International Borrowing Without Commitment and Informational Lags: Choice under Uncertainty

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ABSTRACT. A series of recent studies in economic growth theory have considered a class of models of international borrowing where, in the absence of a perfect investment commitment, the borrowing constraint depends on the historical performances of the country. Thus, a better level of past economic activity gives a higher reputation, thereby increasing the possibility of accessing the international credit market. This note considers this problem in a stochastic setting based on the volatility of the internal net capital. We study how the optimal consumption level and the maximal expected welfare depend on the combined influence of the trajectory of past economic variables and the volatile environment. In particular, we show how the strength of the *history effect* and the relative weight of the historical performance depend on the degree of risk.

1. INTRODUCTION

Modeling constraints on access to the international credit market for small or highly indebted countries is a lively issue at present. A possible approach to address this question was proposed by Boucekkine and Pintus (2012) based on an intuition of Coen and Sachs (1986). They relaxed the unrealistic assumption of commitment to investment by considering the importance of the "historical course" of the economy. In particular they assumed that, in the impossibility for the debtor country to commit to an investment strategy, the lender bases its decisions on past investments, and thus the past path of the capital stock.

The no-commitment delay between the past capital measure and the current borrowing capacity (and thus the current investment possibilities) is the basis of the *history effect* emphasized by Boucekkine and Pintus (2012), which allows their model to replicate a series of macroeconomic instability behaviors, such as growth break and growth reversal phenomena that are recurrent and well documented (e.g., see Jones and Olken, 2008 or Cuberes and Jerzmanowki, 2009), and to justify their relationship with the process of financial integration.

Boucekkine et al. (2013) (and a companion paper by Boucekkine et al., 2011) introduced explicit preferences and optimal saving decisions into the framework of Boucekkine and Pintus's model, which was originally formulated based on the hypothesis of a fixed exogenous saving rate *à la Solow*. In this manner, they studied the welfare implications of financial globalization in the context of the model. They qualitatively replicated the empirical observations of Kaminsky and Schmuler (2008), thereby suggesting that financial globalization

can lead to a short-run consumption (and welfare) drop and a long-run gain. In addition, Boucekkine et al. (2013) emphasized the differential impact of financial integration changing the historical economic path, where countries with the same initial capital stock but different paths achieve highly variable results after integration into the international financial system, thereby further demonstrating the importance of history.

In this note, we propose a stochastic version of the model of Boucekkine et al. (2013). In fact, due to problems of analytical tractability, we employ a constant absolute risk aversion (CARA) utility function whereas Boucekkine et al. (2013) focused on the constant relative risk aversion (CRRA) case. Apart from this (and the new stochastic terms), the two models are identical.

Several determinants of risk need to be considered when a country assesses its borrowing choices. First, there is a series of exogenous factors related to exposure to international credit market volatility: as argued, e.g., by Prasad et al. (2007), at least in the early stages, financial integration is associated with significant increases in the volatilities of both output and consumption. Second, there is the volatility associated with domestic shocks, which, as shown by Loayza et al. (2007), has an important role especially in the setting of the small, typically developing, countries with open economies that we consider in the present study. In this study, we focus on this second series of phenomena and specifically on the macroeconomic volatility that affects the structure of production, due, for example, to production specialization (e.g., see Kraay and Ventura, 2007) or social conflicts (Raddatz, 2007). Thus, the volatility is linked to the level of net capital (capital net of foreign debt) in our model (see Section 2 for details).

A version of the model without any informational lag was studied by Boucekkine et al. (2014), whereas we model the absence of commitment to investment *à la* Boucekkine and Pintus (2012). A certain number of model predictions, such as the positive effect of volatility on precautionary saving and then on the long-run growth rate, can be described using the simpler stochastic one-dimensional model of Boucekkine et al. (2014), but studying the interaction between the *history effect* and the risky environment needs to introduce the no-commitment delay. Thus, we omit the questions that can be answered clearly using the simpler set-up of Boucekkine et al. (2014), and we focus on those that can only be studied in the new context: how does the history effect change with the characteristics of the economy and what is the role of volatility? Furthermore, does the relative importance of remote events or more recent facts change in different contexts? In particular, can we observe some "oblivious" processes?

To answer these questions, we first characterize the optimal planner solution. In Section 3, we provide the explicit expression of the optimal consumption in feedback form, we characterize the optimal capital trajectory as the solution to a suitable stochastic equation (Theorem 3.3), and we determine the welfare that corresponds to the optimal consumption/saving policy (see Proposition 3.5). These results allow us to consider the structure of the optimal policy in detail, by decomposing its expression in terms of the contributions of the present net capital and of the past capital history and by emphasizing the different weights of different past periods (see Section 4.1). We prove that the total strength of the history effect is not reduced by the volatile environment. This is an interesting corroboration of the solidity of the history effect. Is spite of this we show that the relative weights of the "old" history terms decrease when the environment is more volatile (or in a situation where individuals are more risk averse), whereas recent events become increasingly important; thus, the volatility promotes an "oblivious" process.

The methodological contribution. Several previous studies used delay differential equations (i.e., functional differential equations where the variable appears in delayed form[1]) to model several economic phenomena, but Boucekkine and Pintus (2012) were probably the first to introduce an economic model driven by a neutral differential equation (NDE). In the NDE case, the "past" of the variable and that of its derivative are included in the equation. NDEs are harder to study than delay differential equations: the typical regularizing properties of delay differential equations are not valid in the NDE case (e.g., see Hale and Lunel, 1993) and the asymptotic properties are more difficult to prove. However, because systems driven by delay differential equations are already infinite dimensional, *a fortiori* dealing with NDEs involves working in an infinite-dimensional set-up.

A further advance in terms of technical complexity was considered by Boucekkine et al. (2011) and Boucekkine et al. (2013), where they had to deal with an optimal control problem driven by an NDE to study their model. As argued by Kolmanovski and Myshkis (1999) (particularly in Chapter 14), the use of the maximum principle is problematic in the NDE case (indeed most previous studies of the control of NDEs consider robust control and optimal control is very rare). Boucekkine et al. (2011) studied the problem by using the tools of dynamic programming in infinite dimensions. A similar approach was already used for simpler cases of models driven by delay differential equations, see, e.g., Fabbri and Gozzi (2008).

An additional difficulty is considered in the present note. The optimal control problem is now driven by a stochastic NDE (i.e., the state equation (8)), i.e., an NDE with an extra stochastic term. This problem is also approached using dynamic programming in infinite dimensions. Provided that the positivity condition on the net capital trajectory is satisfied, we can write the value function expression explicitly and characterize the explicit solution to the problem in closed-loop form (see Theorem 3.3). To the best of our knowledge, this is the first optimal control problem driven by a stochastic NDE to be solved in the (not only economic) literature[2]. The generalization with respect to the deterministic case is not trivial because the stochastic term in the state equation entails a second order term in the infinite dimensional Hamilton–Jacobi–Bellman equation (by contrast, only the first order Fréchet differential appears in the deterministic case) and a stronger regularity is needed to define the regular solutions. For further details, Appendix A provides the mathematical apparatus and the necessary proofs.

Structure of the note. This note is organized as follows. In Section 2, we introduce the model and its main features. Section 3 presents the analytical results, in Section 4 we discuss the results and their implications, in Section 5 we present two generalizations of our approach while Section 6 gives the conclusions of this study. Appendix A contains the proofs.

2. The model

We consider a small open economy with an aggregate AK production function, where $K(t)$ is the capital input at time t and A is the level of technology. At each time point, the country can borrow on the international credit market at a fixed and exogenous interest rate r.

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[1] Some examples in various domains are: Asea and Zak (1999) or Bambi (2008) in terms of growth models with time-to-build during production, d'Albis et al. (2012) in modeling the learning-by-doing process, Boucekkine et al. (2005) with a vintage capital model, and Feichtinger et al. (1994) with an advertising model.

[2] By contrast, there have been several economic models in the form of optimal control problems driven by stochastic delay differential equations, e.g., see Gozzi et al. (2009), Federico and Tankov (2014), and Fabbri and Federico (2014). However, as in the deterministic case, they are more tractable than optimal control problems driven by stochastic NDEs due to the absence of the delayed derivative term.

We denote by δ the depreciation rate of the capital, $C(t)$ and $D(t)$ are the level of the aggregate consumption and the stock of net foreign debt at time t, respectively, and $N(t)$ is noise (as specified below) that perturbs the economy. We assume that the evolution of the variables satisfies the following equation

$$(1) \qquad \dot{K}(t) - \dot{D}(t) = AK(t) - \delta K(t) - rD(t) - C(t) + N(t).$$

Excluding the noise N, this is simply the deterministic budget constraint of the economy described by Boucekkine et al. (2013).

Following Boucekkine and Pintus (2012) and Boucekkine et al. (2013), and in the spirit of Cohen and Sachs (1986), we assume that the borrowing capacity of the country depends on the past performance of the economy and particularly that, for any $t \geq 0$,

$$(2) \qquad D(t) = \lambda K(t - \tau),$$

for some positive exogenous constant (commitment-delay) $\tau > 0$ and some credit multiplier λ with

$$(3) \qquad \lambda \in [0, 1).$$

Since the model is AK we can rewrite the previous expression as $D(t) = \frac{\lambda}{A} Y(t - \tau)$ and then we can see that (2) is in fact a relation between the debt and the GDP.

