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A Noncooperative Model of Contest Network Formation

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Abstract

Network structure has a significant role in determining the outcomes of many socio-economic relationships, including antagonistic ones. In this paper we study a situation in which agents, embedded in a network, simultaneously play interrelated bilateral contest games with their neighbours. Spillovers between contests induce complex local and global network effects. We first characterize the equilibrium of the game on an arbitrary fixed network. Then we study a network formation model, introducing a novel but intuitive link formation protocol. As links represent negative relationships, link formation is unilateral while link destruction is a bilateral action. The unique stable network topology is the complete k-partite network with partitions of different sizes. This model also provides a micro-foundation for the concept of structural balance, and the main results go in line with theoretical and empirical findings from other disciplines, including biology and sociology.

Key Words: Network formation, game on network, contest, structural balance
JEL: D85, D74

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1 Introduction

There is a number of situations in which agents can increase the probability of favourable outcome of a competitive interaction by means of certain costly actions[1]. We refer to this type of interaction as a contest. To be more precise, a contest is an interaction in which players can exert costly effort in order to (i) extract resources from other players or (ii) to receive a larger share of an exogenous prize. Following (Hillman and Riley, 1989) we refer to (i) as *transferable contest.* Agents often compete with several different opponents simultaneously. The contests an agent is involved in at the same time are related (i.e. an agent spends the same costly resources for each contest) which creates (local) spillovers. Another type of spillovers (global) comes from the fact that an agent's opponent can also be involved in more than one contest, as well as his opponents, and so on. As for how much the agent will spend in a particular contest depends on how much his opponents spend. This creates global spillovers.

This environment can be represented and explained more effectively in the language of networks. Let $G = G(N, L)$ be a network. Link $g_{ij} \in L$ indicates the presence of the contest between i and j. In this paper we shall focus on the case of transferable contests, and discuss the case(i) from above in the Appendix.

In this paper we first study a model on a given network of contests and provide results for the existence and the uniqueness of the equilibrium. This is very important in order to study network formation, which is the main focus of the paper. In the formation model, agents decide with whom to engage in a contest (form negative links), and which contest to end (destroy negative links). We interpret the absence of a negative link as a self enforcing commitment of agents not to engage in the contest[2]. The extension in which friendship (positive) links are explicitly modelled is discussed in the Section 5. As the network changes, the effort that players exert in each particular contest will, in general, change. Thus, the model of network formation can be thought of as a model of coupled evolution of network topology and play on the network. We are interested in stable networks - the networks in which no agent has an incentive to form additional links or destroy existing links. It's important to note that, given the nature of links, link formation is an unilateral decision of an agent (filing a suit, starting a war). On the other hand, for the destruction of a link (ending a litigation process, ending a war) both parties must agree to do it.

As an illustrative example, let us describe a real world contest network which is already studied in the literature - the network of litigations and antitrust disputes studied in (Sytch and Tatarynowicz, 2013). These types of lawsuits are often very intense and have significant consequences on the future of the company. They arise from the plaintiff's claims that an infringement has been made (patent litigation), or that a firm has adopted unfair competitive practices. The U.S. Federal District Courts registered about 10000 antitrust and 29000 patent infringement cases from 2000 to 2010 (Sytch and Tatarynowicz, 2013). These types of litigation have consequences for both contesting parties. A plaintiff demands that a

[1]Examples are armed conflict, litigation, lobbying, sport competitions, job tournaments

[2]In what follows we shall only talk about negative links.

defendant refrains from injurious acts and demands to be compensated for the losses. On the other hand, the plaintiff risks being counter-sued (which often happens). The costs of litigation are very high, reaching more than 5 millions USD per lawsuit, excluding damages and royalties. The transfers to be paid reach sums which are considerably higher. The firms can be, and usually are, involved in more than one litigation process at the same time. For example, in 2003., Lucent Technologies filed suit against Gateway and Dell in U.S. District Court, San Diego, claiming violation of patent rights. Microsoft joined the lawsuit later that year. After this lawsuit was filed, Microsoft and Lucent have filed additional patent lawsuits against each other. Finally, the court ruled that the amount of damages to be awarded is 1.53 billion [3]. Apart from this, Microsoft has fought numerous legal battles with other firms. These include litigation processes with Apple, Netscape, Intel, Sun Microsystems, Stac Electronics and many others. By modelling litigation process as transferable contests, mapping firms to nodes, and litigation processes to links (indicating contests) we construct a contest network. Another example is the network of Massive Multi-player Online Role Playing Games (MMORPG) analysed in (Szell et al., 2010). They analyse data from the complete population of 300.000 players of the game Pardus. It is a game where players live in a virtual world in which they interact with other in a variety of ways, including engaging in a conflict and friendships/alliances. Practically all actions of all players are recorded in the game log files which allows authors to conduct a detailed analysis of the virtual society from as a signed network. They find strong support in favour of structural balance theory (Subsection 1.1). All stable networks in our model also satisfy the structural balance property. Other examples of contest networks include networks of international conflict, patent races, lobbying, school violence etc. In this paper we don't focus on any specific situation. Instead, we abstract from idiosyncrasies and build an abstract model which captures the following qualitative properties of contest-type relations: (i) they are costly (ii) the probability of a favourable outcome increases with own effort and decreases with opponents effort (iii) there are spillovers between contests an agent is involved in (iv) starting a contest is a unilateral decision and ending a contest is a bilateral decision

The results of the model also have important implications for the structural balance theory from social psychology. Let us informally introduce this theory:

1.1 Structural Balance Theory

The theory of (strong) structural balance, originated in (Heider, 1946), applies to situations in which relations between agents can be either negative (antagonistic) or positive (friendship). It states that in groups of three agents, the only socially and psychologically stable structures are those in which all three agents are friends (all links are positive) or two of them are friends, with the third being a common enemy (one positive and two negative links). In other words, a friendship relation is transitive.

As defined in (Heider, 1946), structural balance can be seen as a local property of a network. A natural question then is: 'What are the global properties of networks that satisfy

[3]Details available in (McDougall, 2007).

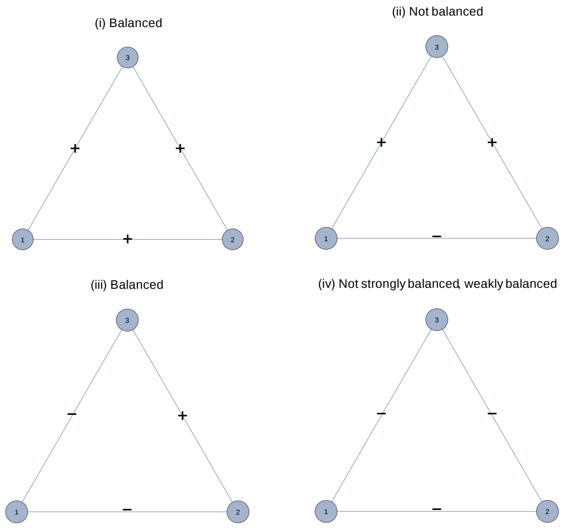


Figure 1: Structural balance

structural balance?'. That is, given a complete network, how can we sign links (indicating positive and negative) so that all triads of nodes in the network are structurally balanced? The Cartwright-Harary Theorem (Cartwright and Harary, 1956) provides the answer to this question. It states that there are two network structures that satisfy structural balance: (i) all agents are friends (there are no negative links) or (ii) agents are divided into two groups, and links within groups are positive and links across groups are negative. With respect to positive links, a network that satisfies structural balance will be the complete network or a network with two components that are cliques. With respect to negative links, it will be either the empty network or the complete bipartite network. It has been argued in (Davis, 1967) that in many contexts we may witness a situation in which all links in a triad are negative. To encompass this type of configuration, the concept of weak structural balance is proposed. The implication for the global structure when allowing for this type of triads is an emergence of the additional balanced network structure. With respect to positive links this is a network with more than 2 components, where each component is a clique. With respect to negative links it is the complete k-partite network.

There is a number of empirical papers that support (weak) structural balance in the real world networks. For example (Antal et al., 2006; Szell et al., 2010; Sytch and Tatarynowicz, 2013). On the theoretical side, the only paper known to us that aims to provide a micro-foundation for structural balance is (Hiller, 2011). However, the interaction between agents in (Hiller, 2011) paper is modelled differently than in our paper, and different equilibrium concept is used. Furthermore, in (Hiller, 2011) agents do not decide how much to invest in negative relations and thus the paper does not say anything about intensities of contests. Given the differences it is very interesting to note that results regarding structural balance from (Hiller, 2011) are in line with the results from this paper, which goes in favour of the robustness of results from both papers.

This paper provides a model of non-cooperative network formation that results in stable networks that are always weakly balanced (they satisfy weak structural balance). The strong structural balance is satisfied in particular cases. It is important to note that the structural balance is a concept concerned only with the sign of links, but does not say anything about the intensities/weights assigned to links. Stable networks in our mode are signed and weighted, therefore the model provides implications that go much beyond structural balance theory. Thus, the micro-foundation of structural balance is just one of the results of the paper.

This paper is related to several different streams of literature. First, there is a wast literature on games on fixed networks. The closest paper to ours is (Franke and Öztürk, 2009). Section 3 of this paper can be seen as a generalization of their main result. However, (Franke and Öztürk, 2009) say nothing about network formation, which is the central issue stressed in this paper. An important property of our model is that agents choose their opponents as well as the effort in each particular contest. This makes our paper close to the literature on the formation of weighted networks, but also to the literature that jointly considers network formation and playing games on the network ((Bloch and Dutta, 2009), (Deroïan, 2009)). However, the contest game has a very different properties than games considered in these papers - it is neither the game of strategic complements nor strategic substitutes. Informally, a contest is a game in which players decide (simultaneously or sequentially) on the level of effort in order to increase the probability of winning the (endogenous or exogenous) prize. The Contest Success Function (CSF) is a function that describes how the efforts determine the probability of winning the contest. An example is litigation, where the prize is the value of transfer between parties.

