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Sunspot Fluctuations in Two-Sector Models: New Results with Additively-Separable Preferences

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WP 2015 - Nr 15

Sunspot fluctuations in two-sector models: new results with additively-separable preferences*

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First version: *November 2009*; Revised: *February 2015*

Abstract: *We analyze local indeterminacy and sunspot-driven fluctuations in the standard two-sector model with additively separable preferences. We provide a detailed theoretical analysis enabling us to derive relevant bifurcation loci and to characterize the steady-state local stability properties as a function of various structural parameters influencing the degree of increasing returns to scale, the amount of intertemporal substitution in consumption, and the elasticity of the aggregate labor supply curve. On the theoretical side, we prove the existence of both a flip and a Hopf bifurcation locus in the corresponding parameter space. We also show that local indeterminacy can be obtained under any labor supply elasticity or under an arbitrarily low elasticity of intertemporal substitution in consumption. On the empirical side, we find that indeterminacy and sunspot fluctuations are robust features of two-sector models, prevailing for most empirically plausible calibrations for these parameters.*

Keywords: *Indeterminacy, sunspots, two-sector model, sector-specific externalities, real business cycles*

Journal of Economic Literature Classification Numbers: C62, E32, O41.

*We thank H. d'Albis, S. Bosi, R. Boucekkine, R. Dos Santos Ferreira, J.P. Drugeon, J.M. Grandmont, C. Le Van, T. Lloyd-Braga, L. Modesto, C. Nourry, T. Seegmuller and Y. Vailakis for useful comments and suggestions.

1 Introduction

The literature on one-sector Real Business Cycle models with productive externalities and increasing returns to scale (IRS) offers today a relatively exhaustive picture of the conditions required for the existence of local indeterminacy and sunspot fluctuations.[1] Local indeterminacy typically requires a large enough elasticity of intertemporal substitution in consumption (EIS), a large enough degree of increasing returns to scale (IRS), and a large enough elasticity of aggregate labor supply. These conditions obviously interact together: for a given labor supply elasticity, a lower degree of IRS must be combined with a larger EIS in consumption in order to obtain indeterminacy, and vice-versa. Likewise, for any given degree of IRS, a lower EIS in consumption must be combined with a larger elasticity of aggregate labor supply for indeterminacy to prevail. Despite these tradeoffs in the relative intensities of these economic mechanisms, a standard conclusion from one sector models is that indeterminacy hardly occurs for empirically plausible calibrations of the parameters unless other features such as a variable capital utilization rate are introduced (Wen [25], Benhabib and Wen [5]).

One noticeable feature of two-sector Real Business Cycle models is that local indeterminacy typically requires much lower degrees of IRS than their one-sector equivalents. This is well known from the canonical two-sector model of Benhabib and Farmer [3] – featuring a separable utility function with a unitary EIS – in which only 7% of IRS are required for indeterminacy compared to about 50% in the corresponding one-sector model of Benhabib and Farmer [2]. However, the literature on two-sector models is far from being as exhaustive on the required combinations in terms of labor supply elasticity, intertemporal substitution effects and externalities consistent with indeterminacy as the literature on one-sector models is. Actually, many of the results obtained in two-sector models have been derived under relatively narrow specifications for technology and/or preference, without much systematic analysis of the interplays between the relevant underlying economic mechanisms, and often through numerical simulations.[2]

Our aim in this paper is to contribute to fill this gap by providing an exten-

[1]See among others Benhabib and Farmer [2], Lloyd-Braga *et al.* [16], Nishimura *et al.* [18], Pintus [20], Wen [25].

[2]For example, in Benhabib and Farmer [3], the utility function is restricted to be logarithmic in consumption (unitary EIS). In Harrison [13], a more general utility function for consumption is considered, but the analysis is restricted to the case of an infinitely elastic labor supply. These analyses thus do not cover the set of empirically credible calibrations for these parameters.

sive analysis of the local stability properties of two-sector optimal growth model, considering a fairly general class of additively separable preferences.[3] Starting from the Benhabib and Farmer [3]'s formulation with increasing social returns to scale but considering a more general additively separable utility function, we analyze the interplays between the degree of IRS, the EIS in consumption and the labor supply elasticity in the emergence of local indeterminacy.

Assuming in a first step a sufficiently elastic labor supply, which includes the range of empirically credible values for this elasticity, we prove that local indeterminacy occurs quite generally, in particular for *arbitrarily low EIS in consumption,* provided that the degree of increasing social returns is larger than some (empirically plausible) lower bound. This conclusion is drastically different from what is known from the previous literature, in which a large enough EIS was always assumed in order to get indeterminacy with empirically plausible amounts of externalities (Garnier *et al.* [8, 9], Harrison [13]). We show that changes in the local stability properties of the model occur through both flip and Hopf bifurcations, and we provide the analytical expressions for these bifurcation values. We also prove that local indeterminacy occurs *no matter how elastic or inelastic the labor supply is*, provided the EIS in consumption and the amount of externalities are in an *intermediary range* still compatible with empirically relevant values. As a result, we show that indeterminacy and sunspot driven fluctuations can occur under a wide range of empirically credible calibrations for all the structural parameters.

The rest of this paper is organized as follows. We present the model and we characterize the intertemporal equilibrium and the steady state in the next Section. In Section 3, the complete set of conditions for indeterminacy are derived and some numerical illustrations are provided. Some concluding remarks are stated in Section 4, whereas all the technical details are given in an Appendix.

2 The model

We consider a standard infinite-horizon two-sector real business-cycle model à la Benhabib and Farmer [3], with productive externalities in the investment sector.[4]

[3]For a thourough analysis of two-sector model with GHH preferences – with no-income effects on labor supply – see Dufourt *et al.* [7].

[4]As is well-known, externalities in the consumption sector tend to increase the aggregate degree of IRS required for indeterminacy Moreover, a constant returns to scale technology in the consumption sector and an increasing returns to scale technology in the investment sector are consistent with the empirical findings of Basu and Fernald [1] and Harrison [14].

2.1 Production

The economy produces a consumption good, c, and an investment good, I, with constant returns to scale Cobb-Douglas technologies at the private level in both sectors and output externalities in the investment sector only. We denote by Y_c and Y_I the outputs of sectors c and I, and by A the external effects. The *production functions at the private level* are thus:

$$Y_{ct} = K_{ct}^{\alpha} L_{ct}^{1-\alpha}, \; Y_{It} = A_t K_{It}^{\alpha} L_{It}^{1-\alpha} \tag{1}$$

where K_{ct} and L_{ct} are capital and labor units allocated to the consumption sector, and K_{It} and L_{It} are capital and labor units allocated to the investment sector. The externality parameter A_t depends on $\bar{K}_{I,t}$ and $\bar{L}_{I,t}$, the average levels of capital and labor in sector I, such that

$$A_t = \bar{K}_{It}^{\alpha\Theta} \bar{L}_{It}^{(1-\alpha)\Theta} \tag{2}$$

with $\Theta \geq 0$. These economy-wide averages are taken as given by individual firms. Assuming that factor markets are perfectly competitive and that capital and labor inputs are perfectly mobile across the two sectors, the first order conditions for profit maximization of the representative firm in each sector are

$$r_t = \frac{\alpha Y_{ct}}{K_{ct}} = p_t \frac{\alpha Y_{It}}{K_{It}}, \quad \omega_t = \frac{(1-\alpha)Y_{ct}}{L_{ct}} = p_t \frac{(1-\alpha)Y_{It}}{L_{It}} \tag{3}$$

where r_t, p_t and ω_t are respectively the rental rate of capital, the price the investment good and the real wage rate at time t, all in terms of the price of the consumption good.

2.2 Preferences

We consider an economy populated by a continuum of unit mass of identical infinitely-lived agents. At each period, a representative agent supplies elastically an amount l_t of labor, consumes c_t and invests I_t so as to accumulate capital. He derives current period utility from consumption and labor according to a standard additively separable utility function given by

$$U(c, l) = \frac{c^{1-\sigma}}{1-\sigma} - \frac{l^{1+\chi}}{1+\chi} \tag{4}$$

with $\sigma \geq 0$ and $\chi \geq 0$ which are respectively the inverse of the EIS in consumption and the inverse of the Frisch wage elasticity of labor supply.