We define

$$(4) \qquad S(t) := K(t) - D(t)$$

as the net capital of the country: the capital net of foreign debt. Of course, by using (2) in (4), we obtain

$$(5) \qquad S(t) = K(t) - \lambda K(t - \tau).$$

As mentioned in the introduction, we assume that the noise $N(t)$ is associated with the net capital level. In particular we will be interested in trajectories along which $S(t) = K(t) - \lambda K(t - \tau)$ remains strictly positive and, similarly to Boucekkine et al. (2014), we assume that $N(t)$ has the form

$$(6) \qquad N(t) := \sqrt{\gamma S(t)} \frac{\mathrm{d}W(t)}{\mathrm{d}t},$$

where $W(t)$ is the standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which generates the filtration $\{\mathcal{F}_s\}_{s \geq 0}$. Focusing on the noise depending on the internal net capital $S(t)$ has the meaning of ignoring the noise coming from fluctuations of the international interest rate and focusing on the volatility due to the internal economic structure. Setting the parameter $\gamma > 0$ allows us to select the strength of the volatility in terms of the net capital. From (5) and (6), we obtain

$$(7) \qquad N(t) = \sqrt{\gamma(K(t) - \lambda K(t - \tau))} \frac{\mathrm{d}W(t)}{\mathrm{d}t}$$

and by using (2) and (7) in (1) we obtain the equation that describes the evolution of the economy after the consumption process $C(\cdot)$ has been selected:

$$(8) \qquad \begin{cases} \mathrm{d}(K(t) - \lambda K(t - \tau)) = [(A - \delta)K(t) - r\lambda K(t - \tau) - C(t)]\,\mathrm{d}t \\ \qquad\qquad\qquad\qquad\qquad\qquad + \sqrt{\gamma(K(t) - \lambda K(t - \tau))}\,\mathrm{d}W(t) \\ K(s) = K_I(s) \qquad \text{for all } s \in [-\tau, 0]. \end{cases}$$

As emphasized in the introduction, (8) is an NDE. Indeed, in the expression of $\dot{K}(t)$ described using this relation, the past of the variable $K(t)$ appears in the form $K(t - \tau)$, as well as a

term that depends on the past of the derivative of K, i.e., $\dot{K}(t-\tau)$. In fact, (8) is a *stochastic NDE* since the stochastic term $\sqrt{\gamma(K(t)-\lambda K(t-\tau))}\frac{dW(t)}{dt}\,dW(t)$ is also included.

Similarly to the case of delay differential equations, we have to consider a whole function as an initial datum, $K_I(s)$ for $s \in [-\tau, 0]$, which is the whole "history" of the variable K in the interval $[-\tau, 0]$. For technical reasons we only consider continuous (and deterministic) initial data $K_I \colon [-\tau, 0] \to \mathbb{R}$. In particular $K_I(\cdot)$ belongs to $L^2(-\tau, 0)$, the space of real square integrable functions defined on $(-\tau, 0)$ (see Appendix A for some details on the structure of the space $L^2(-\tau, 0)$). We also need a certain regularity on the control $C(\cdot)$. More precisely we suppose that $C(\cdot)$ belongs to the set

$$M_{loc}^2 := \left\{ x(\cdot) \colon [0, +\infty) \times \Omega \to \mathbb{R} \ : \ \begin{array}{l} x(\cdot) \text{ is } \mathcal{F}^t - \text{progressively measurable and,} \\ \text{for any } T > 0, \ \left(\mathbb{E}\left(\int_0^T |x(s)|^2 \, ds \right) \right)^{1/2} < +\infty. \end{array} \right\}.$$

After introducing the non-standard equation that describes the evolution of the capital stock, we complete the model in a highly classical manner by assuming that the planner selects the aggregate consumption process $C(t)$ in order to maximize the following welfare functional

$$(9) \qquad J(C(\cdot)) := \mathbb{E}\left[\int_0^\infty e^{-\rho t} \left(-e^{-\eta C(t)} \right) dt \right],$$

where $\eta > 0$ is a fixed parameter that represents the Arrow–Pratt absolute risk aversion coefficient. As highlighted in the introduction, we work with CARA preferences instead of the CRRA case to obtain a closed-form solution to the optimal control problem. Appendix A shows that treating this case is already non-trivial.

Since its square root appears in (8), we need to guarantee the positivity of the net capital $S(t)$ along the admissible trajectories. In fact we will ask for the strict positivity of $S(t) = K(t) - \lambda K(t - \tau)$ to ensure the local Lipschitz continuity of the equation and then the existence-and-uniqueness of the solution of (8), see Appendix A after (33) for details. Thus, we define the set of admissible consumption processes as:

$$\mathcal{U}_{K_I} := \left\{ C(\cdot) \in M_{loc}^2 \ : \ S(\cdot) \text{ remains a.s. strictly positive} \right\}.$$

We denote by

$$(10) \qquad V(K_I) = \sup_{C(\cdot) \in \mathcal{U}_{K_I}} J(C(\cdot))$$

the value function of the problem, which measures the social welfare when the planner follows the optimal policy.

If we select $\tau = 0$, as a special case (apart from a normalization of the parameters), we obtain the exact model studied by Boucekkine et al. (2014). However, they did not consider delayed terms and this was reduced to a standard one-dimensional stochastic optimal control problem. In particular, during the evolution of the economic system described by Boucekkine et al. (2014), there was no role for the value of the capital in the interval $[-\tau, 0)$ and they only needed its initial value $K(0)$.

3. Solution of the model and the results

First, we characterize the solution of the optimal control problem of the planner, i.e., maximizing (9) subject to (8). Later, we consider the implications of the results. All of the proofs are provided in Appendix A, which also gives the Hilbert space set-up that we use to deal with the problem.

We start with the following lemma by introducing the notations used to describe the solution of the problem. First, we characterize the constant $\bar{\xi}$.

Lemma 3.1. *Assume that*

(11) $$A - \delta - r > 0,$$

then the equation

(12) $$\xi = \frac{-(A - \delta)e^{-\xi\tau} + r\lambda}{\left(\frac{1}{2}\gamma + \frac{1}{\eta}\right)(e^{-\xi\tau} - \lambda)\eta}$$

has a unique negative solution $\bar{\xi}$. Moreover, $\bar{\xi}$ is a strictly increasing function of γ, η, r, and δ, and a strictly decreasing function of A.

Remark 3.2. *Both the conditions (3) and (11) (which are the same as those used by Boucekkine et al., 2014) are "technical": we cannot solve the problem if they are violated. Nevertheless, they are verified with reasonable choices of the parameters.*

Indeed: (i) the value of $A = \frac{Y}{K}$ (see Piketty and Zucman, 2014) is, in rich countries, between 17% and 30%, and probably a little more in less developed countries; (ii) the depreciation rate of the capital δ (see Fraumeni, 1997) strongly depends on the nature of capital but at the aggregate level it typically remains (see Kamps, 2004) below 10%; (iii) the global (real) interest rate r, in last decades (IMF, 2014), is a few percentage points. Putting things together we see that (11) is a realistic condition.

Furthermore, since, as we said, A is typically smaller than $30 - 40\%$ and the ratio debt/GDP is only exceptionally greater than 250% (see again IMF, 2014), the value of $\frac{D(t)}{K(t)} = A\frac{D(t)}{Y(t)}$ is almost always smaller than 1. If τ is not too big (some years for example), since the growth rate of the capital and of the GDP is a few percentage points every year, the same is true for $\lambda = \frac{D(t)}{K(t-\tau)} = A\frac{D(t)}{Y(t-\tau)}$ and then (3) is a realistic condition.

Boucekkine and Pintus (2011) (see for instance Table 1 of their paper) propose a wide possible range for the value of the parameter τ. It represents the time interval necessary to past behavior and policy to be well established and clear to international lenders; it also includes the difficulty and the delay to have updated and reliable data for economy of developing countries (see for instance the retard in the data for developing countries in the datasets of IMF, 2014 or WB, 2015). Its order of magnitude should be then thought to be a few years.

When (3) and (11) are satisfied we can ensure the existence of $\bar{\xi}$ and we introduce the following notations:

$$a_0 := -\eta\bar{\xi}$$

and

(13) $$\bar{h} := -\eta\bar{\xi}\frac{(A - \delta - r)\lambda}{e^{-\bar{\xi}\tau} - \lambda}.$$

We observe that a_0 and $\bar{h}$ are positive.

We give the solution of the model in the following two theorems. In the first theorem, we characterize the optimal consumption as a (feedback) function of the state of the problem, i.e., the path of the capital in the last period τ. By using this expression in the state equation (8), we obtain a stochastic NDE, its unique solution is the optimal trajectory of the capital for any $t \geq 0$.

Theorem 3.3. *Assume that (11) is verified. Provided that the corresponding trajectory of S remains strictly positive, the optimal control for the problem (8) - (9) can be expressed in*

feedback form as follows:

$$
(14) \qquad C(t) = \frac{\rho}{a_0} - \frac{1}{\eta} + \frac{1}{\eta}\left[(K(t) - \lambda K(t-\tau))a_0 + \int_{-\tau}^{0} \bar{h}e^{\bar{\xi}s}K(t+s)\,\mathrm{d}s\right].
$$

Moreover, under the same assumptions, the optimal trajectory of K is the only solution of the following stochastic neutral differential equation:

$$
(15)
$$
$$
\begin{cases}
\mathrm{d}(K(t) - \lambda K(t-\tau)) = \left[(A-\delta)K(t) - r\lambda K(t-\tau) - \frac{\rho}{a_0} + \frac{1}{\eta}\right. \\
\qquad \left. - \frac{1}{\eta}\left[(K(t) - \lambda K(t-\tau))a_0 + \int_{-\tau}^{0}\bar{h}e^{\bar{\xi}s}K(t+s)\,\mathrm{d}s\right]\right]\mathrm{d}t + \sqrt{\gamma(K(t)-\lambda K(t-\tau))}\,\mathrm{d}W(t) \\[2mm]
K(s) = K_I(s) \qquad \text{for all } s \in [-\tau, 0].
\end{cases}
$$

Remark 3.4. *Using standard results on the behavior of stochastic neutral delay differential equations (see for example Theorems 4.5 page page 213 and Theorem 4.7 page 216 of Mao, 2007) we can see that, under the hypotheses of Theorem 3.3, the trajectory $K(t)$ satisfies the following estimates:*

(i) *For any $p \geq 2$ there exists two constants c_p^1, c_p^2 (depending also on the initial datum) such that, for any $t \geq 0$,*

$$
\mathbb{E}\left(\sup_{s \in [-\tau, t]} |K(s)|^p\right) \leq c_p^1 \exp\left(t\, c_p^2\right)
$$

(ii) *There exists a constant C such that, $\mathbb{P}$-almost surely, $\limsup_{t\to\infty}\frac{1}{t}\ln|x(t)| \leq C$.*

In the following theorem, we characterize the value function of the problem. It is the supremum (indeed the maximum) of the (welfare) functional (9) by varying the consumption process among all the admissible consumption processes.

