école d'économie d'aix-marseille
aix-marseille school of economics

Working Papers / Documents de travail

Evaluating distributions of opportunities from behind a veil of ignorance: A robust approach

Francesco Andreoli
Mathieu Faure
Nicolas Gravel
Tista Kundu

WP 2021 - Nr 37

Evaluating distributions of opportunities from behind a veil of ignorance: A robust approach*

Francesco Andreoli[†], Mathieu Faure[‡], Nicolas Gravel[§] and Tista Kundu[¶]

July 20th 2021

Abstract

This paper provides a robust criterion for evaluating the allocation of opportunities among various groups. We envisage the problem of comparing these allocations from the view point of an ethical observer placed behind a veil of ignorance with respect to the group in which he/she could end up. We give justification for such an ethical observer to evaluate these allocations of opportunities on the basis of an expected valuation of the expected utility of being in a group assuming an equal probability of falling in every group. We identify a criterion for comparing societies that is agreed upon by all such ethical observers who exhibit aversion to inequality of opportunities. The criterion happens to be a conic extension of zonotope inclusion criterion. We provide various interpretations of this criterion as well as some illustrations of its possible use, notably in the Indian context where we evaluate the inequalities of educational opportunities among castes and genders offered by Indian states.

Keywords: equalizing opportunities, groups, zonotopes, gender, education.

JEL classification numbers: D63, D81, I24

1 Introduction

Improving and equalizing opportunities is considered to be an important social objective by many. In the US, opinion surveys conducted by the Pew research center[1] have consistently found in the last 25 years an agreement by 90% of the

*We gratefully acknowledge the funding of the research that led to this paper by the French *Agence Nationale de la Recherche* through three contracts: (*i*) Measurement of Ordinal and Multidimensional Inequalities (ANR-16-CE41-0005), (*ii*) Challenging Inequalities: A Indo-European perspective (ANR-18-EQUI-0003) and (*iii*) The European University of Research AMSE (ANR-17-EURE-020).

†Department of Economics, University of Verona, Via Cantarane 24, 37129 Verona, Italy and Luxembourg Institute of Socio-Economic Research, LISER. francesco.andreoli@univr.it.

‡Aix-Marseille Univ.,CNRS, AMSE, 2, Boul. Maurice Bourdet, 13001 Marseille France Cedex. mathieu.faure@univ-amu.fr.

§Aix-Marseille Univ., CNRS, AMSE, 2, Boul. Maurice Bourdet, 13001 Marseille, France Cedex. nicolas.gravel@univ-amu.fr.

¶Centre de Sciences Humaines, 2 Dr. APJ Abdul Kalam Road, 110011 Delhi, India. tista.kundu@csh-delhi.com.

[1] see e.g. https://www.pewresearch.org/2011/03/11/the-elusive-90-solution.

respondents on the fact that “our society should do what is necessary to make sure that everyone has an equal opportunity to succeed". A common interpretation of this "equal opportunity to succeed " ideal is through the requirement that the individuals’ probabilities (chances) of reaching outcomes of interest be independent from morally irrelevant characteristics such as skin color, gender, national origin, family background and so on. This of course requires an appropriate identification of what those morally irrelevant characteristics are. But even leaving aside this question, the consensual ideal of an equal opportunity to succeed is a rather poor guide to policy making. For it only indicates what is the destination - equal opportunity - without providing any insight on the way to get there. An example, developed in more details in the last section of this paper, may illustrate this point.

Figure 1 below shows the fraction of low and high caste adults (aged between 30 and 40) who have achieved at most each of the 6th ordered levels of education reported in the 68th round of the Indian NSSO in two neighboring states of India: West-Bengal and Odisha. Education levels are ranked from illiteracy (1) to upper tertiary education (6), and the low and high caste status are defined by the official category of Scheduled Caste (SC) and Scheduled Tribe (ST) (for the low caste) and the others (high caste). As is clear from the picture low and high castes adults *do not* have an "equal opportunity to succeed" in education whatever is the state where they live. Low caste adults are at significantly greater risk of failing to reach any achievement in education - be it the minimal one of literacy - than their high caste counterparts.

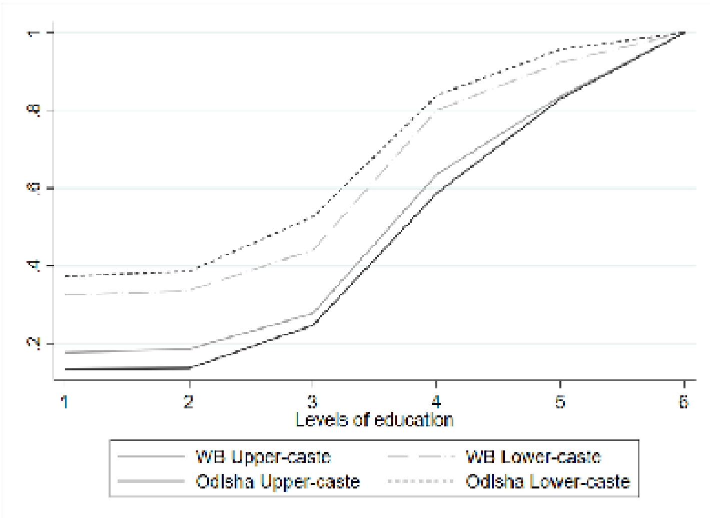

Figure 1: Cumulative distribution of education levels, Low and High caste groups, Odisha and West Bengal, 2012.

Yet, one may want to go beyond the mere observation that educational opportunities are unequally distributed among caste groups. One may, in particular, want to make comparative statements on the extent by which this caste inequality in educational opportunities differs between West Bengal and Odisha. To put it more compactly, one may want to define "opportunity equalization" rather than the mere zero-one "equal/unequal opportunity". In Figure 1 for instance, it could seem that educational opportunities faced by low and high caste adults - clearly unequal in both West Bengal and Odisha - are "more unequal" in the latter than in the former. Indeed, the risk of failing to reach any education level among low caste adults is lower in Odisha than in West Bengal. However, one observes an opposite ranking for the high caste adults who are more at risk of failing to achieve any level of education in West Bengal than in Odisha. Hence, the unfavored group in West Bengal does better than its counterpart in Odisha while the favored group in West Bengal does worse than its Odisha counterpart. Since the average - calculated symmetrically between high and low caste adults - distribution of educational opportunities in the two states is quite - albeit not perfectly - similar, it seems tempting indeed to conclude that the inequalities of educational opportunities between high and low caste adults are larger in Odisha than in West Bengal.

The contribution of this paper is to provide a theoretically justified, robust and implementable definition of "opportunity improvement" and "opportunity equalization" that provides support to the intuitive reasoning sketched above. Like other approaches to the subject found in the literature (see e.g. Roemer and Trannoy (2016) for a survey), our approach rides on the idea that there are many morally arbitrary variables - caste in the above example - that impact unduly on individuals' destiny. The different combinations of values taken by these morally irrelevant variables in the population lead to a partition of this population into groups of individuals for which those values are the same. Such groups are often referred to as *types* in the literature. Everything else being the same, our approach considers important to *improve* the probability of achieving outcomes of interest for some, or for all, the types. Our approach also considers important to *equalize* across types those probabilities of achieving those outcomes. These two objectives require definitions of what is meant by "improving" and by "equalizing" probability distributions. A contribution of this paper is to provide an operational and reasonably robust such definition.

Before detailing this definition, we find useful to distinguish further our approach from that of the literature surveyed in Roemer and Trannoy (2016). The latter stems from the widely discussed Dworkin (1981) cut between the characteristics that affect an individual's outcome for which he or she should be held *responsible* and the morally irrelevant ones that determine the individual's type. The main creed of this literature is that opportunity equalization should be concerned with *equalizing outcomes* among individuals for whom the "responsibility characteristics" are the same. However, no attempt should be made in equalizing outcomes that can be shown to result from the sole "free" exercise of responsibility. In recent years, the "cut" between the variables affecting individuals' achievements has been enlarged to luck and randomness (see e.g. Vallentyne (2002), Lefranc, Pistolesi, and Trannoy (2009) and Lefranc and Trannoy (2017)). Along with few other contributions (e.g. Bénabou and Ok (2001) and Mariotti and Veneziani (2017)), our approach departs from this "cut inspired" literature by being agnostic about the individuals' degree of responsibility for some of their characteristics. Responsibility plays actually no role in our approach, even though one may hold the view that individuals in each group are "responsible" for their success in the life (defined in our approach by their probability of achieving whatever outcome of relevance). Another important difference between our approach and those of the literature surveyed in Roemer and Trannoy (2016) is that we provide a definition of opportunity equalization and improvement, while many contributions to the literature are interested in defining - somewhat binarily - either inequality or (perfect) equality of opportunity. Moreover, most of the contributions to the literature that define opportunity equalization either ride heavily on the Dworkin (1981) cut (like for example Peragine (2004) or other contributions surveyed by Brunori, Ferreira, and Peragine (2021)), or do so by decomposing total outcome inequality (measured by some specific index) into *within* and *between* group inequality (see e.g. Ferreira and Gignoux (2011)), with between-group inequality defined, in the tradition of Shorrocks (1984), with respect to the groups' mean outcomes. A focus on the group means outcome is also a feature of the approach developed by Bénabou and Ok (2001) in the specific context of mobility measurement. Focusing on group mean outcome is restrictive because it discards all information related to the possibly varying riskiness of those outcomes across groups.

The contributions to the literature that we find the closest to our are, for different reasons, Andreoli, Havne, and Lefranc (2019) and Mariotti and Veneziani (2017). Andreoli, Havne, and Lefranc (2019) propose a robust definition of opportunity *equalization* across types based on a sequential (if there are more than two types) comparison of the absolute value of the "gaps" between probability distributions faced by individuals from two types. Their criterion is robust because it applies to societies where the probability distributions faced by the differing types are ordered by a large class of preferences. However, by requiring such a unanimous agreement over the ranking of probability distributions of the various types (for example by first order stochastic dominance), Andreoli, Havne, and Lefranc (2019) restrict the possible use of their criterion to those societies where such a unanimous agreements of the probability distributions is observed. Moreover, while the criterion proposed by Andreoli, Havne, and Lefranc (2019) is sensitive to welfare gaps between distributions, it is not sensitive to their welfare levels. As a result, their criterion may favor a policy that actually reduces the opportunities offered to every group if the reduction in opportunities is not uniform between groups and reduces the gap between them. By contrast, our criterion is sensitive to both the "levels" of opportunities offered to the groups and the "gap" between them, and incorporates in its very definition a trade-off between "overall improvement" in the opportunities offered to people and the unequal sharing of those improvement between the groups. This trade-off is similar in spirit to that underlying the generalized Lorenz curve (see e.g. Shorrocks (1983)) in conventional one-dimensional income inequality measurement. While incomplete, our criterion is applicable to any pair of distributions of opportunities, including those in which there is no unanimity about the ranking of the opportunities offered to the different groups. Mariotti and Veneziani (2017) proposes a justification for comparing allocations of opportunities with only two ordered outcomes (say good and bad) on the basis of the product - over all groups - of the probabilities of occurrence of the good outcome. When applied to such a restricted setting, their complete ranking of allocation of opportunities is compatible with our incomplete one. But our criterion applies to all allocations of opportunities, and not only to those whose outcomes are binary or, even, completely ordered. A last, but in our view not least, feature of our criterion is that it rides on an explicit normative justification for a concern for equality of opportunities that is of different spirit than those based on the Dworkin cut.

The normative basis on which our criterion stands is that of an ethical observer who is behind a *veil of ignorance* with respect to the group she/he might fall in if she/he were to born in a given society. How would such an ethical observer compare the various possible societies? Using results from decision theory under objective ambiguity - and notably Gravel, Marchant, and Sen (2011) and Gravel, Marchant, and Sen (2012) - one can provide arguments for such an ethical observer to do these comparisons on the basis of a *uniform expected valuation of expected utility* criterion. Such a criterion evaluates any list of distributions of opportunities by a *three-step* procedure. In the first step, a utility level is assigned to every conceivable outcome so that each group becomes identified by an expected utility of achieving those outcomes. In a second step, a valuation is assigned to the expected utility of every group by some valuation function. In the third step, a uniform expected valuation is calculated for the whole society under the (uniform) assumption that every group is equally likely. Of course there are many such ethical observers, as many as there are logically conceivable ways to assign utility levels to each outcome, and to assign (in the second step) valuation to expected utility of these outcomes. If one makes the additional assumption that the ethical observer dislikes inequality of opportunity in the sense of preferring a society in which the same average distribution of opportunities is observed in every group to one where the average is unevenly distributed among groups, one can obtain the additional restriction that the valuation function used to evaluate the expected utility of outcomes of every group is a concave function of this expected utility. But this still leaves quite many criteria to consider. The main theoretical contribution of this paper is to provide an empirically operational test that identifies when one distribution of opportunities among groups is better than another for all such ethical observers. The test, explained in detail in the paper, is the inclusion of the quasi-ordering extended zonotopes uniquely associated to the compared societies. The zonotope set of any list of probability distributions is the set of all Minkowski sums of those probability distributions (see e.g. Koshevoy (1995), Koshevoy (1998)). A quasi-ordering extended zonotope is a zonotope set that has been enlarged by a specific collection of translations that capture the assumptions made about the ranking of outcomes faced by members of the groups. Our approach is, indeed, quite general in that respect. If outcomes are completely ordered - as assumed in the education example given above - then the enlargement of the zonotope is made by translations that correspond to all possible ways of generating first order stochastic dominance on the distributions. If at the other extreme, the outcomes are not ordered at all, then the zonotope is not enlarged and the test amounts to checking for simple zonotope inclusion. Between these two extremes, our approach handles any incomplete quasi-ordering of the outcomes by enlargements of the zonotope that are specific to the quasi-ordering assumed. While the extended zonotope inclusion test is theoretically implementable with any number of groups, its actual implementation may sometimes be difficult. However, we are able to provide a precise and finite test for the general criterion in many specific cases of interest. One of them concerns the case - illustrated above - when there are two groups only (e.g. low caste and high caste adults). Another arises when, as assumed in Andreoli, Havne, and Lefranc (2019), the distributions over outcomes are ordered across groups.

We also put our criterion to work by examining the evolution, in the last three decades, of the educational opportunities offered to Indian adults depending upon their caste group and gender. The last three decades have indeed witnessed a significant improvement in access to education in India. Yet in the most recent census of 2011, one still finds 35% of females and 19% of males adult to be illiterate. Moreover the inequalities in educational opportunities between genders and caste groups remain impressive by any standard, even though the reasons for the persistence of those inequalities are not well-understood. An interesting possible explanation for the under-investment in children education made by low caste parents pointed out by Munshi and Rosenzweig (2006) (see also Munshi (2019) for a far reaching discussion of castes in India) is the fear that over education may deprive access to caste-based networks of traditional jobs. Persistent gender-based educational inequalities of opportunities are less specific to India and are observed in many developing countries. Here again however, there are no completely satisfactory explanation of why parents systematically invest less in girls' than in boys education and why this discrimination does not fade away with growth and development (see e.g. Heath and Jayachandran (2018) for a recent survey of the issues). Using data from the Employment-Unemployment schedule of the Indian National Sample Survey (NSS), we specifically analyze the evolution over time of the educational opportunities of low caste - as defined by the Indian official categorization of Scheduled Tribe (ST) and Scheduled Caste (SC) - and high caste (the others) male and female adults in India's 15 largest states and union territories. We compare the allocations of educational opportunities in those states when the groups are formed by castes only (low-caste and high caste) and when the groups are also further divided among genders. Our criterion happen to compare a large fraction of these Indian states. Our criterion shows that the inequality of educational opportunities among castes and genders is strong, and is even stronger among castes than among genders. Indeed, in all states, the educational opportunities offered to each of four groups (high caste male, high caste females, low caste males and low caste female) are all ordered by first order stochastic dominance. This ordering is the same in almost states: high caste male dominates high caste females who dominate low caste male who dominate low caste female. The only exception to this is Kerala where the upper caste ranking of the male and female group is reversed. While the ranking of Indian states in terms of allocation of educational opportunities among groups often follows the ranking of those states in terms of their average opportunities, there are a few cases where excessive inequality of educational opportunities prevent states with relatively favorable average distribution of opportunities to dominate other states with a less favorable - but more equal - such distribution. An interesting example of this is Kerala, an Indian state that is rightly portrayed as standing particularly well on the education front. Yet, the unequal distribution among caste of educational opportunities prevent Kerala from dominating - as per our criterion - the state of Andhra Pradesh whose average distribution of educational opportunities is dominated at the first order of stochastic dominance.

The plan of the remaining of the paper is as follows. The next section describes the general setting in which distributions of opportunities are evaluated and provides a normative foundation to this evaluation through the view point of an ethical observer placed behind the veil of ignorance. Section 3 presents the operational extended zonotope criterion and establishes its equivalence with the ranking of societies made by all opportunity inequality averse uniform expected utility ethical observer. It also indicates how this criterion can be associated with specific elementary transformations and can be easily implemented in the two-group case and, in some cases of interest, for an arbitrary number of cases. Section 4 presents the result of the empirical implementation of the criterion to appraise inequality of educational opportunities between men and women and Section 5 concludes.

2 A framework for evaluating opportunities[2]

We are interested in comparing *societies* on the basis of their performance in providing and distributing "justly" *opportunities* offered to members of some exogenously given - but possibly variable across societies - groups. These groups may be based on caste, religion, race, gender, family background, etc. They may also be based on the fact that the members of those groups have exerted the same level of *responsibility* (if one believes in such a thing). We do not assume that the number of those groups is the same across societies. For instance, we may consider societies formed by one group only. Our approach would then view such one-group societies as achieving (trivially) perfect equality of opportunity, even though they may differ by the levels of those opportunities. The opportunities offered to a group are described from an *ex ante* point of view (see Fleurbaey (2010) and Fleurbaey (2018) for discussions of and alternative approaches to the normative analysis of "socially risky situations") by the *probabilities* of achieving *relevant outcomes* faced by its members. We assume specifically that there are k such outcomes taken from the set $\{1, ..., k\}$. Outcomes are to be interpreted as anything observable that people have reason to value. Examples would include income, education or health levels (expressed in discrete units). They may also be pairs of, say, education and health levels. Hence, the general approach that we propose does not require outcomes to be completely ordered. We may even take the extreme view that they are not ordered at all. For example, we may care about the distribution among males and females of the opportunity of entering in the army. In such an case, there would be only two outcomes (joining or not the army) which are not ordered in an obvious way. Formally, the ranking of outcomes may be viewed as resulting from some quasi-ordering $\geq_{QO}$ on $\{1, ..., k\}$ with the interpretation that $j \geq_{QO} h$ if and only if outcome j is "clearly better" for an agent than outcome h". An extreme form of incomplete ordering is the case just discussed where *none* of the outcomes can be compared with one another. Let us denote by $\geq_{\varnothing}$ this empty quasi-ordering. At the other extreme, one could have the case, commonly assumed in the equality of opportunity literature surveyed in Roemer and Trannoy (2016), of a complete ordering of outcomes (based for example on income levels), that we denote by $\geq_C$. Given the quasi-ordering $\geq_{QO}$, we will always assume that the outcomes $\{1, ..., k\}$ are labeled in a way compatible with $\geq_{QO}$. Hence, if $h < i$ for outcomes h and i in $\{1, ..., k\}$, this is meant to imply that $h \geq_{QO} i$ *does not* hold.

Any society $\mathbf{p}$ is depicted as an $n(\mathbf{p}) \times k$ row-stochastic matrix:

$$\mathbf{p} = \begin{bmatrix} p_{11} & \cdots & p_{1k} \\ \cdots & \cdots & \cdots \\ p_{n(\mathbf{p})1} & \cdots & p_{n(\mathbf{p})k} \end{bmatrix}$$

[2] All possibly non-standard mathematical notations and definitions are provided in the first subsection of the Appendix.

where p_{ij}, for $i = 1, ..., n(\mathbf{p})$ and $j = 1, ..., k$ denotes the probability that an agent from group i achieves outcome j in society $\mathbf{p}$ and $n(\mathbf{p})$ denotes the number of groups in $\mathbf{p}$. For any society $\mathbf{p}$, we denote by p_i the distribution of probabilities (opportunities) associated to group i in $\mathbf{p}$ and by $\overline{p}$ its (symmetric across groups) average distribution of opportunities defined by:

$$\overline{p} = \frac{1}{n(\mathbf{p})} \sum_{i=1}^{n(\mathbf{p})} p_i.$$

If $\mathbf{p}$ is a society in $(\Delta^{k-1})^m$ and $\mathbf{q}$ is a society in $(\Delta^{k-1})^n$, we denote by $(\mathbf{p}, \mathbf{q})$ the society in $(\Delta^{k-1})^{m+n}$ where the m first groups face the opportunities associated to $\mathbf{p}$ and the n last groups face the opportunities associated to $\mathbf{q}$ (in the corresponding order). If ρ is in Δ^{k-1}, we abuse notation in denoting also by ρ the *one group society* in which everyone faces the distribution of opportunities $\rho \in \Delta^{k-1}$. We similarly sometimes abuse notation by using j to denote both the outcome $j \in \{1, ..., k\}$ itself and the degenerate probability distribution $\rho \in \Delta^{k-1}$ defined by $\rho_j = 1$ and $\rho_h = 0$ for all outcome $h \neq j$ that gives j for sure. Finally, we denote by $\mathbb{S}$ the set of all conceivable societies defined by $\mathbb{S} = \bigcup_{n \geq 1} \left(\Delta^{k-1}\right)^n$.

Societies are to be compared by an *ethical observer* who agrees with the ranking of the outcomes provided by $\geq_{QO}$, and who is placed behind a "veil of ignorance" as to the group to which she would belong if she was to live in the considered societies. Such an ethical observer would compare societies by means of some ordering $\succsim$, with asymmetric and symmetric factors $\succ$ and $\sim$. The statement $\mathbf{p} \succsim \mathbf{q}$ is interpreted as meaning that "The ethical observer would weakly prefer being born in society $\mathbf{p}$ than in society $\mathbf{q}$" (and similarly for the statements $\mathbf{p} \succ \mathbf{q}$ (strict preference) and $\mathbf{p} \sim \mathbf{q}$ (indifference). Since the ordering $\succsim$ is defined on the whole set $\mathbb{S}$, it is in particular defined on the set Δ^{k-1} of all conceivable one-group societies and, therefore, on all probability distributions over the k outcomes. Hence the ethical observer is also a decision maker under risk who is capable to order all probability distributions over outcomes.

