Public Goods in Networks: Comparative Statics Results*

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Abstract

We consider public goods games with heterogeneous players interacting on a network and investigate how shocks to players' characteristics and changes in interaction patterns influence individual and total contributions. We introduce a linear system associated to the initial game, in which heterogeneity in players' characteristics is removed and interactions between players are reversed, and show that what matters in determining the effects of a shock on contributions is the sign of the coordinates of its unconstrained solution. When players are identical, we demonstrate that shocks on active players increase contributions, while shocks on strictly inactive players decrease them, contrary to intuition. We also identify a subset of players, called neutral players, who exert no influence on total contributions. Furthermore, we provide precise formulas for the change in total contributions following various types of shocks, and provide conditions to determine whether the shock will have positive or negative consequences on contributions. We show that these conditions always rely on the sign of the associated problem's unconstrained solution coordinates of the players impacted by the shock.

JEL classification: C72; D85; H41

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1 Introduction

The public goods game has garnered significant attention from economists due to critical issues like under-provision and free-riding, which arise when public goods are voluntarily provided by individuals. Understanding how changes in individual characteristics of players, or in their interaction patterns, affect both individual and aggregate contributions is crucial for understanding the determinants of these contributions. This knowledge is essential, in particular, for designing effective public policies aimed at enhancing the provision of public goods.

Conducting these comparative static exercises requires considering heterogeneity among individuals, and in this paper, we explore two main types of heterogeneities. First, individuals may differ in their personal characteristics, such as their preferences for public goods, their ability to produce them, or their wealth. Second, there may be variation in how individuals interact. This can occur because some individuals have different substitution rates between their own contributions and others', or because individuals may benefit from the public goods provided by specific subsets of others, with these subsets varying among individuals. Additionally, interactions can be either reciprocal or non-reciprocal.

Consider the example of charitable donations: wealthy individuals might find it easier to contribute to charity compared to those with less wealth. Some individuals may be more influenced by the warm glow of giving, while others prioritize altruism due to political or religious beliefs. If a charity operates only in a specific neighborhood, then for someone interested in that charity, only the contributions from those who donate to it matter and thus, individuals from different neighborhoods are concerned about contributions from distinct sets of individuals. In this scenario, interactions are likely reciprocal because if one person benefits from another's contribution, the reverse is likely to be true. Another example concerns the cleanliness of the streets. Maintaining cleanliness on one's street does not benefit everyone in society, it only benefits the homeowner, their neighbors, and frequent users of the street. Also, the cost and effort of cleaning can vary based on factors like the homeowner's age, available free time, preference for cleanliness, and the initial condition of the road. In this case, interactions are probably not reciprocal. If someone who frequently uses a street benefits from a homeowner's efforts on that street, the homeowner will not gain anything if they never use the street where the other user lives. Finally, firms within a particular sector investing in research for innovation may benefit solely from research conducted within that same sector, while firms from another sector might benefit from research conducted across different sectors as well. Firms may also be heterogeneous in size, in access to capital for research, or in how much they rely on innovation for success.

Developing effective financial incentives to promote charitable giving, maintain clean streets, or fostering links between firms to boost research efforts requires understanding how changes in individual characteristics and interaction patterns affect the current contributions to the public good.

We consider a general class of games with linear best-replies and strategic substitutes,[1] allowing for both types of heterogeneity, in terms of individual characteristics, and in terms of interaction patterns between agents, where this second type of heterogeneity is modeled as a network. Our primary focus is comparative static analysis, specifically examining how changes in the model parameters' values impact both individual and aggregate actions. Despite the game typically having multiple equilibria,[2] we can still uncover general results that hold for *any* equilibrium.

We begin by examining the impact of a positive shock on a player's characteristics, such as an increase in wealth or an increase in the ability to produce public goods, on both the player's contribution and total contributions (Section 3). We show that, when the interaction matrix is a P-matrix, this shock leads to an increase in the player's contribution. Otherwise, this increase is not guaranteed and an individual might, in fact, decrease their contribution following a positive shock.

Regarding the impact on total contributions, our analysis unveils a straightforward method for determining such effects. We start by associating each equilibrium of the game with an alternative linear problem which only accounts for the heterogeneity of players in terms of their interaction patterns, but not in terms of their individual characteristics. Then, we determine the unconstrained solution to this problem, and we prove that the impact on total contributions of a shock on one player only depends on the sign of this player's coordinate in this solution. If the player receiving the shock has a positive coordinate in the associated problem's unconstrained solution, total contributions will rise; conversely, if the coordinate is negative, it will decrease.

It is worth noting that there is no direct link between the unconstrained solution of this associated problem, and the Nash equilibria of the initial game, which is a constrained solution of another problem. In particular, the signs of the players' coordinates in the former cannot be determined from their actions or their status (active or inactive) in the later. However, the unconstrained solution to that associated problem holds the relevant information.