Proposition 3.5. *Under the same hypothesis as Theorem 3.3, the social welfare obtained by the planner who implements the optimal consumption policy (i.e., the value function V of the optimization problem (8) - (9)) can be expressed as an explicit function of the initial history of the capital path. More precisely, if we introduce*

$$
\bar{\beta} := \frac{-1}{\bar{\xi}}\exp\left(\frac{\rho}{\bar{\xi}} + 1\right) > 0,
$$

V is given by

$$
(16) \qquad V(K_I) = -\bar{\beta}\exp\left(-(K_I(0) - \lambda K_I(-\tau))a_0 - \int_{-\tau}^{0}\bar{h}e^{\bar{\xi}s}K_I(s)\right).
$$

The optimal trajectory of the capital is given by the solution of (15) and it then has a complex behavior. Indeed, already in the nonstochastic case (i.e., if we take $\gamma = 0$), the optimal K is characterized as the solution of an NDE and then, differently from the basic (deterministic, one-dimensional) AK models, its evolution is not, in general, a simple exponential (see Boucekkine et al., 2011, 2013).

Note that we are working with a continuous and deterministic initial datum K_I, thus (16) is well defined.

4. COMMENTS ON THE RESULTS

$V(K_I)$ depends on the initial datum K_I in terms of the expression

$$(17) \qquad \left(-(K_I(0) - \lambda K_I(-\tau))a_0 - \int_{-\tau}^0 \bar{h} e^{\bar{\xi}s} K_I(s) \right).$$

It allows us to separate the weight of different past points of the historical capital path. Formally, this does not differ greatly from the expressions that appear in the value functions of various models driven by delay differential equations (e.g., see Fabbri and Gozzi, 2008; or Boucekkine et al., 2010) but, if we compare (17) with the corresponding expressions in Fabbri and Gozzi (2008) or Boucekkine et al. (2010), we can see that there is a specific role for the value of the past capital at time $-\tau$. This is attributable to the NDE nature of the dynamics of our economy. From an economic viewpoint, this is not too surprising because given (5), the term $K_I(0) - \lambda K_I(-\tau)$ represents the initial value of the net capital.

As we can see in (16), the dependence of the value function on the variable described in (17) is exponential, while, in the papers quoted above, where the structure of the utility function is CRRA, the value function is proportional to a certain power of the expression that correspond to (17). It is, qualitatively, the same difference we find in the (deterministic) one-dimensional versions of the AK model if we vary the structure of the utility function. Differently from the models in the papers mentioned above we have here a stochastic setting; the volatility coefficient γ influences the value of (17) only through the weight $\bar{h} e^{\bar{\xi}s}$. The next subsection is devoted to the study of this dependence.

4.1. **The effect of the volatility on the memory effect.** At time t, consider the expression of the optimal consumption in feedback form given by (14). The dependence on the state K is given in terms of

$$a_0 \left[(K(t) - \lambda K(t - \tau)) + (A - \delta - r)\lambda \int_{-\tau}^0 \frac{e^{\bar{\xi}s}}{e^{-\bar{\xi}\tau} - \lambda} K(s + t) \, \mathrm{d}s \right].$$

Due to (5), this can be rewritten, with the exception of the common factor a_0, which does not affect the relative weights of the various terms, as

$$(18) \qquad \left[S(t) + (A - \delta - r)\lambda \int_{-\tau}^0 \frac{e^{\bar{\xi}s}}{e^{-\bar{\xi}\tau} - \lambda} K(s + t) \, \mathrm{d}s \right] =: S(t) + \mathscr{H}(t).$$

This depends on two elements: the net capital S at time t and a weighted integral of the path of the capital stock in the interval $[t - \tau, t]$, that we denote by $\mathscr{H}(t)$.

To measure the relevance of the history effect on the optimal decision of the policy maker we observe that

(i) the information about the present state of the economy is contained in the variable $S(t)$; its weight in the whole expression, equal to its multiplicative coefficient, is 1
(ii) the weight given to the (information about the) stock of the capital at time $t + s$, where $s \in [-\tau, 0)$, is $(A - \delta - r)\lambda \frac{e^{\bar{\xi}s}}{e^{-\bar{\xi}\tau} - \lambda} \, \mathrm{d}s$

so a measure of the total weight of the information concerning the past behavior of the economy (the strength of the history effect) is given by the variable $\mathcal{I}_{tot}$

$$\mathcal{I}_{tot} := (A - \delta - r)\lambda \int_{-\tau}^0 \frac{e^{\bar{\xi}s}}{e^{-\bar{\xi}\tau} - \lambda} \, \mathrm{d}s = \frac{(A - \delta - r)\lambda}{e^{-\bar{\xi}\tau} - \lambda} \left(\frac{1}{\bar{\xi}} \left(1 - e^{-\bar{\xi}\tau} \right) \right).$$

To measure how the different past periods matter in the total history effect we also introduce the variable $\mathcal{I}_{\tau_1}^{\tau_2}$ measuring the weight of the interval $[\tau_1, \tau_2]$ (with $-\tau \leq \tau_1 \leq \tau_2 \leq 0$):

$$\mathcal{I}_{\tau_1}^{\tau_2} := (A - \delta - r)\lambda \int_{\tau_1}^{\tau_2} \frac{e^{\bar{\xi}s}}{e^{-\bar{\xi}\tau} - \lambda} \, \mathrm{d}s = \frac{(A - \delta - r)\lambda}{e^{-\bar{\xi}\tau} - \lambda} \left(\frac{1}{\bar{\xi}} \left(e^{-\bar{\xi}\tau_2} - e^{-\bar{\xi}\tau_1} \right) \right).$$

and the index $i_{\tau_1}^{\tau_2}$ measuring the relative weight of the interval $[\tau_1, \tau_2]$ in the whole history effect:

$$i_{\tau_1}^{\tau_2} := \frac{\mathcal{I}_{\tau_1}^{\tau_2}}{\mathcal{I}_{tot}}.$$

Observe that, for any $a \in [-\tau, 0]$, $i_{-\tau}^a + i_a^0 = 1$.

We study now the impact of a modification of the parameters on the history effect. We start by γ.

Proposition 4.1. *Increasing the parameter γ increases the relative importance of the recent history and decreases the relative importance of the more ancient events in the sense that, for any $a \in [-\tau, 0]$,*

$$\frac{\mathrm{d}i_{-\tau}^a}{\mathrm{d}\gamma} < 0 \qquad and \qquad \frac{\mathrm{d}i_a^0}{\mathrm{d}\gamma} > 0.$$

Moreover, increasing γ does not reduce the aggregate importance of the history effect on the decision of the policy maker in the sense that

$$\frac{\mathrm{d}}{\mathrm{d}\gamma} \frac{\mathcal{I}_{tot}}{\mathcal{I}_{tot} + 1} > 0.$$

The first result of the proposition follows by noticing that the way different past periods contribute to the history effect depends on the discounting term $e^{\bar{\xi}s}$ and then on the value of $\bar{\xi}$: the higher $\bar{\xi}$ the higher the relative importance of recent events. Since we know, from Lemma 3.1, that $\bar{\xi}$ is an increasing function of the volatility parameter γ, we can conclude that in a more volatile environment the older history decreasingly determines the optimal decisions and the optimal policy depends increasingly on recent events.

In the model, even in a very noisy context, the history of the variable K determines the possibility of borrowing in the future and then it has a key role; the second part of the proposition tells us that the aggregate historic effect does not weaken increasing γ. In particular, even if we consider a extremely big γ, the weight of the total history effect never vanishes.

The impact of the absolute risk aversion parameter η is similar as shown in the following proposition.

Proposition 4.2. *Increasing the parameter η increases the relative importance of the recent history and decreases the relative importance of the more ancient events in the sense that, for any $a \in [-\tau, 0]$,*

$$\frac{\mathrm{d}i_{-\tau}^a}{\mathrm{d}\eta} < 0 \qquad and \qquad \frac{\mathrm{d}i_a^0}{\mathrm{d}\eta} > 0.$$

Moreover, increasing γ does not reduce the aggregate impact of the history effect on the decision of the policy maker in the sense that

$$\frac{\mathrm{d}}{\mathrm{d}\eta} \frac{\mathcal{I}_{tot}}{\mathcal{I}_{tot} + 1} > 0.$$

Indeed the fact that investment decisions are affected by the risk aversion parameter η and by the volatility parameter γ in a similar way is not characteristic of the setting we have here (see for example Boucekkine et al., 2014). It is not too surprising because the risk aversion measures in fact the sensibility of the agent to the volatility parameter.