Denoting by k_t the household's capital stock and by Y_t the GDP, the budget constraint faced by the representative household is

$$c_t + p_t I_t = Y_t = r_t k_t + \omega_t l_t \tag{5}$$

Assuming that capital depreciates at rate $\delta \in (0,1)$ in each period, the law of motion of the capital stock is:

$$k_{t+1} = (1-\delta)k_t + I_t \tag{6}$$

with k_0 given. Combining (5) and (6), the representative household then maximizes its present discounted lifetime utility

$$\begin{array}{rl} \max\limits_{\{c_t, l_t, k_{t+1}\}_{t=0\ldots\infty}} & \sum\limits_{t=0}^{+\infty} \beta^t \left[\frac{c_t^{1-\sigma}}{1-\sigma} - \frac{l_t^{1+\chi}}{1+\chi} \right] \\ s.t. & k_{t+1} = (1-\delta+r_t)k_t + \omega_t l_t - c_t, \quad t = 0\ldots\infty, \\ & k_0 \; given \end{array} \tag{7}$$

with $\beta \in (0,1)$ the discount factor. The first-order conditions are, for $t = 0\ldots\infty$,

$$c_t^{-\sigma} = \beta c_{t+1}^{-\sigma} \left[\frac{r_{t+1}+(1-\delta)p_{t+1}}{p_t} \right] \tag{8}$$

$$\omega_t c_t^{-\sigma} = l_t^{\chi} \tag{9}$$

Equation (8) is the standard Euler equation, and (9) corresponds to the trade-off between consumption and leisure.

2.3 Intertemporal equilibrium and steady state

We consider symmetric perfect-foresight equilibria which consist of prices $\{r_t, p_t, \omega_t\}_{t\geq 0}$ and quantities $\{c_t, l_t, I_t, k_t, Y_{ct}, Y_{It}, K_{ct}, K_{It}, L_{ct}, L_{It}\}_{t\geq 0}$ that satisfy the household's and the firms' first-order conditions as given by (3) and (8)-(9), the technological and budget constraints (1)-(2) and (5)-(6), and the market equilibrium conditions. All firms of sector I being identical, we have $\bar{K}_{It} = K_{It}$ and $\bar{L}_{It} = L_{It}$ for any t. The *production function at the social (aggregate) level* in the investment good sector is defined as

$$Y_{It} = K_{It}^{\alpha(1+\Theta)} L_{It}^{(1-\alpha)(1+\Theta)} \tag{10}$$

We thus have increasing returns at the social level with size given by Θ.

The market clearing conditions for the consumption and investment goods are $c_t = Y_{ct}$ and $I_t = Y_{It}$, while the market clearing conditions for capital and labor yield $K_{ct} + K_{It} = k_t$ and $L_{ct} + L_{It} = l_t$. Any solution that also satisfies the transversality condition

$$\lim_{t\to+\infty} \beta^t c_t^{-\sigma} k_{t+1} = 0$$

is called an equilibrium path.

A steady state is defined by constant equilibrium quantities and prices. We provide the following Proposition:

Proposition 1. *Assume that* $\Theta \neq (1-\alpha)/\alpha$ *and* $\chi \neq \hat{\chi} \equiv \frac{\sigma(1-\alpha)+\alpha\Theta}{1-\alpha(1+\Theta)}$. *Then there exists a unique steady state.*

Proof: See Appendix 5.1.

We can now turn to the analysis of the local stability properties of the model, considering a family of economies parameterized by the EIS in consumption $1/\sigma$, the degree of IRS Θ and the wage elasticity of labor supply $1/\chi$.

3 Local stability analysis

After some manipulations, the two-sector model described above can be reduced to a two-dimensional dynamic system in (k_t, c_t). Linearizing this system in a neighborhood of the steady state yields a Jacobian matrix for which the characteristic polynomial is given in Appendix 5.2. In this Appendix, we also show that for a given value of χ, the Trace and Determinant of the Jacobian matrix are linear functions of σ. This means that when σ is varied over $(0, +\infty)$, the Trace and Determinant move along a line, denoted by Δ_χ, whose location depends on the parameters, in particular on the size of externalities Θ and the (inverse of) the elasticity of the labor supply χ. Thus, we can analyze the local stability properties of the model by using the geometrical methodology described in Grandmont *et al.* [10].

Let us introduce at this stage the following parameter restrictions, which enable us to simplify the analysis by restricting the number of possible configurations:

Assumption 1. *$\alpha \in (1/4, 1/2)$, $\beta \in (\hat{\beta}, 1)$, $\delta < \hat{\delta}$ and $\Theta \in (\underline{\Theta}, \bar{\Theta})$ with $\hat{\beta} \equiv \max\{(1-2\alpha)/[(1-\delta)(1-\alpha)^2], (1-\alpha-\delta)/(1-\delta)(1-\alpha)\}$, $\hat{\delta} \equiv [\beta(1-\alpha) - 2(1-\beta)]/\beta(2-\alpha)$, $\underline{\Theta} = \delta/(1-\delta)$ and $\bar{\Theta} = \alpha/(1-\alpha)$.*[5]

These restrictions are sufficient to consider the whole range of empirically credible values for these parameters. Estimates for the labor share (equal to $1-\alpha$ in the model) in industrialized economies are typically in the range 60-70%. Estimates for the quarterly depreciation rate are typically close to 2.5%, and estimates for the subjective discount factor are typically around 0.99. Using a standard calibration of RBC models compatible with quarterly data, namely $(\alpha, \delta, \beta) = (0.3, 0.025, 0.99)$, Assumption 1 holds and is compatible with mild external effects since $\hat{\beta} \approx 0.837$, $\hat{\delta} \approx 0.4$, $\underline{\Theta} \approx 0.0256$ and $\bar{\Theta} \approx 0.4286$. The interval for Θ largely covers the range of available empirical estimates for the amount of IRS to scale in the US economy.[6]

[5] Note that under Assumption 1, $\bar{\Theta} < (1-\alpha)/\alpha$.

[6] For example, Basu and Fernald [1] obtain a point estimates for the degree of IRS in the durable manufacturing industry in the US economy of 0.33, with standard deviation 0.11.

Note also that Assumption 1 implies that all our results are compatible with standard negative slopes for the capital and labor equilibrium demand functions.

Denoting $\theta = \beta(1-\delta)$, let us also introduce the following bounds on χ and Θ, important for the local stability properties of the model (see below):

$$\underline{\chi} = \frac{\alpha\Theta}{1-\alpha(1+\Theta)}, \quad \tilde{\Theta}_\chi = \frac{\frac{\alpha(1+\beta)[1-\theta(1-\alpha)+\chi]}{1-\theta} + \frac{(1-\alpha)(1-\theta)}{2}\left(1-\frac{\beta\alpha\delta}{1-\theta}\right)}{\alpha(\tilde{\chi}-\chi)}$$

with

$$\tilde{\chi} = \frac{(1-\delta)(1+\beta)(1-\alpha)-\delta\alpha}{\delta\alpha}$$

Since $\tilde{\Theta}_\chi$ is increasing in χ, there are two main cases to consider: (i) χ is not too large, i.e. the elasticity of the labor supply curve is not too small, so that $\tilde{\Theta}_\chi \in (\underline{\Theta}, \bar{\Theta})$; (ii) χ is large (the elasticity of the aggregate labor supply curve is small), so that $\tilde{\Theta}_\chi > \bar{\Theta}$. The next subsection is devoted to case (i) which, we argue, covers all the empirically relevant configurations. Subsection 3.2 will consider instead the case of an arbitrary value for the elasticity of aggregate labor supply, restricting then the range of values considered for the degree of IRS Θ. This second case is mostly important for theoretical purposes as it covers the case of a fixed labor supply.

3.1 Local indeterminacy for small elasticities of intertemporal substitution in consumption

Let us first consider the case of a not too small labor supply elasticity, so that $\tilde{\Theta}_\chi \in (\underline{\Theta}, \bar{\Theta})$. For practical purposes, we assume that $\chi \in [0, 2/3)$, an interval for which the condition $\tilde{\Theta}_\chi \in (\underline{\Theta}, \bar{\Theta})$ is always satisfied under Assumption 1.[7] This interval covers values for the elasticity of the *aggregate* labor supply curve ranging from 3/2 to $+\infty$, which includes the range of empirically credible values for this elasticity according to Prescott and Wallenius [21] and Rogerson and Wallenius [23], who concluded for values typically larger than 2 and probably around 3.[8] This range also covers Hansen's [12] and Rogerson's [22] models of indivisible labor (with employment lotteries), corresponding to $\chi = 0$.

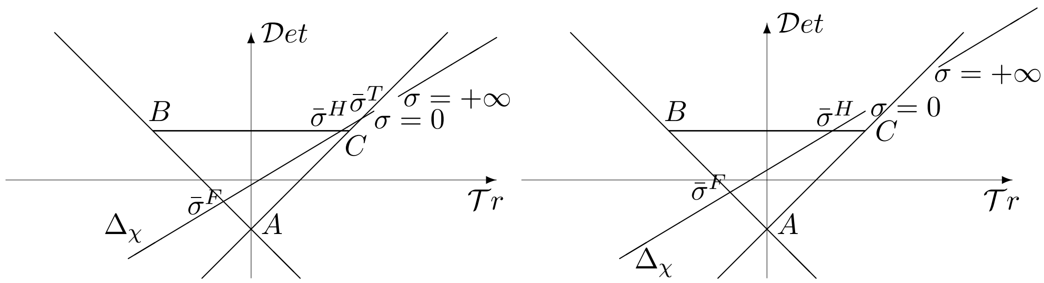
We can now apply the geometrical methodology of Grandmont *et al.* [10]. In appendix 5.2, we show that when σ increases from 0 to $+\infty$, the value of the pair $(\mathcal{T}, \mathcal{D})$ varies along a line Δ_χ, whose starting and ending points depend on the values

[7] Using again as an example the standard calibration $(\alpha, \delta, \beta) = (0.3, 0.025, 0.99)$, we get $\tilde{\Theta}_\chi \approx 0.1033$ when $\chi = 0$, and $\tilde{\Theta}_\chi \approx 0.316$ when $\chi = 2/3$. Note also that the threshold $\underline{\chi}$ lies in $[0, 2/3)$ in all cases.

