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Stochastic stability of endogenous growth: Theory and applications *

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Abstract

We examine the issue of stability of stochastic endogenous growth. First, stochastic stability concepts are introduced and applied to stochastic linear homogenous differential equations to which several stochastic endogenous growth models reduce. Second, we apply the mathematical theory to two models, starting with the stochastic AK model. It's shown that in this case exponential balanced paths, which characterize optimal trajectories in the absence of uncertainty, are not robust to uncertainty: the economy may almost surely collapse at exponential speed even though productivity is initially arbitrarily high. Finally, we revisit the seminal global diversification endogenous growth model (Obstfeld, 1994): taking into account stochastic stability calls for a redefinition of the mean growth concept, which leads to revisit the established wisdom on the growth effect of global diversification.

Keywords: Endogenous growth, stochastic growth, stochastic stability, AK model, Global diversification

JEL classification: O40, C61, C62

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1 Introduction

The neoclassical stochastic growth models has been the subject of a quite visible economic theory literature since the celebrated Brock and Mirman's 1972 paper (see also Mirman and Zilcha, 1975, and the less known contribution of Merton, 1975). This notably includes the study of the existence of stochastic steady states and their stability. In contrast, no such a literature exists for endogenous growth models. This is partly due to the fact that many of these models rely on zero aggregate uncertainty as in the early R&D based models (see for example, Barro and Sala-i-Martin, 1995, chapters 6 and 7). When uncertainty does not vanish by aggregation as in de Hek (1999), the usual treatment consists in applying Merton's portfolio choice methodology (Merton, 1969 and 1971) to track expected growth rate and its volatility. Recently, Steger (2005) and Boucekkine et al. (2014) apply the same methodology to study the stochastic AK model for a closed economy and for a small open capital constrained economy respectively. Precisely, these authors assume the existence of balanced growth paths (as in the deterministic counterparts) and compute the associated expected growth rates and growth volatilities, without addressing the issue of stochastic stability of the selected paths. Earlier and more fundamental papers on stochastic growth have taken the same avenue. The Obstfled (1994) paper on growth and diversification is one of them. We shall show that taking into account stochastic stability calls for a redefinition of the mean growth concept, which leads to revisit the established wisdom on the growth effect of global diversification as exposed in Obstfled's seminal work.

It's important to notice at this stage that one cannot address the issue of stochastic stability of endogenous growth simply by adapting the available proofs in Brock and Mirman (1972) or Merton (1975). For example, strict concavity of the production function is needed in the latter in order to build up the probability measure for stability in distribution, so the strategy cannot be applied to the benchmark stochastic endogenous growth model, the AK model with random output technology. Rather, we take a much simpler and more specific approach exploiting the typical linearities showing

up in (reduced-forms) endogenous growth models. When uncertainty is modelled as a geometric Brownian motion, which is very common in growth theory (see the survey of Jones and Manuelli, 2005, or the textbook of Acemoglu, 2009), we show that the study of stochastic stability of AK-type growth models amounts in such a case to studying stability of a standard stochastic linear differential equation. Relying on the specialized mathematical literature (Mao, 2011, or Khasminskii, 2012), we are able to straightforwardly state stochastic stability theorems. We then apply these theorems to two models. The first one is the standard stochastic AK model. Strikingly enough, we ultimately show that the typical (deterministic) balanced growth paths are hardly stochastically stable in our simple framework. Even more, we show that the trivial equilibrium, $k^* = 0$, is globally stochastically asymptotically stable in the large and almost surely exponentially stable (that's the optimal paths almost surely collapse at exponential speed) even when productivity is arbitrarily high. Kamihigashi (2006) states a similar convergence result for discrete time stochastic growth models. However, his results (see Theorem 2.1, page 233) are based in particular on the assumption that "$\cdots$ for almost every sample path, the gross growth rate over a long horizon is less than one, i.e., the net growth rate over a long horizon is negative". No such assumption is required here, the continuous time framework used allows to reach the same conclusion at a much lower analytical cost.

Even more importantly, through an application to the seminal global diversification model due to Obstfled, we show that accounting for stochastic stability calls for revisiting the concept of mean growth usually chosen. The central and famous result from this model is that if the economy holds *some* risk-free capital, a fall in exogenous risk unambiguously leads to an increase in the share of wealth invested in risky capital. In other words, a portfolio shift toward riskier capital triggers a return effect that dominates the possibly adverse effect of a higher propensity to consume out of total wealth. As a consequence, growth and welfare go up. In contrast, goes the intuition, complete specialization in risky capital results in an ambiguous effect of exogenous risk on growth, that is entirely governed by whether the intertemporal substitution elasticity

is larger or smaller than one.

