficulties can actually be overcome by adopting a sufficiently general specification for GHH preferences and an appropriate calibration for structural parameters. It is at this stage, we

[9]Observe also that in this cross-section, there is no Hopf bifurcation in the (χ, Θ) plane. The Hopf bifurcation emerges only when σ is sufficiently smaller than 1.

[10]See e.g. Benhabib and Farmer [4], Harrison [16], Guo and Harisson [15].

argue, that benefiting from the theoretical characterization of the local stability properties of the model is important as it helps to find calibrations of structural parameters that, at the same time, (i) are consistent with an indeterminate steady-state, thus enabling sunspot shocks to play a role in the business cycle; (ii) leave additional degrees of freedom with respect to the relative intensities of the various underlying economic mechanisms (intertemporal substitution in consumption, labor supply elasticity, etc.) important in the dynamic properties of the business cycle.

We will develop these points more thoroughly in sections 4.2 and 4.3. Before this, we describe shortly the dataset we constructed and the empirical moments we derived in order to evaluate the model.

4.1 Data

One difficulty when evaluating multisector models is to compute empirical series whose definitions are broadly consistent with the corresponding variables in the model. In the case of two-sector models, Baxter [2] made a crucial step in this direction. Using standard input-output tables for the US economy, she shows that most of the SIC one-digit industries are clearly recognizable as producing predominantly either consumption goods or investment goods. She uses this observation to identify two broadly-defined consumption and investment sectors.

Yet, one limitation of the Baxter dataset is that it is based on annual series, while DSGE models are typically evaluated according to their ability to make accurate predictions on the covariations of macroeconomic variables at business cycle frequency (i.e., with quarterly data). For this reason, we constructed our own dataset formed with series taken from various sources but available at quarterly frequency.[11] Our implicit definition for aggregate consumption, aggregate investment and aggregate output is standard, while we followed the Baxter methodology to define series on hours worked in the consumption and investment sectors. More precisely, we use quarterly data from the Bureau of Economic Analysis and define consumption as the sum of personal consumption expenditure in non-durable goods and services, and investment as the sum of private fixed investment and personal consumption expenditures in durable goods.[12] To obtain per capita variables, both series are divided by the population aged 16 and over. Finally, output is the sum of consumption and investment thus defined.

[11] We provide a new Appendix (available as supplementary material downloadable from the web page of the Journal of Economic Theory) in which we give precise details on how we built our dataset. All the series can also be downloaded as supplementary material.

[12] Those data can easily be found in NIPA tables provided by the BEA. See in particular NIPA tables 1.1.4 and 1.1.5.

In order to construct series for hours worked in the consumption and investment sectors, we used data collected by the Current Employment Statistics program and available on the FRED database at the Federal Reserve Bank of Saint-Louis. The CES program provides series on the number of production and non-supervisory employees (together with their average weekly hours of work) at the sectoral level for the period 1964-Q1 to 2014-Q2.[13] Following Baxter's approach, we allocate hours worked in a specific industry either to the consumption or the investment sector according to the predominant final use of the output of this industry as consumption or investment goods (see the NIPA 2012 Input-Output table available in the supplementary material). We thus define hours worked in the investment sector as the total number of (production and non-supervisory) employees in the *Mining and logging, Construction*, *Manufacturing durables* and *Professional and business services* sectors, multiplied by the average weekly hours of work in each sector. Similarly, we define hours worked in the consumption sector as the total number of employees, multiplied by the corresponding series on average weekly hours worked, in the *Manufacturing nondurables*, *Trade, transportation and utilities*, *Information*, *Leisure and hospitality* and *Other services* sectors. To obtain per capita variables, all these series were divided by the population aged 16 and over.

Table 1 provides the main summary statistics (second-order moments), after all series were detrended using the HP filter. The results display all the well-known stylized facts concerning aggregate consumption, investment, output and hours worked, so that we do not comment on them further. The main new features concern the comovements of hours worked in the consumption and investment sectors. As Table 1 indicates, hours worked in the consumption sector appear to be significantly less volatile than output and total hours worked. By contrast, hours worked in the investment sector are twice as volatile as output. Fluctuations in each variable are very persistent (the first-order autocorrelation coefficients are above 0.9) and are strongly positively correlated with fluctuations in output. Finally, hours worked in the consumption and in the investment sectors are very strongly positively correlated (0.96). Note that most of these features are consistent with those obtained by Baxter [2], with the main difference that we obtain substantially larger autocorrelation and contemporaneous correlation coefficients.[14] This is what we expected, taking account of the fact that our dataset is

[13]Industries are defined according to the following classification, which slightly differs from the standard NAICS classification: (1) Mining and Logging, (2) Construction , (3) Manufacturing (durable and nondurable) Goods, (4) Trade, Transportation and Utilities, (5) Information, (6) Professional and Business Services, (7) Leisure and Hospitality, and (8) Other Services.

[14]The Baxter analysis, based on HP-filtered annual data with smoothing coefficient $\lambda = 400$, reports relative volatility coefficient of hours worked in the consumption and investment sector of 0.87 and

I. Volatility (absolute and relative standard deviations)						
(x)	Y	C	pI	L	L_C	L_I
σ_x	1.92	0.89	4.51	2.35	1.65	4.57
$\sigma_{x,Y} =$	1	0.47	2.35	1.22	0.86	2.38
II. Persistence (first-order autocorrelation)						
(x)	Y	C	pI	L	L_C	L_I
ρ_x	0.91	0.86	0.91	0.93	0.93	0.92
III. Covariations (contemporaneous correlations with output)						
(x)	Y	C	pI	L	L_C	L_I
$\rho_{x,y}$	1	0.87	0.98	0.89	0.87	0.88
Sectoral labor comovements: $\text{corr}(L_C, L_I) = 0.96$						

Table 1: US Data - Cyclical properties

quarterly.

4.2 Explaining the puzzles

We now provide a formal analysis enabling us to explain how an exogenous change in expectations (sunspot shock) is transmitted to the economy through various economic mechanisms (intertemporal substitution in consumption, wage elasticity of labor supply, capital accumulation, etc.) important in the business cycle properties of the model. This enables us to understand why most parameter calibrations lead to the four empirical puzzles mentioned above, in particular when the utility function is restricted to be logarithmic in consumption ($\sigma = 1$). In the next section, we make use of this analysis to identify the best strategy to solve these puzzles, and we provide simulation results corroborating our main analysis.