There are two prominent ways to model CSF. The first is to assume that the probability of winning is a function of the ratio of efforts, which is introduced in (Buchanan et al., 1980) . This is the approach we use in this paper[4]. The second way to model CSF assumes that the probability of winning is a function of the difference between efforts. This approach is introduced in (Hirshleifer, 1989). In this paper we consider transferable contests as introduced in (Hillman and Riley, 1989) using the variant of Tullock's specification introduced in (Nti, 1997). An alternative model, which is offered in Appendix B, gives a model formulation as a colonel Blotto game with Tullock CSF.

2 Bilateral Contest Game

In this section we introduce the bilateral contest game which will serve as a building block of the model. There are two players, i and j competing over a prize of exogenous size, normalized to R. In order to increase the probability of wining, players choose a non-negative action (effort, investment). The strategy space is thus given with the set of non-negative real numbers $S = \mathbb{R}_0^+ = [0, +\infty)$. The effort is transformed into the contest specific resource by means of technology function ϕ. One can think of this function as analogue to the production function in a classic market setting. We assume that the technology function $\phi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ satisfies the following properties:

[4] This formulation is also known as Tullock CSF

Assumption 1. *Technology function* $\phi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *is:*

(i) Continuous and twice differentiable

(ii) increasing and (weakly) concave ($\phi' > 0$, $\phi'' \leq 0$)

(iii) $\phi(0) = 0$

The first two assumptions are standard, while the third one states that zero effort implies zero contest input. The probability of wining the prize is defined with contest success function. We choose the Tullock ratio form specification suggested in (Nti, 1997). This means that the probability player i will win, when taking action s_{ij} in contest with player j, choosing action s_{ji} is:

$$p_{ij} = \frac{\phi(s_{ij})}{\phi(s_{ij}) + \phi(s_{ji}) + r} \tag{1}$$

In (1) $r \in \mathbb{R}_0^+$ determines the probability of a draw (no player wins the prize). In the paper we shall maintain the assumption that r is small.

Following (Hillman and Riley, 1989) we consider transferable contest game, that is a game in which the prize is transfer from loser to winner. Assuming the fixed transfer, the payoff function of player i is given with

$$\pi_i(s_{ij}, s_{ji}) = p_{ij}R - p_{ji}R - c(s_{ij})$$

where R is a transfer from loser to winner. We assume that the transfer from i to j is the same as the transfer from j to i, although of course in general this does not have to be the case. The cost function $c : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is assumed to be continuously differentiable, increasing and convex.

The bilateral contest game has the unique and symmetric NE in pure strategies, which is interior for r low enough. In this case, the equilibrium strategy of player i is defined with the following implicit function:

$$\phi'(s_{ij}^*)R = (r + 2\phi(s_{ij}^*))c'(s_{ij}^*)$$

3 Game on a Fixed Network

Let $G = (N, L)$ be an undirected and unweighed network with set of nodes N and set of links L. The nodes represent players, and link $g_{ij} \in L$ indicates that there is contest between players i and j (when $g_{ij} = 1$) [5]. Let us also denote the set of neighbours of agent i as N_i, and let $d_i = |N_i|$ denote the degree of node i. Strategy space of player i is the set $S_i = {\mathbb{R}_0^+}^{d_i}$. A (pure) strategy of player i is d_i-tuple $\mathbf{s}_i = (s_{ij_1}, ..., s_{ij_{d_i}}) \in S_i$. We assume that the size of the transfer R is independent of the network structure and the same for every contest g_{ij} [6] and normalize $R = 1$.

[5] The paper focuses on negative (contest) links. The absence of negative links can be interpreted as a commitment not to initiate contest and thus a positive (friendly) link

[6] We use g_{ij} when we talk about link $g_{ij} \in L$ but also when referring to the contest between players i and j.

The payoff of player i is given with:

$$\pi_i(\mathbf{s_i}, \mathbf{s_{-i}}\ G) = \sum_{j \in N_i} \left(\frac{\phi(s_{ij})}{\phi(s_{ij}) + \phi(s_{ji}) + r} - \frac{\phi(s_{ji})}{\phi(s_{ij}) + \phi(s_{ji}) + r} \right) - c(A_i) \tag{2}$$

In (2), $A_i = \sum_j s_{ij}$ is the total effort of player i, and $\boldsymbol{s}_{-i}$ denotes strategies of players other than i. This specification of cost function generates externalities between the contest that agent i is involved in, making it more interesting to study this model on a network[7]. To be able to study network formation, we need to know if the equilibrium strategies on a fixed network are uniquely determined. In this section we prove the uniqueness of the equilibrium on a fixed network. For the sake of the presentation let us first introduce the following definition:

Definition 1. *A game is n persons concave game if (i) Strategy space of game S is the product of closed, convex and bounded subsets of m dimensional Euclidian space, $S = \{S_1 \times S_2 \times ... \times S_n | S_i \subset E^{m_i}\}$* [8] *and (ii) payoff function of every player* $_2, ..., \mathbf{s}_n)$, *and concave in* $\mathbf{s}_i \in S_i$, *for each fixed value* $\mathbf{s}_{-i} \in S_{-i}$

Let us also introduce the function $\sigma : S \times \mathbb{R}_0^{+n} \to \mathbb{R}$ assigned to n persons concave game given with $\sigma(\mathbf{s}, \mathbf{z}) = \sum_{i=1}^{n} z_i \pi_i(\mathbf{s})$. Then, as proved in (Rosen, 1965):

1. There exists a pure strategy equilibrium of the n persons concave game
2. If function σ is diagonally strictly concave for some $\mathbf{z} \geq 0$ then the equilibrium is unique

Proposition 1. *The contest game on a network has the unique Nash equilibrium which is in pure strategies*

Proof. See Appendix A □

When the probability of a draw is small enough, players will always exert a positive level of effort in the equilibrium. The following proposition states exactly that:

Proposition 2. *The equilibrium is interior when $r > 0$ is small enough*

Proof. See Appendix A □

In what follows, we will assume that r is chosen such that the interiority of the equilibrium is guaranteed. This implies that the equilibrium of the game on a fixed network is defined with the FOC system of equations. Consider now any two connected players i and j. The first order conditions that characterize their behaviour in a contest g_{ij} in the equilibrium are given with:

$$\left(\frac{(r + 2\phi(s_{ji}))\phi'(s_{ij})}{(r + \phi(s_{ij}) + \phi(s_{ji}))^2} - c'(A_i) = 0 \right) \wedge \left(\frac{(r + 2\phi(s_{ij}))\phi'(s_{ji})}{(r + \phi(s_{ij}) + \phi(s_{ji}))^2} - c'(A_j) = 0 \right) \tag{3}$$

[7]Note that the same effect would be present with an appropriate modification of function ϕ

[8]Rosen actually proved more general result when strategy space is 'coupled', that is when $S \subset E^m = E^{m_1} \times E^{m_2} \times ... \times E^{m_n}$ is closed, convex and bounded set. Here we consider special case when strategy space is 'uncoupled'

From (3) we get:

$$\frac{(r + 2\phi(s_{ji}))\phi'(s_{ij})}{(r + 2\phi(s_{ij}))\phi'(s_{ji})} = \frac{c'(A_i)}{c'(A_j)} \tag{4}$$

As $\phi' > 0$ and $\phi'' \leq 0$ and $c'' > 0$ we have: $A_i > A_j \Leftrightarrow \frac{c'(A_i)}{c'(A_j)} > 1 \Leftrightarrow \frac{(r+2\phi(s_{ji}))\phi'(s_{ij})}{(r+2\phi(s_{ij}))\phi'(s_{ji})} \Leftrightarrow s_{ji} > s_{ij}$ where the last equivalence is due to the fact that ϕ is increasing and ϕ' is decreasing function.

This means that in the equilibrium the player with the lower total spending will win a contest with higher probability. This observation reflects the fact that the more 'exhausted' player (one who spends more resources in the equilibrium) performs worse in a particular contest. It is because the additional unit of resources is more costly for him (his marginal costs are higher). Note that this does not necessarily mean that a player involved in more contests will have a higher total spending in the equilibrium, although the total spending is increasing with the number of opponents (keeping everything else fixed). Rather, the player with the worst performance will be the player who is involved in the large number of contests and the contests that he is involved in are more intensive. Which contest will be more intensive, depends on the global position of the players in the network. The second important issue to note is that the contest game is neither the game of strategic substitutes nor strategic complements. The effort of player i will be increasing in the effort of player j if i is winning the contest (has a higher probability of winning). Otherwise it will be decreasing.

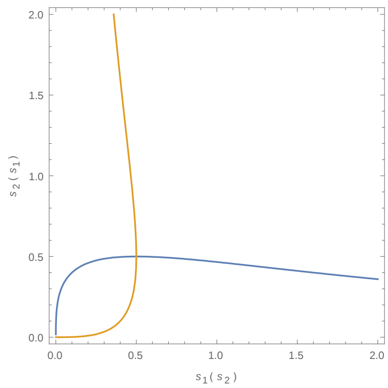


Figure 2: Bilateral contest game - best reply functions

The additional issue worth noting is that the highest total effort in the equilibrium doesn't imply the lowest payoff - although the player with the highest total effort is the 'weakest' player in the network. This is because the equilibrium payoff is determined by the effort and the number of opponents. For example, the total equilibrium efforts of the green and a red agent in Figure 3 are $A_g = 1.11$ and $A_r = 0.95$, but still $\pi_g = -1.63$ and $\pi_r = -1.85$. The reason for this is twofold. The green agent is involved in more contests so the total transfer she can win is higher. The second reason is that red agents lose against blue agents with much higher probability than they win against the green agent.