In this paper we show that such results are based on a definition of mean growth that seems natural absent any concern about stability of the balanced-growth path but that turns out to be misleading if the issue of stability is addressed, as it should be. To understand why a confusion may arise, it is important to stress that Obstfeld considers an AK model where risk is introduced via diffusion equations that involve geometric Brownian motions. In such a setting, the dynamics of the economy that result from optimal portfolio and savings decisions are described by a linear, stochastic differential equation with fixed coefficients and one might wrongly think that two correct concepts of growth are available to freely choose from. The first defines mean growth as the growth rate of average wealth, which is the one used by Obstfeld (1994) and many others (see e.g. Jones and Manuelli, 2005). The alternative definition, which emerges naturally in the quest for stability conditions that is developed below, is to define mean growth as the average growth rate of wealth, as opposed to the growth rate of average wealth. Because by assumption wealth is log-normally distributed, it follows by Jensen's inequality that the average growth rate of wealth - second notion - is lower than the growth rate of average wealth - first notion. Not surprisingly, stochastic stability holds if and only if the average growth rate is positive, a condition that is stronger than the requirement that the growth rate of average wealth be positive. More importantly, we show that very different comparative statics results obtain when one uses the second definition of mean growth, as one should in view of stability conditions. More precisely, mean growth happens to be enhanced by financial integration under conditions that would possibly lead to the opposite conclusion if one were to use the definition of mean growth advocated in Obstfeld. This property is most striking in a specialized economy, where for example a fall in exogenous risk results in larger growth even if the intertemporal substitution elasticity is smaller than one, despite the fact that a portfolio shift does not happen.

This paper is organized as follows. Section 2 presents the mathematical background needed. Section 3 is devoted to the application to the stochastic AK model. Section 4 is

the application to the global diversification model. Section 5 concludes.

2 Stochastic stability of linear stochastic differential equations

Consider the typical linear Ito stochastic differential equation

$$dx(t) = ax(t)dt + bx(t)dB(t), t \geq 0$$

with initial condition $x(0) = x_0$ given, $B(t)$ standard Brownian Motion, a and b constants. The general solution takes the form

$$x(t) = x_0 \ \exp\left\{\left(a - \frac{b^2}{2}\right)t + bB(t)\right\}. \tag{1}$$

Compared to the pure deterministic case (case $b = 0$), an extra negative term, $-\frac{b^2}{2}$ shows up in the deterministic part of the solution . It's therefore easy to figure out why the noise term, $bx(t)dB(t)$, is indeed stabilizing. Incidentally, introducing some specific white noises is one common way to "stabilize" dynamics systems. The pioneering work belongs to Khasminskii (2012) and some more recent results can be found in Appleby et al. (2008) and references therein. Thus, the stability conditions under stochastic environments may well differ from the case with certainty. To tackle seriously this issue, we display some useful preliminary definitions.

For simplicity, we only present results for scalar stochastic differential equations. First let us consider a general stochastic differential equation of the form

$$dx(t) = f(x(t), t)dt + g(x(t), t)dB(t), t \geq t_0 \tag{2}$$

with initial condition $x(t_0) = x_0$ given and $B(t)$ standard Brownian Motion. Functions $f(x(t), t)$ and $g(x(t), t)$ check

$$f(0, t) = 0 \ \text{ and } \ g(0, t) = 0, \ \ \forall t \geq t_0.$$

Thus, solution $x^* = 0$ is a solution corresponding to initial condition $x_0 = 0$. This solution is also called trivial solution or equilibrium solution.Then for the stability concept[1], we take the following definitions from the Definition 4.2.1 and 4.3.1, Mao (2011).[2]

Definition 1 *(i) The equilibrium (or trivial) solution ($x^* = 0$) of equation (2) is said to be stochastically stable or stable in probability if for every pair of $\varepsilon \in (0,1)$ and $r > 0$, there exists a $\delta = \delta(\varepsilon, r) > 0$, such that, probability checks*

$$P\{\mid x(t; x_0, t_0) \mid < r \;\; for \;\; all \;\; t \geq t_0\} \geq 1 - \varepsilon$$

whenever $\mid x_0 \mid < \delta$. Otherwise, it is said to be stochastically unstable.

(ii) The equilibrium solution, $x^* = 0$, of equation (2) is said to be stochastically asymptotically stable if it is stochastically stable and, moreover, for every $\varepsilon \in (0,1)$, there exists a $\delta = \delta(\varepsilon) > 0$, such that,

$$P\{\lim_{t \to +\infty} \mid x(t; x_0, t_0) \mid = 0\} \geq 1 - \varepsilon$$

whenever $x_0 < \delta$.