We focus on ethical observers who use an ordering $\succsim$ for which there is a (continuous) increasing function $\Phi : \mathbb{R} \to \mathbb{R}$ and k real numbers $u_1, ..., u_k$ satisfying, for every distinct outcomes h and i, $i \geq_{QO} h \Longrightarrow u_i \geq u_h$ such that:

$$\mathbf{q} \succsim \mathbf{p} \Longleftrightarrow \sum_{i=1}^{n(\mathbf{q})} \frac{\Phi(\sum_{j=1}^{k} q_{ij} u_j)}{n(\mathbf{q})} \geq \sum_{i=1}^{n(\mathbf{p})} \frac{\Phi(\sum_{j=1}^{k} p_{ij} u_j)}{n(\mathbf{p})} \tag{1}$$

for any two societies $\mathbf{p}$ and $\mathbf{q}$ in $\mathbb{S}$. An ordering satisfying this property *could* therefore be thought of as resulting from a three-step procedure. In the first step, every group in the compared societies is given an *expected utility* that results from the assignment of utility numbers $u_1, ..., u_k$ to outcomes in a way that reflects their ranking by the quasi-ordering $\geq_{QO}$. In the second step, this expected utility is assigned a *valuation* by the ethical observer through some function Φ. In the third step, the ethical observer ranks the societies on the basis of the *expected valuation* of their groups' expected utilities, under the (uniform) assumption that the ethical observer is equally likely to fall in any group. We accordingly call *Uniform Expected Valuator of Expected Utility* (UEVEU) any such ethical observer. There are actually quite a few of them, as many in fact as there are logically conceivable valuation functions Φ and logically conceivable ways of assigning utility to outcomes in a manner consistent with $\geq_{QO}$. Evaluating opportunities by means of a UEVEU criterion has been suggested by Martinez, Schockkaert, and VandeGaer (2001) (see their Equations (1)-(3) p. 528) in connection with mobility measurement. In what follows, we provide an axiomatic characterization - borrowed from Gravel, Marchant, and Sen (2012) and Gravel, Marchant, and Sen (2011) - for using this family of orderings in the case of an observer who is uncertain about the group in which she may fall if she were to be borne in one of the considered societies. What are the axioms that comparisons of societies must satisfy in order to be represented as per (1) for some valuation function Φ and utility numbers $u_1, ..., u_k$ compatible with $\geq_{QO}$?

The first one is the *anonymity* principle according to which the names of the groups do not matter for appraising the opportunities offered to their members. Hence, any permutation between groups of the distributions of opportunities faced by their members is a matter of social indifference.

Axiom 1 *(Anonymity) For every society* $\mathbf{p} \in \mathbb{S}$ *and every* $n(\mathbf{p}) \times n(\mathbf{p})$ *permutation matrix* π*, one has* $\pi.\mathbf{p} \sim \mathbf{p}$.

Anonymity seems plausible when evaluating distributions of opportunities faced by the members of groups formed on the basis of race, gender, and other (morally arbitrary) qualitative characteristics of that sort. However, it may not seems so for groups formed on the basis of a more quantitative attribute like, for example, the income category of the parents. In such a setting, often considered in mobility measurement (see e.g. Atkinson (1981)), it has been suggested that permuting the opportunities offered to children coming from low and from high income groups is not a matter of social indifference. In particular, one may prefer giving the "good opportunities" to the kids coming from low-income families and the "bad opportunities" to those coming from high income families than the other way around. Such a preference is clearly at odd with the requirement of anonymity.

The second axiom is a continuity condition that concerns one-group societies *vis-à-vis* others. This axiom requires that the strict ranking of any one-group society with respect to any other society be robust to “small" changes in the probabilities of achieving any given outcome. Its formal statement is as follows.

Axiom 2 *(Continuity) For every society* $\mathbf{p} \in \mathbb{S}$*, the sets* $B(\mathbf{p}) = \{\rho \in \Delta^{k-1} : \rho \succsim \mathbf{p}\}$*,* $W(p) = \{\rho \in \Delta^{k-1} : \mathbf{p} \succeq \rho\}$ *are both closed in* $\mathbb{R}^k_+$.

We observe that this axiom is mild because it only applies to rankings of one-group societies vis-à-vis any other societies. It does not restrict at all the ranking of societies in which people are partitioned in more than one group.

The next axiom is called *averaging* in Gravel, Marchant, and Sen (2012). It is the only axiom that applies to the ranking of societies involving different numbers of groups. Specifically, the axiom evaluates what happens to the distributions of opportunities in a given society when this society is hypothetically merged with another. To illustrate, consider the distribution of educational opportunities in West Bengal and Odisha discussed in introduction and their partition between low and high caste adults. Suppose, as suggested earlier, that educational opportunities are considered more equally distributed between low and high caste adults in West Bengal than in Odisha. Consider then merging the two states into a larger jurisdictional entity where there will now be four groups: Odisha low caste, Odisha high caste, West Bengal low caste and West Bengal high caste. The averaging axiom requires the ranking of the opportunities offered to this enlarged four-group society to lie in between that associated to the two initial two-groups societies (West Bengal and Odisha). That is, opportunities should be better distributed in West Bengal than in the newly enlarged jurisdiction, and should be better distributed in this enlarged jurisdiction than they were considered to be in Odisha. It says also, conversely, that if a society loses (gains) from identifying new groups with specific distributions of outcome among their members, then this can only be because the distribution of outcomes within those groups is worse (better) than that already present in the original society. The formal statement of this axiom is as follows.

Axiom 3 *(Averaging) For all societies* $\mathbf{p}$ *and* $\mathbf{q}$ *in* $\mathbb{S}$*, we have* $\mathbf{p} \succsim \mathbf{q} \Leftrightarrow \mathbf{p} \succsim (\mathbf{p}, \mathbf{q}) \Leftrightarrow (\mathbf{p}, \mathbf{q}) \succsim \mathbf{q}$.

The next axiom requires the ranking of any two societies with the same number of groups to be robust to the addition, to both societies, of a common distribution of opportunities. That is to say, the ranking of any two societies with the same number of groups should be independent from any groups that they have in common when the distribution of opportunities in each of these groups is the same across societies. Formally, this axiom is stated as follows.

Axiom 4 *(Same Number Group Independence) For all societies* $\mathbf{p}$*,* $\mathbf{p}'$ *and* $\mathbf{p}''$ *in* $\mathbb{S}$ *such that* $n(\mathbf{p}) = n(\mathbf{p}')$*,* $(\mathbf{p}, \mathbf{p}'') \succsim (\mathbf{p}', \mathbf{p}'')$ *if and only if* $\mathbf{p} \succsim \mathbf{p}'$.

The last two axioms deal with the ranking of one-group societies where concerns for inequality of opportunities are by definition absent. The first of these two axioms requires the ranking of one- group societies, which is nothing else than the ranking of probability distributions, to obey the well-known Von Neumann and Morgenstern (1947) independence axiom.

Axiom 5 *(VNM for One-Group societies) For every lotteries* p*,* p' *and* p" $\in \Delta^{k-1}$ *and every number* $\lambda \in [0, 1]$*,* $p \succsim p'$ *if and only if* $\lambda p + (1 - \lambda)p'' \succsim \lambda p' + (1 - \lambda)p''$.

The second of these two axioms ensures the consistency of the ranking of one-group societies, at least when they face no uncertainty at all, with the (possibly incomplete) ranking of outcomes provided by $\geq_{QO}$.

Axiom 6 *(Consistency with* $\geq_{QO}$ *for One-Group societies) For every two distinct outcomes* h *and* $j \in \{1, ..., k\}$ *such that* $j \geq_{QO} h$*, one should have* $j \succsim h$.

It can be checked that any UEVEU ordering satisfies Axioms 1 - 6. Using and adapting results in Gravel, Marchant, and Sen (2012) and Gravel, Marchant, and Sen (2011), one can establish the converse implication. Hence, one has:

Proposition 1 *Let* $\succsim$ *be an ordering on* $\mathbb{S}$ *satisfying Anonymity, Continuity, Averaging, Same Number Group Independence, VNM for one group societies and Consistency with* $\geq_{QO}$ *for one-group societies. Then, there exists a function*

$\Phi : \mathbb{R} \to \mathbb{R}$ and a list of k numbers $u_1, ..., u_k$ satisfying, for every distinct outcomes h and i, $i \geq_{QO} h \Longrightarrow u_i \geq u_h$ such that (1) holds for any two societies $\mathbf{p}$ and $\mathbf{q}$ in $\mathbb{S}$. Furthermore, the function Φ is unique up to a positive affine transformation, and is continuous and increasing.

While we do not provide herein the complete proof of this Proposition, we find useful to sketch the basic line of argument. From Gravel, Marchant, and Sen (2011), one can show that any ordering $\succsim$ of $\mathbb{S}$ satisfying Anonymity, Continuity, Averaging and Same Number Group Independence can be expressed, for any two societies $\mathbf{p}$ and $\mathbf{q}$ in $\mathbb{S}$, by:

$$\mathbf{q} \succsim \mathbf{p} \Longleftrightarrow \sum_{i=1}^{n(\mathbf{q})} \frac{g(q_i)}{n(\mathbf{q})} \geq \sum_{i=1}^{n(\mathbf{p})} \frac{g(p_i)}{n(\mathbf{p})}$$

for some continuous function $g : S^{k-1} \to \mathbb{R}$. Such a numerical representation of $\succsim$ is called Uniform Expected Utility by Gravel, Marchant, and Sen (2012). Hence the sole axioms of Anonymity, Continuity, Averaging and Same number Group Independence force the ethical observer who subscribes to them to rank societies by means of a uniform expectation of the valuation of the *probability distributions* faced by the different groups by some valuation function g. Adding the VNM for One-Group Societies axiom restricts further g to be a monotonic transformation of the expected utility associated to the probability distribution faced by any group for some lists of utility numbers assigned to the outcomes (see e.g. Proposition 6 in Gravel, Marchant, and Sen (2012)). Finally, the Consistency with $\geq_{QO}$for One-Group Societies axiom further restricts these utility numbers to those that are consistent with the quasi-ordering $\geq_{QO}$.

We take the view that our UEVEU ethical observers exhibit aversion to inequality of opportunity, which we define as a preference for a society exhibiting no inequality of opportunity - say because they are made of one single group - over societies who exhibit some inequality of opportunity. This suggests the following notion of comparative aversion to inequality of opportunities among ethical observers.

Definition 1 *Given two orderings $\succsim_1$ and $\succsim_2$ on $\mathbb{S}$, we say that $\succsim_1$ exhibits* at least as much aversion to inequality of opportunity *as $\succsim_2$ if, for every lottery $\rho \in \Delta^{k-1}$ and society $\mathbf{p} \in \mathbb{S}$, we have $\rho \succsim_2 \mathbf{p} \Longrightarrow \rho \succsim_1 \mathbf{p}$.*

In words, an ethical observer who compares societies by means of $\succsim_1$ exhibits at least as much aversion to inequality of opportunities as another who bases his/her comparisons on $\succsim_2$ if any preference that the latter will have for a society with no inequality of opportunities (as compared to any reference society) would also be endorsed by the former. It is not difficult to see that this notion of “comparative aversion to opportunity inequality" can translate, when expressed for UEVEU orderings, into a statement of “comparative concavity" applied to the function Φ of that expression. Specifically, the following proposition can be established (see Gravel, Marchant, and Sen (2012) proposition 5).

Proposition 2 *Let $\succsim_1$ and $\succsim_2$ be two orderings on $\mathbb{S}$ which can be represented as per (1) for, respectively, some functions Φ^1 and Φ^2 and some lists of k utility numbers $u_1^1, ..., u_k^1$ and $u_1^2, ..., u_k^2$. Then $\succsim_1$exhibits at least as much aversion to* *inequality of opportunity as* $\succsim_2$ *if and only if* $u_j^1 = u_j^2$ *for every* $j = 1, ..., k$ *and there exists some real valued concave function* Ψ *having the range of* Φ^2 *in its domain such that, for every* $p \in \Delta^{k-1}$, *one has* $\Phi^1(\sum_{j=1}^{k} p_j u_j^1) = \Psi(\Phi^2(\sum_{j=1}^{k} p_j u_j^1))$.

Hence, for comparisons of societies made by UEVEU ethical observers, the statement "has more aversion to opportunity inequality as" can be translated into "has a more concave valuation function as". While this is reminiscent of standard definition in the context of standard inequality measurement, there is an important difference. In the usual income inequality setting, there is a (natural) benchmark to define “neutrality to income inequality". An ethical observer concerned with distributions of incomes is usually considered as being neutral vis-à-vis income equality if it considers as equivalent all income distributions that have the same per capita income. Given this benchmark, it is standard to define aversion to inequality “in the absolute" by the fact of exhibiting more aversion to income inequality than neutrality to inequality. While we are not aware of the existence of a well-accepted standard of neutrality toward inequality of opportunities, we believe that a plausible candidate for this would be to consider as equivalent all societies that distribute among their groups the same (symmetric) average probability distribution over outcomes. If one agrees with this standard of neutrality with respect to inequality of opportunity, then one can define an ethical observer as exhibiting aversion to inequality of opportunity whenever the observer has more aversion to inequality of opportunities than an observer who exhibits neutrality with respect to equality of opportunities. We do this formally as follows.

Definition 2 *Let* $\succsim$ *be an ordering on* $\mathbb{S}$.

- (i) $\succsim$ *is said to exhibit* neutrality *to equality of opportunities if for any two societies* $\mathbf{p}$ *and* $\mathbf{q}$ *in* $\mathbb{S}$ *such that* $\overline{p} = \overline{q}$, *one has* $\mathbf{p} \sim \mathbf{q}$.
- (ii) $\succsim$ *is said to exhibit* aversion *to inequality of opportunity if there exists an ordering* $\succsim_0$, *exhibiting neutrality to inequality of opportunity, such that* $\succsim$ *exhibits at least as much aversion to inequality of opportunity as* $\succsim_0$.

Combining this definition with Proposition 2, we can establish the following.

Proposition 3 *An ordering* $\succsim$ *on* $\mathbb{S}$ *that can be numerically represented as per (1) for some function* $\Phi : \mathbb{R} \to \mathbb{R}$ *and some lists of* k *utility numbers* $u_1, ..., u_k$ *exhibits aversion to inequality of opportunity if and only if* Φ *is concave.*

The normative dominance approach that we use for comparing societies requires consensus among all opportunity inequality averse UEVEU ethical observers. Let us denote by $\mathcal{U}^{\geq_{QO}} \subset \mathbb{R}^k$ the set of all lists of utility numbers compatible with the quasi-ordering $\geq_{QO}$ defined by:

$$\mathcal{U}^{\geq_{QO}} = \{(u_1, ..., u_k) \in \mathbb{R}^k : \ j \geq_{QO} h \Longrightarrow u_j \geq u_h, \ \forall j, h \in \{1, ..., k\}\} \qquad (2)$$

With this notation, we define UEVEU dominance as follows.

Definition 3 *Given a quasi-ordering $\geq_{QO}$ on $\{1, ..., k\}$, we say that society* $\mathbf{q}$ *dominates society* $\mathbf{p}$ *for all UEVEU ethical observer, which we denote by* $\mathbf{q} \succsim_{UEVEU}^{QO} \mathbf{p}$, *iff*

$$\frac{1}{n(\mathbf{q})}\sum_{i=1}^{n}\Phi\left(\sum_{h=1}^{k}q_{ih}u_h\right) \geq \frac{1}{n(\mathbf{p})}\sum_{i=1}^{n}\Phi\left(\sum_{h=1}^{k}p_{ih}u_h\right) \tag{3}$$

for all increasing and concave functions $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ and all lists of numbers $(u_1, ..., u_k) \in \mathcal{U}^{\geq_{QO}}$.

While we consider plausible to compare distributions of opportunities by UEVEU dominance, it is worth emphasizing that this criterion, except perhaps in the case where no assumption is made on the ranking of outcomes, involves some trade-off between the average opportunities available in the society and the unequal sharing of this average across the groups. Most UEVEU criteria therefore combine a concern for reducing inequalities in opportunity with a concern for improving the opportunities offered to some, or to all, of the groups. This is particularly clear if one considers the case where all outcomes are ordered from the worst (1) to the best (k) ($\geq_{QO}=\geq_C$). In this case, it is clear that any society, however unequal it is in terms of the opportunities it offers to the different groups, will be considered weakly better than a perfectly egalitarian society in which the members of all groups are sure to end in the worst possible outcome (a situation referred to as "hell" by Mariotti and Veneziani (2017))! In the other direction, any society will be considered by any UEVEU criterion to be worse than the egalitarian society in which everyone irrespective of his/her groups is sure to end in the best possible outcome ("heaven"). The only case where a UEVEU criterion will rank societies on the basis of equalizing opportunities only is where there is no *a priori* ranking of the outcomes (that is $\geq_{QO}=\geq_{\varnothing}$). In such a case, there cannot be any ranking of the average opportunities given to societies - because such a ranking depends upon the utility numbers assigned to the outcomes - and the only consideration that could eventually lead to a conclusive ranking of two societies would be the splitting of this average distribution between the groups. However, this case is unlikely to be of a significant empirical interest.

3 An operational definition of opportunities' equalization

3.1 The criterion in the general case

From now on, we focus on societies containing the same number of groups so that $n(\mathbf{p}) = n(\mathbf{q}) = n$ for some integer $n \geq 2$. The value of this fixed n is immaterial for an inequality such as (3) and we therefore abstract from it. Even for a fixed number of groups, the number of functions Φ and combinations of utility numbers $(u_1, ..., u_n) \in \mathcal{U}^{\geq_{QO}}$ that need to be checked in order to establish whether or not $\mathbf{q} \succsim_{UEVEU}^{QO} \mathbf{p}$ is large. Given any two societies, it would therefore be a very exhausting (if not impossible) task of verifying whether one is better than the other for all such UEVEU criteria. In this section, we identify an operational criterion that eases significantly this verification.

We start by observing that for any quasi-ordering $\geq_{QO}$, the set $\mathcal{U}^{\geq_{QO}}$ is a non-empty closed convex cone. The *dual cone*[3] relative to $\mathcal{U}^{\geq_{QO}}$, denoted by $\mathcal{U}_*^{\geq_{QO}}$, is defined by:

$$\mathcal{U}_*^{\geq_{QO}} = \{(v_1, ..., v_k) \in \mathbb{R}^k : \sum_{j=1}^{k} v_j u_j \geq 0 \text{ for all } (u_1, ..., u_k) \in \mathcal{U}^{\geq_{QO}}\} \tag{4}$$

We observe that $\mathcal{U}_*^{\geq_{QO}} = \{\mathbf{0}^k\}$ if and only if $\geq_{QO} = \geq_{\varnothing}$. If no pair of outcomes can be compared, then the only vector v that is dual to the set of all logically conceivable lists of k numbers - that is $\mathbb{R}^k$ - is the zero vector. Another observation about the dual cone $\mathcal{U}_*^{\geq_{QO}}$ is that all the $k-$tuples $(v_1, ..., v_k)$ that it contains have their components that sum to 0. We state this formally as follows.

Remark 1 *Let $(v_1, ..., v_k) \in \mathcal{U}_*^{\geq_{QO}}$ for some quasi-ordering of $\geq_{QO}$ $\{1, ..., k\}$. Then $v_1 + ... + v_k = 0$.*

The dual cone associated to $\mathcal{U}^{\geq_{QO}}$ has an intuitive interpretation. It is the set of all changes in the probability distribution over outcomes that increase expected utility for all utility functions compatible with $\geq_{QO}$. In plain English, it is the set of all clear improvements in the opportunities of achieving the outcomes. This interpretation is supported by the fact that the sum of these changes is zero and, as a result, they produce a new probability distribution over outcomes which cumulates to 1, just like the initial distribution. What exactly these changes in the distribution are depend of course upon the precise definition of the quasi-ordering.

The proposed operational definition of opportunity equalization makes use of the Zonotope set $\mathbf{Z}(\mathbf{p}) \subset \mathbb{R}_+^k$ associated to any society $\mathbf{p} \in \mathbb{S}$, and defined by:

$$\mathbf{Z}(\mathbf{p}) = \left\{ \mathbf{z} = (z_1, \ldots, z_k) : \mathbf{z} = \sum_{i=1}^{n(\mathbf{p})} \theta_i p_i,\ \theta_i \in [0,1]\ \forall i = 1, \ldots, n \right\} \tag{5}$$

A closely related set has been used by Koshevoy (1995) (see also Koshevoy and Mosler (1996)) to define a criterion called by this author *Lorenz majorization.* We use this zonotope set to define what we call Quasi-Ordering Extended Zonotope (QOEZ) dominance between two societies as follows.

Definition 4 *We say that* $\mathbf{q}$ *dominates* $\mathbf{p}$ *for the* $\geq_{QO}$*- extended Zonotope dominance criterion, which we write as* $\mathbf{q} \succsim_Z^{QO} \mathbf{p}$ *, if and only if*

$$Z(\mathbf{q}) + \mathcal{U}_*^{\geq_{QO}} \subseteq Z(\mathbf{p}) + \mathcal{U}_*^{\geq_{QO}}.$$

While QOEZ dominance may be difficult to verify in general, we provide below easily implementable finite procedures in many cases of interest. Koshevoy and Mosler (2007) have also proposed, in a different context, a somewhat similar test based on the inclusion of suitably extended Zonotope sets.

We now establish the following main equivalence, proved in the Appendix, between the ranking of two societies as per QOEZ dominance and the ranking of those societies agreed upon by all opportunity-inequality averse UEVEU ethical observers who compare outcomes by means of the quasi-ordering $\geq_{QO}$.