This solution represents the aggregate *outgoing effect* of players. If a player's outgoing effect is positive, a positive shock on that player will increase total contributions, whereas a shock on a player with negative outgoing effect will decrease total contributions. Incidentally, this implies that players whose coordinates in the unconstrained solution is

[1]We stick to the example of the public good game, which is a prominent member of this class, throughout the paper.

[2]As already noted in Rébillé and Richefort (2014), this class of games forms a Linear Complementary Problem (LCP). Thus, uniqueness is guaranteed whenever the matrix describing how players interact, which we call the interaction matrix, is a P-matrix.

zero, exert no influence on total contributions, regardless of any shocks they may receive. To the best of our knowledge, this is the first paper to identify these players, which we call *neutral players*. They occupy a position in the network such that any shock, whether positive or negative, prompts changes in the contributions of all players, including themselves, but these changes are balanced in such a way that the total sum of contributions remains constant.

Next, we build on the results regarding the effects of a shock on a single player, and show that shocks on several players' individual characteristics can be simply analyzed as the sum of shocks on one player's individual characteristics. We also establish that altering interaction patterns is in fact analogous to making corresponding adjustments to the individual characteristics of several players. This allows us to show (Sections 4 and 5) that the effect of redistributing wealth between players depends on the difference of the unconstrained solutions of the players involved in the transfer; providing baseline public good to several players, adding links to the network, adding new players, or increasing substitution rates will decrease total contributions, only if the targeted players have positive coordinates in the associated game's unconstrained solution. Otherwise, these effects will be, surprisingly, reversed. Similarly, providing public goods through taxes does not necessarily result in an equivalent decrease in private provisions, as found in previous literature. The overall effect once again depends on the signs of the players' coordinates.

Previous literature has explored comparative statics in the public good game, either in a context without heterogeneity in interaction patterns (i.e. when agents interact on a complete network), or without heterogeneity in individual characteristics. Interestingly, as we show, these are the two specific situations where the signs of the unconstrained solution of the associated problem are predetermined. This enables us to derive these findings as corollaries of our own results.

In a complete network, we prove that all the coordinates of this unconstrained solution are positive. Thus, in a complete network, a player experiencing a positive shock such as an increase in wealth or in preferences toward the public good, increases their contribution, and this induces an increase in total contributions (as observed in Corchón (1994) and Cornes and Hartley (2007)); transfers of wealth from high to low substitution rate players increase total contributions (as in Andreoni (1990)); providing a state baseline public good diminishes total contributions, and the entry of a new player also does (as noted by Acemoglu and Jensen (2013)).

Given that the unconstrained solution on arbitrary networks typically comprises both positive and negative components, this underscores the special nature of complete networks, and explains why conclusions drawn there do not extend to arbitrary networks. In fact, we also prove that the interaction matrix of any complete network, despite featuring heterogeneous players, is a P-matrix, explaining why there is always a unique Nash equilibrium, and why a positive shock typically results in increased contributions of the player receiving the shock. This, together with the previous observations, explains why the complete network stands out as a distinct interaction structure, making it non-representative of typical interaction matrices.

In the second case, when players are identical but interact on an arbitrary, incomplete network, we prove that a player's coordinate in the unconstrained solution is positive if and only if this player is active in Nash equilibrium, and therefore it is negative if and only if this player is strictly inactive in Nash equilibrium. This arises because the unconstrained solution of the associated problem is independent of players' characteristics; it solely relies on the interaction patterns among players. Hence, when players are identical, individual characteristics become irrelevant, and the decisive factor in determining the Nash solution is solely the interaction patterns. This enables us to link the sign of the unconstrained solution with the players' status (active or strictly inactive) at equilibrium. As a consequence, we obtain that positive shocks on active players will increase total contributions, in line with Bramoullé et al. (2014) who show that adding links or increasing substitution rates (both can be understood as negative shocks) reduce total contributions. On the contrary, positive shocks on strictly inactive players will decrease them. This is surprising and counter-intuitive. Intuitively, a policymaker aiming to increase total contributions would prioritize targeting free-riders over those already contributing. Paradoxically, however, this would result in a decrease in total contributions.

In the setting where players have identical substitution rates and play on an undirected, unweighted network, and in games that might have non-linear best-responses, Allouch (2015) investigates the aggregate effect of transfers between players, when transfers are confined to the set of contributors and when the equilibrium is unique. They show that transfers increase total contribution if and only if a transfer happens from a player with lower diagonally weighted Bonacich centrality to a player with higher diagonally weighted Bonacich centrality. Our analysis of transfers in section 4.1 coincides when players are identical and the set of contributors is unchanged after the transfer. However, it extends to the case of heterogeneous players, when there are multiple equilibria, and when shocks could potentially change the set of contributors.