(iii) The equilibrium solution, $x^* = 0$, of equation (2) is said to be stochastically asymptotically stable in the large if it is stochastically stable and, moreover, for all x_0

$$P\{\lim_{t \to +\infty} \mid x(t; x_0, t_0) \mid = 0\} = 1.$$

(iv) The equilibrium solution, $x^* = 0$, of equation (2) is said to be almost surely exponential stable if

$$\lim_{t \to +\infty} \sup \frac{\log \mid x(t; x_0, t_0) \mid}{t} < 0 \;\; a.s.$$

for all x_0.

[1]Noticing, it should not be confused with the convergence to an invariant distribution, see for example the setting of Brock and Mirman (1972) or Merton (1075). A non-degenerate distribution would never survive the stability test of our Definition 1.

[2]See also Khasminskii (2012), section 5.3, page 152, section 5.4, page 155, and Definition 1 in section 5.4, page 157.

We show now how the definitions above give rise to neat stability theorems when applied to homogenous linear stochastic differential equations like those arising from endogenous growth theory. Precisely, consider the equation

$$dx(t) = a(t)x(t)dt + b(t)x(t)dB(t), t \geq t_0 \tag{3}$$

with initial condition $x(t_0) = x_0$ given, $B(t)$ standard Brownian Motion, $a(t)$ and $b(t)$ known functions, we have solution as

$$x(t) = x_0 \ \exp\left\{\int_{t_0}^{t}\left(a(s) - \frac{b^2(s)}{2}\right)ds + \int_{t_0}^{t} b(s)dB(s)\right\} \tag{4}$$

Then the following stability results can be demonstrated for the general linear stochastic equation (3). The proof can be found in Mao (2011), examples 4.2.7 and 4.3.8, pages 117-119 and 126-127, respectively.[3]

Proposition 1 *Consider homogenous linear stochastic equation (3) and denote* $\sigma(t) = \int_{t_0}^{t} b^2(s)ds$, *we have*

- *(i)* $\sigma(\infty) < +\infty$, *then the equilibrium solution,* $x^* = 0$, *of equation (3) is stochastically stable if and only if*

  $$\lim_{t\to+\infty} \sup \int_{t_0}^{t} a(s)ds < +\infty.$$

  While the equilibrium solution is stochastically asymptotically stable in the large if and only if

  $$\lim_{t\to+\infty} \int_{t_0}^{t} a(s)ds = -\infty.$$

- *(ii)* $\sigma(\infty) = +\infty$, *then the equilibrium solution,* $x^* = 0$, *of equation (3) is stochastically asymptotically stable in the large if*

  $$\lim_{t\to+\infty} \sup \frac{\int_{t_0}^{t}\left(a(s) - \frac{b^2(s)}{2}\right)ds}{\sqrt{2\sigma(t)\log\log(\sigma(t))}} < -1, \ a.s. \tag{5}$$

[3]See also Khasminskii (2012), section 5.3, page 154, and section 5.5, page 159-160.

- *(iii) Specially, if both $a(t) = a$ and $b(t) = b$ are constants, (5) holds if and only if*

$$a < \frac{b^2}{2}.$$

  That is, equilibrium solution, $x^* = 0$, of (3) is stochastically asymptotically stable in the large if $a < \frac{b^2}{2}$.

- *(iv) The equilibrium solution, $x^* = 0$, of (3) is almost surely exponentially stable if $a < \frac{b^2}{2}$.*

The last two results read that if $a < \frac{b^2}{2}$, then almost all sample paths of the solution will tend to the equilibrium solution $x^* = 0$ and the this convergence is exponentially fast. This is not obviously the case in the deterministic case if $a > 0$. This is the key point behind the striking results on the stochastic stability of balanced growth paths in the AK model shown here below.

3 Application to the stochastic $AK-$growth model

3.1 The stochastic AK model

Consider strictly increasing and strictly concave utility

$$\max_c E_0 \int_0^\infty U(c)e^{-\rho t}dt, \tag{6}$$

subject to

$$dk(t) = (Ak(t) - c(t) - \delta k)dt + bAkdW(t), \quad \forall t \geq 0 \tag{7}$$

where initial condition $k(0) = k_0$ is given, positive constants δ and ρ measure depreciation and time preference, b is volatility and $W(t)$ is one-dimensional Brownian motion. Define Bellman's value-function as

$$V(k,t) = \max_c E_t \int_t^\infty u(c)e^{-\rho t}dt.$$

Then this value function must satisfy the following stochastic Hamilton-Jacobi-Bellman equation

$$\rho V(k) = \max_c \left[U(c(t)) + V_k \cdot (Ak(t) - c(t) - \delta k) + \frac{1}{2} b^2 A^2 k^2 V_{kk} \right] \tag{8}$$

with V_k first order derivative with respect to k. First order condition on the right hand side of (8) yields

$$U'(c) = V_k(k). \tag{9}$$

Due to the strictly concave utility, the implicit function theorem gives the solution of (9), $c^*(t) = c^*(k(t))$, which is optimal to the right hand side of (8). Substituting this optimal choice into (8), it follows

$$\rho V(k) = U(c^*(t)) + V_k \cdot (Ak(t) - c^*(t) - \delta k) + \frac{1}{2} b^2 A^2 k^2 V_{kk}. \tag{10}$$