[3] Which is the negative of what Rockafellar (1970) p. 121 calls the "polar" of $\mathcal{U}^{\geq_{QO}}$

Theorem 1 *The two following statements are equivalent:*

(i) $\mathbf{q} \succsim_Z^{QO} \mathbf{p}$;

(ii) $\mathbf{q} \succsim_{UEVEU}^{QO} \mathbf{p}$.

As mentioned, this theorem may be considered too general for practical purposes. For one thing, it rides on a quasi-ordering $\geq_{QO}$ of outcomes on which little is known *a priori*. We will give below a more ready-to-use version of the theorem in the two extreme cases where outcomes are not ordered at all, and where they are completely ordered. An additional difficulty raised by Theorem 1 is the uncountably infinite size of the set $\mathcal{U}^{\geq_{QO}}$ of lists $(u_1, ..., u_k)$ of utility numbers compatible with $\geq$ with respect to which the "dual cone" $\mathcal{U}_*^{\geq_{QO}}$ of changes $(v_1, ..., v_k)$ in the distribution - that must be added to the Zonotope sets before checking for inclusion - is defined. How can one identify in practice the dual cone of an uncountably infinite set ? In the following proposition, we alleviate this difficulty by showing that for any uncountably infinite set $\mathcal{U}^{\geq_{QO}}$ of lists $(u_1, ..., u_k)$ of utility numbers compatible with $\geq_{QO}$, there is a finite set of lists of utility numbers (each actually taken in the pair $\{0, 1\}$) that generates exactly the same dual cone $\mathcal{U}_*^{\geq_{QO}}$. Hence, this proposition simplifies the computational problem of finding the appropriate dual cone that is relevant for the implementation of the criterion. The established proposition is the following.

Proposition 4 *We have*

$$\mathcal{U}_*^{\geq_{QO}} = \left\{ \mathbf{v} \in \mathbb{R}^k : \sum_{j=1}^{k} v_j u_j \geq 0 \ \forall (u_1, ..., u_k) \in \mathcal{U}^{\geq_{QO}} \cap \{0, 1\}^k \right\}.$$

Another simple, but interesting, implication of the dominance of one society by another in terms of the QOEZ criterion is the dominance of the average distribution of opportunities of the dominating society over that of the dominated one by all list of utility numbers compatible with $\geq_{QO}$. In effect:

Remark 2 *Suppose that* $\mathbf{q} \succsim_Z^{QO} \mathbf{p}$. *Then* $\overline{q} - \overline{p} \in \mathcal{U}_*^{\geq_{QO}}$.

Let us now interpret Theorem 1 in the two extreme cases where no outcomes are comparable, and where all outcomes are ordered as per their rank in the set $\{1, ..., k\}$.

Starting with the first case, and combining standard results on one dimensional inequality measurement and Theorem 3.1 in Koshevoy and Mosler (1996), we easily establish the following result.

Proposition 5 *Suppose that* $\overline{p} = \overline{q}$. *Then the two following statements are equivalent:*

(i) $Z(\mathbf{q}) \subset Z(\mathbf{p})$;

(ii) $\mathbf{q} \succsim_{UEVEU}^{\emptyset} \mathbf{p}$.

We now turn to the case, typically considered in the equality of opportunity measurement literature, where all outcomes are ordered from the worst (1) to the best (k). In that case, the lists of utility numbers $(u_1, ..., u_k) \in \mathbb{R}^k$ over which a unanimity is looked for are those that satisfy $u_1 \leq u_2 \leq ... \leq u_k$. Exploiting the result of Proposition 4, we can limit our attention to those lists of numbers lying in the set $\{0, 1\}$ that satisfy these inequalities. The dual cone of the set of those lists of 0 and 1 bears a close connection with the notion of *first order stochastic dominance* applied to the distributions of outcomes, that we now recall for the sake of completeness.

Definition 5 *For any two distributions p and $q \in \Delta^{k-1}$, we say that q first order stochastically dominates p, denoted $q \succsim^{1st} p$, if and only if one has: $\sum_{h=j}^{k} q_h \geq \sum_{h=j}^{k} p_h$, for any $j = 1, ..., k$.*

We now observe, thanks to Proposition 4, that the dual cone of the set $\mathcal{U}^{\geq_C}$ can be taken to be the set of changes $(v_1, ..., v_k)$ in the distributions of opportunities that produce first order stochastic improvements over the distributions of opportunities to which they are applied. Specifically, using Proposition 4, one can observe the following.

Remark 3 $\mathcal{U}_*^{\geq_C} = \{v \in \mathbb{R}^k : \sum_{j=1}^{k} v_j = 0, \sum_{g=h}^{k} v_g \geq 0$ *for* $h = 2, ..., k\}$.

The connection between $\succsim_Z^C$ and 1st-order stochastic dominance is not surprising from an intuitive point of view. Any ethical observer who agrees on the complete ranking of the outcomes would therefore also agree that a relation of stochastic dominance between two groups indicates that the dominating group has better opportunities than the dominated group (a similar observation is made by Andreoli, Havne, and Lefranc (2019)). As a result, any such ethical observer - provided of course that she dislikes inequality of opportunity - would want to reduce the dispersion between those two distributions when the reduction of the dispersion does not modify the average distribution of opportunities.

3.2 Elementary operations

An alternative understanding of the $\geq_{QO}$- extended Zonotope dominance criteria (for various specifications of $\geq_{QO}$) can be obtained from the *elementary transformations* in the distributions of opportunities among groups that underlie them. While we do not identify exactly all these elementary transformations in the general n-group case - see however the results of the next subsection concerning two-group societies - we can at least identify some of them. We start with the following one, also identified by Kolm (1977) in the more general setting of multidimensional inequality measurement.

Definition 6 (Uniform averaging) *We say that* $\mathbf{q}$ *is obtained from* $\mathbf{p}$ *through a* uniform averaging *operation if there exists an $n \times n$ bistochastic matrix* $\mathbf{b}$ *such that* $\mathbf{q} = \mathbf{b}.\mathbf{p}$

This operation consists in *uniformly averaging* the various distributions of outcomes of the different groups. Specifically, if $\mathbf{q}$ is obtained from $\mathbf{p}$ through a *uniform averaging* operation, then for every group i, the probability q_{ih} that someone from that group achieves outcome h is a weighted average of the probabilities that people from the different groups in $\mathbf{p}$ achieve that outcome. This averaging is "uniform" in the sense that, for any group i, the weights used in the calculation of the average do not depend upon the outcome. To illustrate this point, consider the societies $\mathbf{p}$, $\mathbf{p}'$ and $\mathbf{p}''$ that distribute opportunities of achieving three outcomes between two groups as follows:

$\mathbf{p} =$

	outcome 1	outcome 2	outcome 3
group 1	1/4	1/12	2/3
group 2	2/3	1/4	1/12

$\mathbf{p}' =$

	outcome 1	outcome 2	outcome 3
group 1	17/48	1/8	25/48
group 2	9/16	5/24	11/48

$\mathbf{p}'' =$

	outcome 1	outcome 2	outcome 3
group 1	11/24	1/8	5/12
group 2	11/24	5/24	1/3

Observe that the average probability of achieving the three outcomes in the three societies is the same (namely $(11/24, 1/6, 9/24)$. In societies $\mathbf{p}'$ and $\mathbf{p}''$, it can be checked that the probability for a member of a group to achieve a given outcome is a weighted average, over the two groups in society $\mathbf{p}$, of the probabilities of achieving that same outcome. Hence, both $\mathbf{p}'$ and $\mathbf{p}''$ are obtained from $\mathbf{p}$ as a result of an *averaging* operation. However only society $\mathbf{p}'$ results from $\mathbf{p}$ out of a *uniform* averaging operation that uses the same weights - namely 3/4 and 1/4 for group 1 and 1/4 and 3/4 for group 2 - for determining the probability of achieving any outcome. The property of uniform averaging has been shown by Kolm (1977) - in the context considered by this author- to be equivalent to the ranking of distributions of consumption bundles that would be made by summing all Schur-concave functions. Since the function G defined by $G(\mathbf{p}) = \sum_{i=1}^{n}\Phi(\sum_{h=1}^{k} p_{ih}u_h)$ is concave and symmetric (across groups) if Φ is concave, it is therefore Schur-concave. Hence, thanks to the result by Kolm (1977), and irrespective of the ordering of outcomes, any uniform averaging operation would be considered worth doing by any ethical observer considered in this paper.

The second elementary operation that we examine is what we call an *bilateral equalizing transfer*. Contrary to uniform averaging - which does not use information on the ranking of the outcomes - the operation of bilateral equalizing transfer rides heavily on such an information. The formal definition of such a transfer is as follows.

Definition 7 (Equalizing transfer) *We say that* $\mathbf{q}$ *is obtained from* $\mathbf{p}$ *through a bilateral equalizing transfer if there exist indices* i_1, i_2, i_1' *and* $i_2' \in \{1, ..., n\}$ *and* $v \in \mathcal{U}_*^{\geq QO}$ *such that:*

$$q_{i_1'} = p_{i_1} + v,\ q_{i_2'} = p_{i_2} - v,\ p_{i_2} - p_{i_1} - v \in \mathcal{U}_*^{\geq QO}$$

and $p_j = q_j$ *for all* $j \notin \{i_1, i_2, i_1', i_2'\}$.

In words, a bilateral equalizing transfer is an operation that improves (through some change v) a distribution of opportunity in a group and that deteriorates (through the same v applied in opposite direction) a distribution of opportunities in another group in the case where the distribution of opportunities in the latter group is unambiguously better than that of the other group from the view point of the quasi-ordering $\geq_{QO}$ or equivalently, thanks to Donaldson and Weymark (1998), of all complete rankings of outcomes whose intersection is $\geq_{QO}$. It is intuitively clear that such a reduction in the "expected-utility gap" between the two distributions - provided that it is done in a way that does not affect the average distribution of opportunities in the society - would be recorded favorably by an opportunity inequality-averse ethical observer who evaluates those expected utilities through a (uniform) expectation of a concave function. We observe that such a transformation only concerns two distributions of opportunities in each of the two societies (and leaves the other distributions faced by the other groups unchanged). Hence, by comparison with the uniform averaging operation which concerns the *totality* of the matrix, a bilateral equalizing transfer, as its name suggests, is a *local operation* that concerns only two rows of each of the matrices under comparison.

In order to illustrate this transformation in the case of an incomplete ranking of the outcomes, consider the following binary health-education example where the outcomes are $(0,0)$ (bad health, low education), $(0,1)$ (bad health, high education) $(1,0)$ (good health, low education) and $(1,1)$ (good health, high education) with the quasi-ordering $\geq_{QO}$ defined by:

$(1,1) \geq_{QO} (0,1) \geq_{QO} (0,0)$ and $(1,1) \geq_{QO} (1,0) \geq_{QO} (0,0)$ ($(0,1)$ and $(1,0)$ being incomparable). Assume that there are only two groups, and consider the two distributions:

$\mathbf{p} =$

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
group 1	1/2	1/6	1/6	1/6
group 2	1/4	1/4	1/4	1/4

and:

$\mathbf{q} =$

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
group 1	7/16	19/86	1/6	19/86
group 2	5/16	7/32	1/4	7/32

The first observation is that the distribution of probabilities of achieving the four outcomes in group 1 provides a lower expected utility than that of group 2 in society $\mathbf{p}$. This can be seen by the fact that, for any of the two complete rankings of the four outcomes that are consistent with $\geq_{QO}$, the distribution of outcome in group 2 first order stochastically dominate that in group 2. The second observation is that the move from $\mathbf{p}$ to $\mathbf{q}$ has been done by improving the probability distribution of group 1 by the vector $v = (-1/16, 1/32, 0, 1/32)$ and deteriorating the probability distribution by the corresponding vector $-v = (1/16, -1/32, 0, 1/32)$. Observe that these two "balanced" offsetting changes in the distributions of outcomes have preserved the dominance of group 2 over group 1. Yet the spread of that difference has shrunk, and this carefully constructed shrinking is appraised favorably by an opportunity-inequality averse UEVEU ethical observer.

The last elementary operation that we discuss is not related to reducing inequalities of opportunities. It is rather concerned with improving those opportunities for some, or all, of the groups (up to a permutation of them thanks to the anonymity principle). It is defined as follows.

Definition 8 (Expected Utility Improvement) *We say that* $\mathbf{q}$ *is obtained from* $\mathbf{p}$ *through an* anonymous and unanimous expected utility improvement *if there exists a one-to-one function* $\pi : \{1, ..., n\} \longrightarrow \{1, ..., n\}$ *such that for every* $i \in \{1, ..., n\}$*, there exists* $v_i \in \mathcal{U}_*^{\geq_{QO}}$ *for which one has:*

$$q_{\pi(i)} = p_{\pi(i)} + v_i$$

In words, $\mathbf{q}$ is obtained from $\mathbf{p}$ through an anonymous and unanimous expected utility improvement if one can find a permutation of the groups such that every permuted group in $\mathbf{q}$ faces opportunities that provide a larger expected utility than those faced in $\mathbf{p}$ for all lists of utility numbers compatible with the underlying quasi-ordering. An anonymous and unanimous expected utility improvement reduces to an anonymous and unanimous first-order stochastic increment in the case where the ranking of outcomes is complete. Any such anonymous and unanimous expected utility improvement will clearly be appraised favorably by any UEVEU ethical observer. In the following lemma, we establish formally that performing a favorable transfer *or* a uniform averaging are also elementary operation that are considered worth doing by those same ethical observers.

Lemma 1 *If* $\mathbf{q}$ *is obtained from* $\mathbf{p}$ *through either a* uniform averaging *or a* favorable transfer *then* $\mathbf{q} \succsim_{UEVEU}^{QO} \mathbf{p}$.

3.3 Evaluating distributions of opportunities in practice

The QOEZ dominance criterion proposed as a test for a consensus among all UEVEU ethical observers is in general a complex procedure that is not always immediately usable. In this subsection, we show how it can be checked empirically with a finite number of steps in many cases of interest. However, we also provide an example of two societies where it can be difficult indeed to use QOEZ dominance as an empirical test for the unanimity of all UEVEU ethical observers.

We start by introducing some additional notation. For any $h = 1, \ldots, n$, let $\{J_h^m,\ m = 1, ..., M(h)\}$ be the collection of all subsets of $\{1, ..., n\}$ of cardinality h.[4] Interestingly, given any society $\mathbf{p} \in \mathbb{S}$, it happens that the elements of the Zonotope $\mathbf{Z}(\mathbf{p})$ that also belong to the set $h\Delta^{k-1}$ (for any $h \in \{1, ..., n\}$ are the elements of the convex hull of $\left\{\sum_{i \in J_h^1} p_i, \ldots, \sum_{i \in J_h^{M(h)}} p_i\right\}$. We record this fact (proved in the Appendix) as follows.

Lemma 2 *For any* $h = 1, ..., n$ *we have*

$$Z(\mathbf{p}) \cap h\Delta^{k-1} = Co\left\{\sum_{i \in J_h^1} p_i, \ldots, \sum_{i \in J_h^{M(h)}} p_i\right\}$$

[4] Hence $M(h) = \binom{n}{h}$.

The sets $(Z(\mathbf{p}) + \mathcal{U}_*^{\geq_{QO}}) \cap h\Delta^{k-1}$ (for various h) will be called the *layers* of the extended zonotope $Z(\mathbf{p}) + \mathcal{U}_*^{\geq_{QO}}$ in the sequel. As it turns out, testing inclusion of the Zonotope on all its layers is somewhat simpler than testing the overall Zonotope inclusion even though the two procedures are equivalent, as established in the following lemma.

Lemma 3 *Given any quasi-ordering $\geq_{QO}$ on $\{1, ..., k\}$ and any societies $\mathbf{q}, \mathbf{p} \in \mathbb{S}$, $\mathbf{q} \succsim_Z^{QO} \mathbf{p}$ if and only if for all $h = 1, \ldots, n$ and $m = 1, \ldots, M(h)$ $\exists \tilde{p} \in Co\{\sum_{i \in J_h^1} p_i, \ldots, \sum_{i \in J_h^{M(h)}} p_i\}$ such that:*

$$\frac{1}{h} \sum_{i \in J_h^m} q_i - \frac{1}{h} \tilde{p} \in \mathcal{U}_*^{\geq_{QO}} \tag{6}$$

This lemma says that the test of QOEZ dominance of $\mathbf{q}$ over $\mathbf{p}$ amounts to testing that, for any collection of h groups, and for any weighted average of the probability distributions faced by each of those groups in $\mathbf{q}$, one can find a distribution $\tilde{p}$ in the layer $(Z(\mathbf{p}) + \mathcal{U}_*^{\geq_{QO}}) \cap h\Delta^{k-1}$ that is dominated by it according to the quasi-ordering. We now turn to two specific cases where this procedure leads to an easy verification of QOEZ dominance.

The first of them is when $\geq_{QO} = \geq_{\varnothing}$ (any ranking of outcomes is *a priori* possible). In that case, QOEZ dominance amounts to simple Zonotope inclusion, which can be tested easily and finitely, as indicated in the following remark.

Remark 4 *For any societies $\mathbf{q}, \mathbf{p} \in \mathbb{S}$, $\mathbf{q} \succsim_Z^{\emptyset} \mathbf{p}$ if and only if $\overline{q} = \overline{p}$ and*

$$\sum_{i \in J_h^m} q_i \in Co\left\{ \sum_{i \in J_h^1} p_i, \ldots, \sum_{i \in J_h^{M(h)}} p_i \right\}, \ \forall h = 1, \ldots, n, \ \forall m = 1, \ldots, M(h). \tag{7}$$

While easy to verify, simple zonotope inclusion is unlikely to be observed in practice. For one thing, Remark 4 says nothing on the ranking of any two societies with differing average opportunities. And for those with the same average opportunities, the instances of actual zonotope inclusion are likely to be rather exceptional.

A second class of situations of greater empirical interest that give rise to easy verifications of QOEZ dominance arises when the distributions faced by the different groups can be ordered within each society by the quasi-ordering. In the Indian case discussed in this paper, the various gender and caste based groups that we consider are all comparable by first order stochastic dominance. This case has also been considered in some of the empirical literature on measurement of equality of opportunity (see e.g. Andreoli, Havne, and Lefranc (2019) and Lefranc, Pistolesi, and Trannoy (2009)). If the distributions faced by the different groups can all be compared by the quasi-ordering - which can of course be the complete ordering - then testing for Zonotope inclusion is extremely simple, as shown in the following remark.

Remark 5 *For any quasi-ordering $\geq_{QO}$ on $\{1, ..., k\}$, and any societies $\mathbf{q}$ and $\mathbf{p} \in \mathbb{S}$ such that $q_{i+1} - q_i \in \mathcal{U}_*^{\geq_{QO}}$ and $p_{i+1} - p_i \in \mathcal{U}_*^{\geq_{QO}}$ for all $i = 1, ..., k-1$,* *then* $\mathbf{q} \succsim_Z^{QO} \mathbf{p}$ *if and only if, for any* $h = 1, \ldots, n$

$$\frac{1}{h}\sum_{i=1}^{h} q_i - \frac{1}{h}\sum_{i=1}^{h} p_i \in \mathcal{U}_*^{\geq_{QO}} \tag{8}$$

The test underlying Remark 5 is reminiscent of the sequential logic underlying generalized Lorenz dominance. Indeed, when applied to two societies where the groups can be unambiguously ordered - as per the quasi-ordering - from the worst to the best, the test works as follows. One first compares the expected utility associated to the distributions faced by the worst group in the two societies for all relevant list of utility numbers. If these distributions are not comparable, then the test fails and the two societies are not comparable. If one distribution dominates the other, then one compares the symmetric average of the distributions faced by the two worst groups of two societies. If those symmetric average are not comparable or if they give rise to an opposite ranking than that observed for the worst group, then the two societies are not comparable. In the other case, one then check for the symmetric average of the three worst groups and so on. When applied to the complete ordering of outcomes, the procedure described in Remark 5 bears some relation with the *test of the orthants* proposed by Dardanoni (1993) in the context of mobility measurement. Since groups are ordered from the worst-off ($i = 1$) to the best-off ($i = n$) and realizations from the least ($j = 1$) to the most ($j = k$) desirable, the criterion above requires that $\frac{1}{h}\sum_{i=1}^{h}\sum_{j=1}^{\ell} p_{ij} \geq \frac{1}{h}\sum_{i=1}^{h}\sum_{j=1}^{\ell} q_{ij}$ for any $h = 1, \ldots, n$ and $\ell = 1, \ldots, k$, which implies that worse off groups have systematically larger chances, on average, of attaining less desirable realizations in $\mathbf{p}$ compared to $\mathbf{q}$. Dardanoni (1993) has proposed the orthant test in the context of ranking monotone (squared) mobility matrices with fixed margins. The test in Remark 5 can be seen as an extension of these results to the case in which distributions do not have the same margins and where possibly $n \neq k$.

The easy implementation of QOEZ dominance provided by Remark 5 will be used extensively in the empirical section below.

There are, however, cases where an "easy" procedure for verifying QOEZ dominance is not readily available. The following example provides one such a case.

Example 1 *Assume that* $\geq_{QO} = \geq_C$ *and consider the following societies* $\mathbf{p}$ *and* $\mathbf{q}$*:*

$\mathbf{q} =$

	a	b	c	d
group 1	0.025	0.375	0.35	0.25
group 2	0.15	0.2	0.35	0.3
group 3	0.1	0.35	0.2	0.35

; $\mathbf{p} =$

	a	b	c	d
group 1	0	0.3	0.6	0.1
group 2	0	0.6	0	0.4
group 3	0.3	0	0.3	0.4

.