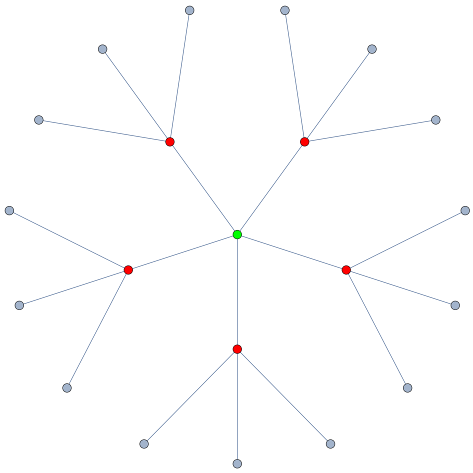


Figure 3: Payoff vs total effort

The identification of the characteristic of a node in the network that would determine the total spending of that node in the equilibrium, although interesting, proved to be a very complex task.

4 Network Formation

The fact that a player with higher total spending in the equilibrium, loses in expectation from a player with lower total equilibrium spending (given that they are connected), gives some hints on how agents behave when contests are determined endogenously. But one must note that the results from the previous section are ex-post, and cannot be directly used in a network formation model. The fact that $A_i^* < A_j^*$ in the equilibrium on network G does not imply that we will still have $A_i^* < A_j^*$ in the equilibrium on network $G + g_{ij}$ (where $+$ denotes addition of the link g_{ij} to the network). When link g_{ij} is created, players i and j will, in general, change their efforts in all other contests that they are involved in. This will, furthermore, result in changes in the equilibrium actions of all opponents of i and j in all of their contests; all according to the system of nonlinear equations defined with (3). Given the general structure of the network, one can see why the effects of link creation, which is a 'fundamental' action of the network formation, are in general case very hard to completely characterize. An example of the global effects caused by adding a link in a very simple network is given in Appendix C.

Because of the above described complexity, we shall assume that agents are not able to fully take into account these effects when making the decision to create or sever a link. Instead, when making this decision, they assume that all other actions in the network will remain constant. If the action is to create a link, the assumption is that equilibrium efforts of that particular contest game will be according to the NE of the bilateral contest game discussed in Section 2, keeping all other actions in the network fixed. We believe that the bounded rationality assumption is justified due to the very complex nature of the spillovers. We shall discuss this issue more formally later.

In what follows we assume that r is sufficiently small, so that the equilibrium of the game on a fixed network is always interior. For simplicity, we shall also assume that ϕ is identity mapping. However, all results hold when ϕ has a general specification from the previous section.

Let us now be more precise. First we introduce the following definition.

Definition 2 (Actions equilibrium). *A network $G(N, L)$ is in the actions equilibrium if all actions s_{ij} and s_{ji} assigned to every link $g_{ij} \in L$ are part of the unique NE equilibrium of the game on a fixed network*

We think of network formation as a discrete dynamic process. Time is indexed with $t \in \mathbb{N} \cup \{0\}$. In period $t = 0$ an arbitrary contest network $G(N, L)$ is given[9].

For every period t:

(i) At the beginning of period t the network from $t - 1$ is in the actions equilibrium

(ii) Random player i is chosen and updates her links according to the link formation protocol, resulting with network G_{t+1}

(iii) The second dynamic process (*action adjustment process*)[10] starts, and in it all agents update their strategies according to the process formally described in Subsection 4.1 (better reply dynamics), until the actions equilibrium is reached

Steps (ii) and (iii) require some further explanation. First let us define the link formation protocol.

Definition 3 (Link formation protocol). *A link g_{ij} will be formed if player i or j decide to form it. A link g_{ij} will be destroyed if both i and j agree to destroy it.*

This means that the link formation is a unilateral and link destruction is a bilateral action. It is natural to define a link formation protocol for contest relations in this fashion. A decision to start a contest (i.e. war, litigation) is unilateral by nature, and the 'attacked' player, whether she decides to fight back or not, cannot change that. To end a contest, it is necessary that both parties agree to do it. To our knowledge, this is the first paper that considers such a link formation protocol.

Let us now clarify what we mean exactly when we say that agents update their connections myopically. When deciding on her connections agent i knows the total spending of all players in the existing network. The effort levels (s_{ij}, s_{ji})[11] assigned to the newly formed link are determined as the solution of a bilateral contest game, keeping all other actions in

[9]Due to the 'zero sum' nature of the contest game, the empty network will always be stable in our model. In order to describe the dynamic process that leads to a non-empty stable networks we assume that, because of some non modelled mutation or a tremble, the initial conditions are given with the non-empty arbitrary network

[10]We shall assume that this process is continuous (it happens on a much faster time scale than network formation process which is discrete).

[11]We omit time index t

the network fixed. For example, in the case of quadratic cost function $c(x) = \frac{1}{2}x^2$, when link g_{ij} is created the corresponding actions s_{ij} and s_{ji} are determined as the solution of:

$$\frac{2s_{ji} - r}{(s_{ij} + s_{ji} + r)^2} = (A_i + s_{ij}) \quad \wedge \quad \frac{2s_{ij} - r}{(s_{ij} + s_{ji} + r)^2} = (A_j + s_{ji})$$

which is:

$$s_{ij} = \frac{2 + A'_i \left(A'_i + A'_j - \sqrt{4 + (A'_i + A'_j)^2} \right)}{2\sqrt{4 + (A'_i + A'_j)^2}} > 0 \tag{5}$$

and symmetric for s_{ji}. Here $A'_i = A_i - r/2$. Player i will wish to form link g_{ij} when:

$$\frac{s_{ij} - s_{ji}}{(s_{ij} + s_{ji} + r)} + A_i^2 - (A_i + s_{ij})^2 > 0$$

and (s_{ij}, s_{ji}) are determined with (5), and analogously for player j.

On the other hand, the existing link g_{ij} will be destroyed if both players agree to destroy it, that is when $\pi_i(\mathbf{s}_i, \mathbf{s}_{-i}, G - g_{ij}) > \pi_i(\mathbf{s}_i, \mathbf{s}_{-i}, G)$ and $\pi_j(\mathbf{s}_j, \mathbf{s}_{-j}, G - g_{ij})) > \pi_i(\mathbf{s}_j, \mathbf{s}_{-j}, G))$. This will be the case when:

$$A_i^2 - (A_i - s_{ij})^2 - \frac{s_{ij} - s_{ji}}{(s_{ij} + s_{ji} + r)} \geq 0 \wedge A_j^2 - (A_j - s_{ji})^2 - \frac{s_{ji} - s_{ij}}{(s_{ij} + s_{ji} + r)} \geq 0$$

We also assume that a player will create a link only if it is strictly beneficial to do so. If a player is indifferent between keeping and destroying link, the link will be destroyed. So, a player prefers to be involved in less contests. This tie-breaking rule does not bear significant implications on the results.

If after some period t^* no player wishes to destroy or create link we say that the process has reached the steady state. Thus, a network is stable when no player can myopically improve her payoff by changing their linking strategy.

Definition 4 (myopically stable network)**.** *A network $G = G(N, L)$ is a myopically stable network if for any player i and any two (possibly empty) sets of nodes $A \subset N$ and $B \subset N$.*

$$\pi_i(G + \{g_{ij}\}_{j \in A} - \{g_{ij}\}_{j \in B}) > \pi_i(G) \Rightarrow (\exists j \in B) : \pi_j(G - g_{ij}) < \pi_j(G)$$

$$\pi_i(G + \{g_{ij}\}_{j \in A}) < \pi_i(G)$$

This definition assumes that no player wishes to change their linking strategy - to destroy or create links. The possibility of replacing link is essential for the results. However, it does not matter if a player can replace only one or more of his links or destroy/create one or more links at the same time. The results will (qualitatively) hold if we consider a process in which an agent in a single period can only create a link, destroy a link, or replace a link. That is if we consider the following definition of stability:

Definition 5 (myopically stable network - alternative)**.** *A network $G = G(N, L)$ is Myopically stable network if the following conditions hold:*

$$\pi_i(G - g_{ij}) > \pi_i(G) \Rightarrow \pi_j(G - g_{ij}) < \pi_j(G) \ (\forall i, j \in N)$$

$$\pi_i(G + g_{ik} - g_{ij}) > \pi_i(G) \Rightarrow \pi_j(G - g_{ij}) < \pi_j(G) \ (\forall i, j, k \in N)$$

$$\pi_i(G + g_{ij}) < \pi_i(G) \ (\forall i \in N)$$

In what follows we provide a characterization of stable networks. Let us consider a network G which is in the actions equilibrium. We can order nodes in increasing order with respect to their total spending ($A_1 < A_2 < ... < A_K$), $K \leq n$ where K is the number of different total spending levels in a network. Note that we use A_i to denote both the total spending of player i and the $i-th$ smallest level of total spending in the network. From context it will be always clear what A_i stands for. Recall also that the equation (4) implies that in any bilateral contest, a node that has larger total spending loses in expectation.

Denote with $\mathcal{A}_i$ the class of nodes that have total spending A_i. Let $K \leq n$ denote the number of classes in network G. When a player $i \in N$ has the total spending A_i we denote that as $i \in \mathcal{A}_i$. We say that player i has control over link g_{ij} if it is beneficial[12] for player j to destroy link g_{ij}. Thus, when player i is in control over a link it is completely up to him if the link will be destroyed.