As usual in the RBC literature, in order to understand the effects of a sunspot shock, it is useful to analyze how the labor market works in this model. Indeed, consider the Euler equation for consumption with generalized GHH preferences, (11), and denote by $R_{t+1} = (r_{t+1} + (1-\delta)p_{t+1})/p_t$ the real rate of return of investing into an

3.37, respectively. The correlation between these two series is 0.87, and the first-order autoregressive coefficients are 0.32 and 0.46, respectively.

additional unit of capital between t and t + 1. Assume that there is a positive change in expectations concerning this rate of return. Clearly, these optimistic expectations must lead consumers to substitute investment for consumption. The extent to which such substitution occurs depends, at the same time, on the degree of consumption smoothing consumers wish to have over the business cycle (which in turn depends on the preference parameters σ and χ, which govern the value of the elasticity of intertemporal substitution in consumption ϵ_{cc}, see (19)), but also on how these consumers adjust their labor supply in response to the shock: a larger initial increase in hours worked implies a smaller decrease in consumption for any given change in the expected real rate of return, and vice versa.[15]

To understand how hours worked l_t adjust, one must therefore refer to the functioning of the labor market. We have shown with equation (20) that l_t fluctuates proportionally with capital, k_t, and the TFP level, z_t. We can also show that output y_t evolves in strict proportion with hours worked at equilibrium. Indeed, using (5), we obtain $Bl_t^{1+\chi} = \omega_t l_t = (1-\alpha)y_t$, so that:

$$y_t = \tfrac{B}{1-\alpha} l_t^{1+\chi}. \tag{25}$$

We can now provide an explanation for the main empirical puzzles associated with the two-sector RBC model submitted to sunspot disturbances.

Autonomous sunspot disturbances. When changes in expectations of the rate of return R_{t+1} occur independently of any change in fundamentals, we can treat the TFP level as constant, $z_t = z$. In this case, (20) and (25) show that, with GHH preferences, the lack of income effect in labor supply implies that total hours worked and output remain *constant* in the immediate aftermath of a sunspot shock. Over time, as consumers substitute investment for consumption, the capital stock accumulates and there is a corresponding gradual increase in hours worked and output, occurring through

[15] To verify this, observe that the log-linearized version of equation (11) is:

$$-\frac{1}{\epsilon_{cc}}(\widehat{c}_t - \xi\widehat{l}_t) = -\frac{1}{\epsilon_{cc}}(E_t\widehat{c}_{t+1} - \xi E_t\widehat{l}_{t+1}) + E_t\widehat{R}_{t+1}.$$

where $\xi \equiv (1-\alpha)/(1-\beta\alpha\delta/(1-\theta))$, see Appendix 1. Along a monotone convergent path to the steady state, the marginal utility of consumption $\widehat{\lambda}_t \equiv \widehat{c}_t - \xi\widehat{l}_t$ gradually reverts back to steady-state. Let us define $\vartheta_t \equiv E_t\widehat{\lambda}_{t+1}/\widehat{\lambda}_t$, with $\vartheta_t < 1$. We then obtain:

$$\widehat{c}_t = -\frac{\epsilon_{cc}}{1-\vartheta_t}E_t R_{t+1} + \xi\widehat{l}_t.$$

Clearly, if the initial reaction of $\widehat{l}_t$ is small (or even zero), an increase in the expected rate of return of investment must generate a significant decrease in consumption, with multiplicative coefficient $\epsilon_{cc}/(1-\vartheta_t)$. As capital accumulates over time, $\widehat{l}_t$ gradually increases away from 0 since there is an increase in labor demand: the reaction of consumption becomes less and less negative, and may even become positive if the increase in hours worked is sufficiently large.

an increase in labor demand.[16] The extent to which these two variables fluctuate along the business cycle depends on χ, the (inverse of the) wage-elasticity of labor supply: larger $\chi's$ imply smaller fluctuations in hours worked and output, and vice versa.

Since indeterminacy requires a sufficiently *inelastic* labor supply (see Proposition 2), following an expected increase in the rate of return of investment, consumption is substituted for investment along a *constant* (in period t) and then weakly volatile (in periods onwards) aggregate output. This implies that consumption and investment must go *in opposite direction* following a sunspot shock, and that the volatilities of consumption and investment are *larger* than the volatility of output. We thus have an explanation for both the *consumption-investment cyclicality puzzle* and the *consumption-volatility puzzle.* Obviously, the consumption-volatility puzzle is magnified when the wage-elasticity of labor supply is low (i.e., when χ is large)

Explaining the *sectoral labor comovement puzzle* also follows directly from this analysis. Since $K_{It}/K_{ct} = L_{It}/L_{ct}$, an increase in the production of the investment good is obtained through both an increase in capital, K_{It}, and hours worked, L_{It}, allocated to the investment sector. Since total hours worked l_t remain constant in the immediate aftermath of the shock, hours worked in the consumption and the investment sectors must also move in the opposite direction.

Finally, we can infer from (25) that there is a close relationship between the preference parameter χ and the *relative* volatility of hours worked with respect to output, σ_l/σ_y. Indeed, using (25), we get: $\sigma_l/\sigma_y = 1/(1+\chi)$. Thus, any calibration for χ significantly greater than 0 must generate a volatility of hours worked that is substantially lower than the volatility of output. Since, in the data, this ratio is close to 1 (see Table 1), we have in this case an explanation for the *hours worked-volatility puzzle.*

Correlated sunspot and technological disturbances. When sunspot shocks are correlated with technological shocks, the analysis differs significantly. Indeed, equation (20) shows that a positive technological shock z_t shifts the aggregate labor demand curve up and thus leads to an increase in total hours worked. Then, when this shock is associated with a change in the expected rate of return R_{t+1} (sunspot shock), the substitution of investment for consumption now occurs along an *increasing* output level. Clearly, if the increase in output is *sufficiently large*, both consumption and investment may now rise in response to the shock, and the rise in consumption may be smaller than the rise in output. This would solve the consumption-investment cyclicality puzzle and

[16] To illustrate these mechanisms, impulse response functions to autonomous and correlated sunspot shocks, obtained under our benchmark calibration described below, are provided in the Appendix available as supplementary material.

the consumption volatility puzzle. As for the sectoral labor comovement puzzle, the same logic applies: if l_t increases sufficiently in response to the shock, L_{It} and L_{ct} may now rise simultaneously.

This line of reasoning of course requires that the variations in hours worked and in output be "sufficiently large". Otherwise, the same logic as above would apply. This shows again that successful calibrations must combine an indeterminate steady-state with a *sufficiently elastic labor supply.* Moreover, a low calibrated value for χ remains a condition for solving the hours worked volatility puzzle since the relative volatility of hours worked, $\sigma_l/\sigma_y = 1/(1+\chi)$, is independent from the assumptions made about the correlation between shocks.