[8] See Prescott and Wallenius [21] for a discussion of the factors that make the wage elasticity of aggregate labor supply significantly different from the corresponding elasticity at the micro level.

of structural parameters, in particular regarding the amount of externalities Θ and the labor supply elasticity, $1/\chi$.[9]

In Appendix 5.3, we show that when externalities are weak, $\Theta \in (\underline{\Theta}, \tilde{\Theta}_\chi)$, we have the following geometrical configurations (see Figure 1,a,b)



(a) $\chi \in [0, \underline{\chi})$ (b) $\chi \in (\underline{\chi}, +\infty)$

Figure 1: Local indeterminacy with $\Theta \in (\underline{\Theta}, \tilde{\Theta}_\chi)$.

Figure 1(a) corresponds to the case of a large (possibly infinite) labor supply elasticity: $\chi \in [0, \underline{\chi})$. The line Δ_χ crosses the triangle ABC in which both characteristic roots have a modulus less than 1 and local indeterminacy arises. Indeed, when σ increases from 0, the value of the pair $(\mathcal{T}, \mathcal{D})$ varies along Δ_χ. The steady state is first saddle-point stable for $\sigma \in [0, \bar{\sigma}^T)$, becomes unstable for $\sigma \in (\bar{\sigma}^T, \bar{\sigma}^H)$, then locally indeterminate when $\sigma \in (\bar{\sigma}^H, \bar{\sigma}^F)$ and is finally saddle-point stable for $\sigma > \bar{\sigma}^F$. When σ crosses $\bar{\sigma}^T$, one positive characteristic root crosses the value 1 and a transcritical bifurcation occurs.[10] When σ crosses $\bar{\sigma}^H$, one pair of complex characteristic roots crosses the unit circle and a Hopf bifurcation occurs generating quasi-periodic endogenous fluctuations. When σ crosses $\bar{\sigma}^F$, one negative characteristic root crosses the value -1 and a flip bifurcation occurs generating period-two cycles.

Figure 1(b) covers the case of a smaller labor supply elasticity: $\chi \in (\underline{\chi}, 2/3)$. As can be observed, the local stability properties of the steady-state are the same, except that the starting point of the line Δ_χ (associated to $\sigma = 0$) is now located above the triangle ABC, where the steady-state is unstable, so that the transcritical bifurcation no longer exists. The steady-state is thus unstable for $\sigma \in (0, \bar{\sigma}^H)$,

[9]The slope of the line is also affected by a change in the values of these parameters. Yet, these changes are sufficiently contained under our parameter restrictions that they do not lead to additional conceptual configurations.

[10]Note that a transcritical bifurcation is usually associated with two-steady-states. However, as proved by Proposition 1, the steady state is unique. It follows that this transcritical bifurcation is degenerate and only associated with a loss of stability of the unique steady state.

locally indeterminate for $\sigma \in (\bar{\sigma}^H, \bar{\sigma}^F)$ and saddle-point stable for $\sigma > \bar{\sigma}^F$.

The Hopf, flip and transcritical bifurcation values are respectively given by (see Appendix 5.3):

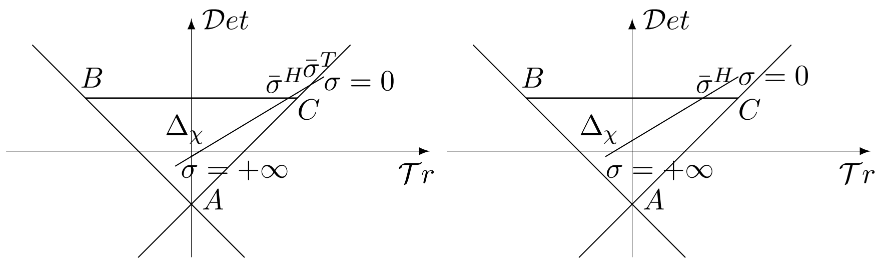
$$\bar{\sigma}^H = \frac{\left(1-\frac{\beta\delta\alpha}{1-\theta}\right)\frac{\Theta(\chi+\alpha)(1-\delta)(1-\beta)}{\delta}}{-\frac{\Theta[(1-\alpha)(1-\theta)-\delta-\chi\delta\alpha]}{\delta}+\frac{\alpha(1-\beta)[1-\theta(1-\alpha)+\chi]}{1-\theta}} \tag{11}$$

$$\bar{\sigma}^F = \frac{\left(1-\frac{\beta\delta\alpha}{1-\theta}\right)\frac{\Theta[\alpha(1+\theta)(2-\delta)+\chi[2(1-\delta)(1+\beta)+\delta\alpha(1-\theta)]]-\delta\chi(1-\alpha)(1-\theta)}{\delta(1-\alpha)}}{2\left[-\frac{\Theta[(1-\delta)(1-\alpha)(1+\beta)-\delta\alpha(1+\chi)]}{\delta(1-\alpha)}+\frac{\alpha(1+\beta)[1-\theta(1-\alpha)+\chi]}{(1-\alpha)(1-\theta)}\right]+(1-\theta)\left(1-\frac{\beta\delta\alpha}{1-\theta}\right)} \tag{12}$$

and

$$\bar{\sigma}^T = \frac{[1-\alpha(1+\Theta)](\underline{\chi}-\chi)}{(1-\alpha)} \tag{13}$$

When the amount of externalities is larger, $\Theta \in (\hat{\Theta}_\chi, \bar{\Theta})$, we get the following geometrical configurations.



(a) $\chi \in [0, \underline{\chi})$ (b) $\chi \in (\underline{\chi}, 2/3)$

Figure 2: Local indeterminacy with $\Theta \in (\tilde{\Theta}_\chi, \bar{\Theta})$.

Comparing Figure 2(a) to Figure 1(a), we observe that the stability properties of the model are the same, except that the line Δ_χ now has a terminal point (for $\sigma = +\infty$) which is located in the interior of the triangle ABC (where the model is locally indeterminate), and no longer crosses the line generated by AB, so that the flip bifurcation no longer occurs. The steady-state is thus locally indeterminate for any $\sigma > \bar{\sigma}^H$, i.e. for *arbitrarily low values for the EIS in consumption.* Likewise, in Figure 2(b) where $\chi \in (\underline{\chi}, 2/3)$, the situation is the same as in Figure 1(b) except that the flip bifurcation no longer exists. Indeterminacy thus also occurs for arbitrarily low values for the EIS.

We summarize these results in the following Proposition:

Proposition 2. *Under Assumption 1, let $\chi \in [0, 2/3)$. Then, there exist $\underline{\delta}$, $\bar{\delta}$, with $0 < \underline{\delta} < \bar{\delta} \leq \hat{\delta}$, and $\underline{\beta} \in [\hat{\beta}, 1)$ such that if $\delta \in (\underline{\delta}, \bar{\delta})$ and $\beta \in (\underline{\beta}, 1)$, the following results hold:*

i) For $\Theta \in (\underline{\Theta}, \tilde{\Theta}_\chi)$, when $\chi \in [0, \underline{\chi})$, the steady state is saddle-point stable when $\sigma \in [0, \bar{\sigma}^T)$, undergoes a transcritical bifurcation at $\sigma = \bar{\sigma}^T$, becomes locally *unstable when* $\sigma \in (\bar{\sigma}^T, \bar{\sigma}^H)$*, undergoes a Hopf bifurcation at* $\sigma = \bar{\sigma}^H$*, becomes locally indeterminate when* $\sigma \in (\bar{\sigma}^H, \bar{\sigma}^F)$*, undergoes a flip bifurcation at* $\sigma = \bar{\sigma}^F$*, and becomes again saddle-point stable when* $\sigma \in (\bar{\sigma}^F, +\infty)$*. When* $\chi \in (\underline{\chi}, 2/3)$*, the transcritical bifurcation disappears so that the model is locally unstable for* $\sigma \in [0, \bar{\sigma}^H)$.

ii) For $\Theta \in (\tilde{\Theta}_\chi, \bar{\Theta})$*, when* $\chi \in [0, \underline{\chi})$*, the steady state is saddle-point stable when* $\sigma \in [0, \bar{\sigma}^T)$*, undergoes a transcritical bifurcation at* $\sigma = \bar{\sigma}^T$*, becomes locally unstable when* $\sigma \in (\bar{\sigma}^T, \bar{\sigma}^H)$*, undergoes a Hopf bifurcation at* $\sigma = \bar{\sigma}^H$*, and becomes locally indeterminate when* $\sigma \in (\bar{\sigma}^H, +\infty)$*. When* $\chi \in (\underline{\chi}, 2/3)$*, the transcritical bifurcation disappears so that the model is locally unstable for* $\sigma \in [0, \bar{\sigma}^H)$.