Using Lemma 3, we can establish that $\mathbf{q} \succsim_Z^C \mathbf{p}$*. Since* $\overline{q} \succsim^{1st} \overline{p} = (0.1, 0.3, 0.3, 0.3)$*, we only have to exhibit six stochastic dominances:*

$$q_1 = \frac{1}{2}p_1 + \frac{1}{3}p_2 + \frac{1}{6}p_3 + (-0.025, 0.025) \succsim^{1st} \frac{1}{2}p_1 + \frac{1}{3}p_2 + \frac{1}{6}p_3;$$

$$q_2 = \frac{1}{3}p_1 + \frac{1}{6}p_2 + \frac{1}{2}p_3;$$

$$q_3 = \frac{1}{6}p_1 + \frac{1}{2}p_2 + \frac{1}{3}p_3;$$

$$q_1 + q_2 \succsim^{1st} \frac{5}{6}p_1 + \frac{1}{2}p_2 + \frac{2}{3}p_3;$$

$$q_1 + q_3 \succeq^{1st} \frac{2}{3}p_1 + \frac{5}{6}p_2 + \frac{1}{2}p_3$$

$$q_2 + q_3 = \frac{1}{2}p_1 + \frac{2}{3}p_2 + \frac{5}{6}p_3$$

This shows that $\mathbf{q} \succsim_Z^C \mathbf{p}$. *However, it would be difficult to identify the candidates that would be stochastically dominated by the various combinations of sums of the* q_i*'s. Take for instance the first dominance. The set*

$$\left\{\tilde{p} \in Co(p_1, p_2, p_3) : \ q_1 \succsim^{1st} \tilde{p}\right\} = \{\lambda p_1 + \mu p_2 + (1 - \lambda - \mu)p_3 : \ \lambda \geq 1/2, \beta \geq 1/3, \lambda + \mu \leq 11/12\}$$

is a convex subset of the interior of $Co(p_1, p_2, p_3)$. *It is far from clear how an element of this set can be find a priori by a finite procedure based on the verification of a finite number of inequalities.*

3.4 Societies with two groups

A case of significant practical interest is when there are only two groups (say low caste and high caste adults). In such a case, we can implement QOEZ dominance by the finite, and somewhat simple, procedure of "majorization", by each of the two distribution of opportunities of the dominating society, of some weighted average of the two distributions of the dominated society exactly in the spirit of Lemma 3. While Example 1 shows how Lemma 3 can be sometimes difficult to apply if there are more than two groups, the difficulty vanishes if there are only two groups. To see how the procedure of Lemma 3 works in this case, consider the family $\mathcal{F}^{\geq_{QO}}$ of sets whose elements form a chain with respect to the quasi-ordering $\geq$. This family is formally defined by:

$$\mathcal{F}^{\geq_{QO}} = \{J \subset \{1, ..., k\} : h \in J \text{ and } j \geq_{QO} h \Longrightarrow j \in J\}$$

This family is closely related to the dual cone $\mathcal{U}_*^{\geq_{QO}}$ of the quasi-ordering $\geq_{QO}$ which can indeed be defined, thanks to Proposition 4, by:

$$\mathcal{U}_*^{\geq_{QO}} = \left\{ v \in \mathbb{R}^k : \sum_{h=1}^{k} v_h = 0 \,\text{and} \sum_{h \in J} v_h \geq 0 \text{ for all } J \in \mathcal{F}^{\geq_{QO}} \right\} \tag{9}$$

The family $\mathcal{F}^{\geq_{QO}}$ is important because it provides the complete (and finite) list of sets of outcomes whose increases in the likelihood are indisputably perceived as improving opportunities. For example, if the quasi-ordering $\geq_{QO}$ is taken to be $\geq_C$(the case where the outcomes are completely ordered), the family $\mathcal{F}^{\geq_{QO}}$would consist in the (anti) cumulated lists of outcome $\{k\}$, $\{k - 1, k\}$,...,$\{1, 2, ..., k\}$ used to check for first order stochastic dominance. For any probability distribution $p \in \Delta^{k-1}$, and any $J \in \mathcal{F}^{\geq_{QO}}$, we let $p(J)$ denote the "cumulated" probability of achieving an outcome in that set defined by $p(J) = \sum_{j \in J} p_j$. The majorization procedure that we propose as a test for QOEZ dominance between two-group societies works as follows. For any two such societies, one first checks if the average distribution of opportunities is better in one society than in the other for all expected utility criteria compatible with the quasi-ordering of outcomes. If no such dominance is observed, then we know from Proposition 2 that the two societies cannot be compared by QOEZ dominance and the test is over. If, on the other hand, such a dominance is observed, then the society with the dominating average is a candidate for being a dominating society thanks to Remark 2. To verify that it is indeed so, one looks, for *each* of the two distributions of opportunities in the (possibly) dominating society, at all the mixtures of the two distributions of opportunities in the (possibly) dominated society that yield the same probability of reaching outcomes in some members of $\mathcal{F}^{\geq_{QO}}$. There may not be any such mixtures in which case one concludes in the absence of dominance. If there are, however, such mixtures, then the verdict of dominance would be obtained if each of the two distributions of opportunities in the (possibly) dominating society dominates at least one such mixture of the two distributions in the dominated society.

The following theorem describes this procedure and shows its equivalence to QOEZ dominance

Theorem 2 *Suppose that $n = 2$. Let $\mathbf{\Lambda}_i$ (for $i = 1, 2$) be defined by*[5]

$$\mathbf{\Lambda}_i = \{1\} \cup \{\lambda \in [0,1] : \exists J \in \mathcal{F}^{\geq_{QO}} \text{ s.t. } q_i(J) = \lambda p_1(J) + (1-\lambda)p_2(J)\}.$$

Then $\mathbf{q} \succsim_Z^{QO} \mathbf{p}$ if and only if $\overline{q} - \overline{p} \in \mathcal{U}_*^{\geq_{QO}}$ and there are $\lambda_i \in \mathbf{\Lambda}_i$ (for $i = 1, 2$) such that $q_1 - (\lambda_1 p_1 + (1-\lambda_1)p_2) \in \mathcal{U}_*^{\geq_{QO}}$ and $q_2 - (\lambda_2 p_1 + (1-\lambda_2)p_2) \in \mathcal{U}_*^{\geq_{QO}}$.

It may be useful to appreciate the simplicity of the procedure described by this theorem through an example of two societies made of two groups where the dominance of one society over the other is not immediately apparent.

Example 2 *Consider the two following societies:*

$\mathbf{p} =$

	1	2	3	4
group 1	16/36	4/36	6/36	10/36
group 2	13/36	3/36	12/36	8/36

and

$\mathbf{q} =$

	1	2	3	4
group 1	16/36	2/36	8/36	10/36
group 2	13/36	5/36	9/36	9/36

Assume that the quasi-ordering of outcomes is the complete ordering $\geq_C$. Observe that

$$\overline{q} - \overline{p} = \frac{1}{72}(0, 0, -1, 1) \in \mathcal{U}_*^{\geq_{QO}},$$

which means that $q_2 + q_1 = p_2 + p_1 + v$, where $v = \frac{1}{36}(0, 0, -1, 1)$. In (almost) plain English, the distribution $\overline{q}$ stochastically dominates the distribution $\overline{p}$. Hence $\mathbf{q}$ is possibly a society that dominates society $\mathbf{p}$ for the criterion $\mathbf{q} \succsim_Z^{QO} \mathbf{p}$. Let us use the procedure described in Theorem 2 to verify that this is indeed the case. The family $\mathcal{F}^{\geq_C}$ here is defined by $\mathcal{F}^{\geq_C} =$

[5] Adding value 1 to this set might seem arbitrary there; the reason for doing so will be made clear in the proof of Theorem 2. One could alternatively decide to add 0 instead of 1, and modify the proof accordingly. Note that these two sets are then finite and non-empty.

$\{\{1,2,3,4\},\{2,3,4\},\{3,4\},\{4\}\}$. *The sets* Λ_1 *and* Λ_2 *are therefore respectively defined as the union of the singleton* $\{1\}$ *and the sets of solutions, in the* $[0,1]$ *interval, of the following equations:*

$$\begin{aligned}
10/36 &= \lambda_{11}10/36 + (1-\lambda_{11})8/36 \Rightarrow \lambda_{11} = 1 \\
18/36 &= \lambda_{12}16/36 + (1-\lambda_{12})20/36 \Rightarrow \lambda_{12} = 1/2 \\
20/36 &= \lambda_{13}20/36 + (1-\lambda_{13})23/36 \Rightarrow \lambda_{13} = 1
\end{aligned}$$

for Λ_1 *and of the equations:*

$$\begin{aligned}
9/36 &= \lambda_{21}10/36 + (1-\lambda_{21})8/36 \Rightarrow \lambda_{21} = 1/2 \\
18/36 &= \lambda_{22}16/36 + (1-\lambda_{22})20/36 \Rightarrow \lambda_{22} = 1/2 \\
23/36 &= \lambda_{23}20/36 + (1-\lambda_{23})23/36 \Rightarrow \lambda_{23} = 0
\end{aligned}$$

for Λ_2. *We thus have* $\Lambda_1 = \{1/2, 1\}$ *and* $\Lambda_2 = \{0, 1/2, 1\}$. *Since* q_1 *1st-order stochastically dominates* p_1*, we have* $q_1 - (\lambda p_1 + (1-\lambda)p_2 \in \mathcal{U}_*^{\geq QO}$ *for* $\lambda = 1 \in \Lambda_1$. *One can also observe that*

$$q_2 = \left(\frac{26}{72}, \frac{10}{72}, \frac{18}{72}, \frac{18}{72}\right)$$

1st order stochastically dominates the mixture of p_1 *and* p_2 *given by:*

$$\frac{p_1}{2} + \frac{p_2}{2} = \left(\frac{29}{72}, \frac{7}{72}, \frac{18}{72}, \frac{18}{72}\right)$$

Hence $\mathbf{q} \succsim_Z^C \mathbf{p}$.

Remark 6 *Interestingly, this example also illustrates that the three aforementioned elementary operations that are considered worth doing by all UEVEU ethical observers are not the only ones that have this property. Indeed, it is not possible to go from* $\mathbf{p}$ *to* $\mathbf{q}$ *by a finite sequence of Uniform averaging, bilateral equalizing transfers and Anonymous expected utility improvements. That no equalizing transfers can be performed to go from* $\mathbf{p}$ *to* $\mathbf{q}$ *is clear since none of the two distributions of opportunities* p_1 *and* p_2 *first order stochastically dominate the other. One can also see that no uniform averaging operation, however small, can be done. Indeed, for any* $\lambda \in [0,1[$, $q_1 - (\lambda p_1 + (1-\lambda)p_2) \notin \mathcal{U}_*^{\geq C}$. *This is so because the probability of achieving the worst outcome for the first group in society* $\mathbf{q}$ *is strictly larger than any mixture of the probabilities of achieving that worst outcome the two groups in society* $\mathbf{p}$ *(*$q_{11} = 16/36 > \lambda 16/36 + (1-\lambda)13/36$ *for all* $0 \leq \lambda < 1$*). Finally, we can show that there is no margin to perform an anonymous and unanimous utility improvement however small on the initial society* $\mathbf{p}$ *in a way that preserve dominance of* $\mathbf{q}$ *over the transformed* $\mathbf{p}$. *See the appendix for the proof.*

There is, however, a particular - but theoretically important - case where two of the three types of the elementary transformations considered in the preceding subsection coincide with the QOEZ dominance criterion. This case is when the two societies offer the same average opportunities to the two groups and therefore only differ in the inequality with which this common average opportunity is split between the two groups. In this case, QOEZ dominance actually coincides with the possibility of going from the dominated to the dominating distribution by a finite sequence of equalizing transfers and uniform averaging operations. The following theorem establishes that fact.

Theorem 3 *Suppose that $n = 2$ and $\overline{p} = \overline{q}$. The three following statements are equivalent:*

1. $\mathbf{q}$ *is obtained from* $\mathbf{p}$ *through a uniform averaging or an equalizing transfer;*
2. $\mathbf{q} \succsim_{UEVEU}^{QO} \mathbf{p}$;
3. $\mathbf{q} \succsim_{Z}^{QO} \mathbf{p}$.

The equivalence established in Theorem 3 *between* the domination of a two-group society by another in terms by the QOEZ criterion *and* the possibility of going from the dominated to the dominating society by either an equalizing transfer or a uniform averaging operation when the average distribution of opportunities is the same provides a simple way to check for dominance in that case. This is at least so if one focuses on the case where the outcomes are completely ordered and where, as a result, the dual cone of the set of lists of utility numbers $(u_1, ..., u_k)$ that are increasing with respect to outcomes is the set of changes v that generates a first-order dominance between distributions. In that case, one can observe the following (obtained as an immediate consequence of Theorem 3 or of Lemma 3 applied to 2 groups).

Remark 7 *Suppose that $\overline{p} = \overline{q}$ and $n = 2$. Assume that either $p_1 \succsim^{1st} p_2$ or $p_2 \succsim^{1st} p_1$. Consider the indexing i_1 and i_2 of the two groups such that $p_{i_2} \succsim^{1st} p_{i_1}$. Then $\mathbf{q} \succsim_{Z}^{C} \mathbf{p}$ if and only if $p_{i_2} \succsim^{1st} q_{i_1} \succsim^{1st} p_{i_1}$ and $p_{i_2} \succsim^{1st} q_{i_2} \succsim^{1st} p_{i_1}$.*

This remark leads itself to a very simple test of opportunity equalization in the two-group case, at least when the average distribution of opportunities is the same, and when one group in one society is stochastically dominated by the other. The test amounts to verifying if, in the other society, the distributions of opportunities of two groups lie "in between" those of the two groups in terms of first order stochastic dominance.

4 Empirical Illustration

While still lagging behind most large economies in the world - including its Chinese neighbor - India has undoubtedly witnessed a significant educational development in the last three decades, that is illustrated on Figures 2-4. The average national literacy rate, that was a mere 16.67% in the first census of independent India in 1951, has reached 64.32% in the 2011 census. Figures 2-4 make however also clear that this undisputable improvement in education has affected unequally the various segments of the hierarchic Indian society. As a result, the educational opportunities offered nowadays to Indian citizens depend heavily upon their caste and gender (among many other characteristics). The aim of this section is to appraise the extent of this dependence by means of the criteria proposed in the previous section. Specifically, we compare the performance of the 14 most populated Indian states in allocating these educational opportunities to their inhabitants on the basis of castes and genders.

It is well-known that casteism in India is a millennium rooted social hierarchical structure that was initially based on occupations, but became gradually hereditary. The traditional Indian society, as described in ancient texts such as the *Manusmriti* (literally, "Manu's laws), was hierarchically divided into four

Figure 2: Adult literacy rate by gender in 1983 and 2012.

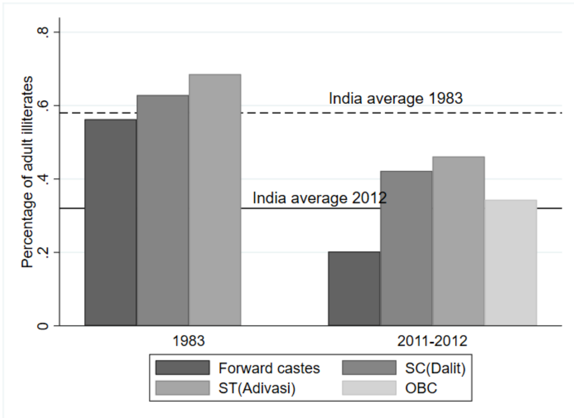

Figure 3: Adult literacy rate by caste group, 1983 and 2012.

major castes, called *Varna*. At at the top of the hierarchy were the '*Brahmins*' (priests) who are related to the most sacred profession of worshiping deities and teaching. They are followed by the '*Khatriyas*' (soldiers), the '*Vaishyas*' (traders) and the '*Shudras*' (servants), where the *Shudras* are almost exclusively occupied to serve the other three upper castes. Outside these four categories, one find also the most marginalized social division of '*Ati-Shudras*' who were entirely secluded from the mainstream Indian society and, in many cases, are considered as 'untouchables' by members of other castes because of their engagement in some of the most 'impure' (but necessary) jobs like burning corpses and manual scavenging. This broad division by occupation gives rise to many further subdivisions in thousands of sub-castes (called *Jati*) with in-built hierarchies that tend to differ greatly across regions of the country. While India has banned the practice of casteism, its government has put into place several corrective affirmative action policies aiming at overcoming the discriminations that people coming from lower caste categories are still massively experiencing. The Indian government has regrouped for this purpose the thousands of caste

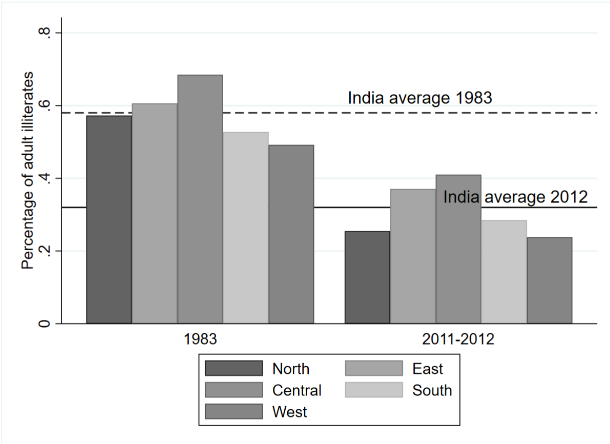

Figure 4: Adult literacy rates by region, 1983 and 2012.

categories into a broad classification that serves as a basis for its corrective policies. At present India has four such caste categories - the Scheduled Tribes (ST or *Adivasi*), the Scheduled Castes (SC or *Dalits*), the Other Backward Classes (OBC) and the forward castes (who are referred as the 'general' category). SC and ST are the most underprivileged caste categories in India and were also among the very first ones to benefit from caste-based affirmative policies. OBC on the other hand was formed later around mid 1980s as a group of socially and economically backward castes who do not fail into the SC/ST categories. However unlike SC/ST, the composition of OBC keeps on changing over the years, to respond duly to the continuous increased demand of many castes who consider themselves as 'socially and economically backward'. The 'general' forward castes are all those that do not belong to any of the other three categories and are excluded from any caste-based benefits.

Important as it is, caste is not the only determinant of inequality of educational achievement in India. Another source of such inequality is gender, as illustrated on Figure 2. Reducing the gender gap in schooling is a major challenge in India as in many other developing part of the world, and has been considered as one of the eight United Nation Millennium Development goals in the last decade. Several policies have been put in place in India to increase girl school attendance at both primary and secondary school and, therefore, to improve women's education level. Examples would include cash transfers to parents (Dhaliwal, Duflo, Glennester, and Tulloch (2013)), free meals at schools (see e.g. Jayamaran and Simroth (1976)) or, in Bihar since 2006, providing girls with bicycles (Muralidharan and Prakash (2017)). While the overall progress in education that India has experienced in the last thirty years has affected both males and females (see Figure 2), it does not seem to have reduced much the discrepancy between the two genders. It is also unclear how the gender inequality in educational opportunities has evolved among castes. With few exceptions - notably Deshpande (2007) and Munshi and Rosenzweig (2006) - there has been relatively few studies that have analyzed educational inequalities in the caste-gender nexus. Deshpande (2007) finds gender educational inequality - measured by some index - to be more important for the low-caste adults than

Education Group	Education Level	years of formal education
1	Illiterate	0
2	literate without formal education	0
3	below primary (4 years or less)	between 1 and 4
4	below secondary (5-10 years)	between 5 and 10
5	below graduate	between 11 and 16
6	graduate and above	more than 16

Table 1: Description of the levels of educational achievement

for the high-castes adults. However Saha (2013) observes the within-caste distribution of education expenditure to be more male-skewed for the upper caste households than for the SC/ST households. This latter finding is in line with the analysis of Munshi and Rosenzweig (2006) who concludes, in the case of Mumbay, that lower caste girls obtain better schooling than boys who tend to be quickly removed from school by their parents and sent to work in traditional low-paid job exploiting the within-caste mutual networks, from which girls do not benefit much because of the legacy of low rate female labor market participation. Hence, the caste-gradient of the gender inequalities in educational opportunities in India is far from clear. It is therefore of some interest to examine empirically with the criteria discussed in the previous sections how these caste and gender inequalities in educational opportunities have evolved over time and across the 14 most populated states of India.

For this sake, we use the earliest and the latest available rounds of the Employment Unemployment survey of the National Sample Survey (NSS) micro database, corresponding to survey years 1983 (round 38) and 2011-2012 (round 68)[6]. This survey records information on education for every member of the household. We however limit our illustration to all Indian adults aged between 30 to 40 years who are currently not attending any educational institution, so as to focus on the prime working age population of the country. We have regrouped the given education levels in 6 mutually exclusive and exhaustive groups, where illiteracy is considered as the worst possible level and having a graduate degree or above is considered as the best one. The interpretation of this regrouping is summarized in 1.

The localization of the 14 states in the main regions of India is shown in Table 2. For the purpose of analyzing castes, we gather SC and ST together as the 'lower caste group' (SC/ST) and everyone else (Non-SC/ST) as the 'upper caste group'.

The distribution of the 30-40 years old adult population between the caste and the gender categories is provided in Table 3. As can be seen, with the noticeable exception of the union territory of Delhi, distribution of the population between males and females is rather symmetric. The relatively low fraction of female adults in the 30-40 years old category in Delhi is difficult to explain. As expected, the distribution of this population between upper and lower caste groups shows some inter-state difference, with Odisha having a high fraction

[6]For 1983, the survey was spanned from the month of January to December of the same year. But as NSS changes the survey year to 'agricultural year', the latest round was surveyed between July 2011 to June 2012.

North	East	Central	South	West
Haryana [HR]	West Bengal [WB]	Uttar Pradesh [UP]	Tamil Nadu [TN]	Maharastra [MH]
Delhi [DL]	Odisha [OD]	Madhya Pradesh [MP]	Karnataka [KT]	Gujrat [GJ]
	Bihar [BH]	Rajasthan [RJ]	Kerala [KR]	
			Andhra Pradesh [AN]	

Table 2: Geographical region of Indian states

	% Male	% Female	% Upper caste	% Lower caste	Sample size
Sample control			age: 30-40 years		
UP	0.49	0.51	0.80	0.20	7911
MH	0.50	0.50	0.80	0.20	6540
BH	0.49	0.51	0.83	0.17	4118
WB	0.49	0.51	0.70	0.30	4977
MP	0.51	0.49	0.69	0.31	4188
TN	0.49	0.51	0.82	0.18	4634
RJ	0.49	0.51	0.71	0.29	3369
KT	0.50	0.50	0.80	0.20	3422
GJ	0.50	0.50	0.79	0.21	2950
AN	0.49	0.51	0.80	0.20	5121
OD	0.49	0.51	0.63	0.37	3427
KR	0.45	0.55	0.90	0.10	3114
HR	0.51	0.49	0.78	0.22	2338
DL	0.54	0.46	0.76	0.24	797

Table 3: Sample summary, Selected Indian states, 2012

(37%) of lower caste adults and Kerala having a low (10%) one.