Note finally that the fact that technological shocks generate an increase in hours worked does not imply that sunspots are unimportant in the business cycle. As explained above, sunspots determine the extent to which expectations about the real rate of return of investment adjust after the observed increase in the TFP level z_t. For a given volatility in hours worked and in output (determined by χ), "optimistic" expectations on R_{t+1} translate into larger increases in investment and smaller increases in consumption, and vice-versa. Thus, sunspot shocks critically affect the relative volatilities of consumption and investment with respect to output. As such, they are a key ingredient for solving the consumption volatility puzzle.

4.3 Simulation results

Armed with the considerations above, we can now turn to data confrontation. Our qualitative discussion suggests that the ability of the model to account for observed stylized facts depends both on the values ascribed to structural parameters (in particular, the wage-elasticity of labor supply) and on the properties of the sunspot shocks (are they an autonomous source of disturbances or do they contribute to amplify the effects of real shocks on the economy?). In order to make more quantitative assessments, we now assume that the TFP level evolves according to:

$$\ln z_{t+1} = \rho \ln z_t + \varepsilon_{t+1} \tag{26}$$

with autoregressive coefficient $\rho \in (0,1)$ and ε_{t+1} a normally distributed technological shock with standard deviation σ_ε. The resulting 3-dimensional dynamic system obtained from log-linearizing (23)-(24) and (26) around the deterministic steady state with $\bar{z} = 1$ can be written as:

$$\begin{pmatrix} \widehat{c}_{t+1} \\ \widehat{k}_{t+1} \\ \widehat{z}_{t+1} \end{pmatrix} = \tilde{J} \begin{pmatrix} \widehat{c}_t \\ \widehat{k}_t \\ \widehat{z}_t \end{pmatrix} + \begin{pmatrix} \eta_{t+1} \\ 0 \\ \varepsilon_{t+1} \end{pmatrix} \tag{27}$$

where $\eta_{t+1} \equiv \widehat{c}_{t+1} - E_t\widehat{c}_{t+1}$, the forecast error of consumption, satisfies $E_t\eta_{t+1} = 0$.

As shown in Appendix 6.3, introducing such an exogenous stochastic TFP level does not change the stability properties of the model. Thus, when the steady-state is a *sink*, the three eigenvalues of the matrix $\tilde{J}$ have modulus lower than 1 and the log-linear system (27) is stable.[17] In this case, the forecast error η_t is not pinned down by economic fundamentals and can be a source of business cycles (the *sunspot* shock). Still, theory does not impose restrictions on the correlation between this sunspot shock and the fundamental disturbance ε_t. We assume in this respect that the forecast error satisfies

$$\eta_t = \iota\varepsilon_t + \upsilon_t$$

where ι is a scalar and υ_t is a white noise shock with variance $\sigma_{\upsilon,}$ uncorrelated with ε_t. The "pure" sunspot shock υ_t reflects *extrinsic uncertainty* in the sense of Cass and Shell (1983), while the scalar ι governs the extent to which households' expectations are influenced by technological shocks and attenuate or amplify the effects of these shocks on the economy.

Following Benhabib and Farmer (1996), we can now evaluate the model considering two alternative assumptions regarding the role of sunspots in the economy: (i) sunspot shocks are purely exogenous and are the only source of disturbances: ($\sigma_\upsilon > 0$ and $\sigma_\varepsilon = 0$, see Table 2), and (ii) sunspot shocks are perfectly correlated with technological shocks, and determine the extent to which agents change their expectations in response to these shocks ($\sigma_\upsilon = 0$ and $\sigma_\varepsilon > 0$, see Table 3).

Calibration. In order to make quantitative assessments, a proper calibration of structural parameters is required. The preference parameter B is just a normalization variable which does not influence the second-order properties of the model. Thus, one only needs to ensure that the condition required for an interior steady state is satisfied (see Proposition 1). Following the standard practice in the Real Business Cycle literature, we set $\beta = 0.99$, implying a net annual return on capital of around 4%, $\delta = 0.025$, implying a 10% annual depreciation rate of physical capital, $\alpha = 0.3$, consistent with a labor share in income of 70%, and $\rho = 0.95$, consistent with the large persistence in

[17] Yet, as concerns the *nonlinear* system from which (27) is an approximation, one must ensure that the model remains in the basin of attraction of the stable steady state. This is particularly true for calibrations that imply that the model is near a subcritical Hopf bifurcation in the parameter space, since in this case the basin of attraction is delimited by an invariant closed curve surrounding the steady-state. Thus, strictly speaking, the distribution of shocks must be truncated to avoid that an extreme realization makes endogenous variables leave the stable neighborhood of the steady state.

estimated Solow residuals in the US economy. We calibrate the degree of increasing return to scale to $\Theta = 0.33$, the point estimate obtained by Basu and Fernald [4] for the degree of IRS in the durable manufacturing industry.[18] Based on this calibration, we obtain from (18) that $\underline{\chi} = 0.17$, and we know from Proposition 2 that $\chi > \underline{\chi}$ is required for indeterminacy. We set $\chi = 1/3$, implying a wage elasticity of aggregate labor supply of 3. This value is the one recommended by Rogerson and Wallenius (2009) and Prescott and Wallenius (2011) to calibrate business cycle models, based on both theoretical considerations and cross-country tax analysis.[19] We also obtain from Proposition 2 that indeterminacy occurs when $\sigma \in (\sigma_H, \sigma_F) = (0.17, 0.35)$ or, using (19), when the elasticity of intertemporal substitution in consumption satisfies $\epsilon_{cc} \in (\epsilon_{ccF}, \epsilon_{ccH}) = (0.86, 1.83)$. There is no agreement in the empirical literature about the precise value of this parameter, since most estimates typically vary between 0 and 2.[20] Yet, indeterminacy clearly occurs for a large set of values in this interval. Based on the most recent estimates by Gruber [14], who concluded for values around 2, our choice is to calibrate ϵ_{cc} close to its maximum value consistent with indeterminacy, namely $\epsilon_{cc} = 1.80$. It is worthwhile to emphasize that with this calibration, the model is located near the Hopf bifurcation locus in the parameter space. As emphasized by Dufourt *et al.* [7], models placed in this configuration – having two complex conjugate eigenvalues with modulus close to 1 – are more likely to account for the persistent and non-monotone dynamics of convergence to the steady-state following transitory shocks observed in the data.

Results. Results obtained using this benchmark calibration are reported in Tables 2 and 3 (Table 2 refers to the case of autonomous sunspots, and Table 3 to the case of correlated sunspots). For comparison purposes, we also provide results obtained when the specification of GHH preferences is restricted to be logarithmic in consumption, as in Guo and Harrison [15]. Note that for this latter specification to be consistent with

[18]This point estimate is roughly consistent with those obtained by Harrison [17] on the degree of externality in the investment sector. Using 2-digit data, Harrison [17] obtains a point estimate of 0.41. With 4-digit data, the point estimate is 0.29. Note that the *aggregate* degree of IRS in the model remains very small (equal to 7%), since the share of investment in GDP is 21%.