In the general case, when both types of heterogeneity are considered, the sign of the associated problem's unconstrained solution is unrelated to the status of players, which makes it more intricate to identify which players will have positive effects on total contributions, and which will have negative effects. However, in each comparative static analysis, we provide the precise formula capturing the change in total contributions. Additionally, we provide an example where total contributions change in a counter-intuitive direction, contrary to the anticipated direction seen in complete networks. These examples are straightforward to construct due to the clear implications we have derived: we consider an arbitrary network and identify the players with negative coordinates in the unconstrained solution of the associated problem. Targeting these players yields the counter-intuitive effect.

2 Model and Shocks

2.1 Model

Consider a game $\mathcal{G} = (N, (X_i)_{i=1,\dots,n}, \mathbf{u})$, where $N = \{1, \dots, n\}$ is the set of players and $X_i = [0, +\infty[$ is the action space of player i, from which he chooses x_i, his contribution to the public good. We denote by X the sum of individual contributions to the public good, i.e. $X = \sum_{i \in N} x_i$. Finally $\mathbf{u} = (u_i)_{i=1,\dots,n}$ is the vector of payoff functions.

Agents are placed on a network represented by a graph $\mathbf{G}$. By convention, we also denote by $\mathbf{G} = (g_{ij})_{i,j \in N}$ the adjacency matrix of the graph with elements g_{ij}. We assume that $g_{ii} = 0$ for all i.

We say that the network is unweighted and undirected when $g_{ij} = g_{ji} \in \{0, 1\}$. We say that the network is weighted if $g_{ij} \in (0, 1]$ if i is linked to j and $g_{ij} = 0$ otherwise. We say that the network is directed if g_{ij} can be different from g_{ji}. When positive, we refer to g_{ij} as an *incoming* link for player i and an *outgoing* link of player j. We also call g_{ij} the *incoming link intensity* of i from j.

In the remainder, we will denote by $\overline{x}_i$ the (weighted) sum of contributions of all neighbors of player i, i.e. $\overline{x}_i = \sum_{j \in N} g_{ij} x_j$.

We will sometimes only consider incoming links of a subset S of players, and delete the incoming links of players in $N \setminus S$. We denote that network by $\mathbf{G}_S = (g_{Sij})_{i,j \in N}$, which is constructed from $\mathbf{G}$, where for player $i \in S$, we set $g_{Sij} = g_{ij}$ for all j, while $g_{Sij} = 0$ for all $i \notin S$. Notice that $\mathbf{G}_S$ is in general directed, even though $\mathbf{G}$ is undirected. For any vector $\mathbf{v}$, we denote by $\mathbf{v}_S$ the vector $(v_{Si})_{i \in N}$ of same size as $\mathbf{v}$, such that $v_{Si} = v_i$ for $i \in S$ and $v_{Si} = 0$ otherwise.

Finally, we will often consider the complete network, in order to contrast our results with previously established results in the literature. The complete network is defined as $g_{ij} = 1$ for all $i \neq j$.

Best-responses

We consider games with payoff functions $u(\cdot)$ that have unique best-responses of the following form

$$\forall i \in N, \;\; BR_i(\mathbf{x}_{-i}) = \max\{q_i - \delta_i \overline{x}_i, 0\} \tag{1}$$

where $\delta_i \in [0, 1]$ represents the substitution rate between agent i's neighbors' actions and agent i's action, and $q_i \in \mathbb{R}_+$ represents the level of contribution that player i would provide if he was isolated. In the remainder of this paper, we call q_i the *needs* of player i. Substitution rates are collected into the matrix $\mathbf{\Delta} = diag(\delta_i)_{i\in N}$ and needs are collected into vector $\mathbf{q}$.

Our framework includes the class of public goods games[3] with payoff function

$$u_i = u_i(x_i, x_i + \gamma_i \overline{x}_i) \tag{2}$$

where $\gamma_i \in [0, 1]$ measures how much a player enjoys the public good provided by his neighbors, and u_i is strictly quasi-concave, increasing in its second argument, and produces a linear best-response function. Note that γ_i and the substitution rate δ_i in (1) are not necessarily the same. γ_i indicates that a player may not fully benefit from the public goods provided by neighbors, but it does not necessarily reflect how much the player adjusts their contribution in response to changes in their neighbors' contributions.