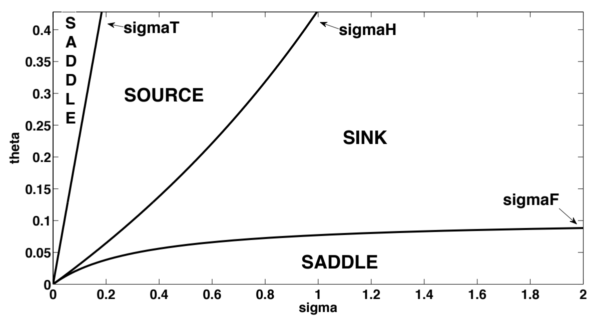
Proof: See Appendix 5.3.

In Proposition 2, the bounds $\underline{\delta}$, $\bar{\delta}$ and $\underline{\beta}$ are complicated expressions of the structural parameters, obtained from second-order polynomials derived in Appendix 5.3. Of course, the conditions $\beta \in (\underline{\beta}, 1)$ and $\delta \in (\underline{\delta}, \bar{\delta})$ are always satisfied under Assumption 1 by our benchmark calibration $(\alpha, \delta, \beta) = (0.3, 0.025, 0.99)$, and are also always satisfied in a significantly large neighborhood of this calibration.

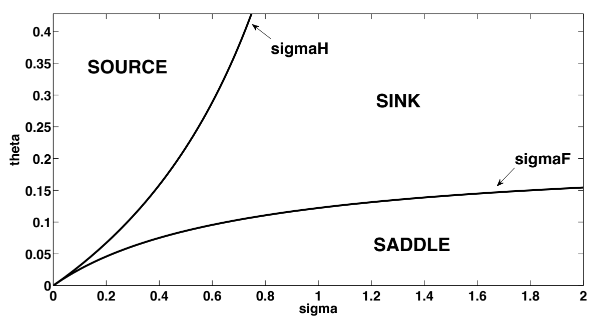
Comments. Proposition 2 provides new clear-cut conditions for the occurrence of local indeterminacy in two-sector RBC models with additively separable preferences, generalizing the results obtained by Benhabib and Farmer [3] in the particular case of utility function that is logarithmic in consumption (unitary EIS). First, we prove the existence of a Hopf and, in some cases, of flip and transcritical bifurcations in the parameter space.[11] Second, we prove that local indeterminacy can arise in two-sector models for an *arbitrarily small* EIS in consumption $1/\sigma$, provided that the amount of IRS in the investment sector is in an intermediary range, namely $\Theta \in (\tilde{\Theta}_\chi, \bar{\Theta})$. Note that this range includes the empirical estimates of Basu and Fernald [1] for the degree of IRS in the US durable manufacturing industry. Third, we easily derive from equations (11) that $\partial\sigma^H/\partial\Theta > 0$ and $\partial\sigma^F/\partial\Theta > 0$. We conclude from case ii) of Proposition 2 (where indeterminacy requires $\sigma \in (\bar{\sigma}^H, +\infty)$) that for any labor supply elasticity $\chi \in (0, 2/3)$, decreasing the amount of externalities actually *favors* the emergence of indeterminacy by increasing the range of values for σ for which the steady-state is locally indeterminate. The conclusion is different in case i) of Proposition 2, where indeterminacy requires $\sigma \in (\bar{\sigma}^H, \bar{\sigma}^F)$, since both bifurcation parameters are decreasing in Θ.

[11] In Benhabib and Farmer [3] the existence of the Hopf bifurcation is mentioned but not proved, while Harrison [13] focuses exclusively on the flip bifurcation through a numerical analysis.

As an application from Proposition 3 , Figures 3(a) and (b) display the determinacy/indeterminacy areas in the (σ, Θ) plane for two different values for the elasticity of the aggregate labor supply curve: $\chi = 0$, which corresponds to Hansen's [12] assumption of indivisible labor, and $\chi = 1/3$, which corresponds to a labor supply elasticity of 3, the value recommended by Prescott and Wallenius [21] and Rogerson and Wallenius [23] to calibrate business cycle models. We observe that indeterminacy prevails for a wide range of empirically plausible values for σ and Θ. For example, when $\chi = 0$ and the amount of IRS is calibrated to $\Theta = 0.33$ (the point estimate obtained by Basu and Fernald [1]) indeterminacy requires $\sigma > \bar{\sigma}^H \approx 0.83$, i.e. an EIS in consumption smaller than 1.2. When $\chi = 1/3$, indeterminacy requires $\sigma > \bar{\sigma}^H \approx 0.65$, i.e. an EIS in consumption smaller than 1.54. This range of values is consistent with most, although not all, empirical estimates for this coefficient.[12]



(a) $\chi = 0$



(b) $\chi = 1/3$

Figure 3: *Indeterminacy areas in the* (σ, Θ) *plane*

[12] There is no agreement in the empirical literature about the precise value of the EIS in consumption, since most estimates typically vary between 0 and 2. See in particular Campbell [6] , Kocherlakota [15] and Vissing-Jorgensen [24] for estimates smaller than 1, and Mulligan [17] and Gruber [11] for estimates ranging between 1 and 2.

3.2 Local indeterminacy for any elasticity of the labor supply

In the previous Section we have shown that by focusing on an empirically realistic subset of values for the wage-elasticity of labor supply, we can get local indeterminacy for arbitrarily low values of the EIS and mild externalities. We now consider the possibility of obtaining local indeterminacy for *any* labor supply elasticity, enabling us in particular to cover the case of a fixed labor supply (obtained as the limiting case where χ tends to $+\infty$). Following such a route requires to restrict the size of externalities, and we consequently introduce the following bound on Θ:

$$\hat{\Theta} = \frac{\delta\alpha[1-\theta(1-\alpha)]}{(1-\alpha)(1-\theta)(1-\delta)} \tag{14}$$

In appendix 5.4, we prove that when $\Theta \in (\underline{\Theta}, \hat{\Theta})$, depending on whether χ is larger or lower than $\underline{\chi}$, we get the same geometrical configurations as those described by Figure 1,a,b. We can then establish the following Proposition:

Proposition 3. *Under Assumption 1, let* $\Theta \in (\underline{\Theta}, \hat{\Theta})$. *Then there exist* $\underline{\delta}$, $\bar{\delta}$, *with* $0 < \underline{\delta} < \bar{\delta} \leq \hat{\delta}$, *and* $\underline{\beta} \in [\hat{\beta}, 1)$ *such that when* $\beta \in (\underline{\beta}, 1)$ *and* $\delta \in (\underline{\delta}, \bar{\delta})$, *the following results hold:*

i) If $\chi \in [0, \underline{\chi})$, *the steady state is saddle-point stable when* $\sigma \in [0, \bar{\sigma}^T)$, *undergoes a transcritical bifurcation at* $\sigma = \bar{\sigma}^T$, *becomes locally unstable when* $\sigma \in (\bar{\sigma}^T, \bar{\sigma}^H)$, *undergoes a Hopf bifurcation at* $\sigma = \bar{\sigma}^H$, *becomes locally indeterminate when* $\sigma \in (\bar{\sigma}^H, \bar{\sigma}^F)$, *undergoes a flip bifurcation at* $\sigma = \bar{\sigma}^F$, *and becomes again saddle-point stable when* $\sigma \in (\bar{\sigma}^F, +\infty)$.

ii) If $\chi \in (\underline{\chi}, +\infty)$, *the steady state is locally unstable when* $\sigma \in [0, \bar{\sigma}^H)$, *undergoes a Hopf bifurcation at* $\sigma = \bar{\sigma}^H$, *becomes locally indeterminate when* $\sigma \in (\bar{\sigma}^H, \bar{\sigma}^F)$, *undergoes a flip bifurcation at* $\sigma = \bar{\sigma}^F$, *and becomes saddle-point stable when* $\sigma \in (\bar{\sigma}^F, +\infty)$.

Proof: See Appendix 5.4.

Proposition 3 also generalizes previous results obtained in the literature on two-sector models with additively separable preferences.[13] Considering the case of a utility function with a unitary EIS in consumption, Benhabib and Farmer [3] show how varying the wage-elasticity of labor supply affects the range of values for which indeterminacy occurs. They show in particular that considering an infinitely elastic labor supply allows to minimize the amount of externalities required for indeterminacy. We prove here that adjusting the EIS in consumption is another way to favor

[13]See in particular Benhabib and Farmer [3] and Harrison [13].

the occurrence of local indeterminacy since, provided the amount of externalities is mild enough (i.e. $\Theta \in (\underline{\Theta}, \hat{\Theta})$), indeterminacy can occur for any elasticity of labor supply.[14] This conclusion thus covers the case of a fixed labor supply.