[19]See Prescott and Wallenius (2011) for a discussion of the factors that make the wage elasticity of aggregate labor supply significantly differ from the corresponding elasticity at the micro level.

[20]While early studies suggest quite low values for this elasticity, more recent estimates provide a much more contrasted view. For example, Mulligan [20] repeatedly obtained estimates above unity – typically in the range $1.1-2.1$ –, and Vissing-Jorgensen and Attanasio [26] found estimates significantly above 1 using the Epstein-Zin preference specification, which enables to separately estimate the IES coefficient from the degree of risk aversion. Gruber [14] also provides robust estimates of this elasticity around 2. In light of all these studies, we assume that a plausible range for ϵ_{cc} is $(0, 2)$.

indeterminacy, the interval (σ_H, σ_F) must now contain 1, which only occurs if the wage elasticity of labor supply is sufficiently low, i.e. if the preference parameter χ is larger than $\chi_F \approx 5.5$.[21] To maximize the model's chances evaluated in this configuration, we calibrate χ to a value close to this minimum value consistent with indeterminacy, namely: $\chi = 6$.

The statistics reported in Tables 2 and 3 are the averaged second-order moments obtained by simulating 200 series of 202 observations each (the length of our dataset), and after detrending all series using the HP filter.[22] As clearly appears, and for the reasons explained in subsection 4.2., the model performs poorly when sunspot shocks are assumed to be an autonomous source of the business cycle (see Table 2). In particular, it exhibits many of the counterfactual predictions associated with two-sector indeterminate RBC models: consumption and investment are much more volatile than output (the consumption volatility puzzle), consumption and investment move in opposite direction, leading to a negative correlation coefficient between investment and output (the consumption/investment cyclicality puzzle), sectoral hours worked also move in opposite direction, leading to a negative correlation between them (the sectoral labor volatility puzzle). Furthermore, in the logarithmic consumption case, the relative volatility of hours worked with output, $1/(1+\chi) = 1/7 \approx 0.14$, is extensively underestimated (in the case of generalized GHH preferences, this elasticity, albeit too low, is much closer to its empirical counterpart: $1/(1+\chi) = 1/(1+1/3) = 0.75$). More generally, although these inaccurate predictions hold for both specifications of individual preferences, they are clearly worse when the utility function is restricted to be logarithmic in consumption. As explained above, the reason is that, when the elasticity of labor supply is low, the reallocation of resources between the investment and the consumption sector occurs along a constant (in the shock period) and then weakly volatile output level. As a result, all the above mentioned puzzles are significantly magnified in this configuration.

When sunspots are assumed to be perfectly correlated with technological shocks, things are substantially different. Now, the outward shift in labor demand generated by positive technological shocks implies that consumption is substituted for investment along an *increasing* output level. The extent to which output increases in response to these shocks depends again on the wage elasticity of labor supply, while the extent to which consumers allocate this extra production between consumption and investment

[21] As shown in Guo and Harrison [15], if the degree of increasing returns to scale in the investment sector is calibrated to $\Theta = 0.3$ (instead of $\Theta = 0.33$ retained here), the minimum value for χ consistent with indeterminacy jumps to $\chi_F \approx 15$.

[22] Note that the results in the logarithmic consumption case differ from those reported by Guo and Harrison [15] because their calibration is slightly different and they did not filter their simulated data.

	Data	Generalized preferences	Logarithmic preferences
Rel. std. dev.:			
Cons. (C)	0.47	3.57	12.41
Inv. (pI)	2.35	12.17	44.50
Hours ($L_C + L_I$)	1.22	0.75	0.14
Hours_cons (L_C)	0.86	3.48	12.18
Hours_inv. (L_I)	2.38	12.18	44.68
Corr. with output			
Cons. (C)	0.87	0.38	0.30
Inv. (pI)	0.98	-0.02	-0.20
Hours ($L_C + L_I$)	0.89	1	1
Hours_cons. (L_C)	0.87	0.31	0.24
Hours_inv. (L_I)	0.88	-0.04	-0.22
Corr. (L_C, L_I)	0.96	-0.96	-0.99

Table 2: Model properties, autonomous sunspots

depends on how their expectations about the real return of investment are affected in this new economic environment (i.e., on the sunspot shock). The unobserved parameter ι is the key parameter in this dimension: a small value for ι implies a small initial response of consumption and a large initial response of investment, and vice versa. Thus, when ι is calibrated in a proper range, the model can generate a correct *initial* ordering in the relative amplitudes of the responses of consumption, investment and output, which is a crucial step toward solving the consumption-volatility puzzle.[23] In the model with generalized GHH preferences, the range of values for which the initial response of consumption is procyclical but smaller than the initial response of output is $\iota \in (0, 2.1)$. In the model with logarithmic-consumption GHH preferences, the corresponding interval is $\iota \in (0, 1.1)$. To make things comparable, we calibrate ι so as to imply, in both versions of the model, that a 1% positive technological shock leads to a 1% drop in the relative price of investment, p_t. Results are reported in Table 3.

As can be observed, the model with generalized GHH preferences is now relatively successful, as it solves all the empirical puzzles traditionally associated with 2 sector

[23]Yet, the ability of the model to actually solve this puzzle depends on the various internal transmission mechanisms (elasticity of labor supply, elasticity of intertemporal substitution in consumption, etc.) at work in the economy.

	Data	Generalized preferences	Logarithmic preferences
<u>Rel. std. dev.</u>:			
Cons. (C)	0.47	0.80	3.42
Inv. (pI)	2.35	2.30	14.94
Hours ($L_C + L_I$)	1.22	0.75	0.14
Hours_cons (L_C)	0.86	0.59	3.81
Hours_inv (L_I)	2.38	2.09	14.43
<u>Corr. with output</u>			
Cons. (C)	0.87	0.92	-0.36
Inv. (pI)	0.98	0.87	0.61
Hours: ($L_C + L_I$)	0.89	1	1
Hours_cons (L_C)	0.87	0.83	-0.55
Hours_inv (L_I)	0.88	0.82	0.58
Corr. (L_C, L_I)	0.96	0.39	-0.99

Table 3: Model properties, correlated sunspots.

RBC models under indeterminacy. The relative volatilities of consumption, investment and output have the correct ordering and are relatively close to their empirical counterparts. The contemporaneous cross-correlations of consumption and investment with output have the correct sign, and so does have the correlation between sectoral hours worked (even though this correlation remains too low compared to the data). The relative volatility of total hours worked remains unchanged at 0.75. By contrast, the model with $\sigma = 1$ remains clearly unsatisfactory, for the same reasons as those described above: although technological shocks imply an outward shift in labor demand, the low elasticity of labor supply implies that variations in total hours worked and output are negligible. The model is then unable to solve the empirical puzzles. This clearly shows that the generalized specification of GHH preferences is the only one that can improve the model's predictions in these dimensions.