A prominent example of such a game is provided in Bramoullé and Kranton (2007), where the payoff function is

$$u_i(\mathbf{x}) = b\left(x_i + \overline{x}_i\right) - cx_i \tag{3}$$

where $c > 0$ is the marginal cost of effort and $b(\cdot)$ is a differentiable, strictly increasing concave function In that case, the best-response is given by

$$\forall i \in N,\ \ BR_i(\mathbf{x}_{-i}) = \max\left\{b'^{-1}(c) - \overline{x}_i, 0\right\}. \tag{4}$$

where $\mathbf{\Delta}$ is the identity matrix, $q_i = b'^{-1}(c)$ and $\delta_i = \gamma_i = 1$ for all i.[4]

Another prominent example is the game of private provision of a public good from Bergstrom et al. (1986), adapted to networks in Allouch (2015) where players have wealth $\mathbf{w} = (w_i)_{i\in N} \in \mathbb{R}_+^N$ that they allocate to the consumption of a private good and a public good, and players have potentially imperfect substitutes. In this literature, usually the model is solved by denoting the private good component as $w_i - x_i$ which only depends on x_i, hence the first argument of payoff function (2), and by expressing the payoff function in terms of the total public goods $x_i + \overline{x}_i$ that a player enjoys.

$$u_i = u_i\left((x_i + \overline{x}_i) - \overline{x}_i, (x_i + \overline{x}_i) - (1 - \gamma_i)\overline{x}_i\right)$$

Differentiating and solving the first order condition with respect to $x_i + \overline{x}_i$ produces a demand function

$$x_i + \overline{x}_i = f_i(\overline{x}_i).$$

[3]Our framework actually encompasses any network game that gives linear best-responses (1), not only the public goods game. For instance, the game discussed in Ballester et al. (2006) with strategic substitute falls within our framework. Along the paper we keep the public goods' interpretation.

[4]The equality $\delta_i = \gamma_i$ comes from the linearity of cost.

where $f_i(\cdot)$ is assumed to be linear in order to fit into our framework. For instance, with a Cobb-Douglas payoff function

$$u_i = \lambda_i log\,(x_i + \gamma_i \overline{x_i}) + (1 - \lambda_i) log(w_i - x_i) \tag{5}$$

where $\lambda_i \in (0, 1)$ is a preference parameter for public good over private consumption, we derive the following demand function

$$x_i + \overline{x}_i = f_i(\overline{x}_i) = (1 - \gamma_i + \lambda_i \gamma_i)\overline{x}_i + \lambda_i w_i \tag{6}$$

and hence the best-response

$$\forall i \in N, \;\; BR_i(\mathbf{x}_{-i}) = \max\{\lambda_i w_i - (1 - \lambda_i)\gamma_i \overline{x}_i, 0\}$$

which generates (1) by setting $q_i = \lambda_i w_i$ and $\delta_i = (1 - \lambda_i)\gamma_i$. Assuming $\gamma_i = 1$ as in the literature (Bergstrom et al. (1986) and Allouch (2015)) results in a substitution rate $\delta_i = 1 - \lambda_i$.

Given $\mathbf{q}$, $\mathbf{\Delta}$ and $\mathbf{G}$, we call the matrix $(\mathbf{I} + \mathbf{\Delta G})$ the *interaction matrix*.[5] Notice that the interaction matrix is symmetric only if $\delta_i g_{ij} = \delta_j g_{ji}$ for every pair of players (i, j). This happens for instance if the network is unweighted, undirected, and players have identical substitution rates. However, this matrix will generally be non-symmetric. In particular, the complete network, although undirected, can yield symmetric or non-symmetric interaction matrices depending on the homogeneity or heterogeneity of substitution rates.

2.2 Shocks

We are interested in the effects of changes in the parameters $(\mathbf{q}, \mathbf{\Delta}, \mathbf{G})$ - and, when relevant, of changes in wealth $\mathbf{w}$ - on equilibrium actions x_i and on $X = \sum_{i \in N} x_i$, the total contributions level of players at equilibrium $\mathbf{x}$.

Changes in $\mathbf{q}$ capture changes in individual characteristics of the players. Indeed, note that q_i is the contribution that a player i would choose if he were in autarky (i.e. linked to no-one). Thus changes in $\mathbf{q}$ can result from changes in costs, in wealth, in the parameter of preference for the public good (λ_i in the above example), in the concavity of the benefit function ($b(\cdot)$ or $log(\cdot)$ in the above examples), or from any change that would modify the preferred level of public good consumption of that player.

In turn, changes in $\mathbf{\Delta}$ or in $\mathbf{G}$ capture changes in the way individuals interact together. These changes result from modifications of the network such as cutting out or adding some links, changing link intensities, from entry of new players, or from changes in levels of substitution.

[5] Throughout the paper we assume that the matrix $(\mathbf{I} + \mathbf{\Delta G})$ is non-degenerate, i.e. 0 is not an eigenvalue.