Figure 5 shows the box plot of the distribution of educational level in the 30-40 years old population in the selected states and union territories of India for 1983 and 2012 (thick shaded areas corresponding to the education levels of the first and the fourth quartiles of the distribution and the white bar corresponding to the median). The spectacular - but unequal across states - improvement in education observed in India in the thirty last years is quite visible on this picture. Progress have been particularly impressive in Rajasthan and Uttar Pradesh. In Rajasthan for instance, 75% of the 30-40 years old population had no formal education in 1983 and more than 50% of this population was illiterate. In 2012, more than 50% of the same population has achieved at least a level of primary education. Progress in Uttar Pradesh have been even more spectacular since 50% of its 30-40 years old population have at least a high school degree in 2012 while the corresponding proportion in 1983 was inferior to 25%. Figure 5 also indicates the rather equal and favorable distribution of education levels in Kerala and the significant and extremely favorable evolution of the distribution of education levels in the union territory of Delhi (where in 2012 75% of its 30-40 years old population has at least a high school degree). The differences among states remain however quite striking, with Andhra Pradesh, Bihar and Rajasthan having half their population with no more than a primary school diploma and Delhi having half of its population with at least a college degree.

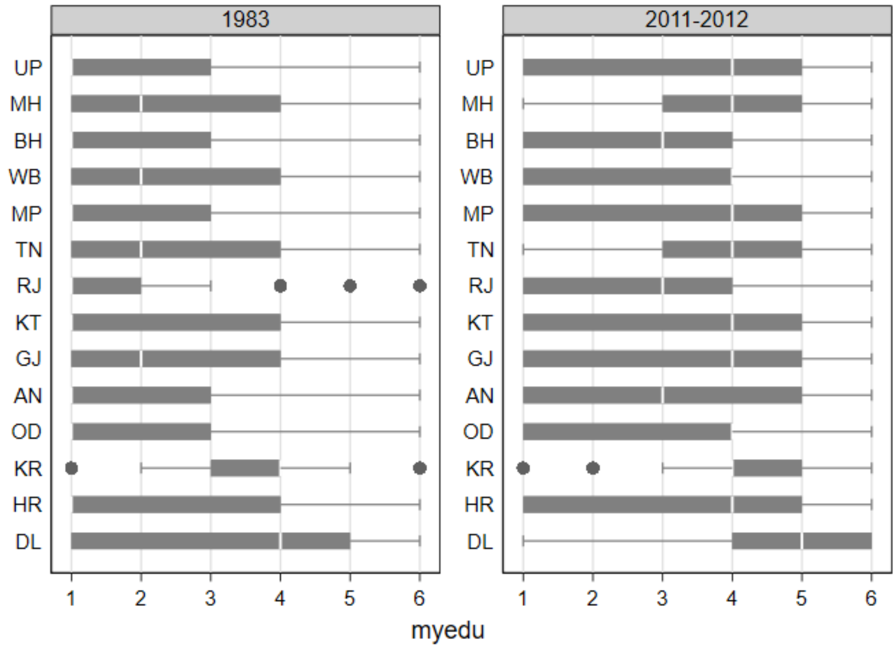

Figure 5: Box plots of the distribution of educational levels in the selected states, 1983 and 2012.

We now turn to the comparisons of the states on the basis of their allocation, between the two caste groups, of their educational opportunities. It should come as no surprise that in every state and union territory, the distribution of education levels in upper caste 30-40 years old adults dominates at the first order the corresponding distribution for their lower-caste counterpart. One can then refer to Remark 5 of the preceding section and check for possible QOEZ dominance among states. This amounts to verifying if the distribution of education levels among lower caste 30-40 years adults in one state stochastically dominates at the first order the corresponding distribution in another state and if this dominance carries over to the (symmetric among the two caste groups) average distribution of education levels observed in the two states. Table 4 provides this dominance matrix[7] for all considered states, with the $\prec_A$ and $\prec_Z$ (resp. $\succ_A$and $\succ_Z$) symbols indicating a 1st-order dominance of the row state by (resp. over) the column state for the symmetric average and the lower caste

[7] The null of first order stochastic dominance of one type over another is tested by the difference in their respective empirical distributions at all levels of education. The test statistic for education level j and groups (r, r') is - $\frac{F(\hat{j},r)-F(\hat{j},r')}{\sqrt{\frac{V(F(\hat{j},r))}{N_r}+\frac{V(F(\hat{j},r'))}{N_{r'}}}}$. The statistical inference is drawn on the basis of the union-intersection criteria proposed by Bishop, Formby, and Thistle (1992) that rejects the dominance of group-r over group-r' if at least one of the test statistics are significantly negative and none of them are significantly positive, where statistical significance is based on the Studentized Maximum Modulus distribution of the test-statistics.

	MH	BH	WB	MP	TN	RJ	KT	GJ	AN	OD	KR	HR	DL
UP	$(\prec_A, \prec_Z)$	$\neq_Z$	$(\prec_A, \prec_Z)$	$(\prec_A, \prec_Z)$	$(\prec_A, \prec_Z)$	$\neq_Z$	$(\prec_A, \prec_Z)$	$\neq_Z$	$(\prec_A, \prec_Z)$	$(\prec_A, \prec_Z)$	$(\prec_A, \prec_Z)$	$(\prec_A, \prec_Z)$	$(\prec_A, \prec_Z)$
MH	-	$(\succ_A, \succ_Z)$	$(\succ_A, \succ_Z)$	$(\succ_A, \succ_Z)$	$(\succ_A, \succ_Z)$	$(\succ_A, \succ_Z)$	$(\succ_A, \succ_Z)$	$(\succ_A, \succ_Z)$	$(\succ_A, \succ_Z)$	$(\succ_A, \succ_Z)$	$\neq_Z$	$\neq_Z$	$\neq_Z$
BH	-	-	$(\prec_A, \prec_Z)$	$(\prec_A, \prec_Z)$	$(\prec_A, \prec_Z)$	$\neq_Z$	$(\prec_A, \prec_Z)$	$(\prec_A, \prec_Z)$	$(\prec_A, \prec_Z)$	$(\prec_A, \prec_Z)$	$(\prec_A, \prec_Z)$	$(\prec_A, \prec_Z)$	$(\prec_A, \prec_Z)$
WB	-	-	-	$(\succ_A, \succ_Z)$	$\neq_Z$	$(\succ_A, \succ_Z)$	$(\prec_A, \prec_Z)$	$\neq_Z$	$\neq_Z$	$(\succ_A, \succ_Z)$	$\neq_Z$	$\neq_Z$	$(\prec_A, \prec_Z)$
MP	-	-	-	-	$\neq_Z$	$(\succ_A, \succ_Z)$	$(\prec_A, \prec_Z)$	$\neq_Z$	$\neq_Z$	$\boldsymbol{(\prec_A, \not\prec_Z)}$	$(\prec_A, \prec_Z)$	$(\prec_A, \prec_Z)$	$(\prec_A, \prec_Z)$
TN	-	-	-	-	-	$(\succ_A, \succ_Z)$	$\neq_Z$	$(\succ_A, \succ_Z)$	$\neq_Z$	$(\succ_A, \succ_Z)$	$(\prec_A, \prec_Z)$	$\neq_Z$	$\neq_Z$
RJ	-	-	-	-	-	-	$(\prec_A, \prec_Z)$	$\neq_Z$	$(\prec_A, \prec_Z)$	$\boldsymbol{(\prec_A, \not\prec_Z)}$	$(\prec_A, \prec_Z)$	$(\prec_A, \prec_Z)$	$(\prec_A, \prec_Z)$
KT	-	-	-	-	-	-	-	$(\succ_A, \succ_Z)$	$(\succ_A, \succ_Z)$	$(\succ_A, \succ_Z)$	$\neq_Z$	$\neq_Z$	$(\prec_A, \prec_Z)$
GJ	-	-	-	-	-	-	-	-	$\neq_Z$	$\neq_Z$	$\neq_Z$	$\neq_Z$	$(\prec_A, \prec_Z)$
AN	-	-	-	-	-	-	-	-	-	$\neq_Z$	$\boldsymbol{(\prec_A, \not\prec_Z)}$	$\boldsymbol{(\prec_A, \not\prec_Z)}$	$(\prec_A, \prec_Z)$
OD	-	-	-	-	-	-	-	-	-	-	$(\prec_A, \prec_Z)$	$\neq_Z$	$(\prec_A, \prec_Z)$
KR	-	-	-	-	-	-	-	-	-	-	-	$\neq_Z$	$\neq_Z$
HR	-	-	-	-	-	-	-	-	-	-	-	-	$(\prec_A, \prec_Z)$

Table 4: QOEZ dominance matrix among states, caste-based allocations of opportunities, 2012

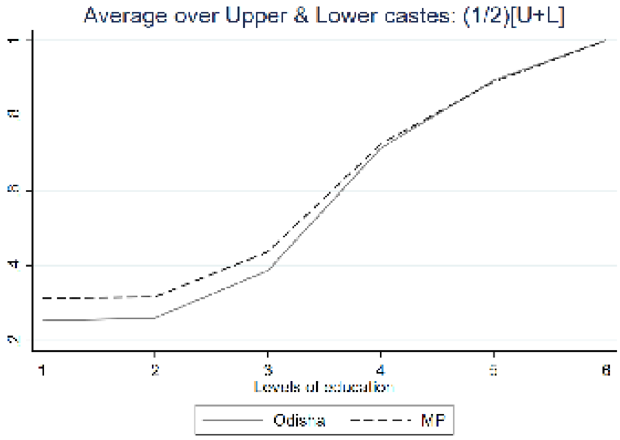

Figure 6: Symmetric across caste average cumulative distributions of education levels in Odisha and Madhya Pradesh, 2012.

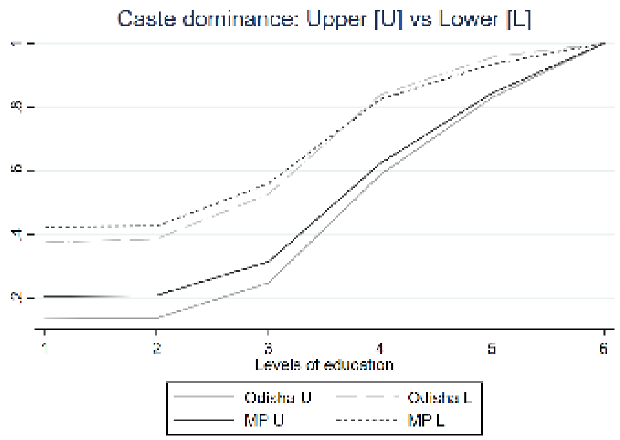

Figure 7: Cumulative distributions of education levels in Upper and Lower castes in Odisha and Madhya Pradesh, 2012.

distribution respectively and the $\neq$ indicating non-comparability - by 1st-order stochastic dominance - of the symmetric average distribution and, therefore, non-comparability by QOEZ dominance). This table shows therefore that there is statistically significant unanimity among all UEVEU educational opportunity averse ethical observers over the ranking of a significant majority - 57 - of all the 91 distinct possible pairs of states. While many of these rankings are more the results of the overall educational opportunitiesoffered by the states than of the unequal sharing of those among castes, caste-based inequalities consideration do play a role in explaining some of the comparisons. This is clear for the comparisons that appear in bold type on Table 4. Two of them, illustrated in Figures 6-9, concern Odisha whose (symmetric among the two caste groups)

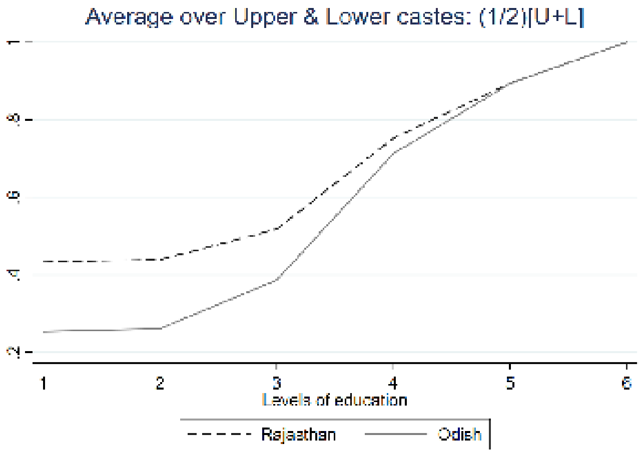

Figure 8: Symmetric across castes average of the cumulative distribution of education levels, Odisha and Rajasthan, 2012.

average distribution of education levels first-order dominates those of Mad-

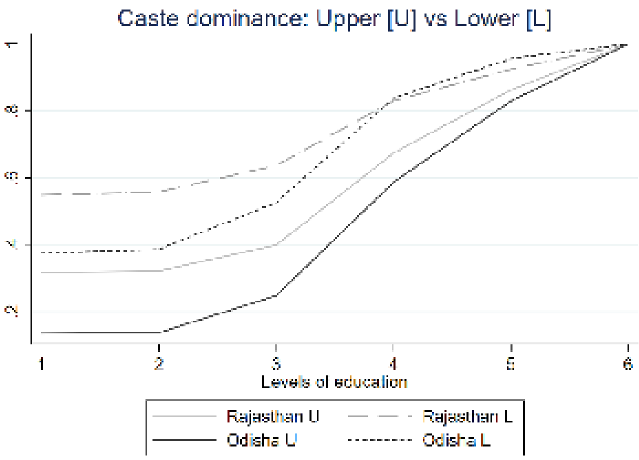

Figure 9: Cumulative distributions of education levels in upper and lower castes in Odisha and Rajasthan, 2012.

hya Pradesh and Rajasthan (the crossing at the top of the curves is not statistically significant) but who *does not* dominate those two states because of the failure of the distribution of education levels of its low caste group to dominate its counterpart in those states (see the crossing in figures 7 and 9). The caste-based inequality of educational opportunity in Odisha is too important to overcome the relatively good average performance of this state.A similar phenomenon is observed for Kerala *vis-à-vis* Andhra Pradesh (see Figures 10 and 11). This may seem as a surprise to those who see Kerala as the "success story"

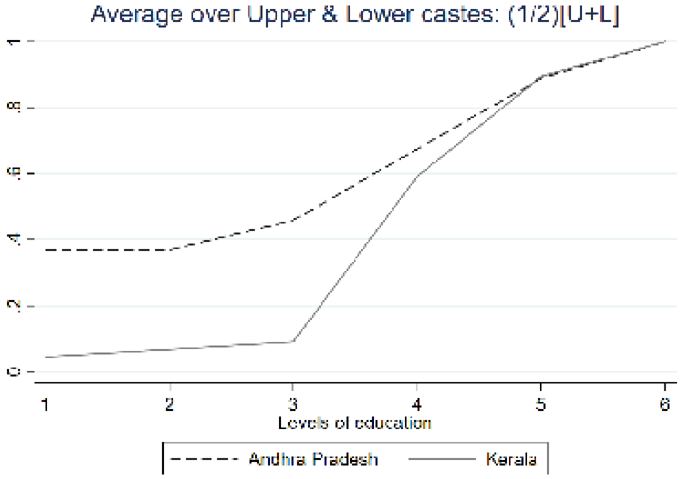

Figure 10: Symmetric across castes average cumulative distributions of education levels, Andhra Pradesh and Kerala, 2012.

of India insofar as health and education are concerned. Yet the overall good performance of Kerala in providing educational opportunities to its inhabitants hides important inequalities between caste groups (that were also noticed, albeit with different tools, by Deshpande (2000) some time ago). As it turns out, the members of the low caste in Kerala do not benefit from educational opportunities to the extent that is sufficient to dominate their counterpart in Andhra Pradesh. *A contrario*, Andhra Pradesh can be seen as a somewhat good performer in so far as caste-based equality of opportunities is concerned. Indeed, while this state does not offer particularly good overall educational opportunities, it does so in a way that does not exhibit large between caste disparities. This therefore prevents it from being QOEZ dominated by Kerala and Haryana.

We now introduce gender as an additional source of (ethically arbitrary) differentiation. Given the illustrative nature of our empirical exercise, we limit our analysis to three pairs of states that are associated with high (Kerala and Maharashtra), medium (Odisha and West Bengal) and low (Andhra Pradesh and Rajasthan) literacy levels. Introducing gender along with caste thus creates four groups: Low Caste Females (LCF), Low Caste Males (LCM), High Caste Females (HCF) and High Caste Males (HCM). The first interesting to note is that while the distributions of education levels of these four groups can be ordered by stochastic dominance in every state, the ordering of those distributions differ across states. In all states, one finds a clear dominance of LCM over LCF. Gender inequality of educational opportunity is therefore a definite feature of low caste populations irrespective of their state. While a similar inequality is often observed in upper caste groups, it is *not* observed in Kerala (see Figures 12-17 at the end of the Appendix). Indeed, in this state, upper caste women face better- as per first-order dominance - educational opportunities than upper caste men. Another noticeable interstate differences is the extent by which caste is a more important source of inequality of educational opportunity than gender. In Odisha and West Bengal, the distribution of educational opportu-

		MH	WB	OD	AN	RJ
KR	(worst type)	$\succ$	$\succ$	$\succ$	$\succ$	$\succ$
	(1/2)(two worst types)	$\succ$	$\nsucc$	$\succ$	$\succ$	$\succ$
	(1/3)(three worst types)	$\nsucc$	$\cdot$	$\succ$	$\succ$	$\succ$
	(1/4)(all four types)	$\cdot$	$\cdot$	$\succ$	$\nsucc$	$\succ$
	Verdict	$\nsucc_Z$	$\nsucc_Z$	$\succ_Z$	$\nsucc_Z$	$\succ_Z$
MH	(worst type)	...	$\succ$	$\succ$	$\succ$	$\succ$
	(1/2)(two worst types)	...	$\nsucc$	$\nsucc$	$\succ$	$\succ$
	(1/3)(three worst types)	...	$\cdot$	$\cdot$	$\succ$	$\succ$
	(1/4)(all four types)	...	$\cdot$	$\cdot$	$\succ$	$\succ$
	Verdict	...	$\nsucc_Z$	$\nsucc_Z$	$\succ_Z$	$\succ_Z$
WB	(worst type)	...	...	$\succ$	$\succ$	$\succ$
	(1/2)(two worst types)	...	...	$\succ$	$\succ$	$\succ$
	(1/3)(three worst types)	...	...	$\succ$	$\nsucc$	$\succ$
	(1/4)(all four types)	...	...	$\succ$	$\cdot$	$\succ$
	Verdict	...	...	$\succ_Z$	$\nsucc_Z$	$\succ_Z$
OD	(worst type)	...	...	...	$\nsucc$	$\succ$
	(1/2)(two worst types)	...	...	...	$\cdot$	$\succ$
	(1/3)(three worst types)	...	...	...	$\cdot$	$\nsucc$
	(1/4)(all four types)	...	...	...	$\cdot$	$\cdot$
	Verdict	...	...	...	$\nsucc_Z$	$\nsucc_Z$
AN	(worst type)	...	...	...	...	$\succ$
	(1/2)(two worst types)	...	...	...	...	$\succ$
	(1/3)(three worst types)	...	...	...	...	$\succ$
	(1/4)(all four types)	...	...	...	...	$\succ$
	Verdict	...	...	...	...	$\succ_Z$

Table 5: Caste and Gender dominance, Selected states, 2012

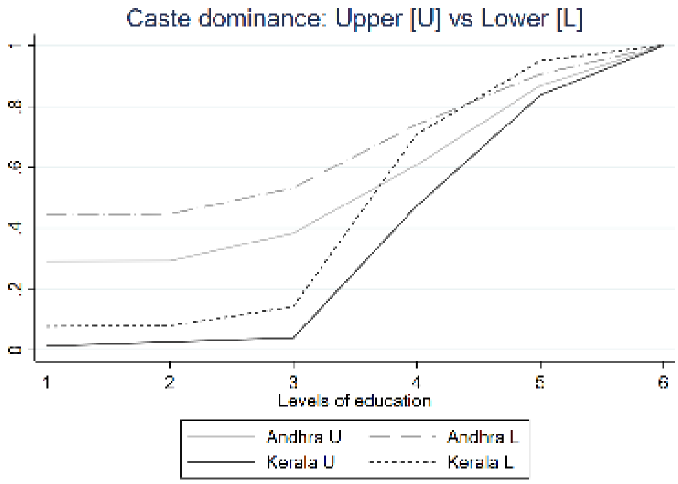

Figure 11: Cumulative distributions of education levels in upper and lower castes in Andhra Pradesh and Kerala, 2012.

nities happens to be better for HCF than for LCM while the opposite ranking holds for the states of Andhra Pradesh, Maharashtra and Rajasthan. Hence, in those three states, women of high caste groups have access to lower educational opportunities than men of low caste groups. One can then consider these states to be more discriminatory against women than they are against low caste. By contrast, Kerala, Odisha and West Bengal are more biased against low caste than again women.

How is the ranking of those states by all UEVEU ethical observers affected by adding genders to castes as a basis for group definitions ? While several such rankings remain unaffected as a result of this addition, one observes nonetheless interesting differences. One of them concerns the relatively prosperous Maharashtra, in which is located the financial capital of the country (Mumbay). As noticed above, this state is characterized by significant gender inequalities of educational opportunity that make high caste women worst than low caste males. As it happens, this bad performance of Maharashtra in terms of gender inequality of opportunities leads to a failure of this state to dominate West Bengal and Odisha, while this dominance was observed when caste was considered as the only source of differing opportunities. A similar loss of dominance is observed for Kerala vis-à-vis Odisha. This latter lost can be explained here by the extreme inter caste inequality of Kerala that is to some extent exacerbated when genders are taken into consideration. Indeed, the severe discrepancies between high caste women in Kerala (the best group of the state) and low caste women (the worst group of the state), when combined with the no-less severe disparities observed between high and low caste males, are so important that they prevent this state from dominating the less educated, but somewhat more equal state of Odisha.