If $A_i > A_j$ in the actions equilibrium we say that player j is stronger than player i or that player i is weaker than player j. We shall refer to A_i as the strength of player i (higher A_i implies weaker player i). It is clear that when i is stronger than j then i controls link g_{ij}. Furthermore, both players i and j shall have control over link g_{ij} if this link is not beneficial for any of them. Abusing the notation, let $\pi_i(s^*_{ij}, g_{ij}) = \frac{s^*_{ij}-s^*_{ji}}{(s^*_{ij}+s^*_{ji}+r)} - c(A^*_i)$ denote the equilibrium payoff of player a from link g_{ab} in the actions equilibrium. Then the following holds:

Proposition 3. *Let $a \in \mathcal{A}_i$, $b \in \mathcal{A}_j$, $c \in \mathcal{A}_k$ and $i < j < k$. Then $s^*_{ab} > s^*_{ac}$, $s^*_{ba} > s^*_{ca}$, furthermore $\pi_a(s^*_{ab}, g_{ab}) < \pi_a(s^*_{ac}, g_{ac})$*

Proof. See Appendix A ☐

Proposition 3 delivers the main intuition of our result. It implies that the contest between two players who are more equal in strength is more costly to win. A strong player spends less when competing with a weaker player and has a higher payoff. The result of this proposition illustrates the incentive that a strong player has to compete with weak players, given that the transfer for every contest is the same. This effect is self-enforcing in the sense that it further weakens the weak player, thus making him a more likely target for other strong players. For the sake of the exposition, let us state the following definition.

Definition 6. *Player $a \in \mathcal{A}_i$ is an attacker (winner) if he has all of his links with players from the family of classes $\overline{\mathcal{A}}_i = \{\mathcal{A}_j | j > i\}$. Player $a \in \mathcal{A}_i$ is mixed type if there exist players b and c such that $g_{ab}, g_{ac} \in G$ and $A_b > A_a > A_c$. Player $a \in \mathcal{A}_i$ is a victim (loser) if he has all of his links with players from classes $\underline{\mathcal{A}}_i = \{\mathcal{A}_j | j < i\}$*

It is clear that every player i must be an attacker, a victim or mixed type. Note also that in a stable network all attackers must have a positive payoff. If this is not true for some attacker i then, since she controls all of her links, she could deviate and destroy these links. This deviation would be profitable. Furthermore, there cannot exist a player in a stable

[12]A link g_{ij} is said to be beneficial for player i if the creation of this link (if it did not exist) makes player i better off and if the destruction of this link (if it does exist) makes player i worse off

network that is not stronger than all opponents of some attacker, and still not be connected to this attacker. This is because the attacker always has an incentive to switch from a strong to a weak opponent. This is the intuition behind Lemma 1.

Lemma 1. *Let $a \in \mathcal{A}$, and $\mathcal{A}$ is the class of attackers. Let b and c be two nodes in the network such that $A_b^* \leq A_c^*$, $g_{ab} \in G$ and $g_{ac} \notin G$. Then the deviation of player a in which she replaces contest g_{ab} with g_{ac} is payoff improving.*

Proof. See Appendix A □

From Lemma 1 we have:

Corollary 1. *If in a stable network player $a \in \mathcal{A}_i$ has a link with player $b \in \mathcal{A}_j$ then she has a link with every player $c \in \mathcal{A}_{j+k}$ $k = 1, 2, ..K - j$*

Proof. See Appendix A □

Lemma 1 implies that if there is a player k in the network that is not stronger than at least one opponent j of player i, and if player i has control over g_{ij} link she will have an incentive to replace link g_{ik} with g_{ij}. This furthermore implies that there cannot be more than one component in a stable non-empty network. Indeed, by Corollary 1 all nodes that have at least one link in the stable network must have a link towards the weakest player (except that player, of course) - as he is the most attractive player to extract resources from. So the following result holds:

Lemma 2. *A stable network must be connected if not empty*

Proof. See Appendix A □

From now on, we always talk about connected networks. The above results follow from the attractiveness of weak players as a victims. Let us focus now on the attackers. By definition, an attacker has a control over all of her links. Thus it is always possible for an attacker to imitate a strategy of another attacker. Building on this observation we can state and prove the following:

Lemma 3. *If in a stable network two players belong to the same class of attackers $\mathcal{A}$ than they have the same neighbourhood*

Proof. See Appendix A □

Since all attackers in the same class have the same neighbourhood it must be that they have the same payoff in a stable network. Suppose that there is more than one class of attackers and that members of different classes have different payoffs. Since attackers have a control over their links, the members of a class with lower payoff have an incentive to match the strategy of members of the class with higher payoff. This process will eventually lead to all attackers in the network having the same payoff. Furthermore, the incentive to attack weaker players will push all attackers to have the same neighbourhood, and thus to become members of the unique class of attackers.

Lemma 4. *There is only one class of attackers in a stable network*

Proof. See Appendix A □

Note that Lemma 4 and Corollary 1 imply that the members of the unique class of attackers are connected to all other nodes in the network (but not with each other). This is due to the fact that class $\mathcal{A}_2$ must be a class of mixed types or victims. In either case, Lemma 4, together with the fact that two players from the same class cannot be connected in a stable network, implies that all members of $\mathcal{A}_1$ and $\mathcal{A}_2$ are connected. Then Corollary 1 implies that attackers are connected to all other nodes in the network which don't belong to $\mathcal{A}_1$.

Let us now say something about mixed types in a stable network. Following the same reasoning as in the case of the attackers, we can conclude that all members of a mixed class must have the same neighbours with respect to players that are weaker than they are. The same thing will hold with respect to the neighbours that are stronger than they are. We formulate and prove this result in Lemma 5.

Lemma 5. *In a stable network all members of all existing mixed type classes $\mathcal{A}$ are connected to all other nodes in the network except nodes from their class.*

Proof. See Appendix A □

It is now immediate:

Corollary 2. *There is only one class of victims and all victims have the same neighborhood*

We have shown so far that we can partition all the players in a stable network in $k < n$ different classes with respect to their equilibrium spending. There is only one class of attackers and only one class of victims. All members of the same class of players have the same neighbourhood. Let us now say something about the size of partitions in a stable network. The results from Lemma 1, Lemma 5 and the previous corollary imply that players will be connected to all other players in the network except members of their own class. So, one could expect that a player that belongs to larger class (has more friends) has a higher payoff in the equilibrium comparing to a member of smaller class. This intuition is correct, and the following Lemma holds.

Lemma 6. *Let $|\mathcal{A}_k|$ denote the number of nodes that belong to class $\mathcal{A}_k$. Then $|\mathcal{A}_k| > |\mathcal{A}_{k+1}|$ $\forall k \in \{1, ..., K\}$*

Proof. See Appendix A □

It is clear that $|\mathcal{A}_k| > |\mathcal{A}_{k+1}|$ is not a sufficient condition for the stability of the network. The difference $|\mathcal{A}_k| > |\mathcal{A}_{k+1}|$ must be large enough so that members of the stronger class do not find it payoff improving to delete links with members of the weaker class. We are now ready to state the main result of the paper.

Proposition 4. *A stable network is either an empty network or a complete k-partite network with partitions of different sizes. The payoff of members of a partition is increasing with the size of the partition and the total spending per node is decreasing with the size of a partition.*

Recall that the complete k-partite network is the only network topology that satisfies weak structural balance property, as discussed before. When the cost function is too steep, or the transfer size is small, the only nonempty stable network would be the complete bipartite network. The complete bipartite network (with respect to negative links) is the only network topology that satisfies the strong structural balance property.

Not all complete k-partite networks will be stable. In order for them to be stable, no player must have an incentive to create or destroy a link. As the only links that can be created are between players from the same partition, no player will wish to create a link. This is because link g_{ij} between players i and j such that $A_i = A_j = A$ cannot be profitable (they will exert the same effort in the equilibrium and thus win and lose the contest with the same probability, and since the effort is costly, have a negative net payoff from contest g_{ij}). No player will wish to destroy a link if all links bring a positive payoff to the winner. Combining equilibrium conditions for players i and j we get that this will be the case when[13]

$$\left(s_{ij} = \frac{c'(A_j)}{c'(A_i)} s_{ji} \wedge \frac{2s_{ji}}{(s_{ij}+s_{ji})^2} = c'(A_i)\right) \Rightarrow s_{ij} = \frac{2c'(A_j)}{(c'(A_i)+c'(A_j))^2} \tag{6}$$

Using (6) we can express the sufficient conditions for stability of a network in terms of the total spending in the eqilibrium, that is we have that a complete k-partite network will be stable when for any contest g_{ij} we have:

$$2\frac{c'(A_j)-c'(A_i)}{c'(A_j)+c'(A_i)} > c(A_i) - c\left(\frac{2c'(A_j)}{(c'(A_i)+c'(A_j))^2}\right)$$

Let us consider the complete bipartite network. Note that, due to symmetry, agents that belong to the same partition will play the same strategy in every contest g_{ij} . Then all contests g_{ij} will result in a positive payoff for winners iff members of the larger partition have a (total) positive payoff in the equilibrium. Denote the two partitions with $\mathcal{X}$ and $\mathcal{Y}$, and sizes of partitions with x and y respectively, and let $x > y$. Then total efforts of members of two partitions can be written as $A_X = ys_X$ and $A_Y = xs_Y$, where $s_i,\ i \in \{\mathcal{X},\mathcal{Y}\}$ is the equilibrium effort level in each particular contest of members of partition i. Using (6) we get that:

$$\pi_X(s_X,s_Y) > 0 \Leftrightarrow y\frac{c'(A_Y)-c'(A_X)}{c'(A_Y)+c'(A_X)} - c(A_X) > 0 \Leftrightarrow \frac{c'(A_Y)}{c'(A_X)} > \frac{y+c(A_X)}{y-c(A_Y)}$$