2.3 Nash equilibria: Existence, Uniqueness

The set of Nash equilibria of games with best-responses (1) is described by the set of all profiles $\mathbf{x}$ such that:

$$\begin{array}{ll}(\text{A}) & q_i - \delta_i \overline{x}_i \geq 0 \implies x_i = q_i - \delta_i \overline{x}_i \\ (\text{SI}) & q_i - \delta_i \overline{x}_i < 0 \implies x_i = 0\end{array}$$

Players satisfying condition (A) are active players, while players satisfying condition (SI) are strictly inactive players, or free-riders. The set of active players at equilibrium $\mathbf{x}$ is denoted by $A(\mathbf{x})$, while the set of strictly inactive players at $\mathbf{x}$ is denoted by $SI(\mathbf{x})$. Note that players such that $q_i - \delta_i \overline{x}_i = 0$ are considered active players. We denote the set of such players as $Z(\mathbf{x})$, with $Z(\mathbf{x}) \subset A(\mathbf{x})$.

As noticed in Rébillé and Richefort (2014) in a more general context than ours, finding the Nash equilibria of such games amounts to solving a Linear Complementarity Problem (LCP). Problem $LCP(\mathbf{b}, \mathbf{C})$ consists of finding vectors $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{v} \in \mathbb{R}^n$ such that

$$\begin{aligned}&\mathbf{Cx} + \mathbf{b} = \mathbf{v}, \\ &\mathbf{v} \geq 0,\ \mathbf{x} \geq 0,\ \mathbf{v}^T\mathbf{x} = 0\end{aligned}$$

where $\mathbf{C} \in \mathbb{R}^{n \times n}$ and $\mathbf{b} \in \mathbb{R}^n$ are given. This class of linear problems are called complementary because of the constraint $\mathbf{v}^T\mathbf{x} = 0$, which implies that if one of the variables is strictly positive then the other is necessarily 0.

By letting $\mathbf{C} = (\mathbf{I} + \mathbf{\Delta G})$ and $\mathbf{b} = -\mathbf{q}$, then $\mathbf{x}$ is a Nash equilibrium of games with best-responses (1) if and only if $\mathbf{x}$ is a solution to $LCP(-\mathbf{q}, (\mathbf{I} + \mathbf{\Delta G}))$, where $\mathbf{v}$ represents the excess public good of players. To see this, just notice that for every player i at a Nash equilibrium, either the excess public good v_i is 0 (and $i \in A(\mathbf{x})$ since i is active if and only if $v_i = 0$), or $v_i > 0$ and $x_i = 0$ (and $i \in SI(\mathbf{x})$).

Existence of a Nash equilibrium is usually proved by Brouwer's fixed point theorem.[6] However, we can also use LCP theory for a different proof. In an LCP, matrices that guarantee existence of at least one solution for each $\mathbf{q}$ are called $Q-$matrices.

Theorem 5.2 (Murty (1972))**.** *Let* $\mathbf{C} \geq 0$. $\mathbf{C}$ *is a Q-matrix if and only if* $c_{ii} > 0$ *for each* $i = 1, .., n$

Corollary 1. *Games with best-responses (1) have at least one Nash equilibrium.*

This corollary is straightforward, since all terms of $(\mathbf{I} + \mathbf{\Delta G})$ are positive and the diagonal terms are all equal to 1.

[6]Although the strategy space is unbounded, the best-response is bounded by q_i, enabling to apply Brouwer's fixed point theorem.

Remark 1. *Unlike the standard existence argument, which relies on Brouwer's fixed point theorem, reformulating the problem of finding all Nash equilibria as an LCP problem enables the use of efficient computational algorithms, such as the complementary pivot algorithm (Lemke (1978)), to actually find at least one equilibrium. This is of course particularly useful when the equilibrium is unique.*

Regarding uniqueness, Rébillé and Richefort (2014) show that the equilibrium of the game with best-responses (1) is unique, for any vector $\mathbf{q}$, if and only if the interaction matrix $(\mathbf{I} + \mathbf{\Delta G})$ is a P-matrix.[7,8] Since a symmetric matrix is a P-matrix if and only if it is positive definite,[9] this yields the result in Bramoullé et al. (2014) according to which there is a unique Nash equilibrium if the absolute value of the lowest eigenvalue of matrix $\mathbf{G}$ is smaller than $1/\delta$, when $\delta_i = \delta$ for all $i \in N$.

Three observations are in order. First, the sufficient condition on the lowest eigenvalue in Bramoullé et al. (2014) becomes necessary if we require uniqueness for any needs vector $\mathbf{q}$ when the interaction matrix is symmetric, as this implies positive definiteness. However, this is no longer true in the non-symmetric case that we mostly consider in the paper, since a non-symmetric matrix can have only positive eigenvalues and yet not be positive definite.[10] Besides, some non-positive definite matrices are also P-matrices.

Second, the P-matrix condition only depends on the interaction patterns between agents, and not on their individual characteristics. Thus comparative statics on $\mathbf{q}$ can be performed without losing the uniqueness of the Nash equilibrium.