Both the parameters γ and η influence the weight $\bar{h}e^{\bar{\xi}s}$ and then on all the $\mathcal{I}^{\tau_2}_{\tau_1}$ and $i^{\tau_2}_{\tau_1}$ only through their impact on the value of $\bar{\xi}$. The relation (12) defining $\bar{\xi}$ can be rewritten as follows

$$\xi = \frac{-(A - \delta)e^{-\xi\tau} + r\lambda}{\left(\frac{1}{2}\gamma\eta + 1\right)\left(e^{-\xi\tau} - \lambda\right)}$$

so the effects of γ and η are complementary. Even if we cannot provide simple expressions for the elasticities of the weight, $\mathcal{I}^{\tau_2}_{\tau_1}$ and $i^{\tau_2}_{\tau_1}$ with respect to γ and η, the expression above tells us that the values of the elasticities with respect to γ are the same the values of the elasticities with respect to η. Indeed, if we denote by V the weight $\bar{h}e^{\bar{\xi}s}$ (at a certain point s) or, fixed τ_1 and τ_2, $\mathcal{I}^{\tau_2}_{\tau_1}$ or $i^{\tau_2}_{\tau_1}$ and by ε_γ (respectively ε_γ) the elasticity of V with respect to γ (respectively η) we have

$$(19) \quad \varepsilon_\gamma = \frac{\mathrm{d}V}{\mathrm{d}\gamma}\frac{\gamma}{V} = \frac{\mathrm{d}V}{\mathrm{d}\bar{\xi}}\frac{\mathrm{d}\bar{\xi}}{\mathrm{d}\left(\frac{1}{2}\gamma\eta\right)}\frac{\mathrm{d}\left(\frac{1}{2}\gamma\eta\right)}{\mathrm{d}\gamma}\frac{\gamma}{V} = \frac{\mathrm{d}V}{\mathrm{d}\bar{\xi}}\frac{\mathrm{d}\bar{\xi}}{\mathrm{d}\left(\frac{1}{2}\gamma\eta\right)}\frac{\left(\frac{1}{2}\gamma\eta\right)}{V}$$

$$= \frac{\mathrm{d}V}{\mathrm{d}\bar{\xi}}\frac{\mathrm{d}\bar{\xi}}{\mathrm{d}\left(\frac{1}{2}\gamma\eta\right)}\frac{\mathrm{d}\left(\frac{1}{2}\gamma\eta\right)}{\mathrm{d}\eta}\frac{\eta}{V} = \frac{\mathrm{d}V}{\mathrm{d}\gamma}\frac{\eta}{V} = \varepsilon_\eta.$$

5. Possible extensions

In this section we look at two possible extensions of the presented approach to cover, on the one hand, models where the level of the variable D at time t depends on the whole history of K in the interval $[t - \tau, t]$ in a distributed way, and, on the other hand, the case where it depends both on $K(t)$ and $K(t - \tau)$.

5.1. **Models with distributed delays.** Consider again a model whose dynamics is described by an equation of the form described in (1). Assume now that, instead of having a dependence of $D(t)$ in terms of $K(t - \tau)$ as in (2), we have the following relation:

$$(20) \qquad\qquad D(t) = \int_{-\tau}^{0} K(t + r)f(r)\,\mathrm{d}r$$

where f is a positive function in $L^2(-\tau, 0)$. The variable $S(t)$ reads now as

$$S(t) = K(t) - \int_{-\tau}^{0} K(t + r)f(r)\,\mathrm{d}r$$

and the noise becomes

$$N(t) := \sqrt{\gamma S(t)}\frac{\mathrm{d}W(t)}{\mathrm{d}t} = \sqrt{\gamma\left(K(t) - \int_{-\tau}^{0} K(t + r)f(r)\,\mathrm{d}r\right)}\frac{\mathrm{d}W(t)}{\mathrm{d}t}.$$

We keep then the same target functional (9) and the same set of controls as before. Similarly to what happens in the problem studied in the previous section, we can solve explicitly the problem as described in the following.

Lemma 5.1. *Suppose that*

$$(21) \qquad\qquad A - \delta - r > \frac{A - \delta}{\int_{-\tau}^{0} f(s)\,\mathrm{d}s} > 0$$

then there exist a unique $\xi < 0$ such that

$$\frac{1}{r}\left(1 - \frac{1}{\int_{-\tau}^{0} e^{-\xi s}f(s)\,\mathrm{d}s}\right) = \frac{1}{\xi^2\eta^2\left(\frac{1}{2}\gamma + \frac{1}{\eta}\right) - (A - \delta)}.$$

For such a value of ξ, we have $\int_{-\tau}^{0} e^{-\xi s}f(s)\,\mathrm{d}s > 1$.

Theorem 5.2. *Let the condition (21) be verified. Denote by β the constant*

$$\beta = -\frac{1}{\xi}\exp\left(1 + \frac{\rho}{\xi}\right)$$

with ξ the negative value found in Lemma 5.1, by a_0 the quantity $-\eta\xi > 0$ and by h the value

$$h = \frac{-\eta\xi r}{\int_{-\tau}^{0} e^{-\xi\theta} f(\theta)\,d\theta - 1} > 0.$$

Provided that the corresponding trajectory of S remains strictly positive, the optimal control for the problem, in the case of the distributed delays problem described above, can be expressed in feedback form as follows:

$$(22) \quad C(t) = \frac{\rho}{a_0} - \frac{1}{\eta} + \frac{1}{\eta}\left[a_0\left(K(t) - \int_{-\tau}^{0} f(\theta)K(t+\theta)\,d\theta\right)\right.$$
$$\left. + \int_{-\tau}^{0} he^{\xi s}\int_{-\tau}^{s} e^{-\xi\theta} f(\theta)\,d\theta\, K(t+s)\,ds\right].$$

Moreover the social welfare obtained by the planner who implements the optimal consumption policy (i.e., the value function V of the optimization problem) is

$$(23) \quad -\beta\exp\left(a_0\left[K(0) - \int_{-\tau}^{0} K(\theta)f(\theta)\,d\theta\right] + \int_{-\tau}^{0} he^{\xi s}\int_{-\tau}^{s} e^{-\xi\theta} f(\theta)\,d\theta\, K(s)\,ds\right).$$

5.2. **A model where $D(t)$ depends on the present and on the past of K.** A second possible variant of the model is the one where the level of debt available for the borrower depends on the present and the past values of the GDP (or of the capital, since the model is AK) in the following way:

$$(24) \qquad\qquad D(t) = \lambda_1 K(t) + \lambda_2 K(t - \tau).$$

The variable $S(t)$ becomes

$$S(t) = K(t) - (\lambda_1 K(t) + \lambda_2 K(t - \tau)) = (1 - \lambda_1)K(t) - \lambda_2 K(t - \tau).$$

The counterpart of (8) is now

$$d((1 - \lambda_1)K(t) - \lambda_2 K(t - \tau)) = [(A - \delta)K(t) - r\lambda_1 K(t) - r\lambda_2 K(t - \tau) - C(t)]\,dt$$
$$+ \sqrt{\gamma\left((1 - \lambda_1)K(t) - \lambda_2 K(t - \tau)\right)}\,dW(t).$$

If we denote $(1 - \lambda_1)K(t)$ by $\tilde{K}$, $\frac{\lambda_2}{1-\lambda_1}$ by $\tilde{\lambda}$, $\frac{A}{1-\lambda_1}$ by $\tilde{A}$ and $\frac{\delta + r\lambda_1}{1-\lambda_1}$ by $\tilde{\delta}$, we can rewrite the previous expression as

$$d(\tilde{K}(t) - \tilde{\lambda}\tilde{K}(t - \tau)) = \left[(\tilde{A} - \tilde{\delta})\tilde{K}(t) - r\tilde{\lambda}\tilde{K}(t - \tau) - C(t)\right]\,dt$$
$$+ \sqrt{\gamma(\tilde{K}(t) - \tilde{\lambda}\tilde{K}(t - \tau))}\,dW(t).$$

In this way the problem has the same form of the one solved in Section 3.

6. CONCLUSIONS

Based on an idea proposed by Coen and Sachs (1986), Boucekkine and Pintus (2012) first introduced a model where, in the absence of investment commitment, the debt possibilities of a country depend on its past capital/GDP path. Under this assumption, they identified the roles of historical performance and trends in the globalization process, where countries with the same initial capital but different paths are affected in diverse ways by their integration in the international financial market. Boucekkine et al. (2013) studied the "neoclassical" counterpart of this model by considering the effect of the historical course on the optimal policy and welfare.

In this note, we considered the volatility of the internal net capital and we demonstrated how the importance of the *history effect* and its composition change in terms of the degree of risk. In particular, we showed that, even if the total strength of the history effect is not reduced by the volatile environment, the relative weights of the older parts of the historical path decrease in a more risky situation whereas the importance of the recent past increase; thus, an "oblivious" process occurs.

The dynamics of the model was described by a stochastic NDE. To the best of our knowledge, the solution of the planner's optimization problem is the first optimal control problem driven by a stochastic NDE that has been solved explicitly in the literature.

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Appendix A. Some results on NDE and the description of the problem in the Hilbert space setting

In this appendix we show how to solve the model i.e. how to study the optimal control problem (8)-(9). The problem is approached using the dynamic programming in infinite dimension. This means that, as a first step, the state equation is reformulated as an equivalent evolution equations in a suitable Hilbert space (introduced in Appendix A.1). In the new (infinite dimensional) state equation (that is (33)) the lags in time disappear and the state equation reads as a standard stochastic evolution equation in the infinite dimensional space. To perform this first step we use first the results of Burns et al. (1983) and Kappel and Zhang (1986) for deterministic NDE (Appendix A.2) and then we introduce the noise (Appendix A.3).

Once we have completed this first step and we have rewritten the functional in the infinite dimensional formalism as well, we treat the problem (Appendix A.4) using the dynamic programming. So we need to write and solve the second-order infinite dimensional Hamilton-Jacobi-Bellman (HJB) equation associated to the problem (that is (35)) and use the solution (that will be proved to be the value function of the problem) to characterize the optimal solution in feedback form (see the proofs of Theorem 3.3 and Proposition 3.5). A similar approach is used in the economic literature for some models driven by deterministic delay differential equations (see e.g. Fabbri and Gozzi, 2008 or Boucekkine et al., 2010). Of course, even if here we use a similar method, the structure of the problem and then the solution is deeply different because: (i) we deal with the infinite dimensional version of an NDE equation (ii) the problem is stochastic so the infinite dimensional HJB is of the second order while in the deterministic case only the first order Fréchet differential appears in the HJB.