With cost function $c(x) = \frac{1}{2}x^2$, $s_X = \sqrt{\frac{\sqrt{x}}{\sqrt{y}\left(\sqrt{x}+\sqrt{y}\right)^2}}$ The payoff of an agent from partition X is then:

$$\pi_X(s_X,s_Y) = b\frac{s_X - s_Y}{s_X+s_Y} - (bs_X)^2 = \frac{x\left(x-y-\sqrt{xy}\right)}{\sqrt{x}+\sqrt{y}}$$

[13] We let $r \to 0$ to simplify the expression. We omitt *, but it should be clear that equation (6) is calculated in the actions equilibrium

and from here

$$\pi_X(s_X, s_Y) > 0 \Leftrightarrow x > y\left(\frac{3+\sqrt{5}}{2}\right)$$

Thus, we have proved the following proposition:

Proposition 5. *For $c(x) = \frac{1}{2}x^2$ and $\phi(x) = x$ a complete bipartite network will be stable when $x > y\left(\frac{3+\sqrt{5}}{2}\right)$ where x and y are sizes of partitions.*

The payoff of players in the larger partition will be increasing with the size of the larger partition, and increasing in the number of players in the smaller partition for $b \leq \frac{b}{6}\left(14 + \sqrt[3]{1475 + 8\sqrt{41}} + \sqrt[3]{1475 - 8\sqrt{41}}\right) \approx 6.07b$, and decreasing otherwise. There are two effects on the payoff of members of the larger partition when increasing the number of players in the smaller partition. The first one is that the contests become more costly, as the members of smaller partition become stronger (spend less in total in equilibrium). The second effect is that there are more opportunities to extract rents. Depending on which effect dominates, the payoff of agents from the larger partition will increase or decease with the size of smaller partition.

4.1 Actions Adjustment Process

As we have discussed in the beginning of Section 4, after the network structure is changed, players update their strategies until the actions equilibrium on the new network is reached. In this subsection we describe this process and prove that it is globally asymptotically stable. The actions adjustment process is defined as follows:

$$\frac{d\mathbf{s}_i}{dt} = \alpha \nabla_i \pi_i(\mathbf{s}),\ \alpha > 0,\ i = 1, .., n \tag{7}$$

where $\pi_i(\mathbf{s}) = \pi_i(\mathbf{s}_1, \mathbf{s}_2, ..., \mathbf{s}_i, ...\mathbf{s}_n)$ and $\nabla_i \pi_i(\mathbf{s}) = \begin{pmatrix} \frac{\partial \pi_i}{\partial s_{i1}} & \frac{\partial \pi_i}{\partial s_{i1}} & ... & \frac{\partial \pi_i}{\partial s_{id_i}} \end{pmatrix}$ is gradient of the payoff function with respect to $\boldsymbol{s_i}$. It is clear that Nash equilibrium is a steady state of this dynamics. We prove in what follows that NE is the globally asymptotically stable state of this dynamic system. Let us define a function $J : \prod_i [0, M]^{d_i} \to \prod_i [0, M]^{d_i}$ with:

$$J(\mathbf{s}) = \begin{pmatrix} \nabla_1 \pi_1(\mathbf{s}) \\ \nabla_2 \pi_2(\mathbf{s}) \\ ... \\ \nabla_n \pi_n(\mathbf{s}) \end{pmatrix}$$

Denoting with G the Jacobian of J with respect to $\mathbf{s}$, we can write system (7) in a more compact form

$$\dot{\mathbf{s}} = \alpha J(\mathbf{s}) \tag{8}$$

To prove global stability we need to show that the rate of change of $||J|| = JJ'$ is always negative (and equal to 0 in equilibrium). So let us check $\frac{d}{dt}||J||$. We get:

$$\frac{d}{dt}JJ' = (G\dot{\mathbf{s}})'J + J'G\dot{\mathbf{s}} = (J'G'J + J'GJ) = J'(G' + G)J$$

The conditions (i)-(iii) discussed in the proof of Proposition 1 imply that $(G' + G)$ is a negative definite matrix. This implies that $\frac{d}{dt}JJ' < 0$ which is what we need to prove.

Thus, if every player adjusts her actions according to the adjustment process in (8), the action adjustment process converges, irrespectively of the initial conditions. The process (8) can be made discrete without losing the convergence properties. The discussion from above proves the following proposition:

Proposition 6. *The action adjustment process is globally asymptotically stable*

This result provides an efficient way to numerically calculate the equilibrium on an arbitrary network.

4.2 Efficiency

It is easy to show that the unique network that maximizes the total utility of the society is the empty network. This is a direct consequence of the transferable nature of the contest game and the fact that the effort is costly. Indeed, the total payoff that society obtains from network G can be expressed as:

$$\begin{aligned} U(G) &= \sum_i \pi_i(\mathbf{s}_i, \mathbf{s}_{-i}; G) \\ &= \sum_i \sum_{j \in N_i} \left(\frac{\phi(s_{ij})}{\phi(s_{ij}) + \phi(s_{ji}) + r} - \frac{\phi(s_{ji})}{\phi(s_{ij}) + \phi(s_{ji}) + r} - c(A_i) \right) \\ &= -\sum_i c(A_i) \end{aligned}$$

that gives:

Proposition 7. *The efficient network is the empty network*

5 Final Remarks

The model in some sense leads to a self-referential characterization of power in the network of contest relations. This means that a player will be strong in the equilibrium if her opponents are weak (recall that we refer to the total spending of a node in the equilibrium as the strength of a player). The previous sentence illustrates the recursive nature of a node's strength in the network emphasised in this model. This is a common feature of global centrality/prestige measures in networks (i.e. Katz centrality, Bonacich centrality, PageRank). The first important distinctive feature of this model is the negative relation between the strength of a node and strength of her neighbours. That is, player is weak if her neighbours are strong. But also note that the strength of a player is the equilibrium outcome. The second important feature is that the contest game is the game of neither strategic substitutes nor complements. Whether the effort of agent i will increase or decrease with effort level of agent j in (i, j) contest depends on if the agent i is winning or losing this particular contest. And finally, the strength cannot be a linear measure of centrality - as the strength of a node is a nonlinear function of the strength of its neighbours. For the quadratic cost function, total spendings of nodes are given with the solution of the following system of equations ($g_{ij} \in \{0,1\}$):

$$\begin{pmatrix} 0 & \frac{g_{12}}{(A_1+A_2)^2} & \frac{g_{13}}{(A_1+A_3)^2} & \cdots & \cdots & \frac{g_{1n}}{(A_1+A_n)^2} \\ \frac{g_{21}}{(A_2+A_1)^2} & 0 & \frac{g_{23}}{(A_2+A_3)^2} & \cdots & \cdots & \frac{g_{2n}}{(A_2+A_n)^2} \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ \frac{g_{n1}}{(A_n+A_1)^2} & \frac{g_{n2}}{(A_n+A_2)^2} & \frac{g_{n3}}{(A_n+A_3)^2} & \cdot & \cdot & 0 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ \vdots \\ A_n \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ \vdots \\ A_n \end{pmatrix}$$

Replacing the transferable contest with the 'classic' contest (one in which players compete to get a larger share of an exogenous prize) would not change the existence and uniqueness results from Section 3. However, as two types of contest have different interpretations, the link formation protocol in the formation model needs to be adjusted. In this case it is not clear why link destruction should be a bilateral decision, if the prize exists independently of the contest. Innovation contest/patent race is a situation which is more natural to model in this way (Baye and Hoppe, 2003). This approach could be naturally extended to hypergraphs. For example, consider a situation in which there are n firms and m markets (possible contests) in which firms can innovate. Then a linking strategy of a firm would be to decide in which of these m contests to participate (creating a hyperlink to other participants in these contests). The existence and the uniqueness results for a fixed hypernetwork will hold if we specify a contest success function for market k as:

$$p_{ik} = \frac{\phi(s_{ik})}{\sum_{j \in (N_k)} \phi(s_{jk})}$$

where N_k is the set of players competing in contest k, p_{ik} is the probability with which i wins the contest k and the rest of the notation is analogue to what we had before. And finally, modelling positive links explicitly such that (i) positive link g^+_{ij} decreases the marginal cost of effort[14], (ii) forming a positive link is bilateral decision, and destroying a positive link is a unilateral decision will not change the qualitative results of our model.

[14]setting $c(A_i) = \frac{1}{d_i^+} A^2$ where d_i^+ is the number of friends agent has

6 Appendix A: Proofs

Proof of Proposition 1: As discussed i Section 3, the payoff function of every player i is continuous and concave in $\mathbf{s}_i$. The strategy space is in general unbounded, but since the transfer R is finite, and the cost function c is strictly increasing, there exists a point $M \in \mathbb{R}$ such that $c(M) > R$. No player will wish to exert the effort larger than M. Therefore we can bound the strategy space from above. Thus, there exist the pure strategy equilibrium of the game on a network as defined in the beginning of Section 3, as the game is n players concave game. To prove the uniqueness we will use the characterization of diagonally strictly concave function proposed in (Goodman, 1980). The function $\sigma(\mathbf{s}, \mathbf{z})$ will be diagonally strictly concave if the payoff functions are such that for every player i: (i) $\pi_i(\mathbf{s})$ is strictly concave in $\mathbf{s}_i$, (ii) $\pi_i(\mathbf{s})$ is convex in $\mathbf{s}_{-i}$ and (iii) $\sigma(\mathbf{s}, \mathbf{z})$ is concave in $\mathbf{s}$ for some $\mathbf{z} \geq 0$. We prove that (i), (ii), (iii) hold. It is clear that the payoff function π_i is twice differentiable on its domain. Furthermore, the payoff function of player i is strictly concave in $\mathbf{s}_i$. To see this, note that:

$$\frac{\partial^2 \pi_i}{\partial {s_{ij}}^2} = \frac{(r + 2\phi(s_{ji}))\left(\phi''(s_{ij})(r + \phi(s_{ij}) + \phi(s_{ji})) - 2\phi'(s_{ij})^2\right)}{(r + \phi(s_{ij}) + \phi(s_{ji}))^3} - c''(A_i) < 0 \tag{9}$$

$$\frac{\partial^2 \pi_i}{\partial s_{ij} \partial s_{ik}} = -c''(A_i) < 0 \; \forall j, k \in N_i$$