Third, the necessary condition states that the solution should be unique for *each* $\mathbf{q}$. This does not rule out uniqueness for *some* $\mathbf{q}$ when $(\mathbf{I} + \mathbf{\Delta G})$ is not a P-matrix.[11] To the best of our knowledge, there is no identified sufficient condition guaranteeing uniqueness

[7] A square matrix is a P-matrix if all its principal minors are strictly positive. This sufficient condition for uniqueness was first proved in Murty (1972). However, this condition was previously identified in other, specific settings. For instance in economics, Arrow and Hahn (1971) prove the uniqueness of a competitive equilibrium by assuming that the Jacobian matrix of excess supply functions has all positive principal minors. More recently, Acemoglu and Tahbaz-Salehi (2020) also prove the uniqueness of a bargaining equilibrium by using the P-matrix property.

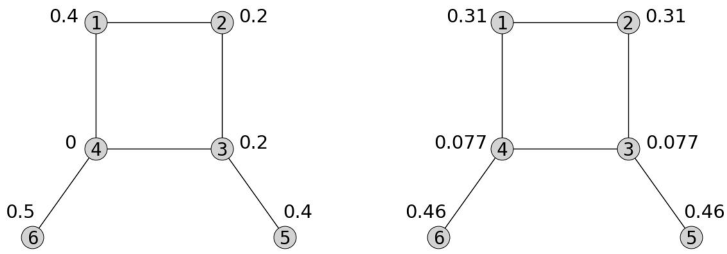
[8] Parise and Ozdaglar (2019) analyse a general public good game on networks, which accommodates cases where the strategy space may be multi-dimensional. They use the theory of variational inequality, and provide a condition for uniqueness (referred to as the P_Υ condition) which is equivalent to the P-matrix condition when the strategy space is one-dimensional.

[9] A real matrix F, whether symmetric or not, is positive definite if $\mathbf{y}^T\mathbf{F}\mathbf{y} > 0$ for all $\mathbf{y} \in \mathbb{R}^n, \mathbf{y} \neq 0$. If $\mathbf{F}$ is symmetric - and only then - it is positive definite if and only if all its eigenvalues are strictly positive. Most matrices we consider in this paper will not be symmetric.

[10] For instance, the reader can check that the complete network with 6 players, $\delta_1 = 1$ and $\delta_i = 0.1$ for $i = 2, ..., 6$, has only positive eigenvalues, yet it is not positive definite.

[11] In fact, if the number of solutions is a constant for every $\mathbf{q}$, then this constant is 1 and $(\mathbf{I} + \mathbf{\Delta G})$ is a P-matrix (Murty (1972), 7.2). Otherwise, even though there is a unique solution for some $\mathbf{q}$, there is always some $\mathbf{q}'$ for which there is more than one solution.

for a given $\mathbf{q}$ when the interaction matrix is not a P-matrix. Furthermore, the number of equilibria is not monotonic in $\mathbf{q}$, as increasing a player's needs can both increase or decrease the number of equilibria. This is illustrated in Figure 1.



(a) $q_4 = 0$ **(b)** $q_4 = 0.5$ **(c)** $q_4 = 2$

Figure 1: Non-monotonicity in the number of equilibria. If $\delta_i = 0.5$ for all i, the network's interaction matrix is not a P-matrix. Assume $q_i = 0.5$ for all players except player 4. In Panel (a), the equilibrium is unique. In Panel (b), there are 3 equilibria: the one of Panel (a), its permutation $\mathbf{x}' = (0.2, 0.4, 0, 0.2, 0.5, 0.4)^T$, and the one represented in Panel (b). Finally, in Panel (c), uniqueness is restored.

The case of the complete network has been extensively analyzed in the literature, as pointed out in the introduction. For each comparative static exercise that follows, we will derive, as corollaries of our results, otherwise established results holding on the complete network. This will help contrast with what happens when the pattern of interactions is incomplete.

Notice that the interaction matrix associated to the complete network is degenerate if, and only if, there are at least two players i and j such that $\delta_i = \delta_j = 1$. Then the two players are perfectly substitutable for all others and between themselves, inducing a continuum of Nash equilibria (see for instance Bervoets and Faure (2019)). Therefore, when we refer to the complete network in what follows, we always assume that at most one player has $\delta_i = 1$.