A.1. **Some definition.** We denote by $L^2(-\tau, 0)$ the space of the real square integrable functions defined on $(-\tau, 0)$. It is a Hilbert space when endowed with the scalar product $\langle f, g \rangle_{L^2} := \int_{-\tau}^{0} f(s)g(s) \, \mathrm{d}s$. We consider the Hilbert space $M^2 := \mathbb{R} \times L^2(-\tau, 0)$ (with the scalar product $\langle (x_0, x_1), (z_0, z_1) \rangle_{M^2} := x_0 z_0 + \langle x_1, z_1 \rangle_{L^2}$). M^2 will be the ambient space where setting our problem. It can be proved (see Burns et al. (1983) Theorem 2.3, page 102) that the operator

$$
(25) \qquad \begin{cases} D(G) := \left\{ (x_0, x_1) \in M^2 \ : \ x_1 \in W^{1,2}(-\tau, 0), \ x_0 = x_1(0) - \lambda x_1(-\tau) \right\} \\[2mm] G(x_0, x_1) := ((A - \delta)x_1(0) - r\lambda x_1(-\tau), \partial x_1) \end{cases}
$$

(being ∂x_1 the derivative of the function x_1 as a real function) is the generator of a C_0-semigroup[3] e^{tG} on M^2.

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[3]Actually, in our specific case, it is a C_0-group (see Burns et al. (1983) Theorem 2.4, page 108). See Bensoussan et al. (2007) for the definitions of C_0-semigroup and C_0-group.

Chosen $(x_0^I, x_1^I) \in M^2$ and P in $L^2_{loc}[0, +\infty)$ we consider the following evolution equation in M^2:

$$(26) \qquad \begin{cases} \dot{x}(t) = Gx(t) - (1,0)P(t) \\ x(0) = (x_0^I, x_1^I). \end{cases}$$

We say that $x \in C([0, +\infty); M^2)$ is a *weak solution* of (26) if, for every $\psi \in D(G^*)$, the function $\langle x(\cdot), G^*\psi \rangle$ belongs to[4] $W^{1,2}_{loc}(0, +\infty; M^2)$ and

$$(27) \qquad \begin{cases} \frac{d}{dt} \langle x(t), \psi \rangle = \langle x(t), G^*\psi \rangle - P(t) \langle (1,0), \psi \rangle \\ \langle x(0), \psi \rangle = \langle (x_0^I, x_1^I), \psi \rangle. \end{cases}$$

It can be proved (see Bonsoussan et al. (2007) Proposition 3.2, page 131[5]) that (26) admits a unique weak solution that can be expressed in the following *mild* form

$$(28) \qquad x(t) := e^{tG}(x_0^I, x_1^I) - \int_0^t e^{(t-s)G}(1,0)P(s)\, ds.$$

A.2. **The NDE in the deterministic case.** Consider now $x_0 \in \mathbb{R}$, $x_1 \in L^2(-\tau, 0)$ and the neutral differential equation[6]

$$(29) \qquad \begin{cases} \dot{K}(t) = \lambda \dot{K}(t - \tau) + (A - \delta)K(t) - r\lambda K(t - \tau) - P(t) \\ K(0) - \lambda K(-\tau) = x_0^I \\ K(s) = x_1^I(s) \qquad s \in [-\tau, 0]. \end{cases}$$

If P is in $L^2_{loc}[0, +\infty)$ (see Burns et al. (1983), page 109) such an NDE has a unique (*generalized* in the sense of Kappel and Zhang (1986)) solution $\phi^{x_0, x_1, P}(\cdot)$.

The nice fact (see Burns et al. (1983), Theorem 3.1 page 110) is that the unique generalized solution $K(\cdot)$ and the unique mild/weak solution $x(\cdot) = (x_0(\cdot), x_1(\cdot))$ of (26) are strictly linked. Indeed if we denote, for any $t \geq 0$,

$$(30) \qquad \begin{cases} K_t \colon [-\tau, 0] \to \mathbb{R} \\ K_t(s) := K(t + s) \end{cases}$$

we have that, for $t \geq 0$,

$$(31) \qquad x(t) = (x_0(t), x_1(t)(\cdot)) = \Big(K(t) - \lambda K(t - \tau), K_t(\cdot) \Big)$$

and then the study of the NDE can be partly reduced to the study of the evolution equation in M^2.

A.3. **The stochastic case.** When P is stochastic of the form $P(t) = C(t) + N(t)\dot{W}(t)$, as in (8), the evolution equation related to the (stochastic) NDE is then

$$(32) \qquad \begin{cases} dx(t) = (Gx(t) - (1,0)C(t))\, dt + (1,0)N(t)\, dW(t) \\ x(0) = (x_0^I, x_1^I) \end{cases}$$

in particular, since, by (31), $x_0(t) = \langle x(t), (1,0) \rangle$ is equal to $S(t) = K(t) - \lambda K(t - \tau)$, when $N(t)$ has the form $N(t) := \sqrt{K(t) - \lambda K(t - \tau)} \frac{dW(t)}{dt} = \sqrt{x_0(t)} \frac{dW(t)}{dt}$, the previous equation becomes

$$(33) \qquad \begin{cases} dx(t) = (Gx(t) - (1,0)C(t))\, dt + (1,0)\sqrt{\langle x(t), (1,0) \rangle}\, dW(t) \\ x(0) = (x_0^I, x_1^I). \end{cases}$$

Using Theorem 3.3 page 97 in Gawarecki and Mandrekar (2010) one can see that such stochastic differential equation in M^2 has a unique solution if the control $C(\cdot)$ belongs to the set of admissible controls

$$\mathcal{U}_{x(0)} := \left\{ C(\cdot) \colon [0, +\infty) \times \Omega \to \mathbb{R} \; : \; \begin{array}{l} C(\cdot) \text{ is } \mathcal{F}^t - \text{progressively measurable} \\ \text{and } x_0(\cdot) = \langle x(\cdot), (1,0) \rangle \text{ remains a.s. strict positive} \end{array} \right\}.$$

Observe that the strict positivity of $\langle x(\cdot), (1,0) \rangle$, that corresponds to the strict positivity of $S(t) = K(t) - \lambda K(t - \tau)$ in the NDE formulation given in the main text, ensures the local Lipschitz continuity of the right

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[4]The set $x \in C([0, +\infty); M^2)$ is the (Banach) space of the M^2-valued continuous functions defined on $[0, +\infty)$ while $W^{1,2}_{loc}(0, +\infty; M^2)$ is the set of the M^2-valued functions defined on $[0, +\infty)$ whose restrictions to $[0, L]$ belong, for any $L > 0$, to the Sobolev space $W^{1,2}(0, L; M^2)$ (i.e. the space of the square integrable, M^2-valued functions defined on $[0, L]$ having square integrable derivative).

[5]Bensoussan et al. (2007) prove the result for an abstract generator of a C_0-semigroup on an abstract Hilbert space that can be specified, as a particular case, as the operator G we are considering and the Hilbert space M^2.

[6]In (8) the value $K(0) - \lambda K(-\tau)$ does not appear explicitly but, since we only consider continuous initial data K_I, it can be derived. Here, in line with the approach of Burns et al. (1983), we emphasize its value.

side of the first line of (33) with respect to the state variable x asked by Theorem 3.3 page 97 of Gawarecki and Mandrekar (2010).

Finally, the optimal control problem described in Section 2 is equivalent to the optimal control problem driven by (33) with functional

$$(34) \qquad J(C(\cdot)) := \mathbb{E}\left[\int_0^\infty e^{-\rho t}\left(-e^{-\eta C(t)}\right)dt\right]$$

and set of admissible controls $\mathcal{U}_{x(0)}$.

A.4. Results for the model.

Proof of Lemma 3.1. The left hand side of

$$\xi = \frac{-(A-\delta)e^{-\xi\tau} + r\lambda}{\left(\frac{1}{2}\gamma + \frac{1}{\eta}\right)(e^{-\xi\tau} - \lambda)\,\eta}$$

is strictly increasing, its limit for $\xi \to -\infty$ is $-\infty$ and its value in 0 is 0 while the right-hand side, when ξ is negative, is decreasing (observe that (11) implies in particular $(A - \delta) > 0$), its limit for $\xi \to -\infty$ is finite and its value in 0 is negative. Then there exists a unique negative root of the equation $\bar{\xi}$.

It is easy to see the right hand side is a strictly increasing function of γ, η, δ and r and a decreasing function of A. This fact gives the second claim. $\qquad\square$

We are ready now to prove Theorem 3.3 and Proposition 3.5. Indeed the two proofs come together using the dynamic programming: we first identify the HJB equation of the system, we look for an explicit solution and we prove that the feedback induced by such a solution is optimal, proving at the same time that the found solution is in fact the value function of the problem. The details of the proof are below.

Proof of Theorem 3.3 and Proposition 3.5. We want to apply the dynamic programming to the problem rewritten in the Hilbert space formulation. So we need first to write the HJB equation of the system.