5 Conclusion

This paper provides an operational definition of what it means for an allocation - between groups- of opportunities - defined as probabilities of achieving outcomes of interest - to be better than another for a reasonably large spectrum of ethical point of views emanating from UEVEU ethical observers. The operational definition that we provided, and which is shown to coincide with the unanimity of all rankings emanating from UEVEU ethical observers who exhibit aversion to inequality of opportunities, is the test of extended zonotope inclusion. The zonotope set of an allocation of opportunities is the convex hull of all partial sums, over groups, of their probability distributions. The extended zonotope of an allocation of opportunities is nothing else than its zonotope set translated by some transformations of the distributions of outcomes that are considered worth doing given some a priori ranking of the outcomes. According to this criterion therefore, undisputable improvements in allocations of opportunities are associated with a shrinking - in the sense of set inclusion - of the associated extended zonotope set. The paper also shows how this Zonotope inclusion test can be made extremely simple in many cases of interest. Among the cases considered are those where there are only two groups between which the opportunities are equalized, and those where the number of groups is arbitrary, but where the groups can all be ordered in terms of the expected utility associated to the probability distribution faced by members, given the a priori ranking of outcomes. However the paper also provides case where the verification of extended zonotope inclusion may be difficult to verify. The paper also identifies elementary transformations of the allocation of opportunities that are considered worth doing by the extended zonotope inclusion criterion and, in the highly specific case where there are two groups between which the same average distribution over outcome is allocated, it identifies them exactly. Last, but not least, the paper illustrates the usefulness of the zonotope inclusion criterion to compare allocation of educational opportunities in a few Indian states based on gender and caste. The empirical analysis in particular emphasizes the importance of between-caste and gender inequality for appraising the varying achievements of those states in terms of opportunities for education that they provide to their inhabitants. The analysis has, among other things, shown that the good average educational performance of Kerala hides important between caste inequalities of opportunities that prevent this state from dominating many others. In the same vein, the analysis has also exhibit significant gender inequalities of opportunities in Maharashtra that also prevent this wealthy and well-educated state to dominate others.

These findings open many ways to future research. One of them is the performing of further empirical analysis. As shown in this paper, the extended zonotope inclusion criterion is a test that can be easily implemented in many empirical cases of interest. It is therefore of importance that this test be implemented in all the contexts that require a robust appraisal of opportunity inequality or, more generally, opportunity misallocation. Another closely related avenue of research would the axiomatic identification of numerical indices that could complete the incomplete comparisons of allocation of opportunities provided by the extended zonotope inclusion criterion, while being compatible with it. From a theoretical view point, it would be also be nice to identify precisely what are the elementary transformations of the allocations of opportunities that lie behind extended zonotope inclusion. While these elementary transformations have been identified in the simple case where there are two groups who share the same symmetric average distributions over outcome, they do not suffice to characterize extended zonotope when there are more than two groups and/or when the average distribution over outcome differs. Another line of inquiry that would be worth pursuing is the identification of simple finite procedures for verifying extended zonotope inclusion that apply to all logically conceivable cases. While Example 1 suggests that such endeavour may be difficult, it is worth in our view giving it another try.

References

Andreoli, F., T. Havne, and A. Lefranc (2019): "Robust Inequality of Opportunities Comparisons: Theory and Application to Early Childhood Policy Evaluation," *Review of Economics and Statistics*, 101, 355–369.

Atkinson, A. B. (1981): "The Measurement of Economic Mobility," in *Inkomensverdeling en Openbare Financien*, ed. by P. Eggelshoven, and L. van Gemerden. Het Spectrum.

Bénabou, R., and E. A. Ok (2001): "Mobility as Progressivity:Ranking income process according to Equality of Opportunity," NBER working paper 8431.

Berge, C. (1959): *Espaces topologiques et fonctions multivoques*. Dunod, Paris.

Bishop, C. M., J. P. Formby, and P. Thistle (1992): "Convergence of the South and the Non-South Income Distributions, 1969-1979," *American Economic Review*, 82, 262–272.

Brunori, P., F. Ferreira, and V. Peragine (2021): "Prioritarism and Equality of Opportunity," in *Prioritarism in Practice*, ed. by M. Adler, and O. Norheim. Cambridge University Press, Cambridge, UK.

Dardanoni, V. (1993): "Measuring Social Mobility," *Journal of Economic Theory*, 61, 372–394.

Deshpande, A. (2000): "Does caste still define disparity ? A look at inequality in Kerala, India," *American Economic Review*, 90, 322–325.

——— (2007): "Overlapping identities under liberalization: gender and caste in India," *Economic Development and Cultural Change*, 55, 735–760.

Dhaliwal, I., E. Duflo, R. Glennester, and C. Tulloch (2013): "Comparative Cost-Effectiveness Analysis to Inform Policy in Developing Countries: A General Framework with Applications for Education," in *Education Policies in Developping Countries*, ed. by P. Glewwe, pp. 285–338. University of Chicago Press, Chicago.

Donaldson, D., and J. A. Weymark (1998): "A Quasi-Ordering is the Intersection of Orderings," *Journal of Economic Theory*, 78, 382–387.

Dworkin, R. (1981): “What is Equality ? Part 1: Equality of Welfare, Part 2: Equality of Ressources,” *Philosophy and Public Affairs*, 10, 185–246.

Ferreira, F. H. G., and J. Gignoux (2011): “The Measurement of Equality of Opportunities: Theory and an Application to Latin America,” *The Review of Income and Wealth*, 57, 622–657.

Fleurbaey, M. (2010): “Assessing Risky Social Situations,” *Journal of Political Economy*, 118, 649–680.

——— (2018): “Welfare Economics, Risk and Uncertainty,” *Canadian Journal of Economics*, 51, 5–40.

Gravel, N., T. Marchant, and A. Sen (2011): “Comparing Societies with Different Numbers of Individuals on the Basis of their Average Advantage,” in *Social Ethics and Normative Econonomics: Essays in Honour of Serge-Christophe Kolm*, ed. by M. Fleurbaey, M. Salles, and J. A. Weymark, pp. 261–277. Springer Verlag.

——— (2012): “Uniform Utility Criteria for Decision Making under Ignorance or Objective Ambiguity,” *Journal of Mathematical Psychology*, 56, 297–315.

Heath, R., and S. Jayachandran (2018): “The causes and consequences of Increase Female Education and Labor Force Participation in Developping Countries,” in *The Oxford Handbook of the Economics of Women*, ed. by S. L. Averett, L. M. Argys, and S. D. Hauffman. New York, Oxford University Press.

Jayamaran, R., and D. Simroth (1976): “The impact of school lunches on primary school enrollment: evidence from India’s midday meal scheme,” *The Scandinavian Journal of Economics*, 117, 1176–1203.

Kolm, S. C. (1977): “Multidimensional Egalitarianisms,” *Quarterly Journal of Economics*, 91, 1–13.

Koshevoy, G. (1995): “Multivariate Lorenz majorization,” *Social Choice and Welfare*, 12, 93–102.

——— (1998): “The Lorenz zonotope and multivariate majorizations,” *Social Choice and Welfare*, 15, 1–14.

Koshevoy, G., and K. Mosler (1996): “The Lorenz zonoid of a multivariate distribution,” *Journal of the American Statistical Association*, 91, 873–882.

——— (2007): “Multivariate Dominance Basded on Zonoids,” *AStA Advances in Statistical Analysis*, 91, 57–76.

Lefranc, A., N. Pistolesi, and A. Trannoy (2009): “Equality of Opportunity and Luck: Definition and Testable Condition with an application to income in France,” *Journal of Public Economics*, 93, 1189–1207.

Lefranc, A., and A. Trannoy (2017): “Equality of opportunity, moral hazard and the timing of luck,” *Social Choice and Welfare*, 49, 469–497.

Mariotti, M., and R. Veneziani (2017): "Opportunities as Chances: Maximizing the probability that everybody succeeds," *Economic Journal*, 128, 1609–1633.

Martinez, M., E. Schockkaert, and D. VandeGaer (2001): "Three Meanings of Intergenerational Mobility," *Economica*, 68, 519–537.

Munshi, K. (2019): "Caste and the Indian Economy," *Journal of the Economic Literature*, 57, 781–834.

Munshi, K., and M. Rosenzweig (2006): "Traditional Institutions meet the Modern World: Caste, Gender and Schooling Choice in a globalized economy," *American Economic Review*, 96, 1225–1252.

Muralidharan, K., and N. Prakash (2017): "Cycling to School: Increasing Secondary School Enrollment for Girls in India," *American Economic Journal: Applied Economics*, 9, 321–350.

Neumann, J. V., and O. Morgenstern (1947): *Theory of Games and Economic Behavior*. Princeton University Press, Princeton, NJ, USA.

Peragine, V. (2004): "Ranking income distributions according to equality of opportunity," *Journal of Economic Inequality*, 2, 11–30.

Rockafellar, R. T. (1970): *Convex Analysis*. Princeton University Press, Princeton, NJ, USA.

Roemer, J. E., and A. Trannoy (2016): "Equality of Opportunity: Theory and Measurement," *Journal of Economic Literature*, 54, 1288–1332.

Saha, A. (2013): "An Assessment of Gender Discrimination in Schooling Expenditures on Education in India," *Oxford Development Studies*, 41, 220–238.

Shorrocks, A. F. (1983): "Ranking Income Distributions," *Economica*, 50, 3–17.

——— (1984): "Inequality Decomposition by Population Subgroups," *Econometrica*, 52, 1369–1386.

Vallentyne, P. (2002): "Brute Luck, Option Luck and Equality of Initial Opportunities," *Ethics*, 112, 529–557.

6 Appendix: Mathematical notation and Proofs

6.1 Mathematical Notation

The (possibly) non standard mathematical notations and definitions used in this paper are as follows. The set of integers and real numbers are denoted by $\mathbb{N}$ and $\mathbb{R}$ respectively, while the set of non-negative (resp. strictly positive) integers and real numbers are denoted respectively by $\mathbb{N}_+$ (resp. $\mathbb{N}_{++}$), and $\mathbb{R}_+$ (resp. $\mathbb{R}_{++}$). The $m-$fold Cartesian product of a set $\mathbf{A}$ (for any strictly positive integer m) is denoted by $\mathbf{A}^m$and the cardinality of a set $\mathbf{A}$ (when this set is finite) is denoted by $\#\mathbf{A}$. Our notation for vectors inequalities is $\geqq$, $\geq$ and $>$. The $k-1$ dimensional simplex in $[0,1]^k$ (for any $k \in \mathbb{N}_+\backslash\{1\}$) is denoted by Δ^{k-1} and is defined by $\Delta^{k-1} = \{(p_1,...,p_k) \in [0,1]^k : \sum_{h=1}^{k} p_h = 1\}$ and the convex hull of a collection of n vectors $\{v^1,...,v^n\}$ in $\mathbb{R}^k$ is denoted by $Co\{v^1,...,v^n\}$and is defined by $Co\{v^1,...,v^n\} = \{(x_1,...,x_k) \in \mathbb{R}^k : x_h = \sum_{i=1}^{n} \lambda_i v_h^i$ for all $h = 1,..,k$ for some $(\lambda_1,...,\lambda_n) \in \Delta^{n-1}\}$. A matrix $\mathbf{m} \in \mathbb{R}_+^{nk}$ (with representative element $m_{ih} \geq 0$ for $i = 1,...,n$ and $h = 1,...,k$) is row-stochastic if it satisfies $\sum_{h=1}^{k} m_{ih} = 1$ for all i and is bistochastic if it is row-stochastic and satisfies, for every h, $\sum_{i=1}^{n} m_{ih} = 1$. A permutation matrix π is a bistochastic matrix that satisfies the additional property that $\pi_{ih} \in \{0,1\}$ for every $i = 1,...,n$ and $h = 1,...,k$. The dot product of matrices $\mathbf{a} \in \mathbb{R}^{nk}$and $\boldsymbol{b} \in \mathbb{R}^{km}$ (for any three strictly positive integers k, m and n) is denoted by $\mathbf{a}.\mathbf{b}$ (where $\mathbf{a}.\mathbf{b} \in \mathbb{R}^{nm}$ is the matrix whose typical element $a.b_{ij}$ (for $i = 1,...,n$ and $j = 1,...,m$) is defined by $a.b_{ij} = \sum_{h=1}^{k} a_{ih} b_{hj}$). Given any vector $v \in \mathbb{R}^k$ (viewed as a matrix in $\mathbb{R}^{k1}$or, alternatively, a column vector), we denote by $v_{(.)}$ its ordered permutation defined by: $v_{(.)} = \pi.v$ for some permutation matrix $\pi \in \mathbb{R}^{kk}$such that $v_{(i)} \leq v_{(i+1)}$ for every $i = 1,..,k-1$. A function $h : \mathbf{A} \to \mathbb{R}$ where $\mathbf{A}$ is a subset of $\mathbb{R}^k$ is increasing if $a \geqq b$ for a and $b \in \mathbf{A}$ implies $h(a) \geq h(b)$, is concave if $h(\lambda a + (1-\lambda)b) \geq \lambda h(a) + (1-\lambda)h(b)$ for every $\lambda \in [0,1]$ and every a,$b \in \mathbf{A}$ and is Schur-concave if for every $a \in \mathbf{A}$, and every bistochastic matrix $\mathbf{b} \in \mathbb{R}^{kk}$, $h(\mathbf{b}.a) \geq h(a)$. Given two vectors u and $v \in \mathbb{R}^k$, we say that u weakly Lorenz dominates v if the inequality $\sum_{g=1}^{h} u_{(g)} \geq \sum_{g=1}^{h} v_{(g)}$ holds for all $h = 1,...,k$ and we say that u strictly Lorenz dominates v if u weakly Lorenz dominates v and v does not weakly Lorenz dominate u. By a binary relation $\succsim$ on a set $\mathbf{\Omega}$, we mean a subset of $\mathbf{\Omega} \times \mathbf{\Omega}$. Following the convention in economics, we write $x \succsim y$ instead of $(x,y) \in \succsim$. Given a binary relation $\succsim$, we define its symmetric factor $\sim$ by $x \sim y \iff x \succsim y$ and $y \succsim x$ and its asymmetric factor $\succ$ by $x \succ y \iff x \succsim y$ and not $(y \succsim x)$. A binary relation $\succsim$on $\mathbf{\Omega}$ is reflexive if the statement $x \succsim x$ holds for every x in $\mathbf{\Omega}$, is transitive if $x \succsim z$always follows $x \succsim y$ and $y \succsim z$ for any $x, y, z \in \mathbf{\Omega}$, is complete if $x \succsim y$ or $y \succsim x$ holds for every distinct x and y in $\mathbf{\Omega}$ and is antisymmetric if $x \succsim y$ and $y \succsim x$ implies $x = y$ for any two x and y in $\mathbf{\Omega}$. A reflexive, transitive and complete binary relation is called an ordering and a reflexive and transitive binary relation is called a quasi-ordering.

6.2 Proofs

6.2.1 Remark 1.

Let $\mathbf{C}$ be the vector sub-space of $\mathbb{R}^k$, generated by the vector $(1,...,1)$. Observe that $\mathbf{C}$ is a convex cone, and is contained in $\mathcal{U}^{\geq QO}$. Hence, by standard results, $\mathcal{U}_*^{\geq QO} \subset \mathbf{C}_* = \left\{(v_1,...,v_k) \in \mathbb{R}^k : \sum_{j=1}^{k} v_k = 0\right\}$.

Before turning to the proof of the other results, we state and prove the following lemma that will be extensively used in the sequel.

Lemma 4 *For every* $\mathbf{p} \in \mathbb{S}$*, one has* $Z(\mathbf{p}) = Co\left(\{\sum_{i=1}^{n} \alpha_i p_i : (\alpha_1, ..., \alpha_n) \in \{0,1\}^n\}\right)$.

Proof. *Proof. Let* $\theta_1, ..., \theta_n \in [0,1]^n$*, and suppose without loss of generality that* $\theta_1 \leq \theta_2 \leq ... \leq \theta_n$*. Then*

$$\sum_{i=1}^{n} \theta_i p_i = \theta_1 \sum_{i=1}^{n} p_i + \sum_{l=2}^{n} ((\theta_l - \theta_{l-1}) \sum_{i=l}^{n} p_i) + (1 - \theta_l)\mathbf{0}.$$

The right-hand side of the equality being a convex combination of the set $\{\mathbf{0}, \sum_{i=l}^{n} p_i : \ l = 1, ..., n\}$*, we get the result.* ■

6.2.2 Theorem 1.

We first show that Statement 2 of Theorem 1 implies Statement 1 of that theorem and, therefore, that $Z(q) + \mathcal{U}_*^{\geq QO} \subseteq Z(p) + \mathcal{U}_*^{\geq QO}$. Since $\mathcal{U}_*^{\geq QO}$ is a cone, it amounts to showing that $Z(q) \subseteq Z(p) + \mathcal{U}_*^{\geq QO}$. By Lemma 4, it is sufficient to show that, for any $\alpha_1, ..., \alpha_n \in \{0,1\}^n$, there exists $v \in \mathcal{U}_*^{\geq QO}$ and $\theta_1, ..., \theta_n \in [0,1]^n$ such that:

$$\sum_{i=1}^{n} \alpha_i q_i = \sum_{i=1}^{n} \theta_i p_i + v. \tag{10}$$

Note that since $\sum_{j=1}^{k} v_j = 0$ (by Remark 1) and p_i and q_i both belong to Δ^{k-1}, we necessarily have $\sum_{i=1}^{n} \theta_i = m$, where $m = \#\{i : \alpha_i = 1\}$. Hence, by re-indexing the distributions q_i (for $i = 1, ..., n$) in such a way that $\alpha_i = 1$ for $i = 1, ..., m$, Expression (10) can be equivalently written as:

$$\frac{1}{m} \sum_{i=1}^{m} q_i = \sum_{i=1}^{n} \frac{\theta_i}{m} p_i + \frac{1}{m} v.$$

Let the set $\mathbf{D}$ be defined by:

$$\mathbf{D} := \overline{q} - Co\{p_1, ..., p_n\}$$

We need to show that $\mathbf{D} \cap \mathcal{U}_*^{\geq QO} \neq \varnothing$. Suppose by contradiction that $\mathbf{D} \cap \mathcal{U}_*^{\geq QO} = \varnothing$. Since $\mathbf{D}$ is a convex polytope[8] and $\mathcal{U}_*^{\geq QO}$ is a closed convex cone, one can conclude from Theorem 2 at p. 80 of Berge (1959) that there are vectors $(d_1^*, ...d_k^*) \in \mathbf{D}$ and $(v_1^*, ...v_k^*) \in \mathcal{U}_*^{\geq QO}$ such that:

$$\left(\sum_{h=1}^{k} (d_h^* - v_h^*)^2\right)^{1/2} = \min_{(d_1,...,d_k) \in D, (v_1,...,v_k) \in \mathcal{U}_*^{\geq QO}} \left(\sum_{h=1}^{k} (d_h - v_h)^2\right)^{1/2}$$

by continuity of the Euclidian norm, and using the fact that the set $\mathbf{D} \times \mathcal{U}_*^{\geq QO}$ on which it is minimized can be made compact by taking a suitable intersection of $\mathcal{U}_*^{\geq QO}$ with some closed ball in $\mathbb{R}^k$. Define the vector $(\widehat{v}_1, ..., \widehat{v}_k)$ by $\widehat{v}_h = v_h^* - d_h^*$ for $h = 1, ..., k$.

[8] seeRockafellar (1970), p. 12. A convex polytope is the convex hull of a finite family of points, called the *vertices* or *extreme points* of this set.