The inequality in 9 holds due to the properties of function ϕ stated in Assumption 1, and the strict convexity of function c. The Hessian H_i of function π_i with respect to $\mathbf{s}_i$ is the sum of diagonal matrix H_{i1} with diagonal elements equal to:

$$\frac{(r + 2\phi(s_{ji}))\left(\phi''(s_{ij})(r + \phi(s_{ij}) + \phi(s_{ji})) - 2\phi'(s_{ij})^2\right)}{(r + \phi(s_{ij}) + \phi(s_{ji}))^3} < 0$$

and matrix H_{i2} which has all elements equal to $-c''(A_i) < 0$. Matrix H_{i1} is negative definite and matrix H_{i2} is negative semidefinite, thus Hessian $H_i = H_{i1} + H_{i2}$ is negative definite. Thus (i) holds. We also have that:

$$\frac{\partial^2 \pi_i}{\partial s_{ji}^2} = \frac{(r + 2\phi(s_{ij}))\left(2\phi'(s_{ji})^2 - \phi''(s_{ji})(r + \phi(s_{ij}) + \phi(s_{ji}))\right)}{(r + \phi(s_{ij}) + \phi(s_{ji}))^3} > 0$$

when there is link g_{ij}. Furthermore, $(\forall g_{jk} \in L : k \neq i)$, $\frac{\partial^2 \pi_i}{\partial s_{jk}^2} = 0$ and $\frac{\partial^2 \pi_i}{\partial s_{jk} \partial s_{lt}} = 0$ for any other combination of players, j, k, l and t. Thus, the Hessian of π_i with respect to $\mathbf{s}_{-i}$ is a diagonal matrix with all entries positive or zero and therefore positive semi-definite, so (ii) also holds. To prove the concavity of $\sigma(\mathbf{s}, \mathbf{z})$ in $\mathbf{s}$ we choose $\boldsymbol{z} = \mathbf{1}$. Then:

$$\sigma(\mathbf{s}, \mathbf{1}) = \sum_i \sum_{j \in N_i} \left(\frac{\phi(s_{ij})}{\phi(s_{ij}) + \phi(s_{ji}) + r} - \frac{\phi(s_{ji})}{\phi(s_{ij}) + \phi(s_{ji}) + r} - c(A_i) \right) = -\sum_i c(A_i)$$

This equality holds since the every summand $\frac{\phi(s_{ij})}{\phi(s_{ij}) + \phi(s_{ji}) + r}$ appears exactly once with a positive sign (as a part of payoff function π_i) and exactly once with a negative sign (as a part of function π_j). Function $-\sum_i c(A_i)$ is strictly concave due to the strict convexity of function c.. Hence, (iii) also holds, and the proof is completed. □

Proof of Proposition 2: Consider two arbitrary connected players i and j. Let us first prove that in the equilibrium it cannot be $s_{ij} = s_{ji} = 0 \ \forall r > 0$[15]. Assume otherwise. Then the payoff for both players in contest g_{ij} will be 0. Consider the deviation of player i from $s_{ij} = 0$ to $s_{ij} = r$. Now the probability of wining for player i becomes $p_{ij} = \frac{\phi(r)}{\phi(r)+r} = \alpha > 0$ and the probability of losing is still 0. This deviation is profitable as long as $c(\tilde{A}_i) - c(A_i) < \alpha$, where $\tilde{A}_i = A_i + r$. As c is continuous, we can always find such r so that $|c(\tilde{A}_i) - c(A_i)| < \alpha$ when $|\tilde{A}_i - A_i| \leq r$. Therefore, in this case we can always find such r so that the deviation from $s_{ij} = 0$ to $s_{ij} = r$ is profitable. Thus, it cannot be that $s_{ij} = s_{ji} = 0$ in the equilibrium. Let us now prove that for two arbitrary connected players i and j it cannot be that $s_{ij} \neq 0$ $\wedge s_{ji} = 0 \ \forall r > 0$. Again, suppose this is the case. Then necessary conditions for equilibrium imply:

$$\frac{\partial \pi_i}{\partial s_{ij}}|_{(s_{ij},0)} = \frac{(r + 2\phi(0))\phi'(s_{ij})}{(r + \phi(s_{ij}) + \phi(0))^2} - c'(A_i) = \frac{r\phi'(s_{ij})}{(r + \phi(s_{ij}))^2} - c'(A_i) = 0 \tag{10}$$

But, we can always find r small enough such that (10) cannot hold for any value $s_{ij} > 0$ and $A_i > 0$. Indeed, since the reward is finite and the number of nodes in the network is finite, then A_i must be finite for any node i in the network. For any cost function $c \in C^2$ satisfying assumptions for cost function stated in Section 2, we can find $U > 0$ such that $c'(A_i) < U$ for every finite A_i. Furthermore, we can always choose $r > 0$ small enough such that $\frac{r\phi'(s_{ij})}{(r+\phi(s_{ij}))^2} > U \ \forall s_{ij} \in [0, M]$, since $\frac{r\phi'(s_{ij})}{(r+\phi(s_{ij}))^2} \to \infty$ when $r \to 0$ for any fixed s_{ij}. □

Proof of Proposition 3: Recall that FOC for any a, b contest are given with:

$$\frac{2s^*_{ab} + r}{(s^*_{ab} + s^*_{ba} + r)^2} = c'(A^*_b) \text{ and } \frac{2s^*_{ba} + r}{(s^*_{ab} + s^*_{ba} + r)^2} = c'(A^*_a) \tag{11}$$

Expressing s^*_{ab} and s^*_{ba} from (11) we get that, in equilibrium:

$$s^*_{ab} = \frac{2c'(A^*_b)}{(c'(A^*_a) + c'(A^*_b))^2} - \frac{r}{2}, \ s^*_{ba} = \frac{2c'(A^*_a)}{(c'(A^*_a) + c'(A^*_b))^2} - \frac{r}{2} \tag{12}$$

Define $f(x, y) = \frac{2c'(x)}{(c'(y)+c'(x))^2} - \frac{r}{2}$. then:

$$\frac{\partial f(x, y)}{\partial x} = \frac{2\left(-c'(x) + c'(y)\right) c''(x)}{\left(c'(x) + c'(y)\right)^3} \lesseqgtr 0 \text{ when } x \gtreqless y \text{ and } \frac{\partial f(x, y)}{\partial y} = -\frac{4c'(x)c''(y)}{\left(c'(x) + c'(y)\right)^3} < 0$$

Together with (12) and $A^*_a < A^*_b < A^*_c$ this implies that $s^*_{ab} > s^*_{ac}$ and $s^*_{ba} > s^*_{ca}$. To prove that $\pi_a(s^*_{ab}, g_{ab}) < \pi_a(s^*_{ac}, g_{ac})$ we use (12) and (after some algebra) get:

$$\pi_a(s^*_{ab}, g_{ab}) = \frac{s^*_{ab} - s^*_{ba}}{(s^*_{ab} + s^*_{ba} + r)} = 1 - \frac{2c'(A_a)}{c'(A_a) + c'(A_b)}$$

It is clear that π_a is strictly increasing in A_b due to the strict convexity of function c. Thus, $A^*_c > A^*_b > A^*_a \Rightarrow \pi_a(s^*_{ac}, g_{ac}) > \pi_a(s^*_{ab}, g_{ab})$ □

Proof of Lemmma 1: From (12) we have that s_{ac} in in case of the deviation is given with:

$$s^*_{ac} = \frac{2c'(A^*_c + s^*_{ca})}{(c'(A^*_a - s^*_{ab} + s^*_{ac}) + c'(A^*_c + s^*_{ca}))^2} - \frac{r}{2}$$

and we can write:

$$s^*_{ab} = \frac{2c'(A^*_b - s^*_{ba} + s^*_{ba})}{(c'(A^*_a - s^*_{ab} + s^*_{ab}) + c'(A^*_b - s^*_{ba} + s^*_{ba}))^2} - \frac{r}{2}$$

[15]We omit * with equilibrium actions in the rest of the proof, but it is clear when s_{ij} denotes action in equilibrium

Because of the interiority of the equilibrium, $A_c^* + s_{ca}^* > A_c^* \geq A_b^* > A_b^* - s_{ba}^*$. Since $A_c^* + s_{ca} > A_b^*$ the Proposition 3 implies that this deviation is profitable. □

Proof of Corollary 1: Let us assume otherwise. If link g_{ab} is not profitable for player a then, as noted before, it is not profitable for player b. Then link g_{ab} cannot be part of a stable network. So it must be that link g_{ab} is profitable for player a. Let $c \in \mathcal{A}_{j+k}$ be a node such that link g_{ac} does not exist. Then, from the Lemma1 the deviation of player a in which she destroys link g_{ab} and creates link g_{ac} will be profitable. □

Proof of Lemma 2: We use the proof by contradiction. So assume that claim doesn't hold [16]. Then there are at least two components. Choose two arbitrary components from the network and denote them with C_1 and C_2. Let us denote two players with the highest total spending in these components with $h_1 \in \mathcal{A}_{C_1}$ and $h_2 \in \mathcal{A}_{C_2}$. Assume, without loss of generality, that $A_{C_1} \geq A_{C_2}$. Then, for any player which attacks h_2 (and there must be at least one) it is profitable to attack player h_1 instead. □

Proof of Lemma 3: Consider two nodes $a, b \in \mathcal{A}$. Let us first prove that they must have the same degree. Suppose that this is not true. So suppose that the network is stable and WLOG that $d_b > d_a$ where d_i denotes degree of node i. Let N_i denote the neighbourhood of player i. It cannot be that $N_a \subset N_b$ because then the total spending of a and b could not be equal (they would not belong to the same class). If $N_a = N_b$, the proof is completed. If not, there must exist some node $h \in N_a \backslash N_b$ and some node $k \in N_b \backslash N_a$. Suppose, WLOG, that $A_k \geq A_h$. Than it would be better for player a to replace link g_{ah} with link g_{ak} according to the Lemma 1. This is in contradiction with the assumption that the network is stable. So it must be $d_a = d_b$.