4 Concluding comments

Although two-sector infinite-horizon models require smaller degrees of increasing returns to scale for indeterminacy than aggregate models, they are usually criticized on the fact that they rely on too large values for the EIS in consumption, and too large values for the elasticity of the labor supply with respect to empirically plausible estimates. However, most of the contributions are based on relatively narrow specifications for technology and preferences and/or often rely on numerical simulations, preventing from getting a full picture of the configurations giving rise to local indeterminacy and sunspot fluctuations. We have proved that local indeterminacy occurs through flip and Hopf bifurcations for any value of the elasticity of the labor supply, and can even be compatible with an arbitrarily low EIS in consumption. Moreover, the existence of expectation-driven fluctuations is consistent with a mild amount of increasing returns.

5 Appendix

5.1 Proof of Proposition 1

Equation (6) evaluated at the steady state gives $I = Y_I = \delta k$. Moreover, we derive from (3) that $r/p = \alpha Y_I/K_I = \alpha\delta k/K_I$. It follows from (8) that $K_I = \beta\delta\alpha k/(1-\theta)$ with $\theta = \beta(1-\delta)$. Merging equations (3) gives $L_I/K_I = L_c/K_c = l/k$ and thus we get from (10)

$$\bar{k} = l^{\frac{(1-\alpha)(1+\Theta)}{1-\alpha(1+\Theta)}} \delta^{\frac{\Theta}{1-\alpha(1+\Theta)}} \left(\frac{\beta\alpha}{1-\theta}\right)^{\frac{1+\Theta}{1-\alpha(1+\Theta)}} \equiv l^{\frac{(1-\alpha)(1+\Theta)}{1-\alpha(1+\Theta)}} \bar{\kappa} \tag{15}$$

assuming of course that $\Theta \neq (1-\alpha)/\alpha$. Consider now $c = Y_c = K_c^\alpha L_c^{1-\alpha} = (k - K_I)\left(\frac{l}{k}\right)^{1-\alpha}$. Substituting (15) into this expression gives

$$\bar{c} \;=\; \left(1 - \frac{\beta\delta\alpha}{1-\theta}\right) \bar{\kappa}^\alpha l^{\frac{1-\alpha}{1-\alpha(1+\Theta)}} \equiv \bar{\psi} l^{\frac{1-\alpha}{1-\alpha(1+\Theta)}} \tag{16}$$

From the expression of the prices in (3) we derive

[14] For an analysis of the role of the labor supply elasticity in two sector models where technologies have constant social returns to scale, see Benhabib and Nishimura [4], Garnier *et al.* [8, 9] and Nishimura and Venditti [19].

$$\bar{\omega} = (1-\alpha)\bar{\kappa}^{\alpha} l^{\frac{\alpha\Theta}{1-\alpha(1+\Theta)}} \equiv \bar{\phi} l^{\frac{\alpha\Theta}{1-\alpha(1+\Theta)}}, \quad \bar{r} = \alpha l^{\frac{-(1-\alpha)\Theta}{1-\alpha(1+\Theta)}} \bar{\kappa}^{\alpha-1}, \quad \bar{p} = \frac{\beta}{1-\theta}\bar{r} \tag{17}$$

Equation (9) now gives using (15)-(17):

$$(1-\alpha)\left(1-\frac{\beta\delta\alpha}{1-\theta}\right)^{\sigma} \delta^{\frac{\Theta\alpha(1+\sigma)}{1-\alpha(1+\Theta)}} \left(\frac{\beta\alpha}{1-\theta}\right)^{\frac{(1+\Theta)\alpha(1+\sigma)}{1-\alpha(1+\Theta)}} = l^{\chi-\hat{\chi}} \tag{18}$$

with

$$\hat{\chi} \equiv \frac{\sigma(1-\alpha)+\alpha\Theta}{1-\alpha(1+\Theta)}$$

Assuming now that $\chi \neq \hat{\chi}$, we derive

$$l = (1-\alpha)^{\frac{1}{\chi-\hat{\chi}}} \left(1-\frac{\beta\delta\alpha}{1-\theta}\right)^{\frac{\sigma}{\chi-\hat{\chi}}} \delta^{\frac{\Theta\alpha(1+\sigma)}{[1-\alpha(1+\Theta)](\chi-\hat{\chi})}} \left(\frac{\beta\alpha}{1-\theta}\right)^{\frac{(1+\Theta)\alpha(1+\sigma)}{[1-\alpha(1+\Theta)](\chi-\hat{\chi})}} \tag{19}$$

□

5.2 Computation of the linearized dynamical system

Let us introduce the following elasticities:

$$\epsilon_{cc} = -\frac{U_1(c,l)}{U_{11}(c,l)c} = \frac{1}{\sigma}, \quad \epsilon_{ll} = -\frac{U_2(c,l)}{U_{22}(c,l)l} = \frac{1}{\chi} \tag{20}$$

Consider the first-order condition (9) together with the expression of the wage rate as given by (3). We easily get

$$l_t = (1-\alpha)^{\frac{1}{\alpha+\chi}} k_t^{\frac{\alpha}{\alpha+\chi}} c_t^{\frac{-\sigma}{\alpha+\chi}} \equiv l(k_t, c_t)$$

Substituting this function into the expressions of the prices (3) allows to get $r_t = r(k_t, c_t)$, $\omega_t = \omega(k_t, c_t)$ and $p_t = p(k_t, c_t)$. Consider now the first-order condition (8) with the capital accumulation equation. We have the following two equations

$$\begin{aligned} c_{t+1}^{-\sigma}\left[r(k_{t+1}, c_{t+1}) + (1-\delta)p(k_{t+1}, c_{t+1})\right] &= \beta p(k_t, c_t) c_t^{-\sigma} \\ k_{t+1} &= (1-\delta)k_t + \frac{r(k_t,c_t)k_t + \omega(k_t,c_t)l(k_t,c_t) - c_t}{p(k_t,c_t)} \end{aligned}$$

Using (17), total differentiation of these equations in a neighborhood of the steady state gives after simplifications the following linear system

$$A\begin{pmatrix} \widehat{c}_{t+1} \\ \widehat{k}_{t+1} \end{pmatrix} = B\begin{pmatrix} \widehat{c}_t \\ \widehat{k}_t \end{pmatrix}$$

with

$$A = \begin{pmatrix} \sigma - \frac{\sigma(1-\alpha)(1-\theta)}{\alpha+\chi} + \Theta\theta\left[1 - \frac{\sigma(1-\alpha)(1-\theta)\left(1-\frac{\beta\delta\alpha}{1-\theta}\right)}{(\alpha+\chi)\beta\delta\alpha}\right] & \frac{\chi(1-\alpha)(1-\theta)}{\alpha+\chi} + \frac{\Theta(1-\delta)(1-\theta)\left(1-\frac{\chi(1-\alpha)(1-\theta)}{\alpha+\chi}\right)}{\delta\alpha} \\ 0 & 1 \end{pmatrix}$$

and

$$B = \begin{pmatrix} \sigma + \Theta\left[1 - \frac{\sigma(1-\alpha)(1-\theta)\left(1-\frac{\beta\delta\alpha}{1-\theta}\right)}{(\alpha+\chi)\beta\delta\alpha}\right] & \frac{\Theta(1-\theta)\left(1-\frac{\chi(1-\alpha)(1-\theta)}{\alpha+\chi}\right)}{\beta\delta\alpha} \\ -\frac{1-\theta}{\beta(\alpha+\chi)}\left[\frac{\sigma(1-\alpha)}{\alpha} + 1 - \frac{\beta\delta\alpha}{1-\theta}\right] & \frac{1}{\beta}\left[1 + \frac{(1-\alpha)(1-\theta)}{\alpha+\chi}\right] \end{pmatrix}$$