Then the hyperplane passing through $(v_1^*, ...v_k^*)$ and orthogonal to $(\widehat{v}_1, ..., \widehat{v}_k)$ strongly separates $\mathbf{D}$ and $\mathcal{U}_*^{\geq QO}$ in the sense that:

$$\inf_{(v_1,....,v_k)\in\mathcal{U}_*^{\geq QO}} \sum_{h=1}^{k} v_h\widehat{v}_h \geq \sum_{h=1}^{k} v_h^*\widehat{v}_h > \sup_{(d_1,...,d_k)\in D} \sum_{h=1}^{k} d_h\widehat{v}_h \tag{11}$$

Since $(0, ..., 0) \in \mathcal{U}_*^{\geq QO}$ one must have that:

$$0 \geq \sum_{h=1}^{k} v_h^*\widehat{v}_h$$

Moreover since $(\lambda v_1^*, ..., \lambda v_k^*) \in \mathcal{U}_*^{\geq QO}$ for every number $\lambda > 0$, one must also have that:

$$\sum_{h=1}^{k} v_h^*\widehat{v}_h \geq 0$$

Indeed, assuming $\sum_{h=1}^{k} v_h^*\widehat{v}_h < 0$ would be contradictory, after taking a suitably large λ, with the strict inequality (11). These two last inequalities enable therefore one to rewrite Inequality (11) more precisely as:

$$\inf_{(v_1,....,v_k)\in\mathcal{U}_*^{\geq QO}} \sum_{h=1}^{k} v_h\widehat{v}_h \geq 0 > \sup_{(d_1,...,d_k)\in D} \sum_{h=1}^{k} d_h\widehat{v}_h \tag{12}$$

By the first of these two inequalities, we conclude that $(\widehat{v}_1, ..., \widehat{v}_k)$ belongs to the dual cone of the set $\mathcal{U}_*^{\geq QO}$, which is itself the dual cone of the set $\mathcal{U}^{\geq QO}$. By the bipolar theorem for convex cones (see for example Theorem 14.1 in Rockafellar (1970)), it therefore follows that the dual cone of $\mathcal{U}_*^{\geq QO}$is $\mathcal{U}^{\geq QO}$ so that $(\widehat{v}_1, ..., \widehat{v}_k) \in \mathcal{U}^{\geq QO}$. Now since Statement 2 of the theorem holds, we know that the inequality

$$\sum_{i=1}^{n} \Phi\left(\sum_{h=1}^{k} q_{ih}u_h\right) \geq \sum_{i=1}^{n} \Phi\left(\sum_{h=1}^{k} p_{ih}u_h\right)$$

holds for all concave Φ and all lists of real numbers $(u_1, ..., u_k) \in \mathcal{U}^{\geq QO}$ .By the Hardy-Littlewood-Polya theorem (see for example Berge (1959), p. 191), this is equivalent to the requirement that the list of n numbers

$$\left(\sum_{h=1}^{k} q_{1h}u_h, ..., \sum_{h=1}^{k} q_{nh}u_h\right)$$

Lorenz dominates the list of $n$numbers:

$$\left(\sum_{h=1}^{k} p_{1h}u_h, ..., \sum_{h=1}^{k} p_{nh}u_h\right)$$

for all list of real numbers $(u_1, ..., u_k) \in \mathcal{U}^{\geq QO}$. In particular this is true for $(\hat{u}_1, ..., \hat{u}_k)$, and thus there exists an indexing $i_1(\widehat{u}), ..., i_n(\widehat{u})$[9] such that:

$$\sum_{h=1}^{k} p_{i_1(\widehat{u})h}\widehat{u}_h \leq \sum_{h=1}^{k} p_{i_2(\widehat{u})h}\widehat{u}_h \leq ... \leq \sum_{h=1}^{k} p_{i_n(\widehat{u})h}\widehat{u}_h$$

[9](which depends of course upon the k-tuple $(\widehat{u}_1, ..., \widehat{u}_k)$)

and:

$$\sum_{i=1}^{m}\sum_{h=1}^{k}q_{ih}\widehat{u}_h \geq \sum_{j=1}^{m}\sum_{h=1}^{k}p_{i_j(\widehat{u})h}\widehat{u}_h. \tag{13}$$

However, by the second inequality of Expression (12),we have (remembering the definition of $\mathbf{D}$):

$$0 > \sum_{h=1}^{k}\overline{q}_h\widehat{u}_h - \sum_{h=1}^{k}p_{ih}\widehat{u}_h \tag{14}$$

for all $i = 1, ..., n$. It follows therefore from Inequalities (13) and (14) that:

$$\sum_{h=1}^{k}p_{ih}\widehat{u}_h > \sum_{h=1}^{k}\overline{q}_h\widehat{u}_h \geq \frac{1}{m}\sum_{j=1}^{m}\sum_{h=1}^{k}p_{i_j(\widehat{u})h}\widehat{u}_h, \text{ for } i = 1, ..., n,$$

which is not possible. This concludes the proof of the first implication.

Let us now prove the reverse implication. Suppose that Statement 1 of the Theorem holds and pick any $(u_1, ..., u_k) \in \mathcal{U}^{\geq QO}$. We must show, using again the Hardy-Littlewood-Polya theorem, that the list of n numbers

$$\left(\sum_{h=1}^{k}q_{1h}u_h, ..., \sum_{h=1}^{k}q_{nh}u_h\right)$$

Lorenz dominates the list of n numbers

$$\left(\sum_{h=1}^{k}p_{1h}u_h, ..., \sum_{h=1}^{k}p_{nh}u_h\right).$$

Without loss of generality (since the ranking of societies is anonymous), we can write the indices of the rows of the two matrices $\mathbf{q}$ and $\mathbf{p}$ in such a way that the two lists are increasingly ordered so that:

$$\sum_{h=1}^{k}q_{1h}u_h \leq ... \leq \sum_{h=1}^{k}q_{nh}u_h \text{ and} \sum_{h=1}^{k}p_{1h}u_h \leq \sum_{h=1}^{k}p_{nh}u_h.$$

Hence, we need to show that for any $n_0 \leq n - 1$,

$$\sum_{i=1}^{n_0}\sum_{h=1}^{k}p_{1h}u_h \leq \sum_{i=1}^{n_0}\sum_{h=1}^{k}q_{1h}u_h$$

Since statement 1 of the theorem holds, we know that there exists $v \in \mathcal{U}_*^{\geq QO}$ and $\theta_1, ..., \theta_n \in [0, 1]$ which can be in such a way that $\sum_{i=1}^{n}\theta_i = n_0 \leq n$ such that:

$$\sum_{i=1}^{n_0}q_i = \sum_{l=1}^{n}\theta_l p_l + v$$

It thus follows that:

$$\begin{aligned}
\sum_{i=1}^{n_0}\sum_{h=1}^{k} q_{1h}u_h &= \sum_{j=1}^{n}\theta_h\sum_{h=1}^{k}p_{jh}u_h + \sum_{h=1}^{k}v_h u_h \\
&\geq \sum_{j=1}^{n}\theta_h\sum_{h=1}^{k}p_{jh}u_h \text{ (since } v\in\mathcal{U}_*^{\geq_{QO}}) \\
&\geq \sum_{j=1}^{n_0}\theta_j\sum_{h=1}^{k}p_{jh}u_h + \sum_{g=n_0+1}^{n}\theta_g\sum_{h=1}^{k}p_{n_0h}u_h \text{ (rows are ordered)} \\
&= \sum_{j=1}^{n_0}\theta_j\sum_{h=1}^{k}p_{jh}u_h + \sum_{j=1}^{n_0}[1-\theta_j]\sum_{h=1}^{k}p_{n_0h}u_h \text{ (since } \sum_{j=1}^{n}\theta_i = n_0) \\
&\geq \sum_{j=1}^{n_0}\theta_j\sum_{h=1}^{k}p_{jh}u_h
\end{aligned}$$

as required.

6.2.3 Proposition 4

The fact that

$$\mathcal{U}_*^{\geq_{QO}} \subseteq \left\{(v_1,...,v_k)\in\mathbb{R}^k : \sum_{j=1}^{k}v_ju_j \geq 0 \ \forall(u_1,...,u_k)\in\mathcal{U}^{\geq_{QO}}\cap\{0,1\}^k\right\}$$

directly follows from the fact that $\mathcal{U}^{\geq_{QO}}\cap\{0,1\}^k \subset \mathcal{U}^{\geq_{QO}}$.

In order to prove the reverse inclusion, consider any $(v_1,...,v_k)$ satisfying $\sum_{h=1}^{k}v_h = 0$ and $\sum_{j=1}^{k}v_ju_j \geq 0$ for all $(u_1,...,u_k)\in\mathcal{U}^{\geq_{QO}}\cap\{0,1\}^k$. We must show that it satisfies also $\sum_{j=1}^{k}v_ju_j \geq 0$ for any $(u_1,...,u_k)\in\mathcal{U}^{\geq_{QO}}$. Consider therefore any such $(u_1,...,u_k)\in\mathcal{U}^{\geq_{QO}}$. By continuity of the map $(u_1,...,u_k)\mapsto\sum_{j=1}^{k}v_ju_j$, we may assume without loss of generality that $u_h \neq u_i$ for any two distinct h and i in $\{1,...,k\}$. Let $j:\{1,...,k\}\longrightarrow\{1,...,k\}$ be a one-to-one function such that such that $u_{j(1)} < u_{j(2)} < ... < u_{j(k)}$. We have:

$$\sum_{j=1}^{k}v_ju_j = \sum_{h=1}^{k}v_{j(h)}u_{j(h)} = \sum_{h=2}^{k}v_{j(h)}(u_{j(h)}-u_{j(1)}) \tag{15}$$

since $\sum_{h=1}^{k}v_h = 0$. Using Abel decomposition formula, one can alternatively write this equality as:

$$\sum_{j=1}^{k}v_ju_j = \sum_{h=2}^{k}(u_{j(h)}-u_{j(h-1)})\sum_{g=h}^{k}v_{j(g)}$$

Now, for any $h=2,...,k$, let $w^h\in\{0,1\}^k$ be defined by:

$$\begin{aligned} w^h_{j(g)} &:= 0 \text{ if } g<h \text{ and,} \\ &:= 1 \text{ if } g\geq h. \end{aligned}$$

We observe that, for any $h \in \{2,...,k\}$, $(w^h_{j(1)},...,w^h_{j(k)}) \in \mathcal{U}^{\geq_{QO}}$. Indeed, if $l >^{QO} g$ for two distinct outcomes g and l in $\{1,...,k\}$, then $u_l > u_g$ by definition of $(u_1,...,u_k) \in \mathcal{U}^{\geq_{QO}}$. Given h, three cases are possible:
(i) $l < h$. In this case, one has $w^h_{j(g)} = 0 = w^h_{j(l)}$ from the definition of w^h.
(ii) $g < h \leq l$. In this case, $w^h_{j(g)} = 0 < 1 = w^h_{j(l)}$ holds from the definition of w^h and the required weak inequality $w^h_{j(g)} \leq w^h_{j(l)}$ is also satisfied.
(iii) $h \leq g < l$. In this case $w^h_{j(g)} = 1 = w^h_{j(l)}$ holds from the definition of w^h.
Hence, in all the three cases, the required weak inequality $w^h_{j(g)} \leq w^h_{j(l)}$ is satisfied. Since $(w^h_{j(1)},...,w^h_{j(k)}) \in \mathcal{U}^{\geq_{QO}} \cap \{0,1\}^k$ for any $h = 2,...,k$, we have $\sum_{g=1}^k v_{j(g)} w^h_{j(g)} = \sum_{g=h}^k v_{j(g)} \geq 0$ for any such h. But this implies that $\sum_{h=2}^k v_{j(h)}(u_{j(h)} - u_{j(1)}) \geq 0$for any such h which, thanks to Equality (15), establishes the result.

6.2.4 Remark 2

Suppose that $q \succsim_Z^{QO} p$ and, as a result, that $Z(q) + \mathcal{U}_*^{\geq_{QO}} \subset Z(p) + \mathcal{U}_*^{\geq_{QO}}$. Since in particular $\sum_{i=1}^n q_i \in Z(q) + U_*^{\geq^{QO}}$, there is a collection of $n$numbers $\theta_1,...,\theta_n$in the $[0,1]$ interval and a vector $v \in \mathcal{U}_*^{\geq_{QO}}$such that:

$$\sum_{i=1}^n q_i = \sum_{i=1}^n \theta_i p_i + v.$$

or, writing this equality for outcome j :

$$\sum_{i=1}^n q_{ij} = \sum_{i=1}^n \theta_i p_{ij} + v_j.$$

Summing over all outcomes, and exploiting the fact that $\sum_{j=1}^k v_j = 0$ (Remark 1) and $\sum_{j=1}^k p_{ij} = \sum_{j=1}^k q_{ij} = 1$for any i) one has:

$$\sum_{j=1}^n \sum_{i=1}^n q_{ij} = n = \sum_{i=1}^n \theta_i \sum_{j=1}^n p_{ij} + \sum_{j=1}^n v_j = \sum_{i=1}^n \theta_i$$

which implies that $\theta_i = 1$ for all i. Hence:

$$\sum_{i=1}^n q_i = \sum_{i=1}^n p_i + v$$

and

$$\sum_{i=1}^n q_i - \sum_{i=1}^n p_i = v \in \mathcal{U}_*^{\geq_{QO}}$$

as required.

6.2.5 Proposition 5

Observing that the inequality:

$$\sum_{i=1}^n \Phi\left(\sum_{h=1}^k q_{ih} u_h\right) \geq \sum_{i=1}^n \Phi\left(\sum_{h=1}^k p_{ih} u_h\right)$$

holds for all concave $\Phi$and all lists of real numbers $u_1, ..., u_k$ is equivalent, thanks to the Hardy-Littlewood-Polya theorem (see for example Berge (1959) p. 191), to the requirement that the list of n numbers $(\sum_{h=1}^{k} q_{1h}u_h, ..., \sum_{j=1}^{k} q_{nh}u_h)$ Lorenz dominates the list of n numbers $(\sum_{h=1}^{k} p_{1h}u_h, ..., \sum_{h=1}^{k} p_{nh}u_h)$ for all list of real numbers $u_1, ..., u_k$. This latter requirement is in turn equivalent to the requirement that the matrix $\mathbf{q}$ price majorizes (using Kolm (1977) terminology) the matrix $\mathbf{p}$ for all "price" vectors $(u_1, ..., u_k)$. Koshevoy (1995) (Theorem 1) proves that the fact for a matrix $\mathbf{q} \in \mathbb{R}^{nd}$ to price majorize a matrix $\mathbf{p} \in \mathbb{R}^{nd}$ is equivalent to observing:

$$\begin{aligned}
\overline{Z}(\mathbf{q}) &= \left\{ \mathbf{z} \in \mathbb{R}^{k+1} : \ \mathbf{z} = \sum_{i=1}^{n} \theta_i \left(\frac{1}{n}, q_{i1}, ..., q_{ik} \right), \ \theta_i \in [0,1] \ \forall i = 1, \ldots, n \right\} \\
&\subseteq \left\{ \mathbf{z} \in \mathbb{R}^{k+1} : \ \mathbf{z} = \sum_{i=1}^{n} \theta_i \left(\frac{1}{n}, p_{i1}, ..., p_{ik} \right), \ \theta_i \in [0,1] \ \forall i = 1, \ldots, n \right\} \\
&= \overline{Z}(\mathbf{p})
\end{aligned}$$

Observe that the set $\overline{\mathbf{Z}}(\mathbf{a})$ (for any matrix $\mathbf{a} \in \mathbb{R}^{nd}$) defined in Koshevoy (1995) is somewhat similar to the set defined in Equation 5 above, with the exception that it takes the Minkowski sums over the "population share extended" vectors $(1/n, p_{i1}, ..., p_{ik})$ rather than over the vectors $(p_{i1}, ..., p_{ik})$ themselves. Hence we only need to prove that $\overline{\mathbf{Z}}(\mathbf{q}) \subseteq \overline{\mathbf{Z}}(\mathbf{p})$ is equivalent to $\mathbf{Z}(\mathbf{q}) \subseteq \mathbf{Z}(\mathbf{p})$ to complete the argument. The fact that $\overline{\mathbf{Z}}(\mathbf{q}) \subseteq \overline{\mathbf{Z}}(\mathbf{p})$ implies $\mathbf{Z}(\mathbf{q}) \subseteq \mathbf{Z}(\mathbf{p})$ is obvious. To establish the other direction assume that $\mathbf{Z}(\mathbf{q}) \subseteq \mathbf{Z}(\mathbf{p})$. This means that for any list of numbers $\theta_1, ..., \theta_n$ in the $[0, 1]$ interval, one can find a list of numbers $\theta'_1, ..., \theta'_n$ in the $[0, 1]$ interval such that:

$$\sum_{i=1}^{n} \theta_i q_i = \sum_{i=1}^{n} \theta'{}_i p_i$$

Observe that this equality implies that for any $j = 1, ..., k$ one has:

$$\sum_{i=1}^{n} \theta_i q_{ij} = \sum_{i=1}^{n} \theta'{}_i p_{ij}$$

Summing these equalities over all $j$yields (exploiting the fact that the probability distributions lie in Δ^{k-1}):

$$\sum_{i=1}^{n} \theta_i \sum_{j=1}^{k} q_{ij} = \sum_{i=1}^{n} \theta_i = \sum_{i=1}^{n} \theta'{}_i \sum_{j=1}^{k} q_{ij} = \sum_{i=1}^{n} \theta'{}_i$$

But this implies that for any for any list of numbers $\theta_1, ..., \theta_n$in the $[0, 1]$ interval, one can find a list of numbers $\theta'_1, ..., \theta'_n$ in that same interval such that;

$$\sum_{i=1}^{n} \theta_i \left(\frac{1}{n}, q_{i1}, ..., q_{ik} \right) = \sum_{i=1}^{n} \theta'_i \left(\frac{1}{n}, p_{i1}, ..., p_{ik} \right)$$

That is, this implies that $\overline{\mathbf{Z}}(\mathbf{q}) \subseteq \overline{\mathbf{Z}}(\mathbf{p})$ holds, as required.

6.2.6 Lemma 1

For uniform averaging, we simply observe that the function $\Psi : \Delta^{k-1} \to \mathbb{R}$ defined, for every $(s_1, ..., s_k) \in \Delta^{k-1}$ by:

$$\Psi(s_1, ..., s_k) = \Phi\left(\sum_{h=1}^{k} s_h u_h\right)$$

is concave if Φ is concave irrespective of what the real numbers $(u_1, ..., u_k)$ are. Hence, by virtue of Theorem 3 in Kolm (1977),

$$\sum_{i=1}^{n}\Phi\left(\sum_{h=1}^{k} q_{ih}u_h\right) \geq \sum_{i=1}^{n}\Phi\left(\sum_{h=1}^{k} p_{ih}u_h\right)$$

if there exists a bistochastic matrix $n\times n$ bistochastic matrix $\mathbf{b}$ such that $\mathbf{q} = \mathbf{b}.\mathbf{p}$.

Assume now that $(u_1, ..., u_k) \in \mathcal{U}^{\geq_{Q}O}$ and that $\mathbf{q}$ results from from $\mathbf{p}$ through a favorable transfer as per Definition 7. We must show that:

$$\sum_{i=1}^{n}\Phi\left(\sum_{h=1}^{k} q_{ih}u_h\right) \geq \sum_{i=1}^{n}\Phi\left(\sum_{h=1}^{k} p_{ih}u_h\right).$$

Since all rows others than i_1 and i_2 in the matrix $\mathbf{p}$ and others than i_1' and i_2' in the matrix $\mathbf{q}$ are unaffected by the change, we have:

$$\sum_{i=1}^{n}\Phi\left(\sum_{h=1}^{k} q_{ih}u_h\right) \geq \sum_{i=1}^{n}\Phi\left(\sum_{h=1}^{k} p_{ih}u_h\right)$$

$$\iff$$

$$\Phi\left(\sum_{h=1}^{k} q_{i_1' h}u_h\right) + \Phi\left(\sum_{h=1}^{k} q_{i_2' h}u_h\right) \geq \Phi\left(\sum_{h=1}^{k} p_{i_1 h}u_h\right) + \Phi\left(\sum_{h=1}^{k} p_{i_2 h}u_h\right) \tag{16}$$

We now observe that the vector $(\sum_{h=1}^{k} q_{i_1' h}u_h, \sum_{h=1}^{k} q_{i_2' h}u_h)$ Lorenz-dominates the vector $(\sum_{h=1}^{k} p_{i_1 h}u_h), \sum_{h=1}^{k} p_{i_2 h}u_h)$. Indeed, one has:

$$\sum_{h=1}^{k} p_{i_1 h}u_h \leq \sum_{h=1}^{k} p_{i_2 h}u_h - \sum_{h=1}^{k} v_h u_h = \sum_{h=1}^{k} q_{i_2' h}u_h \leq \sum_{h=1}^{k} p_{i_2 h}u_h$$

and:

$$\sum_{h=1}^{k} p_{i_1 h}u_h \leq \sum_{h=1}^{k} p_{i_1 h}u_h + \sum_{h=1}^{k} v_h u_h = \sum_{h=1}^{k} q_{i_1' h}u_h \leq \sum_{h=1}^{k} p_{i_2 h}u_h$$

Inequality (16) then follows from the Hardy-Littlewood-Polya Theorem.

6.2.7 Lemma 2

Let $h = 1, ..., n$. The set $\mathbf{Z}(\mathbf{p}) \cap h\Delta^{k-1}$ is the intersection of a convex polytope, $\mathbf{Z}(\mathbf{p})$, and an affine subspace of $\mathbb{R}^k$, $\{x \in \mathbb{R}^k : \sum_{j=1}^{k} x_i = h\}$. Hence $\mathbf{Z}(\mathbf{p}) \cap h\Delta^{k-1}$ is also a convex polytope. Since $\mathbf{Z}(\mathbf{p})\cap h\Delta^{k-1}$contains the points $\sum_{i\in J_h^1} p_i,\ldots,\sum_{i\in J_h^{M(h)}} p_i$, we have:

$$Co\left\{\sum_{i\in J_h^1} p_i,\ldots,\sum_{i\in J_h^{M(h)}} p_i\right\}\subset Z(\mathbf{p})\cap h\Delta^{k-1}=Co\{x^1,...,x^P\},$$

where x^p is an extreme point of $\mathbf{Z}(\mathbf{p})\cap h\Delta^{k-1}$, for $p=1,...,P$. The proof will be complete if we show that $x^p\in\left\{\sum_{i\in J_h^1} p_i,\ldots,\sum_{i\in J_h^{M(h)}} p_i\right\}$for $p=1,...,\mathbf{P}$. Suppose by contradiction that $x^p\notin\left\{\sum_{i\in J_h^1} p_i,\ldots,\sum_{i\in J_h^{M(h)}} p_i\right\}$. Then $\mathbf{x}^p=\sum_{i=1}^n \theta_i p_i$, where we can assume without loss of generality that $\theta_1\in\,]0,1[$. Since $\sum_{i=1}^n \theta_i= h$ there must exist another parameter, θ_2 say, such that $\theta_2\in\,]0,1[$. We then have

$$x^p=\frac{1}{2}(x_+ + x_-),$$

where $x_+:=(\theta_1+\varepsilon)p_1+(\theta_2-\epsilon)p_2+p_3+...+\theta_n p_n$and $x_-:=(\theta_1-\epsilon)p_1+(\theta_2+\epsilon)p_2+p_3+...+\theta_n p_n$ both belong to $\mathbf{Z}(\boldsymbol{p})\cap h\Delta^{k-1}$, provided that ε is small enough. Hence x^p is not an extreme point of $\mathbf{Z}(\mathbf{p})\cap h\Delta^{k-1}$, a contradiction.