Let us now prove that $N_a = N_b$. Again, assume this is not the case. Than we can find two nodes $h \in N_a \backslash N_b$ $k \in N_b \backslash N_a$ such that, WLOG., $A_k \geq A_h$. But then it would be better for player a to replace link g_{ah} with link g_{ak} according to Lemma 1. Thus, network G cannot be stable. The assumption that $N_a \neq N_b$ led us to the contradiction and thus must be rejected. □

Proof of Lemma 4: We again use proof by contradiction. Suppose there are two different classes of attackers and denote them with $\mathcal{A}_1$ and $\mathcal{A}_2$ and let $A_2 > A_1$. Since players in $\mathcal{A}_1$ and $\mathcal{A}_2$ are attackers they have control over all of their links. Since Lemma 3 implies that all members of the same class of attackers have the same neighbourhood, we restrict our attention to the representative nodes $a \in \mathcal{A}_1$ and $b \in \mathcal{A}_2$. Let us first prove that it must be $\pi_a = \pi_b$. Assume this is not the case. Then it must be that $N_a \neq N_b$. Since $A_2 > A_1$ there are two possible situations that we need to consider.

(i) $N_a \subset N_b$ then if $\pi_a > \pi_b$ player b could mimic player a (as he is the attacker), and if $\pi_b > \pi_a$ the opposite will hold.[17]

(ii) $N_a \not\subset N_b \implies (\exists k \in N_a \backslash N_b \wedge \exists h \in N_b \backslash N_a)$. But then, if $A_k \geq A_h$ Lemma 1 implies that b has a profitable deviation, and if not, the same Lemma implies that a has a profitable

[16] We omit *, but it is clear from the context that we are considering the equilibrium strategies

[17] Recall that we assume that when a player is indifferent between two actions he prefers to have less links.

deviation. Hence, have proved that in a stable network it must be $\pi_a = \pi_b$. Since $A_2 > A_1$ then it must be that $d_b > d_a$ or the distribution of total spending of $a's$ and $b's$ opponents is different. We show that in both cases there exists a feasible deviation which makes one of the players better off. Let us first consider the case when $d_b > d_a$. If $N_a \subset N_b$ we have (i) from above. So there must exist nodes $k \in N_a \backslash N_b$ and $h \in N_b \backslash N_a$. If $A_k \geq A_h$ then player b would be better off by replacing contest g_{bd} with g_{bc}. If not, player a can make an analogous profitable deviation.

If $d_a = d_b$ then, since $A_2 > A_1$, strengths (total equilibrium spending) of $a's$ opponents are different than strengths of $b's$ opponents. Let q be the strongest node from $(N_a \cup N_b) \backslash (N_a \cap N_b) \neq \emptyset$. If link g_{aq} exists, then it is profitable for a to switch from q to any node in the set $N_b \backslash N_a$. If g_{bq} exists, then the profitable deviation is switching from q to some node in $N_a \backslash N_b$, and the proof is completed. □

Proof of Lemma 5. If there are only two classes of nodes in network $\mathcal{A}_1$ and $\mathcal{A}_2$ then there are no mixed types. Suppose there are more than two classes in the network. First consider the strongest mixed type class ($\mathcal{A}_2$). A node $m \in \mathcal{A}_2$ must be connected to all nodes in the class of winners $\mathcal{A}_1$. This is because a mixed type m must be connected with at least one stronger player, which must be a winner because of the choice of m. Lemma 4 implies then that m must be connected to all players from the class $\mathcal{A}_1$. Let us now prove that all members of the class $\mathcal{A}_2$ have the same neighborhood. Suppose not. Let $\{m_1, m_2\} \subset \mathcal{A}_2 \wedge N_{m_1} \neq N_{m_2}$. We have $(\mathcal{A}_1 \subset N_{m_1} \wedge \mathcal{A}_1 \subset N_{m_2}) \implies ((N_{m_1}/N_{m_2}) \cup (N_{m_2}/N_{m_1})) \cap \mathcal{A}_1 = \emptyset$. Thus, if they differ, the neighborhoods of m_1 and m_2 must differ only in the part where m_1 and m_2 have control over their links. It cannot be $N_{m_1} \subset N_{m_2} \vee N_{m_2} \subset N_{m_1}$ because then it cannot be $A_{m_1} = A_{m_2}$. Consider two nodes, $k \in N_{m_1} \backslash N_{m_2}$ and $l \in N_{m_2} \backslash N_{m_1}$. Note that sets $N_{m_1} \backslash N_{m_2}$ and $N_{m_2} \backslash N_{m_1}$ cannot be empty. If $A_k \geq A_l$ then m_2 has a profitable deviation (switching from $g_{m_2 l}$ to $g_{m_2 k}$). If not, then m_1 has an analogue profitable deviation.

Let $\mathcal{A}_3$ be the third strongest class in the network. If this is the weakest class (if $K = 3$) then, by definition, all players from $m \in \mathcal{A}_2$ must be connected to some of the players of $\mathcal{A}_3$, because otherwise they would not be mixed types. Note that if player $i \in \mathcal{A}_3$ is connected to some player from class $\mathcal{A}_2$ that he is connected to all players from class $\mathcal{A}_2$ since we have shown that all members of class $\mathcal{A}_2$ have the same neighborhood. If there exists some player $j \in \mathcal{A}_3$ who is not connected to a player from $\mathcal{A}_2$ then he is connected only to players from $\mathcal{A}_1$ but then it cannot be $A_i = A_j$, that is, i and j cannot belong to the same class. Thus, if $K = 3$ the claim holds.

If not, then $\mathcal{A}_3$ is a mixed type class. Corollary 1 implies that all members of $\mathcal{A}_1$ must be connected to all members of $\mathcal{A}_3$ since they are connected to all the members of $\mathcal{A}_2$ and $A_2 < A_3$. Suppose it does not exist link g_{ij} such that $i \in \mathcal{A}_2$ and $j \in \mathcal{A}_3$. Since all players from $\mathcal{A}_2$ have the same neighborhood there aren't any links between members of class $\mathcal{A}_2$ and $\mathcal{A}_3$. This means that players from $\mathcal{A}_3$ lose only in contest with players from $\mathcal{A}_1$, so they have control over all of their links except those that connect them to players in $\mathcal{A}_1$. Furthermore, $A_2 < A_3 \implies N_i \neq N_j$. As before, we first consider the case when $\pi_i \neq \pi_j$.

(i) $N_i \subset N_j$ then j can destroy links towards all players N_j/N_i and have the same payoff as i (if $\pi_i \geq \pi_j$), or player i can create links to all players in N_j/N_i (if $\pi_i < \pi_j$)

(ii) $N_i \not\subset N_j \implies (\exists k \in N_i \backslash N_j \wedge \exists h \in N_j \backslash N_h)$. But then, if $A_k \geq A_h$ Lemma 1 implies that j has a profitable deviation, and if not, the same Lemma implies that i has a profitable deviation.

If $\pi_i = \pi_j$ since $A_2 > A_1$ then it must be that $d_j > d_i$ or that the distribution of total spending of $i's$ and $j's$ opponents is different. We show that in the both cases there exists a profitable deviation.

Let us first consider the case when $d_i > d_j$. If $N_i \subset N_j$ we have (i) from above. If not we have an analogue of (ii).

If $d_i = d_j$ then, since $A_2 > A_1$, the strengths (total equilibrium spending) of $i's$ opponents are different than the strength of $j's$ opponents. Let q be the strongest node from $(N_a \cup N_b) \backslash (N_a \cap N_b) \neq \emptyset$. If link g_{iq} exists, then it is profitable for i to switch from q to any node in the set $N_j \backslash N_i$. If g_{jq} exists, then the profitable deviation is switching from q to some node in $N_i \backslash N_j$. Thus, we have shown that it cannot happen that there are no links between $\mathcal{A}_2$ and $\mathcal{A}_3$, thus every player from $\mathcal{A}_2$ is connected to every player from $\mathcal{A}_3$.