To find an explicit solution, we take CRRA–Constant Relative Risk Aversion utility:

$$U(c) = \frac{c^\gamma}{\gamma}, \ 0 < \gamma < 1.$$

It's worth pointing out here that such a range of values for γ implies that $U(0) = 0$, that's instantaneous utility is bounded from below. Therefore, consumption going to zero is not ruled out from the beginning. Moreover, the assumed γ-values imply that the intertemporal elasticity of substitution (equal to $\frac{1}{1-\gamma}$) is above unity, which has the typical economic implications on the relative size of the income vs substitution effects. This will reveal important for the stochastic stability results obtained in Section 3.2. The first order condition yields the optimal choice

$$c^* = V_k^{\frac{1}{\gamma-1}}.$$

Substituting into the HJB equation (10), we have

$$\rho V(k) = V_k \cdot (A - \delta)k + \frac{1-\gamma}{\gamma} V_k^{\frac{\gamma}{\gamma-1}} + \frac{1}{2} b^2 A^2 k^2 V_{kk}. \tag{11}$$

Parameterizing the solution as

$$V(k) = H^{1-\gamma}\frac{k^{\gamma}}{\gamma},$$

with constant H undetermined, and substituting into (6), it is easy to obtain

$$\frac{1}{H} = \frac{\rho}{1-\gamma} + \frac{b^2A^2\gamma}{2} - \frac{\gamma(A-\delta)}{1-\gamma}. \tag{12}$$

Thus, the optimal choice is

$$c^* = \frac{k}{H}.$$

Then the dynamics of optimal capital accumulation follow

$$dk(t) = \left(A - \delta - \frac{1}{H}\right)k(t)dt + bAkdW(t) \tag{13}$$

which is a linear stochastic differential equation and the explicit solution is

$$k(t) = k(0)\ \exp\left\{\left[\left(A - \delta - \frac{1}{H}\right) - \frac{b^2A^2}{2}\right]t + bAW(t)\right\}. \tag{14}$$

Two observations are in order here. First of all, it is worth pointing out that in the absence of uncertainty, that's when $b = 0$, one gets the typical results: in particular, for any initial condition $k(0) > 0$, the economy jumps on the optimal path given by (14) under $b = 0$, and the growth rate is exactly $\frac{A-\delta-\rho}{1-\gamma}$. The growth rate is strictly positive if and only if $A > \delta + \rho$ given that $0 < \gamma < 1$. Since there are no transitional dynamics, the convergence speed to the balanced growth path is infinite. Second, as already mentioned in Section 2, it is easy to see from the explicit solution above that due to the extra negative term, $-\frac{b^2A^2}{2}$, the stability conditions may differ from the deterministic case.

3.2 Stochastic stability of the AK model

It is easy to check in $AK-$model, $\sigma(t) = \int_0^t bAds = bAt$. Therefore, $\sigma(\infty) = +\infty$.

From Proposition 1, the capital stock tends to equilibrium $k^* = 0$ if

$$\left(A - \delta - \frac{1}{H}\right) < \frac{b^2 A^2}{2}.$$

Substituting $\frac{1}{H}$ from (12) into the above inequality, we have

$$A - \delta - \frac{\rho}{1-\gamma} - \frac{b^2 A^2 \gamma}{2} + \frac{\gamma(A-\delta)}{1-\gamma} < \frac{b^2 A^2}{2},$$

which is equivalent to

$$F(A) \equiv \frac{b^2(1-\gamma^2)A^2}{2} - A + (\rho + \delta) > 0, \quad \text{with} \;\; 1 - \gamma^2 > 0. \tag{15}$$

Obviously, $F(A)$ is a second degree polynomial in term of A and opens upward. Denote $\Delta = 1 - 2b^2\,(1-\gamma^2)(\rho+\delta)$.