Given $p = (p_0, p_1) \in M^2$ with $p_0 > 0$ and $C \in \mathbb{R}$ we denote by

$$H_{CV}(p, C) := \left(-\langle(1,0), p\rangle\, C - e^{-\eta C}\right)$$

the *current value Hamiltonian* of the system and by

$$H(p) := \sup_{C \geq 0} H_{CV}(p, C) = \frac{1}{\eta}\langle(1,0), p\rangle\left[-1 + \ln\left(\frac{1}{\eta}\langle(1,0), p\rangle\right)\right]$$

its *Hamiltonian*. The (infinite dimensional) HJB related to the problem (33)-(34) is defined as follows:

$$(35) \qquad \rho v(x) = \langle x, G^* Dv(x)\rangle + \frac{1}{2}\gamma\langle x, (1,0)\rangle\, D^2 v(x)\left[(1,0),(1,0)\right] + H(Dv(x)).$$

Observe that, in fact, for a fixed p, $H_{CV}(p, C)$, has a unique point of maximum, as a function of C, and it is given by

$$C = -\frac{1}{\eta}\ln\left(\frac{1}{\eta}\langle(1,0), p\rangle\right).$$

We look for a solution of the HJB of the form

$$(36) \qquad v(x) = -\beta e^{-\langle a, x\rangle}$$

where β is some positive constant and $a = (a_0, a_1)$ an element of M_2 with a_1 of the form

$$(37) \qquad a_1(s) = h e^{\xi s}$$

for some real constants h and ξ. Whenever v is a function of the form (36)-(37) we can compute explicitly its Fréchet derivatives obtaining $Dv(x) = \beta e^{-\langle a, x\rangle} a$ and $D^2 v(x) = -\beta e^{-\langle a, x\rangle} a \otimes a$.

As shown in Boucekkine et al. (2011) Proposition 5.3, the expression of the adjoint G^* of G is given by

$$(38) \qquad \begin{cases} D(G^*) = \left\{(y_0, y_1) \in M^2 \ : \ y_1 \in W^{1,2}(-\tau, 0) \text{ and } (A - \delta - r)\lambda y_0 + \lambda y_1(0) - y_1(-\tau) = 0\right\} \\ G^*(y_0, y_1) = ((A - \delta)y_0 + y_1(0), -\partial y_1) \end{cases}$$

so $Dv(x) \in D(G^*)$ if and only if $(A - \delta - r)\lambda a_0 + \lambda a_1(0) - a_1(-\tau) = 0$. So $Dv(x) \in D(G^*)$ for all $x \in M_2$ if and only if

$$(39) \qquad a_0 = h\frac{e^{-\xi\tau} - \lambda}{(A - \delta - r)\lambda}.$$

In order to be able to give a meaning to all the terms of the HJB equation we assume that such a condition is verified, in this case we can compute $G^* Dv(x)$ and we obtain, thanks to (38),

$$G^* Dv(x) = \beta e^{-\langle a,x\rangle} G^*(a) = \beta e^{-\langle a,x\rangle} \left(\frac{(A-\delta)e^{-\xi\tau} - r\lambda}{(A-\delta-r)\lambda} h, -h\xi e^{\xi s} \right).$$

Summarizing we are trying to find a solution of (35) of the form (36)-(37) such that a_0, ξ and h satisfy (39). Let us write explicitly the terms appearing in (35) in this case:

$$\rho v(x) = -\rho\beta e^{-\langle a,x\rangle},$$

$$(40) \quad \langle x, G^* Dv(x)\rangle = \beta e^{-\langle a,x\rangle} \left\langle (x_0, x_1), \left(\frac{(A-\delta)e^{-\xi\tau} - r\lambda}{(A-\delta-r)\lambda} h, -h\xi e^{\xi s} \right) \right\rangle$$

$$= \beta e^{-\langle a,x\rangle} h \frac{(A-\delta)e^{-\xi\tau} - r\lambda}{(A-\delta-r)\lambda} x_0 - \beta e^{-\langle a,x\rangle} h\xi \left\langle x_1, e^{\xi s} \right\rangle_{L^2},$$

$$(41) \quad \frac{1}{2}\gamma \langle x, (1,0)\rangle D^2 v(x)\left[(1,0),(1,0)\right] = \frac{1}{2}\gamma x_0 \left(-\beta e^{-\langle a,x\rangle} \langle a, (1,0)\rangle^2 \right)$$

$$= -\beta e^{-\langle a,x\rangle} \frac{1}{2}\gamma \left(h\frac{e^{-\xi\tau} - \lambda}{(A-\delta-r)\lambda} \right)^2 x_0$$

and

$$(42) \quad H(Dv(x)) = \frac{1}{\eta} \left\langle (1,0), \beta e^{-\langle a,x\rangle} a \right\rangle \left[-1 + \ln\left(\frac{1}{\eta} \left\langle (1,0), \beta e^{-\langle a,x\rangle} a \right\rangle \right) \right]$$

$$= \frac{1}{\eta} \beta e^{-\langle a,x\rangle} a_0 \left[-1 + \ln\left(\frac{\beta a_0}{\eta} \right) - \langle x, a\rangle \right] = \beta e^{-\langle a,x\rangle} \frac{1}{\eta} \left(h\frac{e^{-\xi\tau} - \lambda}{(A-\delta-r)\lambda} \right) \left(-1 + \ln\left(\frac{\beta h}{\eta} \frac{e^{-\xi\tau} - \lambda}{(A-\delta-r)\lambda} \right) \right)$$

$$= -\beta e^{-\langle a,x\rangle} \frac{1}{\eta} \left(h\frac{e^{-\xi\tau} - \lambda}{(A-\delta-r)\lambda} \right)^2 x_0 - \beta e^{-\langle a,x\rangle} \frac{1}{\eta} \left(h\frac{e^{-\xi\tau} - \lambda}{(A-\delta-r)\lambda} \right) h \left\langle x_1, e^{\xi s} \right\rangle_{L^2}.$$

Now we substitute such expressions in (35), we obtain (simplifying the multiplicative term $\beta e^{-\langle a,x\rangle}$ and arranging a little the terms):

$$(43) \qquad\qquad 0 = x_0 A_1 + h \left\langle x_1, e^{\xi s} \right\rangle_{L^2} A_2 + A_3$$

where

$$A_1 := \left(\frac{(A-\delta)e^{-\xi\tau} - r\lambda}{(A-\delta-r)\lambda} h - \left(\frac{1}{2}\gamma + \frac{1}{\eta} \right) \left(h\frac{e^{-\xi\tau} - \lambda}{(A-\delta-r)\lambda} \right)^2 \right)$$

$$A_2 := \left(-\xi - \frac{1}{\eta} \left(h\frac{e^{-\xi\tau} - \lambda}{(A-\delta-r)\lambda} \right) \right)$$

$$A_3 := \left(\rho + \frac{1}{\eta} \left(h\frac{e^{-\xi\tau} - \lambda}{(A-\delta-r)\lambda} \right) \left(-1 + \ln\left(\frac{\beta h}{\eta} \frac{e^{-\xi\tau} - \lambda}{(A-\delta-r)\lambda} \right) \right) \right).$$

Since (43) has to be verified for all choice of x, then there exists a solution of (35) of the form (36)-(37) such that a_0, ξ and h satisfy (39) if and only if A_1, A_2 and A_3 vanish. $A_2 = 0$ implies

$$(44) \qquad\qquad \xi = -\frac{1}{\eta} \left(h\frac{e^{-\xi\tau} - \lambda}{(A-\delta-r)\lambda} \right) \qquad \text{i.e. } \xi = -\frac{a_0}{\eta}.$$

Using this fact in A_1 we have $A_1 = \frac{h}{(A-\delta-r)\lambda} \left((A-\delta)e^{-\xi\tau} - r\lambda + \left(\frac{1}{2}\gamma + \frac{1}{\eta} \right) \left(e^{-\xi\tau} - \lambda \right) \xi\eta \right)$ that is zero if ξ is a solution of $\xi = \frac{-(A-\delta)e^{-\xi\tau} + r\lambda}{\left(\frac{1}{2}\gamma + \frac{1}{\eta} \right)\left(e^{-\xi\tau} - \lambda \right)\eta}$. As shown in Lemma 3.1, thanks to (11) and (3) such an equation has a unique negative solution that we denoted by $\bar\xi$. Then we take $\xi = \bar\xi$. Thanks to (44) we can find the value of h: $h = \bar h := -\eta\bar\xi \frac{(A-\delta-r)\lambda}{e^{-\xi\tau} - \lambda}$. The last parameter is β and we can determine it using the condition $A_3 = 0$. It gives $0 = \rho + \bar\xi - \bar\xi \ln\left(-\beta\bar\xi \right)$ and then $\beta = \bar\beta := \frac{-1}{\bar\xi} \exp\left(\frac{\rho}{\bar\xi} + 1 \right)$. Eventually, we have proved that, called $\bar a = (\bar a_0, \bar a_1) := \left(\bar h\frac{e^{-\bar\xi\tau} - \lambda}{(A-\delta-r)\lambda}, \bar h e^{\bar\xi s} \right) = \left(-\eta\bar\xi, \bar h e^{\bar\xi s} \right)$, the function

$$(45) \qquad\qquad\qquad v(x) = -\bar\beta e^{-\langle \bar a, x\rangle}$$

is a solution of the HJB.

In the next steps of the proofs we will prove that such a solution can be used to find the optimal control in feedback form and that it is indeed the value function of the problem.

The feedback associated to (45) is defined as follows:

$$(46) \quad \begin{cases} \phi\colon M^2 \to \mathbb{R} \\ \phi(K) := \arg\max_C H(Dv(x), C) = -\frac{1}{\eta}\ln\left(\frac{1}{\eta}\langle(1,0), Dv(x)\rangle\right) = -\frac{1}{\eta}\ln\left(\frac{1}{\eta}\left\langle(1,0), \bar{\beta}e^{-\langle\bar{a},x\rangle}\bar{a}\right\rangle\right) \end{cases}$$

The related trajectory in M^2 is the solution of the following stochastic evolution equation in M^2 (found using the feedback (46) in (33))

$$(47) \quad \begin{cases} \mathrm{d}x(t) = \left(Gx(t) + (1,0)\frac{1}{\eta}\ln\left(\frac{1}{\eta}\left\langle(1,0), \bar{\beta}e^{-\langle\bar{a},x\rangle}\bar{a}\right\rangle\right)\right)\mathrm{d}t + (1,0)\sqrt{\langle x(t),(1,0)\rangle}\,\mathrm{d}W(t) \\ x(0) = (x_0^I, x_1^I). \end{cases}$$

Observe that, by hypothesis, the control defined by the feedback is admissible (i.e. it belongs to $\mathcal{U}_{x(0)}$) that is $S(t) = \langle x(t), (1,0)\rangle$ remains strictly positive along the trajectory driven by the feedback and then the term $\sqrt{\langle x(t),(1,0)\rangle}$ in the previous equation is well defined (see Remark A.1 on that).