5 Concluding comments

Although multi-sector Real Business Cycle models are based on a pervasive feature of the data and require smaller degrees of increasing returns to scale for indeterminacy than aggregate models, they are usually associated with several empirical shortcomings:

the labor comovement and consumption-investment cyclicality puzzles, and the consumption and hours worked volatility puzzles. Many contributions have tried to solve these shortcomings and the traditional strategy is to increase the degree of increasing returns. The main contribution of this paper is to show that a strong interaction between the theoretical and the empirical analysis of the model allows to significantly improve its empirical predictions. This is obtained by finding better compromises between the various economic mechanisms important in both the local stability and business cycle properties of the model.

Considering generalized GHH preferences, we have provided a detailed theoretical analysis of the local stability properties of the steady state in order to get a full picture of the configurations giving rise to local indeterminacy and sunspot fluctuations. We have shown that local indeterminacy occurs through flip and Hopf bifurcations for a large set of values of the EIS in consumption provided that the labor supply is sufficiently inelastic. Moreover, the existence of expectations-driven fluctuations is consistent with a mild amount of increasing returns.

Building on this detailed theoretical analysis, we have been able to find, among the set of admissible parameter configurations consistent with sunspot equilibria, the ones that provide the best fit of the data. We have shown that a properly calibrated model is able to solve most empirical puzzles traditionally associated with two-sector RBC models.

6 Appendix

6.1 Proof of Proposition 1

Without loss of generality, let $z_t = 1$. Equation (11) evaluated at the steady state gives $\bar{r}/\bar{p} = (1-\theta)/\beta$, with $\theta = \beta(1-\delta)$. We also derive from (4) that $\bar{r}/\bar{p} = \alpha\delta\bar{k}/\bar{K}_I$ and thus $\bar{K}_I/\bar{k} = \beta\delta\alpha/(1-\theta)$. From (4)–(5) and (16)–(17) , we derive that $\overline{K}_I/\overline{L}_I = \overline{K}_c/\overline{L}_c = \overline{k}/\overline{l}$ and thus that $\overline{L}_I/\bar{l} = \bar{K}_I/\bar{k} = \beta\delta\alpha/(1-\theta)$. Observing from (9) and (13) that $\delta = \bar{\imath}/\bar{k} = \overline{a}^{\alpha(1+\Theta)-1}\left(\overline{L}_I/\bar{l}\right)\overline{L}_I^{\Theta}$, we can express the capital-labor ratio as

$$\bar{a} \equiv \frac{\bar{k}}{\bar{l}} = \delta^{\frac{\Theta}{1-\alpha(1+\Theta)}}\left(\frac{\beta\alpha}{1-\theta}\right)^{\frac{1+\Theta}{1-\alpha(1+\Theta)}}\bar{l}^{\frac{\Theta}{1-\alpha(1+\Theta)}} \tag{28}$$

assuming of course that $\Theta \neq \bar{\Theta} \equiv (1-\alpha)/\alpha$.

Combining now the labor demand equation $\bar{\omega} = (1-\alpha)\bar{a}^{\alpha}$ (with $\bar{a}$ given by (28)) and the labor supply equation (12), we obtain the steady-state level of hours worked:

$$\bar{l} = \left(\frac{1-\alpha}{B}\right)^{\frac{1-\alpha(1+\Theta)}{\chi[1-\alpha(1+\Theta)]-\alpha\Theta}}\delta^{\frac{\alpha\Theta}{\chi[1-\alpha(1+\Theta)]-\alpha\Theta}}\left(\frac{\beta\alpha}{1-\theta}\right)^{\frac{\alpha(1+\Theta)}{\chi[1-\alpha(1+\Theta)]-\alpha\Theta}} \tag{29}$$

Under $\Theta \neq \bar{\Theta} \equiv (1-\alpha)/\alpha$, let $\underline{\chi} \equiv \alpha\Theta/(1-\alpha(1+\Theta))$. An interior steady-state in which (12) holds also requires $\bar{l} < \ell$, or equivalently:

$$B > (1-\alpha)\,\delta^{\underline{\chi}}\left(\frac{\beta\alpha}{1-\theta}\right)^{\frac{\underline{\chi}(1+\Theta)}{\Theta}}\ell^{\chi-\underline{\chi}} \equiv \hat{B} \text{ when } \chi > \underline{\chi}$$

or

$$B < \hat{B} \text{ when } \chi < \underline{\chi}$$

Once $\bar{l}$ is obtained, it is straightforward to derive the steady-state values of all other endogenous variables. For example, using $\bar{k} = \bar{a}\bar{l}$, we get

$$\bar{k} = \delta^{\frac{\Theta}{1-\alpha(1+\Theta)}}\left(\frac{\beta\alpha}{1-\theta}\right)^{\frac{1+\Theta}{1-\alpha(1+\Theta)}}\bar{l}^{\frac{(1-\alpha)(1+\Theta)}{1-\alpha(1+\Theta)}} \tag{30}$$

Likewise, we derive from (1) that $\bar{c} = \bar{a}^\alpha \overline{L}_c = \bar{a}^\alpha(L - \overline{L}_I) = \bar{a}^\alpha(1-\beta\delta\alpha/(1-\theta))\bar{l}$, and thus:

$$\bar{c} \;=\; \left(1-\frac{\beta\delta\alpha}{1-\theta}\right)\delta^{\frac{\alpha\Theta}{1-\alpha(1+\Theta)}}\left(\frac{\beta\alpha}{1-\theta}\right)^{\frac{\alpha(1+\Theta)}{1-\alpha(1+\Theta)}}\bar{l}^{\frac{1-\alpha}{1-\alpha(1+\Theta)}} \tag{31}$$

□

6.2 Proof of Proposition 2

Let us introduce the following elasticities:

$$\epsilon_{cc} = -\frac{U_1(c,l)}{U_{11}(c,l)c}, \quad \epsilon_{lc} = -\frac{U_2(c,l)}{U_{21}(c,l)c}, \quad \epsilon_{cl} = -\frac{U_1(c,l)}{U_{12}(c,l)l}, \quad \epsilon_{ll} = -\frac{U_2(c,l)}{U_{22}(c,l)l} \tag{32}$$