Thus, (a) if $\Delta < 0$, that is, $b^2 > \frac{1}{2(1-\gamma^2)(\rho+\delta)}$, we have $F(A) > 0$, for any A; (b) if $\Delta \geq 0$, i.e., $b^2 \leq \frac{1}{2(1-\gamma^2)(\rho+\delta)}$ then $F(A) > 0$ for $A \in (0, A_1) \cup (A_2, +\infty)$, with A_i, $i = 1, 2$, are the two positive roots of $F(A) = 0$.

The above analysis is concluded in the following:

Proposition 2 *Consider problem (6) with constraint (7). The equilibrium $k^* = 0$ is (globally) stochastically asymptotically stable in the large and almost surely exponentially stable, if and only if one of the two following conditions hold: (a) $b^2 > \frac{1}{2(1-\gamma^2)(\rho+\delta)}$ and for any $A > 0$; or (b) $b^2 \leq \frac{1}{2(1-\gamma^2)(\rho+\delta)}$ and $A \in (0, A_1) \cup (A_2, +\infty)$, with*

$$A_1 = \frac{1 - \sqrt{1 - 2(\delta+\rho)b^2(1-\gamma^2)}}{b^2(1-\gamma^2)}, \quad A_2 = \frac{1 + \sqrt{1 - 2(\delta+\rho)b^2(1-\gamma^2)}}{b^2(1-\gamma^2)}.$$

The final proposition is striking at first glance. In contrast to the deterministic case, where the economy will optimally jump on an exponentially increasing path provided $A > \rho + \delta$, it turns out that under uncertainty, our economy almost surely collapses (at an exponential speed) for a large class of parameterizations. Two engines are driving

this result. First, the size of uncertainty as captured by parameter b matters: a too large uncertainty in the sense of condition (a) of Proposition 2 will destroy the usual deterministic growth paths even if productivity is initially very high (so even if $A >> \delta + \rho$). Second, since $0 < \gamma < 1$, we are in the typical case where uncertainty boosts contemporaneous consumption at the expense of savings and growth because the inherent income effects are dominated by the intertemporal substitution effects. In such a case, even if uncertainty is not large in the sense of condition (b) of Proposition 2, the usual deterministic growth paths are not robust to uncertainty. To understand more clearly the associated productivity values, it is interesting to come back to the parametric case considered by Steger (2005). Steger sets $b = 1$ and $\delta = 0$. Then, the first part of condition (b) holds for ρ small enough. Indeed, condition $1 \leq \frac{1}{2\rho(1-\gamma^2)}$ is fulfilled for ρ going to zero and given $0 < \gamma < 1$. The second part of condition (b) is more interesting. For ρ close to zero, and using elementary approximation, one can easily show that $A_1 \approx \rho$ and $A_2 \approx \frac{2-\rho(1-\gamma^2)}{1-\gamma^2}$. Condition (b) states that the economy collapses almost surely and at an exponential speed either if $A < A_1$ or $A > A_2$. Condition $A < A_1$, which amounts to $A < \rho$, is compatible with the deterministic counterpart as exponentially increasing paths require $A > \rho$ when $\delta = 0$. However, $A > A_2$ is not since $A_2 > \rho$ for ρ small enough: exponentially optimal increasing paths exist in the deterministic case but not in the stochastic counterpart where the economy optimally almost surely collapses. In

such a case, balanced growth is not robust to uncertainty.[4]

4 Risk-Taking, Global Diversification and Growth

4.1 Stochastic Stability and the Definition of Mean Growth

Obstfeld (1994) considers an AK version of the optimal portfolio model developed in Merton (1969). The following equation describes optimal wealth accumulation and is identical to equation [14] derived in Obstfeld (1994)[5]:

$$dW = [\omega\alpha + (1-\omega)i - \mu]Wdt + \omega\sigma W dz, \tag{17}$$

where $\alpha(>0)$ and $i(>0)$ are the mean returns of risky capital and risk-free bonds, respectively, $\mu(>0)$ is the average propensity to consume out of total wealth, z is a Wiener process and $\sigma^2(\geq 0)$ is the exogenous variance of the return on risky capital .

[4]One intuitive way to understand why $k(t)$ converges to zero as $t \to +\infty$ is to look at convergence in probability, which is weaker than almost sure convergence used in this paper. Evidently, $k(t)$ in (14) goes to zero in probability when

$$\left[\left(A - \delta - \frac{1}{H}\right) - \frac{b^2A^2}{2}\right]t + bAW(t) \to -\infty$$

in probability. For any t, this random variable has the same distribution as

$$Y(t) = \left[\left(A - \delta - \frac{1}{H}\right) - \frac{b^2A^2}{2}\right]t + bA\sqrt{t}Z; \tag{16}$$

where Z is standard normal distribution. To show that $Y(t) \to -\infty$, it is sufficient to show that $\frac{Y(t)}{\sqrt{t}} \to -\infty$. This ratio satisfies

$$\frac{Y(t)}{\sqrt{t}} = \left[\left(A - \delta - \frac{1}{H}\right) - \frac{b^2A^2}{2}\right]\sqrt{t} + bAZ.$$

If condition $\left[\left(A - \delta - \frac{1}{H}\right) - \frac{b^2A^2}{2}\right] < 0$ holds, the mean of this random variable converges to $-\infty$ as $t \to +\infty$ while the variance stays constant. Hence probability mass converges to $-\infty$.