Proceeding in the same way, we can show that all players from $\mathcal{A}_k$ must be connected to all players from $\mathcal{A}_{k+1}$. Since the number of nodes is finite, the number of classes is finite and this procedure reaches $\mathcal{A}_K$ in a finite number of steps. □

Proof of Lemma 6: Suppose the claim doesn't hold. Note that FOC imply that $s^*_{ij} = s^*_{ih}$ $\forall\{i,j,h\} \in N \wedge \{j,h\} \in \mathcal{A}_l$. If $|\mathcal{A}_k| < |\mathcal{A}_{k+1}|$. Lemma5 implies that $A_k = \sum\limits_{i \neq k} |\mathcal{A}_i| s_{ki}$ and $A_{k+1} = \sum\limits_{i \neq k+1} |\mathcal{A}_i| s_{k'i}$ for any two nodes $k \in \mathcal{A}_k$ and $k' \in \mathcal{A}_{k+1}$. Recall that s^*_{ij} is strictly decreasing in A^*_i. This implies that $s^*_{kj} > s^*_{k'j}$ $\forall j \in \{1,..,K\} \backslash \{k,k'\}$. Also, $A_k < A_{k+1} \Rightarrow s_{kk'} > s_{k'k}$. But then $|\mathcal{A}_k| < |\mathcal{A}_{k+1}| \Rightarrow (A_k = \sum\limits_{i \neq k} |\mathcal{A}_i s_{ki} > A_{k+1} = \sum\limits_{i \neq k+1} |\mathcal{A}_i| s_{k'i})$, contradiction! It must be $|\mathcal{A}_k| > |\mathcal{A}_{k+1}|$ □

7 Appendix B: An Alternative Formulation

Suppose that, instead of a contest game with a general convex cost function, we consider Colonel Blotto game with Tullock CSF. That is, each player is endowed with the equal amount of resources (time) and the strategy is to distribute the resources across different contests. Note that this also defines convex cost function (resources are free up to some point and then prohibitively costly)

To fix ideas, let the budget constraint be defined with: $\sum_{j \in N_i} s_{ij} = 1 \; \forall i, j$. Keeping the same CSF, the existence, uniqueness and the interiority are guaranteed by the results from (Rosen, 1965). Let λ_i denote the Lagrange multiplier associated to the budget constraint for agent i. The first order conditions that characterize behaviour in contest g_{ij} are given with:

$$\frac{(r+2\phi(s_{ji}^*))\phi'(s_{ij}^*)}{(r+\phi(s_{ij}^*)+\phi(s_{ji}^*))^2} - \lambda_i = 0$$

$$\frac{(r+2\phi(s_{ij}^*))\phi'(s_{ji})^*}{(r+\phi(s_{ij}^*)+\phi(s_{ji}^*))^2} - \lambda_j = 0$$

$$\sum_{k\in N_i} s_{ik}^* = 1, \ \sum_{k\in N_j} s_{jk}^* = 1$$

which gives:

$$\frac{(r+2\phi(s_{ji}^*))\phi'(s_{ij}^*)}{(r+2\phi(s_{ij}^*))\phi'(s_{ji}^*)} = \frac{\lambda_i}{\lambda_j} \tag{13}$$

Thus, the role of λ_i is analogous to the role of A_i^*. Higher A_i^* implies a higher marginal cost of an additional unit of effort, and λ_i is the shadow price of the resource for player i in this formulation. The analysis in the paper will go through using this specification.

8 Appendix C: Numerical Example

To illustrate the complexity of global effects in the network, we present a very simple example which still exhibits very complex behaviour. Consider a network:

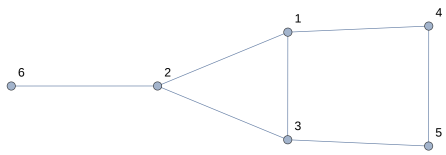


Figure 4: Example: Initial Network

Calculating the equilibrium actions, we get that the vector of total equilibrium efforts (strengths of nodes) $\boldsymbol{A}$ is given with: $\boldsymbol{A} = (0.864534, 0.854561, 0.864534, 0.705285, 0.705285, 0.479887)$, and the corresponding payoffs are: $\boldsymbol{\pi} = (-0.854, -0.999, -0.854, -0.395, -0.395, 0.050)$ Deleting link g_{13} the network becomes

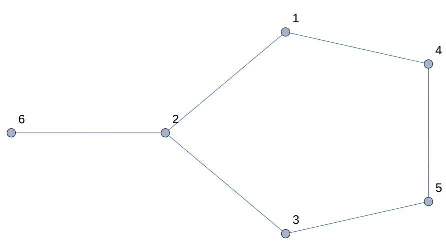


Figure 5: Example: Resulting Network

with $\hat{\boldsymbol{A}} = (0.70554, 0.852104, 0.70554, 0.707107, 0.707107, 0.480112)$ and the corresponding payoffs are: $\hat{\boldsymbol{\pi}} = (-0.402, -1.193, -0.402, -0.501, -0.501, 0.048)$

The considered local change will obviously have a global effect - the total spending and the payoff of all players change. The direct effect on the payoff of players 1 and 3 will be large, but the effect on their total spending will not be as large. Before the elimination of g_{13}, players 1 and 3 were losing in all of their contests. After the elimination of g_{13} they are winning in all of their contests. It is also interesting to note the effect on node 6. Even though the opponents of node 2 become stronger node 2 itself becomes stronger. This is because the contests of node 2 aren't as intensive as before, and therefore 2 exerts less contest effort. This affects node 6 negatively, and 6 is spending more and at the same time has a lower payoff.

Suppose that Figure 4 is the starting network in the formation process. There are two possible stable networks. The first one is the empty network and the second one is the network on Figure 6.

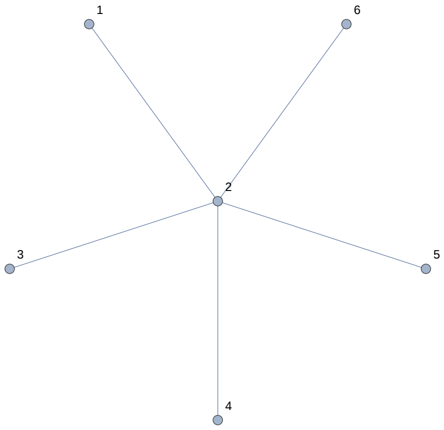
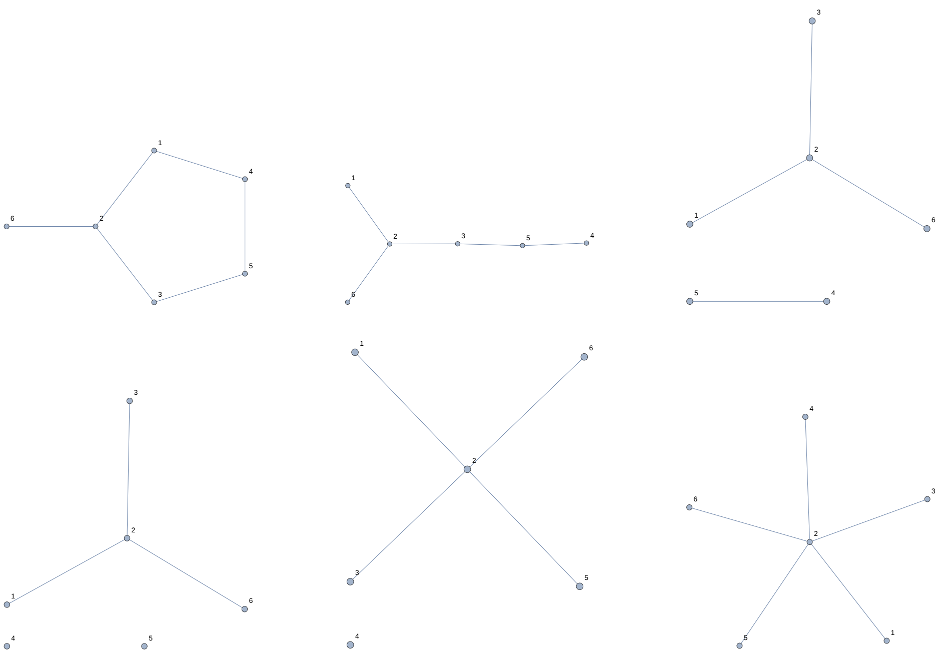


Figure 6: Example: Resulting Network

With a possible path towards it:



References

Antal, T., Krapivsky, P. L., and Redner, S. (2006). Social balance on networks: The dynamics of friendship and enmity. *Physica D: Nonlinear Phenomena*, 224(1):130–136.

Baye, M. R. and Hoppe, H. C. (2003). The strategic equivalence of rent-seeking, innovation, and patent-race games. *Games and Economic Behavior*, 44(2):217–226.

Bloch, F. and Dutta, B. (2009). Communication networks with endogenous link strength. *Games and Economic Behavior*, 66(1):39–56.

Buchanan, J. M., Tollison, R. D., and Tullock, G. (1980). *Toward a theory of the rent-seeking society*. Number 4. Texas A & M Univ Pr.

Cartwright, D. and Harary, F. (1956). Structural balance: a generalization of heider's theory. *Psychological review*, 63(5):277.

Davis, J. A. (1967). Clustering and structural balance in graphs. *Human relations*.

Deroïan, F. (2009). Endogenous link strength in directed communication networks. *Mathematical Social Sciences*, 57(1):110–116.

Franke, J. and Öztürk, T. (2009). *Conflict networks*. Number 116. Ruhr economic papers.

Goodman, J. C. (1980). Note on existence and uniqueness of equilibrium points for concave n-person games. *Econometrica*, 48(1).

Heider, F. (1946). Attitudes and cognitive organization. *The Journal of psychology*, 21(1):107–112.

Hiller, T. (2011). Alliance formation and coercion in networks. Technical report, Nota di lavoro//Fondazione Eni Enrico Mattei: Climate Change and Sustainable Development Series.

Hillman, A. L. and Riley, J. G. (1989). Politically contestable rents and transfers*. *Economics & Politics*, 1(1):17–39.

Hirshleifer, J. (1989). Conflict and rent-seeking success functions: Ratio vs. difference models of relative success. *Public choice*, 63(2):101–112.

McDougall, P. (2007). Microsoft's court battles with alcatel-lucent far from over. *Information Week, http://www.informationweek.com/*.

Nti, K. O. (1997). Comparative statics of contests and rent-seeking games. *International Economic Review*, pages 43–59.

Rosen, J. B. (1965). Existence and uniqueness of equilibrium points for concave n-person games. *Econometrica: Journal of the Econometric Society*, pages 520–534.

Sytch, M. and Tatarynowicz, A. (2013). Friends and foes: The dynamics of dual social structures. *Academy of Management Journal*.

Szell, M., Lambiotte, R., and Thurner, S. (2010). Multirelational organization of large-scale social networks in an online world. *Proceedings of the National Academy of Sciences*, 107(31):13636–13641.