Proposition 1. *Let* $\mathbf{G}$ *be the complete network, and let* $\mathbf{\Delta} = diag(\delta_i)_{i=1,\ldots,n}$ *with* $\delta_i \in (0, 1]$ *for all* i*, and* $\delta_i = 1$ *for at most one player* i*. Then* $(\mathbf{I} + \mathbf{\Delta}\mathbf{G})$ *is a* P*-matrix. Therefore, the complete network has a unique Nash equilibrium for each needs vector* $\mathbf{q}$ *whether the interaction matrix is symmetric or not.*

2.4 Equilibrium Interaction Matrix and Unconstrained Solutions

Notice that if $\mathbf{x}$ is a Nash equilibrium with sets of active agents $A(\mathbf{x})$ and strictly inactive agents $SI(\mathbf{x})$, and if $(\mathbf{I} + \mathbf{\Delta}\mathbf{G}_{A(\mathbf{x})})$ is non-degenerate, then $\mathbf{x}$ is the unique equilibrium with sets $A(\mathbf{x})$ and $SI(\mathbf{x})$. This equilibrium $\mathbf{x}$ is the (unique) unconstrained solution to

$$(\mathbf{I} + \mathbf{\Delta G}_{A(\mathbf{x})})\mathbf{x} = \mathbf{q}_{A(\mathbf{x})} \tag{7}$$

where $q_{A(\mathbf{x}),i} = q_i$ if $i \in A(\mathbf{x})$, and $q_{A(\mathbf{x}),i} = 0$ is $i \in N \setminus A(\mathbf{x})$. In other words, this solution is found by deleting incoming links of SI players and by setting their needs to 0. Of course the set $A(\mathbf{x})$ is usually not known when searching for $\mathbf{x}$.

Definition 1. *Let* $\mathbf{x}$ *be a Nash equilibrium of the game with interaction matrix* $(\mathbf{I} + \mathbf{\Delta G})$*, with sets of active and strictly inactive players* $A(\mathbf{x})$ *and* $SI(\mathbf{x})$*. We call matrix* $(\mathbf{I} + \mathbf{\Delta G}_{A(\mathbf{x})})$ *the equilibrium interaction matrix of* $\mathbf{x}$*.*

When doing comparative statics on $\mathbf{x}$, the solution of equation (7), we will associate to $\mathbf{x}$ the solution of another problem: let $(\mathbf{I} + \mathbf{\Delta G}_{A(\mathbf{x})})^T$ be the transposed equilibrium matrix. This is the matrix in which active players' initial incoming links and outgoing links are interchanged. Assume now that needs of players are homogenized and normalized to 1 and consider the following problem:

$$(\mathbf{I} + \mathbf{\Delta G}_{A(\mathbf{x})})^T \mathbf{x} = \mathbf{1} \tag{8}$$

This problem has a unique *unconstrained* solution, that we call $\mathbf{x}^{unc}((\mathbf{\Delta G}_{A(\mathbf{x})})^T, \mathbf{1})$, and that we denote by the shortcut $\mathbf{x}^{unc}$ when there is no ambiguity, given by:

$$\mathbf{x}^{unc} = \left[(\mathbf{I} + \mathbf{\Delta G}_{A(\mathbf{x})})^T\right]^{-1} \mathbf{1}$$

Notice that this vector is determined only by the interaction patterns between players, i.e. the network and the substitution rates, and does not depend on their individual characteristics. As we will see along the next sections, this vector contains critical information for understanding the impact of shocks on total contributions in games with general non-symmetric interaction matrices and with players having different needs. Importantly, note that $\mathbf{x}$ and $\mathbf{x}^{unc}$ are unrelated in the sense that the values of one do not help predicting the values of the other, and while the coordinates of $\mathbf{x}$ are all positive, the sign of the coordinates of $\mathbf{x}^{unc}((\mathbf{\Delta G}_{A(\mathbf{x})})^T, \mathbf{1})$ - which can be positive or negative - will tell us whether these shocks have positive or negative impacts. Notice also that the unconstrained solution to the initial problem (i.e. $\mathbf{x}^{unc}(\mathbf{\Delta G}, \mathbf{1})$) is unrelated to this one, in particular because the signs of the unconstrained solution do not inform us on which player is active and which player is strictly inactive.[12]

[12]While it might seem intuitive to associate positive coordinates in the unconstrained solution of the initial problem with active players and negative coordinates with strictly inactive ones, this assumption is unfounded. There is no straightforward rule for identifying active or strictly inactive players based solely on these coordinates.

The only exception is a recent paper by Zheng et al. (2016), which, when adapted to our framework, However, whenever players have identical needs and the interaction matrix is symmetric[13] - and only in that case - the status of players coincides with the sign of their associated problem's unconstrained solution's coordinate, and a simple connection exists for active players between their equilibrium action and their unconstrained solution's coordinate.