[5]For clarity, we use square brackets to label equations that appear in Obstfeld (1994) and round brackets for equations in this paper.

In equation (17), $\omega(\in [0,1])$ denotes the share of wealth invested in risky capital and its expression is given in equation [11], that is:

$$\omega \equiv \frac{\alpha - i}{R\sigma^2} > 0. \tag{18}$$

Two cases occur, depending on whether ω is smaller than or equal to one. We will refer to the first case as incomplete specialization - when the economy holds some risk-free bond - and to the second case as complete specialization - when the economy has all its wealth invested in risky capital. It is important to notice a major difference between the two configurations: when $\omega < 1$, Obstfeld (1994) shows that i equals r, the mean return on risk-free capital such that $r < \alpha$, so that a fall in exogenous risk σ^2 always results in a portfolio shift away from risk-free capital, that is, ω goes up. This first case occurs when $R\sigma^2 > \alpha - r$, that is if (utility adjusted) risk is large enough to prevent complete specialization. If, however, $R\sigma^2 < \alpha - r$, specialization is complete because risk is small enough to ensure $\omega = 1$. In that case a fall in exogenous risk triggers a rise in risk-free return $i = \alpha - R\sigma^2$ that compensates for the fall in σ^2 so that the economy keeps all its wealth in risky capital and enjoys lower risk.

A straightforward application of Proposition 1 in Section 2 leads to the following lemma:

Lemma 1 (Stochastic Stability of the Balanced-Growth Path) *Wealth tends exponentially to infinity, along a balanced growth path, with probability one when time tends to infinity if and only if* $\omega\alpha + (1-\omega)i - \mu > \frac{1}{2}\omega^2\sigma^2$.

Obstfeld (1994) defines mean growth - g in his notation - as the *growth rate of average wealth*, which is given in view of equation (17) by $\omega\alpha + (1-\omega)i - \mu$ or, equivalently after plugging the expression of the share invested in risky capital, by $g(\mathbf{E}[W]) = \varepsilon(i - \delta) + (1+\varepsilon)(\alpha - i)^2/(2R\sigma^2)$, where $\varepsilon(> 0)$ is the elasticity of intertemporal substitution in consumption and $R(> 0)$ is relative risk aversion (see equation [16] in Obstfeld, 1994). Lemma 1 shows that $g(\mathbf{E}[W]) > 0$ is necessary but not sufficient for stochastic stability of exponential growth at a positive rate. In other words, assuming $g(\mathbf{E}[W]) > 0$ would

result in convergence to *zero* wealth with probability one provided that $g(\mathbf{E}[W]) < \frac{\omega^2\sigma^2}{2}$. This is an example of the well-known fact that noise, if big enough, can significantly alter and sometimes overturn convergence as already shown in the AK case above (Section 3).

Lemma 1 therefore suggests that mean growth should be defined as the *average of the wealth growth rate*[6], that is, $\mathbf{E}[g(W)] \equiv \omega\alpha + (1-\omega)i - \mu - \frac{1}{2}\omega^2\sigma^2$ which can be simplified, using the expression of ω in (18), to:

$$\mathbf{E}[g(W)] = \varepsilon(i-\delta) + \frac{(\alpha-i)^2}{2R\sigma^2}\left(1+\varepsilon-\frac{1}{R}\right) \tag{19}$$

where $\delta > 0$ is the subjective rate of time preference. A few comments are in order. Because wealth is assumed to be log-normally distributed, the property that $\mathbf{E}[g(W)] < g(\mathbf{E}[W])$ follows, of course, from Jensen's inequality: the expected value of the log of wealth is smaller than the log of expected wealth and a similar inequality applies to their derivatives with respect to time. More importantly, one goes from the first definition of mean growth, used by Obstfeld (1994), to the second, more appropriate, one by subtracting half of the (endogenous) variance of wealth, that is, $(\alpha-i)^2/(2R^2\sigma^2)$, hence the additional term $-1/R$ in equation (19). Therefore, comparative statics results are expected to be very different, as we show next.

4.2 Comparative Statics of Mean Growth

Straightforward computations lead to the following main result of this note.