We get at the steady state

$$\epsilon_{cc} = \epsilon_{lc}, \;\; \frac{1}{\epsilon_{ll}} - \frac{1}{\epsilon_{cl}} = -\chi < 0 \text{ with } \epsilon_{cc} = \frac{1}{\sigma}\left(1 - \frac{1-\alpha}{(1+\chi)\left(1-\frac{\beta\delta\alpha}{1-\theta}\right)}\right) \tag{33}$$

and

$$\epsilon_{cl} = -\epsilon_{cc}\frac{\left(1-\frac{\beta\alpha\delta}{1-\theta}\right)}{1-\alpha} = \frac{1}{\sigma(1-\alpha)}\left[\frac{1-\alpha}{1+\chi} - \left(1-\frac{\beta\delta\alpha}{1-\theta}\right)\right] \tag{34}$$

Considering the first-order condition (11), the households' budget constraint (8), the capital accumulation equation (9) and the equilibrium value of labor as given by (20) with $z_t = 1$, we get the following system of three equations

$$\begin{aligned}
\left(c_t - \frac{Bl_t^{1+\chi}}{1+\chi}\right)^{-\sigma} &= \beta\left(c_{t+1} - \frac{Bl_{t+1}^{1+\chi}}{1+\chi}\right)^{-\sigma}\left[\frac{r_{t+1}+(1-\delta)p_{t+1}}{p_t}\right] \\
l_t &= \left(\frac{1-\alpha}{B}\right)^{\frac{1}{\alpha+\chi}} k_t^{\frac{\alpha}{\alpha+\chi}} \\
c_t + p_t[k_{t+1} - (1-\delta)k_t] &= r_t k_t + \omega_t l_t
\end{aligned}$$

Using the price equations at the equilibrium (21)-(22), total differentiation of this system in a neighborhood of the steady state gives after simplifications:

$$\begin{aligned}
&\begin{pmatrix} \frac{1}{\epsilon_{cc}} - \frac{(1-\delta)(1-\theta)\Theta\left(1-\frac{\beta\alpha\delta}{1-\theta}\right)}{\delta\alpha} & \frac{\frac{\alpha}{\epsilon_{cl}} + (1-\alpha)(1-\theta)\chi + \frac{(1-\theta)(1-\delta)\Theta(1+\chi)}{\delta}}{\chi+\alpha} \\ 0 & \frac{\beta\alpha}{(1-\theta)(1+\Theta)} \end{pmatrix}\begin{pmatrix} \widehat{c}_{t+1} \\ \widehat{k}_{t+1} \end{pmatrix} \\
= &\begin{pmatrix} \frac{1}{\epsilon_{cc}} - \frac{(1-\theta)\Theta\left(1-\frac{\beta\alpha\delta}{1-\theta}\right)}{\beta\delta\alpha} & \frac{\frac{\alpha}{\epsilon_{cl}} + \frac{(1-\theta)\Theta(1+\chi)}{\beta\delta}}{\chi+\alpha} \\ -\left(1-\frac{\beta\delta\alpha}{1-\theta}\right) & \alpha\left[\frac{1+\chi}{\chi+\alpha} + \frac{(1-\delta)\beta}{(1-\theta)(1+\Theta)}\right] \end{pmatrix}\begin{pmatrix} \widehat{c}_t \\ \widehat{k}_t \end{pmatrix}
\end{aligned} \tag{35}$$

$$\Leftrightarrow \;\; M\begin{pmatrix} \widehat{c}_{t+1} \\ \widehat{k}_{t+1} \end{pmatrix} = N\begin{pmatrix} \widehat{c}_t \\ \widehat{k}_t \end{pmatrix}$$

Considering the expression of ϵ_{cc} as given by (33), the matrix M is invertible if and only if

$$\sigma \neq \frac{\Theta(1-\delta)(1-\theta)\left[\left(1-\frac{\beta\delta\alpha}{1-\theta}\right)-\frac{1-\alpha}{1+\chi}\right]}{\delta\alpha} \equiv \sigma_\infty \tag{36}$$

Under this condition, let us denote $J = M^{-1}N$. We then get

$$\begin{pmatrix} \widehat{c}_{t+1} \\ \widehat{k}_{t+1} \end{pmatrix} = J \begin{pmatrix} \widehat{c}_t \\ \widehat{k}_t \end{pmatrix} = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \begin{pmatrix} \widehat{c}_t \\ \widehat{k}_t \end{pmatrix} \tag{37}$$

with

$$\begin{aligned}
J_{11} &= \frac{\frac{1}{\epsilon_{cc}}-\frac{(1-\theta)\Theta\left(1-\frac{\beta\alpha\delta}{1-\theta}\right)}{\beta\delta\alpha}}{\frac{1}{\epsilon_{cc}}-\frac{(1-\theta)\Theta(1-\delta)\left(1-\frac{\beta\alpha\delta}{1-\theta}\right)}{\delta\alpha}} + \frac{\frac{(1-\theta)(1+\Theta)}{\beta\alpha(\chi+\alpha)}\left(1-\frac{\beta\delta\alpha}{1-\theta}\right)}{\frac{\alpha}{\epsilon_{cl}}+(1-\alpha)(1-\theta)\chi+\frac{(1-\theta)(1-\delta)\Theta(1+\chi)}{\delta}} \\
J_{12} &= \frac{\frac{1}{\chi+\alpha}\left[\frac{\alpha}{\epsilon_{cl}}+\frac{(1-\theta)\Theta(1+\chi)}{\beta\delta}\right]}{\frac{1}{\epsilon_{cc}}-\frac{(1-\theta)\Theta(1-\delta)\left(1-\frac{\beta\alpha\delta}{1-\theta}\right)}{\delta\alpha}} - \frac{\frac{(1-\theta)(1+\Theta)}{\beta(\chi+\alpha)}\left[\frac{1+\chi}{\chi+\alpha}+\frac{(1-\delta)\beta}{(1-\theta)(1+\Theta)}\right]}{\frac{\alpha}{\epsilon_{cl}}+(1-\alpha)(1-\theta)\chi+\frac{(1-\theta)(1-\delta)\Theta(1+\chi)}{\delta}} \\
J_{21} &= -\frac{(1-\theta)(1+\Theta)}{\beta\alpha}\left(1-\frac{\beta\delta\alpha}{1-\theta}\right), \; J_{22} = \frac{(1-\theta)(1+\Theta)}{\beta}\left[\frac{1+\chi}{\chi+\alpha}+\frac{(1-\delta)\beta}{(1-\theta)(1+\Theta)}\right]
\end{aligned} \tag{38}$$