It follows that the linearized system is

$$\begin{pmatrix} \widehat{c}_{t+1} \\ \widehat{k}_{t+1} \end{pmatrix} = J \begin{pmatrix} \widehat{c}_t \\ \widehat{k}_t \end{pmatrix}$$

with $J = A^{-1}B$. Denoting by $\mathcal{T}$ and $\mathcal{D}$ the Trace and Determinant of J, tedious but straightforward computations allow therefore to compute the following characteristic polynomial $P_\sigma(\lambda) = \lambda^2 - \mathcal{T}\lambda + \mathcal{D} = 0$ with

$$\begin{aligned} \mathcal{D} = \mathcal{D}_\chi(\sigma) &= \frac{\sigma\left[-\frac{1-\delta}{\delta}\Theta + (1+\Theta)\alpha + \frac{(\chi+\alpha)\alpha}{(1-\alpha)}\left(\frac{1+\Theta(1-\theta)}{1-\theta}\right)\right] - \left(1-\frac{\beta\alpha\delta}{1-\theta}\right)\frac{(1-\delta)\Theta(\chi+\alpha)}{\delta(1-\alpha)}}{\sigma\left[-\frac{\theta\Theta}{\delta} + \beta\alpha + \frac{(\chi+\alpha)\beta\alpha}{(1-\theta)(1-\alpha)}\right] - \left(1-\frac{\beta\alpha\delta}{1-\theta}\right)\frac{\theta\Theta(\chi+\alpha)}{\delta(1-\alpha)}} \\ \mathcal{T} = \mathcal{T}_\chi(\sigma) &= 1 + \mathcal{D}_\chi(\sigma) + \frac{(1-\theta)\left(1-\frac{\beta\alpha\delta}{1-\theta}\right)\left[\sigma + \frac{\chi(1-\alpha(1+\Theta))-\Theta\alpha}{1-\alpha}\right]}{\sigma\left[-\frac{\theta\Theta}{\delta} + \beta\alpha + \frac{(\chi+\alpha)\beta\alpha}{(1-\theta)(1-\alpha)}\right] - \left(1-\frac{\beta\alpha\delta}{1-\theta}\right)\frac{\theta\Theta(\chi+\alpha)}{\delta(1-\alpha)}} \end{aligned}$$

We derive from this that when σ is varied over the interval $[0, +\infty)$, $\mathcal{D}$ and $\mathcal{T}$ are linked through a linear relationship $\mathcal{D} = \Delta_\chi(\mathcal{T}) = \mathcal{T}\mathcal{S}_\chi + \mathcal{C}$ with a slope $\mathcal{S}_\chi = (\partial\mathcal{D}_\chi(\sigma)/\partial\sigma)/(\partial\mathcal{T}_\chi(\sigma)/\partial\sigma)$. Let us introduce the following notation:

$$\mathcal{N} = \frac{\left(1-\frac{\beta\alpha\delta}{1-\theta}\right)}{\left\{\sigma\left[-\frac{\theta\Theta}{\delta} + \beta\alpha + \frac{(\chi+\alpha)\beta\alpha}{(1-\theta)(1-\alpha)}\right] - \left(1-\frac{\beta\alpha\delta}{1-\theta}\right)\frac{\theta\Theta(\chi+\alpha)}{\delta(1-\alpha)}\right\}^2}$$

We easily derive that

$$\begin{aligned} \frac{\partial\mathcal{D}_\chi(\sigma)}{\partial\sigma} &= -\mathcal{N}\frac{\theta\Theta^2(\chi+\alpha)\alpha(1+\chi)}{\delta(1-\alpha)^2} < 0 \\ \frac{\partial\mathcal{T}_\chi(\sigma)}{\partial\sigma} &= -\mathcal{N}\beta\alpha\frac{\Theta(\chi+\alpha)(1+\chi)\left\{\frac{(1-\alpha)(1-\delta)(1-\theta)(1+\Theta)}{\chi+\alpha} + (1-\delta)(1+\Theta) - 1\right\} + \chi\delta(1-\alpha)(1+\Theta)(1+\chi-\theta(1-\alpha))}{\delta(1-\alpha)^2} \end{aligned}$$

and thus

$$\mathcal{S}_\chi = \frac{(1-\delta)\Theta^2(\chi+\alpha)(1+\chi)}{\Theta(1+\chi)\left\{(1-\alpha)(1-\delta)(1-\theta)(1+\Theta) + (\chi+\alpha)[(1-\delta)(1+\Theta)-1]\right\} + \chi\delta(1-\alpha)(1+\Theta)(1+\chi-\theta(1-\alpha))}$$

Under Assumption 1, we conclude that if $\Theta > \underline{\Theta}$, then $\partial\mathcal{T}_\chi(\sigma)/\partial\sigma < 0$ and thus $\mathcal{S}_\chi > 0$. We also derive that $\mathcal{S}_\chi < 1$ if and only if

$$\begin{aligned} &\chi^2\delta\left(\Theta\alpha - (1-\alpha)\right) - \chi\Big\{\Theta^2(1-\alpha)(1-\delta)(1-\theta) \\ &+ \Theta\Big[(1-\alpha)(1-\theta) - \delta[1+\alpha(1-\theta(1-\alpha))]\Big] + \delta(1-\alpha)(1-\theta(1-\alpha))\Big\} \\ &- \Theta\Big\{(1-\alpha)(1+\Theta)(1-\theta)(1-\delta) - \delta\alpha\Big\} \equiv g(\chi) < 0 \end{aligned} \tag{21}$$

Under Assumption 1, there exists $\tilde{\delta} \leq \hat{\delta}$ such that $g(\chi) < 0$ for any $\chi \geq 0$ if $\delta < \tilde{\delta}$.

In other words, $\Delta_\chi(\mathcal{T})$ corresponds to a half-line in the $(\mathcal{T}, \mathcal{D})$ plane with the starting point $(\mathcal{T}_\chi(+\infty), \mathcal{D}_\chi(+\infty))$ obtained when $\sigma = +\infty$ such that

$$\begin{aligned} \mathcal{D}_\chi(+\infty) &= \frac{-\frac{(1-\delta)\Theta}{\delta} + (1+\Theta)\alpha + \frac{(\chi+\alpha)\alpha}{(1-\alpha)}\left(\frac{1+\Theta(1-\theta)}{1-\theta}\right)}{-\frac{\theta\Theta}{\delta} + \beta\alpha + \frac{\beta\alpha(\chi+\alpha)}{(1-\theta)(1-\alpha)}} \\ \mathcal{T}_\chi(+\infty) &= 1 + \mathcal{D}_\chi(+\infty) + \frac{(1-\theta)\left(1-\frac{\beta\alpha\delta}{1-\theta}\right)}{-\frac{\theta\Theta}{\delta} + \beta\alpha + \frac{\beta\alpha(\chi+\alpha)}{(1-\theta)(1-\alpha)}} \end{aligned}$$

and the end-point $(\mathcal{T}_\chi(0), \mathcal{D}_\chi(0))$ obtained when $\sigma = 0$ such that:

$$\mathcal{D}_\chi(0) = \frac{1}{\beta}, \quad \mathcal{T}_\chi(0) = 1 + \mathcal{D}_\chi(0) - \delta(1-\theta)\frac{\chi(1-\alpha(1+\Theta))-\alpha\Theta}{\theta\Theta(\chi+\alpha)}$$

For the starting point, when $\sigma = +\infty$, i.e. $\epsilon_{cc} = 0$, we get:

$$\begin{aligned} P_{+\infty}(1) &= -\frac{(1-\theta)\left(1-\frac{\beta\alpha\delta}{1-\theta}\right)}{-\frac{\theta\Theta}{\delta}+\beta\alpha+\frac{\beta\alpha(\chi+\alpha)}{(1-\theta)(1-\alpha)}} \\ P_{+\infty}(-1) &= \frac{2\left[-\frac{(1-\delta)(1+\beta)\Theta}{\delta}+(1+\beta+\Theta)\alpha+\frac{(\chi+\alpha)\alpha}{(1-\alpha)}\left(\frac{1+\beta+\Theta(1-\theta)}{1-\theta}\right)\right]+(1-\theta)\left(1-\frac{\beta\alpha\delta}{1-\theta}\right)}{-\frac{\theta\Theta}{\delta}+\beta\alpha+\frac{\beta\alpha(\chi+\alpha)}{(1-\theta)(1-\alpha)}} \end{aligned}$$

For the end point, when $\sigma = 0$, i.e. $\epsilon_{cc} = +\infty$, we get:

$$\begin{aligned} P_0(1) &= \frac{(1-\theta)\delta[\chi(1-\alpha(1+\Theta))-\alpha\Theta]}{(\chi+\alpha)\theta\Theta} \\ P_0(-1) &= \frac{\chi\{\Theta[2(1-\theta(1-\delta))+\delta(1-\theta)\alpha]-\delta(1-\theta)(1-\alpha)\}+\alpha(1+\theta)(2-\delta)\Theta}{(\chi+\alpha)\theta\Theta} \end{aligned}$$

It follows immediately that $P_0(1) > 0$ if and only if $\chi > \alpha\Theta/[1-\alpha(1+\Theta)] \equiv \underline{\chi}$, while it can be easily shown that $P_0(-1) > 0$ if $\Theta > \underline{\Theta}$.