Proposition 3 (Comparative Statics of Mean Growth) *The dynamics of wealth accumulation defined in equation (17) has two regimes:*

(i) *if $R\sigma^2 > \alpha - r$ (incomplete specialization): the average growth rate $\mathbf{E}[g(W)]$ is a decreasing*

[6]In fact, Obstfeld (1994) uses later in his paper this notion for measurement purpose, e.g. in page 1321, although not for the comparative statics analysis developed at the outset.

function of exogenous risk σ^2 if and only if $R(1+\varepsilon) > 1$, that is, for large values of either risk aversion or of the intertemporal substitution elasticity.

(ii) if $\alpha - r > R\sigma^2$ (complete specialization): the average growth rate $\mathbf{E}[g(W)]$ is a decreasing function of exogenous risk σ^2 if and only if $R(1-\varepsilon) < 1$, that is, for small values of risk aversion and large values for the intertemporal substitution elasticity.

Not surprisingly, comparing Proposition 3 and results in Obstfeld (p. 1315, 1994) shows important differences. As shown in case (i) of Proposition 3, incomplete specialization results in larger growth when exogenous risk falls down only for large enough values of either risk aversion or of the intertemporal substitution elasticity. In contrast, Obstfeld (1994) claims that a portfolio shift unambiguously improves growth, independent of R and ε. When the correct definition of mean growth is used, this is no longer true. In addition, case (ii) of Proposition 3 shows that the results obtained by Obstfeld (1994) for complete specialization can be overturned under reasonable assumptions on parameters. A striking example is the case of unitary intertemporal substitution elasticity, that is, $\varepsilon = 1$. Whereas this case implies that the growth rate of average wealth is independent of exogenous risk in Obstfeld (1994) (see his equation [17]), case (ii) in Proposition 3 shows that the average growth rate is in fact a decreasing function of exogenous risk for all values of risk aversion. This property suggests that international financial integration is likely to boost growth in economies that invest all their wealth in risky capital.

More generally, conditions ensuring that a fall in exogenous risk boosts growth for both complete and incomplete specialization become clearer under the assumption that the intertemporal substitution elasticity is smaller than one, which seems to accord better with empirical measures. Remember that in Obstfeld (1994), in this case growth unambiguously goes up under incomplete specialization whereas growth *slows down* in specialized economies, following financial integration. In contrast, Proposition 3 shows that using the correct definition of mean growth delivers a more contrasted picture: when $\varepsilon < 1$, a fall in exogenous risk leads to an increase in mean growth

provided that relative risk aversion takes on moderate values, that is, if and only if $1/(1-\varepsilon) > R > 1/(1+\varepsilon)$. For example, the latter inequalities are met when $R = 1$. The bottom line is that because it leads to smaller exogenous risk, financial integration is expected to improve mean growth for both complete and incomplete specialization under reasonable parameter values.

So as to clarify the intuition behind the striking differences with results reported in Obstfeld (1994), we now focus on the case such that $\varepsilon = 1$, which leads to the well-known result that the average propensity to consume out of total wealth is then given by the impatience rate, that is, $\mu = \delta$. This assumption neutralizes the effect of exogenous risk on the consumption-wealth ratio, which has been described in earlier papers and in Obstfeld (1994) in particular. We now explain how a fall in exogenous risk affects mean growth. Again, two cases arise depending on the level of exogenous risk.

(i) if $R\sigma^2 > \alpha - r$, specialization is incomplete because exogenous risk is so large that the economy holds some risk-free capital (that is, $\omega < 1$). It follows that the risk-free interest rate $i = r$ and that the expression for mean growth simplifies to:

$$\mathbf{E}[g(W)] = r - \delta + \underbrace{\frac{(\alpha - i)^2}{R\sigma^2}}_{\text{mean return effect}} - \underbrace{\frac{(\alpha - i)^2}{2R^2\sigma^2}}_{\text{variance effect}} . \tag{20}$$

The expression for mean growth in equation (20) reveals that two conflicting effects are at work. The return effect is such that a fall in exogenous risk σ^2 boosts welfare growth because, as shown by Obstfeld (1994), the portfolio shift away from risk-free capital increases growth under the assumption that risky capital has a larger mean return than risk-free capital. However, although ignored by Obstfeld (1994), a variance effect also materializes, essentially because a larger share in the risky asset implies that the endogenous variance of wealth goes *up* when exogenous risk goes down. Stochastic stability of the balanced-growth path requires the variance effect to be not too large but such a condition does not exclude that mean growth be a decreasing function of exogenous risk, then overturning Obstfeld's result, if risk aversion is less than one half.