Using (34), we then derive after simplifications the characteristic polynomial $P_{\epsilon_{cc}}(\lambda) = \lambda^2 - \mathcal{T}\lambda + \mathcal{D}$ with

$$\begin{aligned}
\mathcal{D} = \mathcal{D}_\chi(\sigma) &= \frac{(1-\theta)\delta\left\{\sigma\alpha\left(\frac{1+\Theta(1-\theta)}{1-\theta}\right)-\frac{1-\delta}{\delta}\Theta\left[\left(1-\frac{\beta\delta\alpha}{1-\theta}\right)-\frac{1-\alpha}{1+\chi}\right]\right\}}{\beta\left\{\sigma\delta\alpha-(1-\delta)(1-\theta)\Theta\left[\left(1-\frac{\beta\delta\alpha}{1-\theta}\right)-\frac{1-\alpha}{1+\chi}\right]\right\}} \\
\mathcal{T} = \mathcal{T}_\chi(\sigma) &= 1+\mathcal{D}_\chi(\sigma)+\frac{(1-\theta)^2\delta[\chi(1-\alpha(1+\Theta))-\alpha\Theta]\left[\left(1-\frac{\beta\delta\alpha}{1-\theta}\right)-\frac{1-\alpha}{1+\chi}\right]}{\beta(\chi+\alpha)\left\{\sigma\delta\alpha-(1-\delta)(1-\theta)\Theta\left[\left(1-\frac{\beta\delta\alpha}{1-\theta}\right)-\frac{1-\alpha}{1+\chi}\right]\right\}}
\end{aligned} \tag{39}$$

We conclude that when σ is varied over the interval $[0, +\infty)$, $\mathcal{D}$ and $\mathcal{T}$ are linked through a linear relationship $\mathcal{D} = \Delta_\chi(\mathcal{T}) = \mathcal{T}\mathcal{S}_\chi + \mathcal{C}$ with a slope

$$\mathcal{S}_\chi = \frac{(\chi+\alpha)(1-\delta)\Theta^2}{\chi\delta(1-\alpha)(1+\Theta)-(\chi+\alpha)\Theta[1-(1-\delta)(1+\Theta)]}$$

In other words, $\Delta_\chi(\mathcal{T})$ corresponds to a half-line in the $(\mathcal{T}, \mathcal{D})$ plane with a starting point $(\mathcal{T}_\chi(+\infty), \mathcal{D}_\chi(+\infty))$ obtained when $\sigma = +\infty$ such that:

$$\begin{aligned}
\mathcal{D}_\chi(+\infty) &= \frac{(1-\theta)(1+\Theta)+\theta}{\beta} = \mathcal{D}(+\infty) \\
\mathcal{T}_\chi(+\infty) &= \frac{(1-\theta)(1+\Theta)+(\theta+\beta)}{\beta} = \mathcal{T}(+\infty) = 1+\mathcal{D}(+\infty)
\end{aligned}$$

It follows that when $\sigma = +\infty$, $P_{+\infty}(1) = 0$ and $P_{+\infty}(-1) = 2(1+\mathcal{D}(+\infty))$. Note also that $\mathcal{D}(+\infty) > 1$.

When ϵ_{cc} increases the $\Delta_\chi(\mathcal{T})$ half-line crosses the triangle ABC depending on the slope and on the location of the end-point $(\mathcal{T}_\chi(0), \mathcal{D}_\chi(0))$ obtained when $\sigma = 0$. Assume from now on that $\alpha < 1/2$ and $\Theta \in (\underline{\Theta}, \bar{\Theta})$ with $\underline{\Theta} \equiv \delta/(1-\delta)$ and $\bar{\Theta} = (1-\alpha)/\alpha$. It follows that the slope satisfies $\mathcal{S}_\chi > 0$ for any χ. Moreover, we have $\mathcal{S}_\chi \leq 1$ if and only if $\chi \geq \alpha\Theta/[1-\alpha(1+\Theta)] \equiv \underline{\chi}$. Notice also that $\partial\mathcal{S}_\chi/\partial\chi < 0$ with $\lim_{\chi\to+\infty}\mathcal{S}_\chi = (1-\delta)\Theta^2/\left[(1-\delta)\Theta^2+(1-\alpha(1+\Theta))\delta\right] < 1$ and $\partial\mathcal{D}_\chi(\sigma)/\partial\sigma > 0$. We have indeed:

$$\begin{aligned}\mathcal{D}_\chi(0) = \lim_{\sigma\to 0}\mathcal{D}_\chi(\sigma) &= \tfrac{1}{\beta} = \mathcal{D}(0)\\ \mathcal{T}_\chi(0) = \lim_{\sigma\to 0}\mathcal{T}_\chi(\sigma) &= 1+\tfrac{1}{\beta}-\delta(1-\theta)\tfrac{\chi(1-\alpha(1+\Theta))-\alpha\Theta}{(\chi+\alpha)\theta\Theta}\end{aligned}$$

We then derive that when $\sigma = 0$

$$\begin{aligned}P_0(1) &= \tfrac{(1-\theta)\delta[\chi(1-\alpha(1+\Theta))-\alpha\Theta]}{(\chi+\alpha)\theta\Theta}\\ P_0(-1) &= \tfrac{\chi\{\Theta[2(1-\theta(1-\delta))+\delta(1-\theta)\alpha]-\delta(1-\theta)(1-\alpha)\}+\alpha(1+\theta)(2-\delta)\Theta}{(\chi+\alpha)\theta\Theta}\end{aligned}$$

It follows that $P_0(1) > 0$ if and only if $\chi > \underline{\chi}$. Moreover, it can be easily shown that when $\Theta > \underline{\Theta}$, $P_0(-1) > 0$ for any $\chi \geq 0$.

The Hopf bifurcation value $\bar{\sigma}^H > 0$ such that $\mathcal{D}_\chi(\sigma) = 1$ is given by

$$\bar{\sigma}^H = \tfrac{\Theta(1-\delta)(1-\beta)(1-\theta)\left[\left(1-\frac{\beta\delta\alpha}{1-\theta}\right)-\frac{1-\alpha}{1+\chi}\right]}{\delta\alpha[1-\beta+(1-\theta)\Theta]}$$

Similarly, the flip bifurcation value $\bar{\sigma}^F > 0$ such that $P_\sigma(-1) = 1 + \mathcal{T}_\chi(\sigma) + \mathcal{D}_\chi(\sigma) = 0$ is given by

$$\bar{\sigma}^F = \tfrac{(1-\theta)\left[(\chi+\alpha)\Theta[2(1-\delta)(1+\beta)+(1-\theta)\delta]-(1-\theta)\delta\chi(1-\alpha)(1+\Theta)\right]\left[\left(1-\frac{\beta\delta\alpha}{1-\theta}\right)-\frac{1-\alpha}{1+\chi}\right]}{2(\chi+\alpha)\delta\alpha[1+\beta+(1-\theta)\Theta]}$$

We easily derive that

$$\lim_{\chi\to+\infty}\mathcal{T}_\chi(\bar{\sigma}^H) = 2-\tfrac{\delta[1-\beta+(1-\theta)\Theta](1-\alpha(1+\Theta))}{\theta\Theta^2} \leq 2$$