6.2.8 Remark 4

The fact that $\overline{q}=\overline{p}$ must hold is an immediate implication of Remark 2. We now observe that if $\mathbf{Z}(\mathbf{q})\subseteq\mathbf{Z}(\mathbf{p})$, then $\mathbf{Z}(\mathbf{q})\cap h\Delta^{k-1}\subseteq\mathbf{Z}(\mathbf{p})\cap h\Delta^{k-1}\forall h$. Thus we obtain the direct implication since, by Lemma 2, we have $\mathbf{Z}(\mathbf{p})\cap h\Delta^{k-1}=Co\left(\left\{\sum_{i\in J_h^1} p_i,\ldots,\sum_{i\in J_h^{M(h)}} p_i\right\}\right)$. Now (7) implies that $\sum_{i\in J_h^m} q_i\in\mathbf{Z}(\mathbf{q})$ for any $h=1,...,n$ and any $m=1,...,M(h)$, which implies in turn that $\sum_{i=1}^n \alpha_i p_i\in\mathbf{Z}(\mathbf{q})$, for any $\alpha_1,...,\alpha_n\in\{0,1\}^n$. The reverse implication is obtained as an immediate consequence of Lemma 4.

6.2.9 Lemma 3

As noted in several instances, $\mathbf{q}\succsim_Z^{QO}\mathbf{p}$ if and only if, for all $\alpha_1,...,\alpha_n\in\{0,1\}^n$, there exist $v\in\mathcal{U}_*^{\geq QO}$ and $\theta_1,...,\theta_n\in[0,1]^n$ such that:

$$\sum_{i=1}^n \alpha_i q_i=\sum_{i=1}^n \theta_i p_i+v.$$

This is in turn equivalent to having that, for all $h=1,\ldots,n$ and all $m=1,\ldots,M(h)$, there exist $v\in\mathcal{U}_*^{\geq QO}$ and $\theta_1,...,\theta_n\in[0,1]^n$such that $\sum_{i=1}^n \theta_i=h$ and:

$$\frac{1}{h}\sum_{i\in J_h^m} q_i=\frac{1}{h}\sum_{i=1}^n \theta_i p_i+\frac{1}{h}v.$$

Since:

$$\left\{\frac{1}{h}\sum_{i=1}^n \theta_i p_i:\theta_1,...,\theta_n\in[0,1]^n,\ \sum_{i=1}^n \theta_i=h\right\}=Co\left\{\sum_{i\in J_h^1} p_i,\ldots,\sum_{i\in J_h^{M(h)}} p_i\right\}$$

(by Lemma 2), this concludes the proof.

6.2.10 Proof of Remark 5

Let $h = 1, ..., n$. If $q \succsim_Z^{QO} p$ then there exists some $\tilde{p} \in Co\{\sum_{i \in J_h^1} p_i, \ldots, \sum_{i \in J_h^{M(h)}} p_i\}$ such that:

$$\frac{1}{h}\sum_{i=1}^{h} q_i - \frac{1}{h}\tilde{p} \in \mathcal{U}_*^{\geq QO}$$

by Lemma 3. Since $\frac{1}{h}\tilde{p} - \frac{1}{h}\sum_{i=1}^{h} p_i \in \mathcal{U}_*^{\geq QO}$ we obtain (8).
Now suppose that (8) holds. Since $\frac{1}{h}\sum_{i \in J_h^m} q_i - \frac{1}{h}\sum_{i=1}^{h} q_i \in \mathcal{U}_*^{\geq QO}$, we have:

$$\frac{1}{h}\sum_{i \in J_h^m} q_i - \frac{1}{h}\sum_{i=1}^{h} p_i = \frac{1}{h}\sum_{i \in J_h^m} q_i - \frac{1}{h}\sum_{i=1}^{h} q_i + \frac{1}{h}\sum_{i=1}^{h} q_i - \frac{1}{h}\sum_{i=1}^{h} p_i \in \mathcal{U}_*^{\geq QO}$$

and this concludes the proof.

6.2.11 Theorem 2

Using the reasoning in the proof of Proposition 2, one can observe that if $n(\mathbf{p}) = n(q) = 2$, the statement $\mathbf{q} \succsim_Z^{QO} \mathbf{p}$ is equivalent to the requirement that $\overline{q} - \overline{p} \in \mathcal{U}_*^{\geq QO}$ and that there exist θ_1 and $\theta_2 \in [0, 1]$ such that $q_1 - (\theta_1 p_1 + (1 - \theta_1)p_2) \in \mathcal{U}_*^{\geq QO}$ and $q_2 - (\theta_2 p_1 + (1 - \theta_2)p_2)$. Since these θ_1 and θ_2 may belong respectively to $\mathbf{\Lambda}_1$ and $\mathbf{\Lambda}_2$, this establishes one direction of the implication.

For the other direction, it is sufficient to prove that the statement $\mathbf{q} \succsim_Z^{QO} \mathbf{p}$ implies the existence of $\lambda_1 \in \mathbf{\Lambda}_1$ such that $q_1 - (\lambda_1 p_1 + (1 - \lambda_1)p_2) \in \mathcal{U}_*^{\geq QO}$ (the argument being similar for λ_2). If $q_1 - p_1 \in \mathcal{U}_*^{\geq QO}$, then one selects $\lambda_1 = 1 \in \mathbf{\Lambda}_1$ and the proof is over. If $q_1 - p_1 \notin \mathcal{U}_*^{\geq QO}$, then we know that since $q_1 - (\theta_1 p_1 + (1 - \theta_1)p_2) \in \mathcal{U}_*^{\geq QO}$ for some $\theta_1 \in [0, 1]$, there exists some $v_1 \in \mathcal{U}_*^{\geq QO}$ such that:

$$q_1 = \theta_1 p_1 + (1 - \theta_1)p_2 + v_1$$

Let $\mathbf{D}(q_1)$ denote the (compact) set of distributions of opportunities that are weakly dominated by q_1, with respect to the quasi-ordering, defined by:

$$\mathbf{D}(q_1) = \{x \in \Delta^{k-1} : q_1 - x \in \mathcal{U}_*^{\geq QO}\}$$

Consider the continuous map $x : [0, 1] \to [0, 1]$ defined by:

$$x(t) = tp_1 + (1 - t)p_2$$

Since $q_1 - p_1 \notin \mathcal{U}_*^{\geq QO}$ one has that $x(1) \notin \mathbf{D}(q_1)$ while $x(\theta_1) \in \mathbf{D}(q_1)$. Let $\overline{\theta}_1$be defined by:

$$\overline{\theta}_1 = \max\{t \geq \theta_1 : x(t) \in \mathbf{D}(q_1)\} \tag{17}$$

We then have $\overline{\theta}_1 \in [\theta_1, 1[$ and $x(\overline{\theta}_1) \in \mathbf{D}(q_1)$. We therefore have:

$$q_1 = \overline{\theta}_1 p_1 + (1 - \overline{\theta}_1)p_2 + \overline{v}_1$$

for some $\overline{v}_1 \in \mathcal{U}_*^{\geq QO}$. Also observe that $\overline{v}_1$ must be such that $\sum_{j \in J} \overline{v}_{1j} = 0$ for some $J \in \mathcal{F}^{\geq QO}$. Indeed, using Expression (9), assuming that $\sum_{j \in J} \overline{v}_{1j} > 0$ for all

$J \in \mathcal{F}^{\geq QO}$ would imply the possibility of increasing a bit the t above $\overline{\theta}_1$ while maintaining $x(t)$ in the set $\mathbf{D}(q_1)$ in the maximization described by Expression (17), and will therefore be contradictory. Hence for the set J where $\sum_{j \in J} \overline{v}_{1j} = 0$, one has $q_1(J) = \overline{\theta}_1 p_1(J) + (1 - \overline{\theta}_1 p_2(J)$ and this completes the proof.

6.2.12 Remark 6.

Let the transformed society $\mathbf{p}'$ be defined by $p_1' = p_1 + w_1$ and $p_2' = p_2 + w_2$ for some $w_1, w_2 \in \mathcal{U}_*^{\geq C}$. We claim that if $\mathbf{q} \succsim_Z^C \mathbf{p}'$ then $w_1 + w_2 = 0$. Suppose indeed that:

$$q_1 - (\theta_1 p_1' + (1 - \theta_1) p_2') \in \mathcal{U}_*^{\geq C}, \ \ q_2 - (\theta_2 p_1' + (1 - \theta_2) p_2') \in \mathcal{U}_*^{\geq C}$$

and:

$$q_2 + q_1 - (p_1' + p_2') \in \mathcal{U}_*^{\geq QO_C}.$$

Then it follows that $\theta_1 = 1$, as we have seen in the argument that we just made about the impossibility of performing a uniform averaging. This implies that $q_1 - p_1 - w_1 \in \mathcal{U}_*^{\geq C}$, that is $\frac{1}{36}(0, -2, 2, 0) - w_1 \in \mathcal{U}_*^{\geq C}$. Secondly $q_2 - (\theta_2 p_1 + (1 - \theta_2) p_2) = \frac{1}{36}(-3\theta_2, 2 - \theta_2, -3 + 6\theta_2, 1 - 2\theta_2)$. This vector belongs to $\mathcal{U}_*^{\geq C}$ if and only if $\theta_2 = 1/2$ and it is then equal to $\frac{1}{72}(-3, 3, 0, 0)$. To sum up we have:

$$\frac{1}{36}(0, -2, 2, 0) - w_1 \in \mathcal{U}_*^{\geq C}, \ \ \frac{1}{72}(-3, 3, 0, 0) - \frac{1}{2}(w_1 + w_2) \in \mathcal{U}_*^{\geq C}$$

and:

$$\frac{1}{36}(0, 0, -1, 1) - (w_1 + w_2) \in \mathcal{U}_*^{\geq C}.$$

Now $w_1 + w_2 = (a, b, c, d)$ is by assumption an element of $\mathcal{U}_*^{\geq C}$. The condition $\frac{1}{36}(-3, 3, 0, 0) - (w_1 + w_2) \in \mathcal{U}_*^{\geq C}$ implies that $c = d = 0$. On the other hand the condition $\frac{1}{36}(0, 0, -1, 1) - (w_1 + w_2) \in \mathcal{U}_*^{\geq C}$ implies that $a = b = 0$. Thus $w_1 + w_2 = 0$ and, actually, $w_1 = w_2 = 0$.

6.2.13 Theorem 3

The fact that Statement 1 implies Statement 2 has been proved (for any number of groups) by Lemma 1 while the implication of Statement 3 by Statement 1 has been established by Theorem 1. We therefore only need to prove that Statement 3 implies Statement 1. Suppose therefore that $\mathbf{q} \succsim_Z^{QO} \mathbf{p}$.

Consider first the case where $p_2 - p_1 \in \mathcal{U}_*^{\geq QO}$.[10] Then:

$$Z(\mathbf{p}) + \mathcal{U}_*^{\geq QO} \subseteq \{\theta p_1 + v : \ \theta \in [0, 2], \ v \in \mathcal{U}_*^{\geq QO}\}.$$

Since $\mathbf{q} \succsim_Z^{QO} \mathbf{p}$ we have:

$$q_1 = \theta_1 p_1 + v_1; \ \ q_2 = \theta_2 p_1 + v_2,$$

where $\theta_1, \theta_2 \in [0, 2]$ and $v_1, v_2 \in \mathcal{U}_*^{\geq QO}$. Now, q_1 and q_2 being both in Δ^{k-1} and v_1 and v_2 having both their components summing to zero, we must have $\theta_1 = \theta_2 = 1$.

[10] The case where $p_1 - p_2 \in \mathcal{U}_*^{\geq QO}$ is similar.

As a result $q_1 = p_1 + v_1$ and $q_2 = p_1 + v_2$. Since $p_1 + p_2 = q_1 + q_2$, we have $p_2 = p_1 + v_1 + v_2$. Hence:

$$q_1 = p_1 + v_1,\ q_2 = p_2 - v_1 \text{ and } p_2 - p_1 - v_1 = v_2 \in \mathcal{U}_*^{\geq QO}$$

which means that $\mathbf{q}$ has been obtained from $\mathbf{p}$ through a favorable transfer.

Consider now the case where neither $p_2 - p_1 \in \mathcal{U}_*^{\geq QO}$ nor $p_1 - p_2 \in \mathcal{U}_*^{\geq QO}$. Since $\mathbf{Z}(\mathbf{q}) + \mathcal{U}_*^{\geq QO} \subseteq \mathbf{Z}(\mathbf{p}) + \mathcal{U}_*^{\geq QO}$ and both q_1 and $q_2 \in \mathbf{Z}(\mathbf{q}) + \mathcal{U}_*^{\geq QO}$, there are numbers θ_1^1, θ_1^1, θ_1^2 and $\theta_2^2 \in [0,1]$ satisfying $\theta_1^1 + \theta_2^1 = \theta_1^2 + \theta_2^2 = 1$ such that:

$$q_1 = \theta_1^1 p_1 + \theta_2^1 p_2 + v_1 \text{ and } q_2 = \theta_1^2 p_1 + \theta_2^2 p_2 + v_2,$$

for some v_1 and $v_2 \in \mathcal{U}_*^{\geq QO}$. Since $p_1 + p_2 = q_1 + q_2$ we then have:

$$\begin{aligned} v_1 + v_2 &= q_1 - \theta_1^1 p_1 - \theta_2^1 p_2 + q_2 - \theta_1^2 p_1 - \theta_2^2 p_2 \\ &= p_1 + p_2 - \theta_1^1 p_1 - \theta_2^1 p_2 - \theta_1^2 p_1 - \theta_2^2 p_2 \\ &= (1 - \theta_1^1 - \theta_1^2)(p_1 - p_2). \end{aligned} \tag{18}$$

Now, since neither $p_2 - p_1 \in \mathcal{U}_*^{\geq QO}$ nor $p_1 - p_2 \in \mathcal{U}_*^{\geq QO}$ $\langle\wr\updownarrow\lceil$ while $v_1 + v_2 \in \mathcal{U}_*^{\geq QO}$, the only way by which Equality (18) can hold is if $(1 - \theta_1^1 - \theta_1^2) = 0$ and, as a result, $v_1 + v_2 = 0$. Setting in that case $\theta_1 = \theta_1^1 = \theta_1^2$, we must therefore have:

$$q_1 = \theta_1 p_1 + (1 - \theta_1) p_2;\ \ q_2 = (1 - \theta_1) p_1 + \theta_1 p_2$$

so that $\mathbf{q} = \mathbf{m}.\mathbf{p}$ for the bistochastic matrix $\mathbf{m} = \begin{bmatrix} \theta_1 & 1 - \theta_1 \\ 1 - \theta_1 & \theta_1 \end{bmatrix}$. Hence $\mathbf{q}$ can be obtained from $\mathbf{p}$ through a uniform averaging operation in that case.

6.3 Distributions of educational opportunities among genders and castes

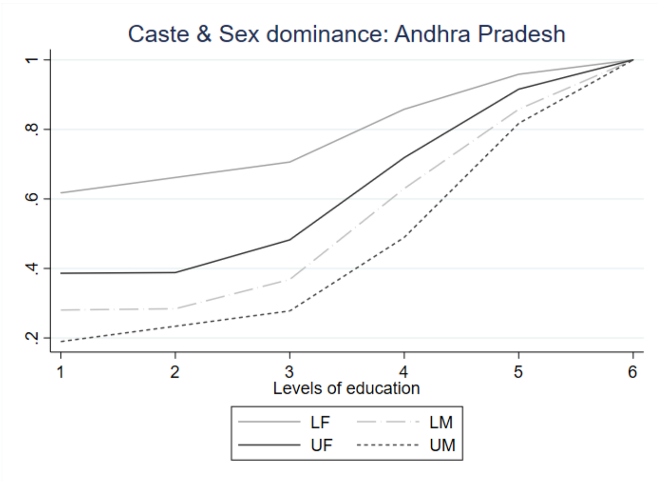

Figure 12: Cumulative distribution of education levels in gender and caste groups, Andhra Pradesh, 2012.

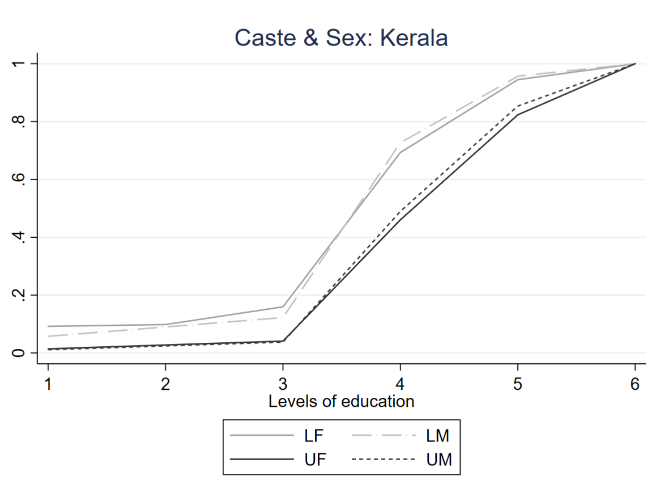

Figure 13: Cumulative distributions of education levels in gender and caste groups, Kerala, 2012.

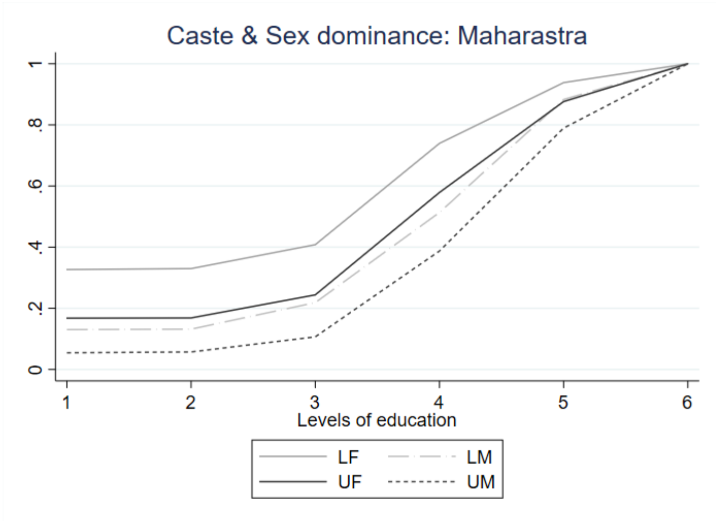

Figure 14: Cumulative distributions of education levels in gender and caste groups, Maharashtra, 2012.

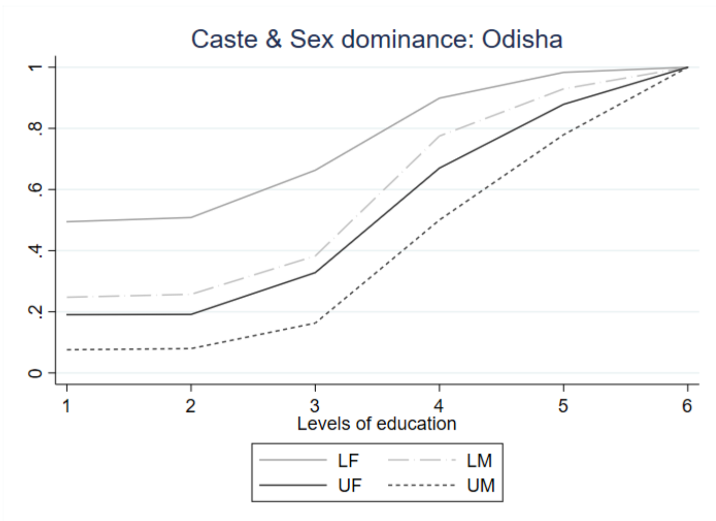

Figure 15: Cumulative distributions of education levels in gender and caste groups, Odisha, 2012.

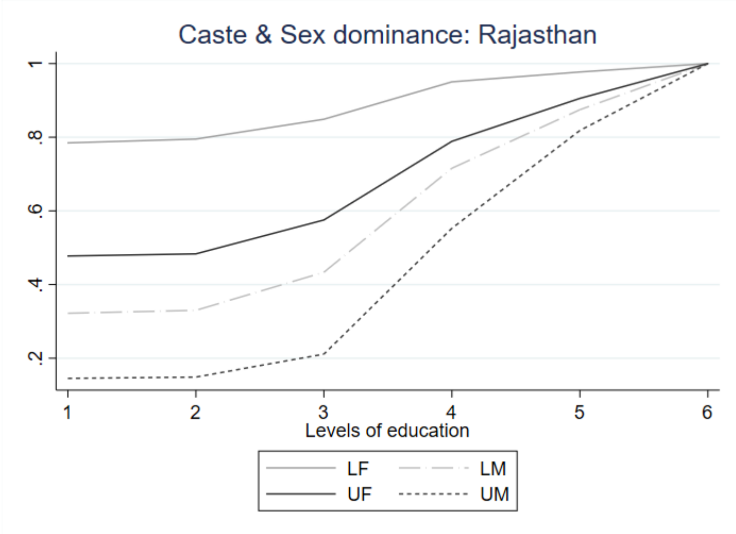

Figure 16: Cumulative distributions of education levels in gender and caste groups, Rajasthan, 2012.

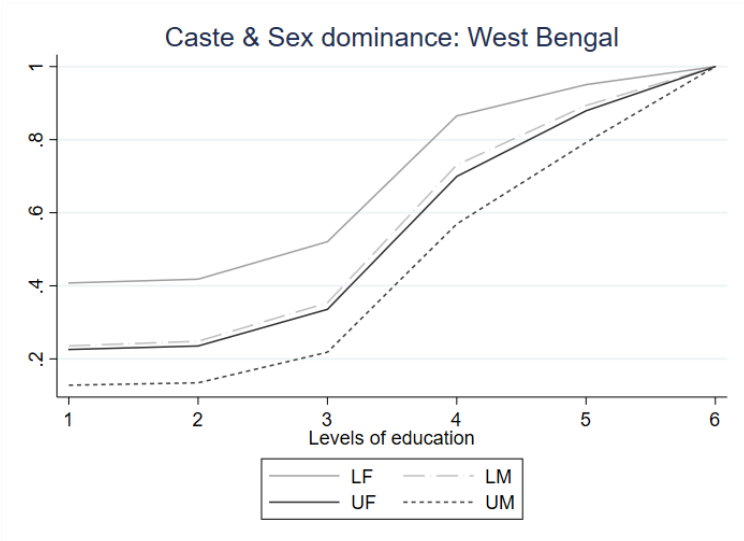

Figure 17: Cumulative distributions of education levels in gender and caste groups, West Bengal, 2012.