Let us prove that the feedback defined in (46) is optimal, namely that the solution $x^*(\cdot)$ of (47) is indeed the trajectory of the system along the optimal path and that the corresponding control $C^*(t) := \phi(x^*(t))$ is the optimal control of the problem.

Define $\omega(t,x)$ the following function

$$(48) \quad \begin{cases} \omega\colon [0,+\infty) \times M^2 \to \mathbb{R} \\ \omega(t,x) := e^{-\rho t}v(x). \end{cases}$$

Consider an admissible control $\tilde{C}(\cdot) \in \mathcal{U}_{x(0)}$ and the related trajectory $\tilde{x}(\cdot)$. Chosen $T > 0$. We have, using the Ito formula (see Gawarecki and Mandrekar (2010), Theorem 2.9 page 62[7]),

$$(49) \quad \mathbb{E}\left[\int_0^T e^{-\rho t}\left(-e^{-\eta\tilde{C}(t)}\right)\mathrm{d}t\right] - v(x(0)) + \mathbb{E}\left[\omega(T,\tilde{x}(T))\right]$$

$$= \mathbb{E}\left[\int_0^T e^{-\rho t}\left(-e^{-\eta\tilde{C}(t)}\right)\mathrm{d}t\right] - \mathbb{E}\left[\omega(0,\tilde{x}(0)) - \omega(T,\tilde{x}(T))\right]$$

$$= \int_0^T e^{-\rho t}\left(-e^{-\eta\tilde{C}(t)}\right)\mathrm{d}t + \mathbb{E}\left[\int_0^T \frac{\partial\omega}{\partial t}(t,\tilde{x}(t)) + \langle G^*D\omega(t,\tilde{x}(t)),\tilde{x}(t)\rangle - \left\langle D\omega(t,\tilde{x}(t)),(1,0)\tilde{C}(t)\right\rangle \right.$$

$$\left. + \frac{1}{2}D^2\omega(t,\tilde{x}(t))[(1,0),(1,0)]\langle\tilde{x}(t),(1,0)\rangle\right]$$

$$= \int_0^T e^{-\rho t}H_{CV}(Dv(\tilde{x}(t),\tilde{C}(t)))\,\mathrm{d}t$$

$$+ \mathbb{E}\left[\int_0^T \frac{\partial\omega}{\partial t}(t,\tilde{x}(t)) + \langle G^*D\omega(t,\tilde{x}(t)),\tilde{x}(t)\rangle + \frac{1}{2}D^2\omega(t,\tilde{x}(t))[(1,0),(1,0)]\langle\tilde{x}(t),(1,0)\rangle\right].$$

Since $v(\cdot)$ is a solution of (35) we have

$$(50) \quad \frac{\partial\omega}{\partial t}(t,\tilde{x}(t)) = -\rho e^{-\rho t}v(\tilde{x}(t)) = -e^{-\rho t}\left(\rho v(\tilde{x}(t))\right)$$

$$= -e^{-\rho t}\left[\langle\tilde{x}(t), G^*Dv(\tilde{x}(t))\rangle + \frac{1}{2}\gamma\langle\tilde{x}(t),(1,0)\rangle D^2v(\tilde{x}(t))[(1,0),(1,0)] + H(Dv(\tilde{x}(t)))\right].$$

Using last expression in (49) we get

(51)

$$\mathbb{E}\left[\int_0^T e^{-\rho t}\left(-e^{-\eta\tilde{C}(t)}\right)\mathrm{d}t - v(x(0)) + \omega(T,\tilde{x}(T))\right] = \mathbb{E}\left[\int_0^T e^{-\rho t}\left(H_{CV}(Dv(\tilde{x}(t),\tilde{C}(t))) - H(Dv(\tilde{x}(t)))\right)\right]$$

$$= \mathbb{E}\left[\int_0^T e^{-\rho t}\left(H_{CV}(Dv(\tilde{x}(t),\tilde{C}(t))) - \sup_{C\geq 0} H_{CV}(Dv(\tilde{x}(t)),C)\right)\right] \leq 0.$$

We can observe that

$$\mathbb{E}\left[\int_0^T e^{-\rho t}\left(-e^{-\eta\tilde{C}(t)}\right)\mathrm{d}t\right]$$

---

[7]Indeed here we need a slightly extended version of the result of Gawarecki and Mandrekar (2010) that can be easily obtained (thanks to an approximation argument) using that Dv is $D(G^*)$-valued and uniformly continuous on bounded subsets as $D(G^*)$-valued function.

is a decreasing function of T (the integrand is always negative) so it admits a limit (possibly equal to $-\infty$) for $T \to +\infty$. Since we are looking for an optimal solution we can restrict our attention to the set of controls $\tilde{C}(\cdot)$ s.t. such a limit is finite (the proof will show that the control induced by the feedback satisfies this condition and then such a set is non-void), moreover one can also see that, along the admissible trajectories, $\omega(T, \tilde{x}(T)) \xrightarrow{T \to +\infty} 0$. So we can pass to the limit in (51) and we find

$$
(52) \qquad J(C(\cdot)) - v(x(0)) = \mathbb{E}\left[\int_0^{+\infty} e^{-\rho t}\left(-e^{-\eta \tilde{C}(t)}\right) \mathrm{d}t\right] - v(x(0)) \le 0
$$

In other words, for all admissible controls $\tilde{C}(\cdot)$, one has

$$
(53) \qquad J(C(\cdot)) \le v(x(0)).
$$

Since $C^*(t)$ satisfied (46) for all $t \ge 0$, then along the trajectory x^* driven by C^* the integrand in the right hand side of (52) is always zero and then $J(C(\cdot)) - v(x(0)) = 0$ i.e.

$$
J(C^*(\cdot)) = v(x(0)).
$$

This fact together with (53), since $\tilde{C}(\cdot)$ is a generic admissible control, proves that

$$
(54) \qquad v(x(0)) = J(C^*(\cdot)) = \sup_{\tilde{C}(\cdot) \in \mathcal{U}_{x(0)}} J(\tilde{C}(\cdot))
$$

and then (using the second equality of such an expression) the optimality of $C^*(\cdot)$.

This also means that v is the value function of the problem, indeed

$$
v(x(0)) = \sup_{\tilde{C}(\cdot) \in \mathcal{U}_{x(0)}} J(\tilde{C}(\cdot))
$$

and the right hand side is the definition of value function.

$\square$

Remark A.1 (On the positivity). *Let us observe what happens when τ converges 0. In this case, from (12) one can see that $\bar{\xi} \xrightarrow{\tau \to 0} \frac{-(A-\delta)+r\lambda}{\left(\frac{1}{2}\gamma+\frac{1}{\eta}\right)(1-\lambda)\eta} =: \mu$, $a_0 \xrightarrow{\tau \to 0} \frac{1}{1-\lambda}\frac{A-\delta-r\lambda}{\frac{1}{2}\gamma+\frac{1}{\eta}}$ and $\bar{\beta} \xrightarrow{\tau \to 0} -\frac{1}{\mu}\exp\left(\frac{\rho}{\mu}+1\right)$.*

Finally (since the trajectories of K are a.s. continuous so $K(-\tau)$ converges to $K(0)$ and the integral term goes to zero when τ vanishes), the feedback defined in (14) converges to $C(t) = \frac{\alpha}{\eta}K(t) - \frac{1}{\eta} + \frac{\rho(1-\lambda)}{\alpha}$. This is indeed the expression of the feedback control found by Boucekkine et al. (2014). If we use this expression in the state equation of the case $\tau = 0$, we can identify a sufficient condition on the parameters that ensures that the trajectory driven by the feedback remains positive. Observe that in the case $\tau = 0$ there is not delay and $S(t) = (1-\lambda)K(t)$ so S remains positive and then the related control is admissible. It can be seen that, for $\tau = 0$, a sufficient condition for the positivity (if the initial capital is positive) is $\frac{1}{\eta(1-\lambda)} - \frac{\rho\left(\frac{1}{2}\gamma+\frac{1}{\eta}\right)}{A-\delta-r\lambda} > \frac{\gamma}{2(1-\lambda)}$ (it is the same kind of condition one has e.g., for the Cox-Ingersoll-Ross interest rate model, see e.g. Theorem 2.2 and Remark 2.2 (page 79) by Mishura and Posashkova, 2008).

In the case $\tau > 0$ the situation is more complex. Since the system is driven by a stochastic neutral differential equation, and the literature about positivity of the solution of a stochastic NDE equation is almost void, we cannot quote specific results. So we could not identify, explicitly, a restriction on the set of the parameters that ensures that the trajectory of S along the trajectory driven by the feedback remains positive in the general case.

Proof of Propositions 4.1 and 4.2. For $a \in (-\tau, 0)$, we have

$$
i_a^0 = \frac{\mathcal{I}_a^0}{\mathcal{I}_{tot}} = \frac{e^{a\bar{\xi}} - 1}{e^{\xi\tau} - 1}.
$$

Looking at the derivative of such an expression with respect to $\bar{\xi}$ one can easily realize that it is positive. Moreover we know, from Lemma 3.1, that $\bar{\xi}$ is increasing in the volatility parameter γ and the absolute risk aversion η. This gives the first claim.