□

5.3 Proof of Proposition 2

We assume first that $\chi \in [0, \underline{\chi})$. We know that the end point satisfies $\mathcal{D}_\chi(0) = 1/\beta > 1$, $P_0(1) < 0$ and $P_0(-1) > 0$. The starting point can be written:

$$\begin{aligned} \mathcal{D}_\chi(+\infty) &= \frac{\delta\alpha(\chi_1-\chi)(\Theta-\Theta_1)}{\beta(1-\alpha)(1-\delta)(\Theta-\hat{\Theta}_\chi)} \\ P_{+\infty}(1) &= \frac{\delta(1-\theta)\left(1-\frac{\beta\alpha\delta}{1-\theta}\right)}{\beta(1-\delta)(\Theta-\hat{\Theta}_\chi)} \\ P_{+\infty}(-1) &= \frac{2\delta\alpha(\tilde{\chi}-\chi)(\Theta-\tilde{\Theta}_\chi)}{\beta(1-\alpha)(1-\delta)(\Theta-\hat{\Theta}_\chi)} \end{aligned}$$

with

$$\hat{\Theta}_\chi = \frac{\delta\alpha[1-\theta(1-\alpha)+\chi]}{(1-\alpha)(1-\theta)(1-\delta)}, \quad \tilde{\Theta}_\chi = \frac{\frac{\alpha(1+\beta)[1-\theta(1-\alpha)+\chi]}{1-\theta}+\frac{(1-\alpha)(1-\theta)}{2}\left(1-\frac{\beta\alpha\delta}{1-\theta}\right)}{\alpha(\tilde{\chi}-\chi)}, \quad \Theta_1 = \frac{1-\theta(1-\alpha)+\chi}{(1-\theta)(\chi_1-\chi)}$$

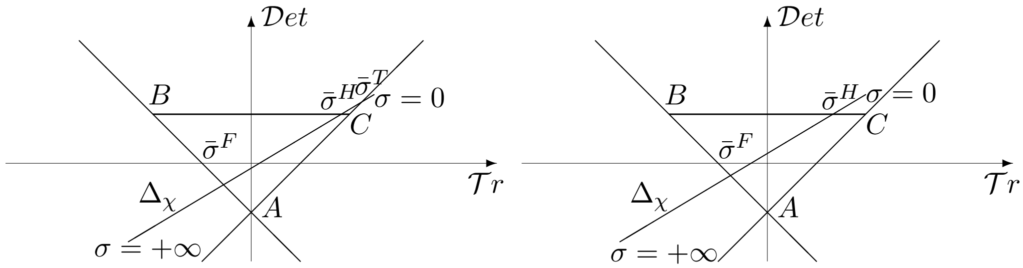
and

$$\chi_1 = \frac{1-\alpha-\delta}{\delta\alpha}, \quad \tilde{\chi} = \frac{(1-\delta)(1+\beta)(1-\alpha)-\delta\alpha}{\delta\alpha}$$

Under Assumption 1 we get $\underline{\chi} < 2/3 < \chi_1 < \tilde{\chi}$ and $\underline{\Theta} < \hat{\Theta}_\chi < \tilde{\Theta}_\chi < \Theta_1 < \bar{\Theta}$ for any $\chi \in [0, 2/3)$. Let us denote by $\bar{\sigma}^H$ the value of σ that solves $\mathcal{D}_\chi(\sigma) = 1$.

a) Assume first that $\underline{\Theta} < \Theta < \hat{\Theta}_\chi$. We get $\mathcal{D}_\chi(+\infty) > 1/\beta = \mathcal{D}_\chi(0)$, $P_{+\infty}(1) < 0$ and $P_{+\infty}(-1) > 0$. Therefore, assuming $\beta \in (\tilde{\beta}, 1)$ and $\delta \in (\delta_0, \tilde{\delta})$, provided $\mathcal{T}_\chi(\bar{\sigma}^H) \in (-2, 2)$, we conclude that the Δ_χ line is located as in Figure 1(a).

b) Assume then that $\hat{\Theta}_\chi < \Theta < \Theta_1$. We get $\mathcal{D}_\chi(+\infty) < 0$, $P_{+\infty}(1) > 0$ and $P_{+\infty}(-1) < 0$ if $\Theta \in (\hat{\Theta}_\chi, \tilde{\Theta}_\chi)$ while $P_{+\infty}(-1) > 0$ if $\Theta \in (\tilde{\Theta}_\chi, \Theta_1)$. Provided $\mathcal{T}_\chi(\bar{\sigma}^H) \in (-2, 2)$, we conclude that when $\Theta \in (\hat{\Theta}_\chi, \tilde{\Theta}_\chi)$, the Δ_χ line is located as in case (a) of the following Figure:



(a) $\chi \in [0, \underline{\chi})$ (b) $\chi \in (\underline{\chi}, 2/3)$

Figure 4: Local indeterminacy with $\Theta \in (\underline{\Theta}, \hat{\Theta}_\chi)$.

and when $\Theta \in (\tilde{\Theta}_\chi, \Theta_1)$, the Δ_χ line is located as in Figure 2(a). It is worth noting that the local stability properties of the steady state in Figure 4(a) are exactly the same as those implied by Figure 1(a).

c) Assume finally that $\Theta \in (\Theta_1, \bar{\Theta})$. We get under Assumption 1 $\mathcal{D}_\chi(+\infty) \in (0,1)$ and $P_{+\infty}(-1) > 0$. Therefore, provided $\mathcal{T}_\chi(\bar\sigma^H) \in (-2,2)$, the Δ_χ line is again located as in Figure 2(a).

We need now to prove that $\mathcal{T}_\chi(\bar\sigma^H) \in (-2,2)$. Solving $\mathcal{D}_\chi(\sigma) = 1$ gives

$$\bar\sigma^H = \frac{\left(1-\frac{\beta\delta\alpha}{1-\theta}\right)\Theta(\chi+\alpha)(1-\delta)(1-\beta)}{-\Theta[(1-\alpha)(1-\theta)-\delta-\chi\delta\alpha]+\frac{\delta\alpha(1-\beta)[1-\theta(1-\alpha)+\chi]}{1-\theta}} \tag{22}$$

Under Assumption 1, there exists $\delta_0 \in (0,1)$ as given by

$$\delta_0 = (1-\beta)\frac{-\{\Theta[1+\chi\alpha-2\beta(1-\alpha)]+\alpha[1+\chi-\beta(1-\alpha)]\}+\sqrt{\Lambda}}{2\beta\{\Theta[1+\chi\alpha-\beta(1-\alpha)]+\alpha(1-\alpha)(1-\beta)\}} \tag{23}$$

with

$$\begin{aligned}\Lambda &= \{\Theta\,[1+\chi\alpha-2\beta(1-\alpha)]+\alpha\,[1+\chi-\beta(1-\alpha)]\}^2 \\ &+ 4\Theta\beta(1-\alpha)\,\{\Theta\,[1+\chi\alpha-\beta(1-\alpha)]+\alpha(1-\alpha)(1-\beta)\}\end{aligned}$$

such that $\bar\sigma^H > 0$ if and only if $\delta \in (\delta_0, 1)$. Moreover, there exists $\tilde\beta \in [\hat\beta, 1)$ such that $\delta_0 < \tilde\delta \leq \hat\delta$ when $\beta \in (\tilde\beta, 1)$. Assuming $\beta \in (\tilde\beta, 1)$ and $\delta \in (\delta_0, \tilde\delta)$, we then derive

$$\mathcal{T}_\chi(\bar\sigma^H) = 2 - \frac{(1-\theta)\left(1-\frac{\beta\alpha\delta}{1-\theta}\right)\left[\bar\sigma^H+\frac{[1-\alpha(1+\Theta)](\chi-\underline{\chi})}{1-\alpha}\right]}{\bar\sigma^H\left[\frac{\theta(\Theta-\hat\Theta)}{\delta}-\frac{\chi\beta\alpha}{(1-\theta)(1-\alpha)}\right]+\left(1-\frac{\beta\alpha\delta}{1-\theta}\right)\frac{\theta\Theta(\chi+\alpha)}{\delta(1-\alpha)}}$$

Under Assumption 1, the denominator of the ratio in $\mathcal{T}_\chi(\bar\sigma^H)$ is positive for all $\chi \geq 0$.