(ii) if $\alpha - r > R\sigma^2$, specialization is complete ($\omega = 1$). It follows that the risk-free interest rate adjusts to ensure $i = \alpha - R\sigma^2 > r$ and that the expression for mean growth simplifies to:

$$\mathbf{E}[g(W)] = \alpha - \delta - \underbrace{\frac{\sigma^2}{2}}_{\text{variance effect}} . \tag{21}$$

Equation (21) makes clear what happens when specialization is complete. In contrast to case (i), there is no return effect because the economy already benefits from full specialization so that a fall in σ^2 has no effect on the mean return - there is no portfolio shift. However, a variance effect still occurs but it now has an opposite effect on mean growth compared to case (i). This is because the endogenous variance of wealth now goes *down* when exogenous risk goes down, as the risk-free return goes up to ensure that specialization remains complete in the face of a fall in risk. Quite interestingly, an analysis based on the alternative but misleading notion of mean growth, as in Obstfeld (1994), predicts that growth is not affected by such a fall in risk.

Relaxing the assumption that $\varepsilon = 1$ delivers similar intuitions. In case (i) the return effect dominates the variance effect so that a fall in exogenous risk fosters growth if and only if risk aversion is large enough. In case (ii) there is no return effect and the variance effect, now working in opposite direction, implies that growth improves after a fall in exogenous risk only if risk aversion is not too large when the intertemporal substitution elasticity is smaller than unity. In other words, our results about specialized economies accord with the well-documented trade-off between growth and volatility under reasonable assumptions about attitudes toward risk, for example if relative risk aversion equals one. In contrast, incomplete specialization leads to a positive relationship between the mean growth and variance of wealth under unitary risk aversion. Overall these results suggest that taking into account the variance effect on mean growth, which has been ignored by Obstfeld (1994), yields the prediction that the effects of financial integration on economies that specialize in risky capital do not qualitatively differ from those on economies that hold some risk-free capital if reasonable parameter values are assigned to risk aversion and intertemporal substitution.

To make the comparison even more transparent, we now reproduce and extend in Table 1 a numerical example given in Obstfeld (1994). More precisely, Table 1 starts with the Example 1 that is presented in pages 1318-1319 of Obstfeld (1994) and that assumes $R = 4$ and $\varepsilon = 1/2$ in particular. Table 1 compares the magnitudes of both definitions of mean growth under this parameterization and also, for robustness purpose, when $R = 1$ while all other parameter values are unchanged.

Table 1. Numerical Values of $\mathbf{E}[g(W)]$ in Left Panel and $g(\mathbf{E}[W])$ in Right Panel

	$R = 4$	$R = 1$
Autarky	1.41%	1.25%
Integration	1.75%	1.38%

	$R = 4$	$R = 1$
Autarky	1.69%	1.75%
Integration	2.00%	1.63%

In line with the analytical characterization outlined above, comparing both panels in Table 1 confirms that the mean growth rate of wealth is lower than the growth rate of mean wealth. More interestingly, comparing the rightmost columns of both panels reveals that, when $R = 1$, the conclusion regarding growth that is obtained by Obstfeld (1994) is overturned when the appropriate concept of mean growth is adopted. In fact, while the right panel predicts that growth falls (by about 12 basis points) after integration in the case of full specialization, it turns out that growth actually *goes up* (by about 13 basis points) as depicted in the left panel that uses the appropriate definition of mean growth. Let us stress that although welfare computations reported in Obstfeld (1994) are not altered at all by stability considerations, the examples in Table 1 further confirm that different comparative statics properties obtain when the stability-related concept of mean growth is used, as it should be. Aside from theoretical concerns, this is also relevant for empirical research, which typically aims at measuring the growth gains from international financial integration.

5 Conclusion

The economic literature is extremely scarce on the stability of stochastic endogenous growth models in contrast to the neoclassical growth model. This paper presents a simple mathematical apparatus to appraise this task in continuous time settings. We show why stability of balanced growth paths inherent in the AK-like growth models need not be robust to uncertainty, the key mathematical mechanism behind being the stabilizing properties of stochastic noise. We notably argue that accounting for stochastic stability is most important in practice, and we illustrate this by revisiting the seminal global diversification model due to Obstfled (1994). Concretely, we show, by way of analytical results and numerical examples, that the comparative statics results derived in Obstfeld (1994) are misleading because they are based on an inappropriate notion of mean growth: conditions ensuring that the exponential balanced-growth path is stable, in the stochastic sense, reveal that mean growth should be defined as the average growth rate of wealth, as opposed to the growth rate of average wealth. With such a definition in hand, international financial integration leads to very different comparative statics results and that it is much more likely to boost growth, both for fully specialized economies that invest all their wealth in risky capital and for economies that hold some risk-free capital.

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