It follows that $\lim_{\chi\to+\infty}\mathcal{T}_\chi(\bar{\sigma}^H) > -2$ if and only if

$$\begin{aligned}h(\delta) &\equiv \delta^2\beta\Theta(1-\alpha(1+\Theta)) + \delta\left[(1-\beta)(1+\Theta)(1-\alpha(1+\Theta)) + 4\beta\Theta^2\right]\\ &- 4\beta\Theta^2 < 0\end{aligned}$$

Let $\Omega = \left[(1-\beta)(1+\Theta)(1-\alpha(1+\Theta)) + 4\beta\Theta^2\right]^2 + 4\beta^2\Theta^3(1-\alpha(1+\Theta))$. Therefore, there exists $\bar{\delta} \in (0,1)$ as given by

$$\bar{\delta} = \tfrac{\sqrt{\Omega}-\left[(1-\beta)(1+\Theta)(1-\alpha(1+\Theta))+4\beta\Theta^2\right]}{2\beta\Theta(1-\alpha(1+\Theta))} \tag{40}$$

such that when $\delta \in (0,\bar{\delta})$, $\lim_{\chi\to+\infty}\mathcal{T}_\chi(\bar{\sigma}^H) \in (-2,2)$. Moreover, we have $\partial\mathcal{T}_\chi(\bar{\sigma}^H)/\partial\chi < 0$ and $\mathcal{T}_{\underline{\chi}}(\bar{\sigma}^H) = 2$, so that $\mathcal{T}_\chi(\bar{\sigma}^H) \in (-2,2)$ for any $\chi > \underline{\chi}$. Therefore, the Δ_χ line, when $\chi > \underline{\chi}$, is located as in Figure 1. On the contrary, when $\chi \in [0,\underline{\chi})$, the Δ_χ line is located as in Figure 2 and the steady state is either saddle-point stable or unstable. It is worth noting that when σ crosses the singular value σ_∞ for which the matrix M is non-invertible, $\mathcal{D}$ changes its sign as it crosses infinity but in the neighborhood of σ_∞ the steady state remains locally unstable or saddle-point stable depending on whether $\chi > \underline{\chi}$ or $\chi < \underline{\chi}$.

□

6.3 The three-dimensional stochastic dynamic system

Consider the stochastic dynamic system as given by (23)-(24) and (26). Log-linearizing this system around the deterministic steady state with $\bar{z} = 1$, and denoting $\eta_{t+1} = \widehat{c}_{t+1} - E_t\widehat{c}_{t+1}$, $\varepsilon_{t+1} = \widehat{z}_{t+1} - E_t\widehat{z}_{t+1}$, with $E_t\eta_{t+1} = E_t\varepsilon_{t+1} = 0$, we derive:

$$\tilde{M}\begin{pmatrix} \widehat{c}_{t+1} \\ \widehat{k}_{t+1} \\ \widehat{z}_{t+1} \end{pmatrix} = \tilde{N}\begin{pmatrix} \widehat{c}_{t} \\ \widehat{k}_{t} \\ \widehat{z}_{t} \end{pmatrix} + \tilde{M}\begin{pmatrix} \eta_{t+1} \\ 0 \\ \varepsilon_{t+1} \end{pmatrix}$$

We obviously have

$$\tilde{M} = \begin{pmatrix} \multicolumn{2}{c}{\multirow{2}{*}{M}} & \tilde{M}_{13} \\ & & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \tilde{N} = \begin{pmatrix} \multicolumn{2}{c}{\multirow{2}{*}{N}} & \tilde{N}_{13} \\ & & \tilde{N}_{23} \\ 0 & 0 & \rho \end{pmatrix}$$

with M, N as given by (35) and where

$$\begin{aligned}
\tilde{M}_{13} &= \frac{\bar{p}}{\beta}\left(\bar{c} - \frac{B\bar{l}^{1+\chi}}{1+\chi}\right)^{-1-\sigma}\left\{\left(\bar{c} - \frac{B\bar{l}^{1+\chi}}{1+\chi}\right)\left[\frac{(1-\alpha)(1-\theta)}{\alpha+\chi} - \beta(1-\delta)\Theta\right] + \sigma\frac{B\bar{l}^{1+\chi}}{1+\chi}\right\} \\
\tilde{N}_{13} &= \frac{\bar{p}}{\beta}\left(\bar{c} - \frac{B\bar{l}^{1+\chi}}{1+\chi}\right)^{-1-\sigma}\left\{\sigma\frac{B\bar{l}^{1+\chi}}{1+\chi} - \Theta\left(\bar{c} - \frac{B\bar{l}^{1+\chi}}{1+\chi}\right)\right\} \\
\tilde{N}_{23} &= \frac{\bar{y}}{\bar{p}}\left(\frac{1+\chi}{\alpha+\chi} + \Theta\right)
\end{aligned}$$

Assuming that (36) holds, the matrix $\tilde{M}$ is invertible with $|\tilde{M}| = |M|$ and we get

$$\begin{pmatrix} \widehat{c}_{t+1} \\ \widehat{k}_{t+1} \\ \widehat{z}_{t+1} \end{pmatrix} = \tilde{J}\begin{pmatrix} \widehat{c}_{t} \\ \widehat{k}_{t} \\ \widehat{z}_{t} \end{pmatrix} + \begin{pmatrix} \eta_{t+1} \\ 0 \\ \varepsilon_{t+1} \end{pmatrix} \tag{41}$$

with

$$\tilde{J} \equiv \tilde{M}^{-1}\tilde{N} = \begin{pmatrix} \multicolumn{2}{c}{\multirow{2}{*}{J}} & \tilde{J}_{13} \\ & & \tilde{J}_{23} \\ 0 & 0 & \rho \end{pmatrix}$$

and J as given by (37). Obvious computations then give

$$\mathcal{D}(\tilde{J}) = \rho\mathcal{D},\ \mathcal{T}(\tilde{J}) = \rho + \mathcal{T} \text{ and } \mathcal{S}(\tilde{J}) = \mathcal{D} + \rho\mathcal{T}$$

with $\mathcal{D}(\tilde{J})$, $\mathcal{T}(\tilde{J})$, $\mathcal{S}(\tilde{J})$ respectively the determinant, trace and sum of principal minors of $\tilde{J}$, and $\mathcal{D}, \mathcal{T}$ as given by (39). It follows therefore that the degree-three characteristic polynomial associated with (41) is $\tilde{P}(\lambda) = (\lambda - \rho)P_{\epsilon_{cc}}(\lambda)$, so that two eigenvalues of matrix $\tilde{J}$ are the same as those of the 2-dimensional dynamic system described by the matrix J, while the third eigenvalue is equal to ρ.

□

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