Let $\underline{\beta} = \tilde\beta$ and $\beta \in (\underline{\beta}, 1)$. As $\chi \in [0, \underline{\chi})$, we need to study the numerator of the ratio in $\mathcal{T}_\chi(\bar\sigma^H)$. Note that

$$\bar\sigma^H + \frac{[1-\alpha(1+\Theta)](\chi-\underline{\chi})}{1-\alpha} > \bar\sigma^H - \frac{\alpha\Theta}{1-\alpha} = \frac{\left(1-\frac{\beta\delta\alpha}{1-\theta}\right)\Theta(\chi+\alpha)(1-\delta)(1-\beta)(1-\alpha)-\alpha\Theta den\bar\sigma^H}{(1-\alpha)den\bar\sigma^H}$$

with $den\bar\sigma^H$ the denominator of σ^H. We have shown previously that $den\bar\sigma^H > 0$ if and only if $\delta \in (\delta_0, 1)$, with $\lim_{\delta\to\delta_0} den\bar\sigma^H = 0$. Moreover, we derive from the expression of $\bar{\sigma}^H$ that $\lim_{\delta\to 1}\bar{\sigma}^H = 0$. Therefore, we conclude that there exists $\delta_1 \in (\delta_0, \tilde{\delta}]$ such that the numerator of the ratio in $\mathcal{T}_\chi(\bar{\sigma}^H)$ is positive for all $\chi \geq 0$ if $\delta \in (\delta_0, \delta_1)$. It follows that $\mathcal{T}_\chi(\bar{\sigma}^H) < 2$ for all $\chi \geq 0$ if $\delta \in (\delta_0, \delta_1)$. Let us then denote $\bar{\delta} = \delta_1$. As in case i), since $\lim_{\delta\to 0}\hat{\Theta} = 0$, we get $\lim_{\delta\to 0}\mathcal{S} = 0$ and thus $\lim_{\delta\to 0}\mathcal{T}_\chi(\bar{\sigma}^H) = -\infty$. Therefore, there exists $\underline{\delta} \in [\delta_0, \bar{\delta})$ such that when $\delta \in (\underline{\delta}, \bar{\delta})$, $\mathcal{T}_\chi(\bar{\sigma}^H) \in (-2, 2)$.

Finally, we may compute the bifurcation values of σ. The Hopf bifurcation value $\bar{\sigma}^H$ is given by (11), while the flip bifurcation value $\bar{\sigma}^F$ is such that $P_\sigma(-1) = 1 + \mathcal{T}_\chi(\sigma) + \mathcal{D}_\chi(\sigma) = 0$, which leads to the expression given by (12). The transcritical bifurcation value $\bar{\sigma}^T$ is such that $P_\sigma(-1) = 1 - \mathcal{T}_\chi(\sigma) + \mathcal{D}_\chi(\sigma) = 0$, leading to the expression given by (13).

We assume now that $\chi \in (\underline{\chi}, 2/3)$. The same proof as in the case $\chi \in [0, \underline{\chi})$ applies except that now $P_0(1) > 0$. Provided $\mathcal{T}_\chi(\bar{\sigma}^H) \in (-2, 2)$, it follows that depending on the value of Θ, the Δ_χ line is again located as in Figures 1(b), 4(b) or 2(b).

The last step consists finally in showing that $\mathcal{T}_\chi(\bar{\sigma}^H) \in (-2, 2)$. Let us note first that $\mathcal{T}_\chi(\bar{\sigma}^H)$ can be written as follows:

$$\mathcal{T}_\chi(\bar{\sigma}^H) \;=\; 2 - \frac{(1-\theta)\left(1-\frac{\beta\alpha\delta}{1-\theta}\right)\left[\bar{\sigma}^H + \frac{[1-\alpha(1+\Theta)](\chi-\underline{\chi})}{1-\alpha}\right]}{\bar{\sigma}^H(\Theta-\hat{\Theta}_\chi)+\left(1-\frac{\beta\alpha\delta}{1-\theta}\right)\frac{\theta\Theta(\chi+\alpha)}{\delta(1-\alpha)}}$$

As we have shown previously, $\bar{\sigma}^H > 0$ if and only if $\delta \in (\delta_0, 1)$ with δ_0 as given by (23). Moreover, there exists $\tilde{\beta} \in [\hat{\beta}, 1)$ such that $\delta_0 < \tilde{\delta} \leq \hat{\delta}$ when $\beta \in (\tilde{\beta}, 1)$. From now on, let $\beta \in (\tilde{\beta}, 1)$ and $\delta \in (\delta_0, \tilde{\delta})$. Under these restrictions, when $\Theta \in (\underline{\Theta}, \hat{\Theta}_\chi)$, Assumption 1 ensures that the denominator of the ratio in $\mathcal{T}_\chi(\bar{\sigma}^H)$ is positive. Moreover, when $\chi \in (\underline{\chi}, 2/3)$, the numerator of the ratio in $\mathcal{T}_\chi(\bar{\sigma}^H)$ is positive. It follows that $\mathcal{T}_\chi(\bar{\sigma}^H) < 2$. Let us then denote $\bar{\delta} = \tilde{\delta}$ and $\underline{\beta} = \tilde{\beta}$. Note that, as $\lim_{\delta\to 0}\hat{\Theta}_\chi = 0$, we get $\lim_{\delta\to 0}\mathcal{S} = 0$ and thus $\lim_{\delta\to 0}\mathcal{T}_\chi(\bar{\sigma}^H) = -\infty$. Therefore, there exists $\underline{\delta} \in [\delta_0, \bar{\delta})$ such that when $\beta \in (\underline{\beta}, 1)$ and $\delta \in (\underline{\delta}, \bar{\delta})$, $\mathcal{T}_\chi(\bar{\sigma}^H) \in (-2, 2)$.

Assume now that $\Theta > \hat{\Theta}_\chi$. The denominator of the ratio in $\mathcal{T}_\chi(\bar{\sigma}^H)$ is positive and thus $\mathcal{T}_\chi(\bar{\sigma}^H) < 2$ for all $\chi \in (\underline{\chi}, 2/3)$. We have shown previously that $den\bar{\sigma}^H > 0$ if and only if $\delta \in (\delta_0, 1)$, with $\lim_{\delta\to\delta_0} den\bar{\sigma}^H = 0$. Moreover, we derive from the expression of $\bar{\sigma}^H$ that $\lim_{\delta\to 1}\bar{\sigma}^H = 0$. Therefore, we conclude that there exists $\delta_1 \in (\delta_0, \tilde{\delta}]$ such that the numerator of the ratio in $\mathcal{T}_0(\bar{\sigma}^H)$ is positive and $\mathcal{T}_0(\bar{\sigma}^H) < 2$ if $\delta \in (\delta_0, \delta_1)$. Let us then denote $\bar{\delta} = \delta_1$ and consider the case $\delta = \delta_0$. As $den\bar{\sigma}^H = 0$ we derive

$$\Theta = \frac{\delta_0\alpha(1-\beta)[1-\theta(1-\alpha)+\chi]}{[(1-\alpha)(1-\theta)-\delta_0(1+\chi\alpha)](1-\theta)} \equiv \Theta_{\delta_0}$$

and $\bar{\sigma}^H = \infty$ with $\Theta_{\delta_0} > \hat{\Theta}_\chi$ for any $\chi \in (\underline{\chi}, 2/3)$. It follows therefore that

$$\mathcal{T}_\chi(\bar{\sigma}^H)\Big|_{\delta=\delta_0} \quad = \quad 2 - \frac{(1-\theta)\left(1-\frac{\beta\alpha\delta_0}{1-\theta}\right)}{\Theta_{\delta_0}-\hat{\Theta}_\chi}$$

Straightforward computations show that $\mathcal{T}_\chi(\bar{\sigma}^H)|_{\delta=\delta_0} > -2$ is equivalent to

$$\begin{aligned} 4(\delta_0\alpha)^2\left[1-\theta(1-\alpha)+\chi\right](1+\chi) \ > \ & [1-\theta-\beta\delta_0\alpha](1-\theta)(1-\alpha)(1-\delta_0) \\ & \times \ [(1-\alpha)(1-\theta)-\delta_0(1+\chi\alpha)] \end{aligned}$$

As this equality is satisfied when $\beta = 1$, there exists $\underline{\beta} \in [\tilde{\beta}, 1)$ such that $\mathcal{T}_\chi(\bar{\sigma}^H)|_{\delta=\delta_0} > -2$ for any $\beta \in (\underline{\beta}, 1)$. Denoting $\underline{\delta} = \delta_0$ we conclude that when $\delta \in (\underline{\delta}, \bar{\delta})$ and $\beta \in (\underline{\beta}, 1)$, $\mathcal{T}_\chi(\bar{\sigma}^H) \in (-2, 2)$.

□

5.4 Proof of Proposition 3

From the proof of Proposition 2, let $\hat{\Theta} = \hat{\Theta}_0$ and assume that $\Theta \in (\underline{\Theta}, \hat{\Theta})$. It follows easily that $\mathcal{D}_\chi(+\infty) > 1/\beta = \mathcal{D}_\chi(0)$, $P_{+\infty}(1) < 0$ and $P_{+\infty}(-1) > 0$ for any $\chi \geq 0$. Therefore, assuming $\beta \in (\tilde{\beta}, 1)$ and $\delta \in (\delta_0, \tilde{\delta})$, provided $\mathcal{T}_\chi(\bar{\sigma}^H) \in (-2, 2)$, we conclude that i) when $\chi > \underline{\chi}$, the Δ_χ line is located as in Figure 4(a), and ii) when $\chi < \underline{\chi}$, the Δ_χ line is located as in Figure 4(b). Depending on whether χ is larger or lower than $\underline{\chi}$, the arguments to prove that $\mathcal{T}_\chi(\bar{\sigma}^H) \in (-2, 2)$ are the same as those presented in the proof of Proposition 2.

□

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