For the second claim one can easily verify that the derivative w.r.t. $\bar{\xi}$ of $\mathcal{I}_{tot}$ (as long as the term $(A-\delta-r)$ remains positive) is positive. Since the parameters γ and η influence the expression of $\mathcal{I}$ only through the value of $\bar{\xi}$, we can conclude, thanks to Lemma 3.1, that $\mathcal{I}$ is increasing in γ and η. $\square$

Proof of Lemma 5.1. One can easily prove the statement observing that:

(i) since f is positive, the function

$$j_l(\xi) := \frac{1}{r}\left(1 - \frac{1}{\int_{-\tau}^0 e^{-\xi s} f(s)\, \mathrm{d}s}\right)$$

is always decreasing on $(-\infty, 0]$, it has limit equal to $\frac{1}{r} > 0$ when $\xi \to -\infty$ and takes the value $\frac{1}{r}\left(1 - \frac{1}{\int_{-\tau}^0 f(s)\, \mathrm{d}s}\right)$ in 0

(ii) the function

$$j_r(\xi) := \frac{1}{\xi^2 \eta^2 \left(\frac{1}{2}\gamma + \frac{1}{\eta}\right) - (A - \delta)}$$

has limit equal to 0 when $\xi \to -\infty$, it is always increasing on $\left(\infty, -\sqrt{\frac{A-\delta}{\eta^2\left(\frac{1}{2}\gamma+\frac{1}{\eta}\right)}}\right)$, it has left limit equal to $+\infty$ when $\xi \to -\sqrt{\frac{A-\delta}{\eta^2\left(\frac{1}{2}\gamma+\frac{1}{\eta}\right)}}$ and, thanks to (21), it is always lower than $j_l(0)$ on $\left(-\sqrt{\frac{A-\delta}{\eta^2\left(\frac{1}{2}\gamma+\frac{1}{\eta}\right)}}, 0\right)$.

So there is a unique point ξ where $j_l(\xi) = j_r(\xi)$ and it is in the interval $\left(-\infty, -\sqrt{\frac{A-\delta}{\eta^2\left(\frac{1}{2}\gamma+\frac{1}{\eta}\right)}}\right)$. Since in this interval $j_r(\xi)$ is positive then $j_l(\xi)$ is positive a well and we have the last claim. $\square$

Proof of Theorem 5.2. The structure of the proof is the same of Theorem 3.5 and Proposition 3.3 so we sketch it here underlining the differences.

Again the controlled system is rewritten in the infinite dimensional framework. As we did in (31) introducing the variable x, following again Burns et al. (1983), for $t \geq 0$, we consider here the variable $\tilde{x}$ defined as follows:

$$(55) \qquad \tilde{x}(t) = (\tilde{x}_0(t), \tilde{x}_1(t)) = \left(K(t) - \int_{-\tau}^0 K(t+\theta)f(\theta)\, \mathrm{d}\theta, K_t\right)$$

where K_t is again the "history" of the solution of the neutral differential equation K.

Thanks to Theorem 3.1 of Burns et al. (1983) we know that $\tilde{x}$ is the solution of the following equation:

$$(56) \qquad \begin{cases} \mathrm{d}\tilde{x}(t) = \left(\tilde{G}x(t) - (1,0)C(t)\right)\mathrm{d}t + (1,0)\sqrt{\langle \tilde{x}(t), (1,0)\rangle}\, \mathrm{d}W(t) \\ \tilde{x}(0) = (x_0^I, x_1^I). \end{cases}$$

The only difference with respect to (33) is the expression of the generator of the strongly continuous semigroup (called here $\tilde{G}$), that is now given by

$$(57) \qquad \begin{cases} D(\tilde{G}) := \left\{(x_0, x_1) \in M^2 \; : \; x_1 \in W^{1,2}(-\tau, 0),\; x_0 = x_1(0) - \int_{-\tau}^0 x_1(\theta)f(\theta)\, \mathrm{d}\theta\right\} \\ \tilde{G}(x_0, x_1) := \left((A-\delta)x_1(0) - r\int_{-\tau}^0 x_1(\theta)f(\theta)\, \mathrm{d}\theta, \partial x_1\right). \end{cases}$$

After some computations (in line for example with the proof of Proposition 5.3 by of Boucekkine et al., 2011) one can verify that the adjoint of this operator is given by

$$(58) \qquad \begin{cases} D(\tilde{G}^*) := \left\{(y_0, y_1) \in M^2 \; : \; y_1 \in W^{1,2}(-\tau, 0),\; y_1(-\tau) = 0\right\} \\ \tilde{G}^*(y_0, y_1) := ((A-\delta)y_0 + y_1(0), -\partial y_1 + (y_1(0) - ry_0)f). \end{cases}$$

Since the target functional of the considered distributed problem is the same of the problem considered in Section 3 the HJB equation has the same form, apart for the presence of the operator $\tilde{G}^*$ instead of G^*, so it has now th following expression:

$$(59) \qquad \rho v(x) = \left\langle x, \tilde{G}^* Dv(x)\right\rangle + \frac{1}{2}\gamma \langle x, (1,0)\rangle D^2 v(x)[(1,0),(1,0)] + H(Dv(x)).$$

where,

$$H(p) := \sup_{C \geq 0}\left(-\langle(1,0), p\rangle C - e^{-\eta C}\right) = \frac{1}{\eta}\langle(1,0), p\rangle\left[-1 + \ln\left(\frac{1}{\eta}\langle(1,0), p\rangle\right)\right].$$

Following the same approach above, but changing a bit of the form of the solution, we look for a solution of the HJB of the following form

$$v(x) = -\beta e^{-\langle a, x \rangle} \tag{60}$$

where β is some positive constant and $a = (a_0, a_1)$ an element of M_2 with a_1 of the form

$$a_1(s) = h e^{\xi s} \int_{-\tau}^{s} e^{-\xi \theta} f(\theta) \, \mathrm{d}\theta, \qquad \forall s \in [\tau, 0], \tag{61}$$

for some real constants h and ξ (one can easily verify that all the couples of this form belongs to $D(\tilde{G}^*)$). As before we can compute explicitly the Fréchet derivatives of a function v of this form obtaining $Dv(x) = \beta e^{-\langle a, x \rangle} a$ and $D^2 v(x) = -\beta e^{-\langle a, x \rangle} a \otimes a$. Using (60) in (59) we can see that (59) is verified if and only if

$$-\rho \beta e^{-\langle a, x \rangle} = \beta e^{-\langle a, x \rangle} \langle (x_0, x_1), ((A - \delta) a_0 + a_1(0), -\partial a_1 + (a_1(0) - r a_0) f) \rangle \tag{62}$$

$$+ \frac{1}{2} \gamma x_0 \left(-\beta e^{-\langle a, x \rangle} a_0^2 \right) + \frac{1}{\eta} \beta e^{-\langle a, x \rangle} a_0 \left(-1 + \ln \left(\frac{1}{\eta} \beta e^{-\langle a, x \rangle} a_0 \right) \right)$$

that is (after some computations)

$$0 = x_0 H_1 + \langle x_1, H_2 \rangle_{L^2} + H_3 := x_0 \left[(A - \delta) a_0 + a_1(0) - \frac{1}{2} \gamma a_0^2 - \frac{1}{\eta} a_0^2 \right] \tag{63}$$

$$+ \left\langle x_1, \left[-\partial a_1 + (a_1(0) - r a_0) f - \frac{1}{\eta} a_0 a_1 \right] \right\rangle_{L^2} + \left[\rho - \frac{1}{\eta} a_0 \left(1 + \ln(a_0) + \ln(\beta) - \ln(\eta) \right) \right].$$

Since (63) needs to be verified for any choice of x we need to have $H_1 = 0$, $H_2 = 0$ (as an L^2 function) and $H_3 = 0$. Using (61) we can see that the condition $H_2 = 0$ is verified if and only if, for almost every $s \in [-\delta, 0)$,

$$0 = \left(-h \xi e^{\xi s} \int_{-\tau}^{s} e^{-\xi \theta} f(\theta) \, \mathrm{d}\theta + h f(s) \right) + \left(h \int_{-\tau}^{0} e^{-\xi \theta} f(\theta) \, \mathrm{d}\theta - r a_0 \right) f - \frac{1}{\eta} a_0 h e^{\xi s} \int_{-\tau}^{s} e^{-\xi \theta} f(\theta) \, \mathrm{d}\theta \tag{64}$$

that is verified if and only if the two following conditions are satisfied:

$$\begin{cases} \xi = -\dfrac{a_0}{\eta} \\ h = \dfrac{a_0 r}{\int_{-\tau}^{0} e^{-\xi \theta} f(\theta) \, \mathrm{d}\theta - 1}. \end{cases} \tag{65}$$

Similarly using (61) and the condition $H_1 = 0$ we get

$$\left((A - \delta) a_0 + h \int_{-\tau}^{0} e^{-\xi \theta} f(\theta) \, \mathrm{d}\theta - \frac{1}{2} \gamma a_0^2 - \frac{1}{\eta} a_0^2 \right) = 0$$

while the condition $H_3 = 0$ becomes

$$\ln(\beta) = (1 + \ln(a_0) - \ln(\eta)) - \frac{\eta}{a_0} \rho.$$

Using these expression together with (65) we get the following condition for (defining) ξ

$$\frac{1}{r} \left(1 - \frac{1}{\int_{-\tau}^{0} e^{-\xi \theta} f(\theta) \, \mathrm{d}\theta} \right) = \frac{1}{\left(\xi^2 \eta^2 \left(\frac{1}{2} \gamma + \frac{1}{\eta} \right) - (A - \delta) \right)}$$

and the following relationships:

$$\begin{cases} a_0 = -\eta \xi \\ h = \dfrac{-\eta \xi r}{\int_{-\tau}^{0} e^{-\xi \theta} f(\theta) \, \mathrm{d}\theta - 1} \\ \beta = -\dfrac{1}{\xi} \exp \left(1 + \dfrac{\rho}{\xi} \right). \end{cases}$$

We have then found an explicit solution of the HJB equation. The arguments to show that it is the value function of the problem and to prove the optimality of the feedback are exactly the same of the proof of Theorem 3.5 and Proposition 3.3. $\qquad \square$

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