Proposition 2. *Let* $\mathbf{x}$ *be an equilibrium of the game with parameters* $(\mathbf{q}, \mathbf{\Delta}, \mathbf{G})$*, where* $q_i = q$ *and* $\delta_i = \delta$ *for all* $i \in N$*, and* $\mathbf{G}$ *is undirected. Let* $\mathbf{x}^{unc} = \mathbf{x}^{unc}((\mathbf{\Delta G}_{A(\mathbf{x})})^T, \mathbf{1})$*. Then*

- $i \in A(\mathbf{x}) \setminus Z(\mathbf{x}) \iff x_i^{unc} > 0$
- $i \in Z(\mathbf{x}) \iff x_i^{unc} = 0$
- $i \in SI(\mathbf{x}) \iff x_i^{unc} < 0$

Moreover, for $i \in A(\mathbf{x})$*,* $x_i = q x_i^{unc}$*.*

When players have identical characteristics in an undirected network, equilibria are such that the status of players (active or strictly inactive) coincides with the sign of their associated game's unconstrained solution's coordinate. To better understand the first and the second points, notice that in the general case, an active player i's contribution is given by $x_i = \sum_{j \in A(\mathbf{x})} q_j m_{ij}$ where $\mathbf{M} = (\mathbf{I} + \mathbf{\Delta G}_{A(\mathbf{x})})^{-1}$, a *weighted row sum* of the inverse equilibrium matrix restricted to active players. On the other hand, $x_i^{unc}((\mathbf{\Delta G}_{A(\mathbf{x})})^T, \mathbf{1})$ is the *unweighted column sum* of $(\mathbf{I} + \mathbf{\Delta G}_{A(\mathbf{x})})^{-1}$, i.e. $x_i^{unc} = \sum_{j \in N} m_{ji}$. However, in the case where the interaction matrix $(\mathbf{I} + \mathbf{\Delta G})$ is symmetric, we can show that $\sum_{j \in N} m_{ji} = \sum_{j \in A(\mathbf{x})} m_{ij}$. Since also needs are assumed to be homogeneous, i.e. $q_j = q$ for all j, we get the conclusion that for active players, contributions are proportional to the unconstrained solution of the associated problem. The rationale behind the third point, which may be more intricate, will be discussed in Proposition 5.

Notice that the correspondence between the status of a player and the sign of his unconstrained solution does not hold true when heterogeneity is introduced.

Finally, we state the following proposition which will allow us to derive results about the complete network as corollaries of our more general results.

Proposition 3. *Let* $\mathbf{G}$ *be a complete network and let* $\mathbf{x}$ *be the unique equilibrium of the game with parameters* $(\mathbf{q}, \mathbf{\Delta}, \mathbf{G})$*. Let* $\mathbf{x}^{unc} = \mathbf{x}^{unc}((\mathbf{\Delta G}_{A(\mathbf{x})})^T, \mathbf{1})$*. Then,*

- *If* $\delta_i \in (0, 1)$ *for all* i*,* $x_i^{unc} > 0$ *for all* i
- *If* $\delta_i = 1$ *for* $i \in A(\mathbf{x})$ *and* $\delta_j = (0, 1)$ *for all* $j \neq i$*,* $x_i^{unc} = 1$ *and* $x_j^{unc} = 0$

establishes that in cases where the interaction matrix is positive definite, player i is active at equilibrium if his unconstrained coordinate is strictly positive, while it is strictly negative if player i is strictly inactive. To our knowledge, this represents the only established connection between the unconstrained solution and an equilibrium.

[13]That is, when players have the same substitution rates and the network is undirected.

Since comparative statics will heavily rely on the signs of the coordinates of vector $\mathbf{x}^{unc}$, this proposition illustrates how the complete network is in fact a very special network, and how results holding on this network cannot be extended to arbitrary networks.

3 Shocks on Individual Characteristics

Since we wish to compare contributions before and after shocks, we consider an equilibrium $\mathbf{x}$ before the shock and $\mathbf{x}'$ after and define the following sets of players:

$$a(\mathbf{x}, \mathbf{x}') := \{i \in N; i \in A(\mathbf{x}) \cap SI(\mathbf{x}')\}$$
$$si(\mathbf{x}, \mathbf{x}') := \{i \in N; i \in SI(\mathbf{x}) \cap A(\mathbf{x}')\}$$

The first is the set of active players who become strictly inactive and the second is the set of strictly inactive players who become active.

3.1 Shocks and Individual Contribution

Here we analyze how a player's contribution is affected by an increase in his needs. We emphasize the following counter-intuitive observation: in the public good game with global and symmetric interactions (i.e. played on a complete network with the same substitution rates), it was established earlier that increasing the needs of one player always induces an increase in this player's contribution. However, this ceases to be true once a network structure is introduced, as illustrated in Figure 2, where an increase in player 1's needs induces a decrease in his contribution. This may seem counter-intuitive, but it is due to the complex pattern of interactions and substitutions between players.

(a) $q_1 = 0.5$ **(b)** $q_1 = 0.5125$

Figure 2: In Panel (a), a Nash equilibrium with homogeneous needs and substitution rates ($q_i = 0.5$ and $\delta_i = 0.5$ for all i). In Panel (b) needs of player 1 are increased and his contribution decreases.

However, once the matrix $(\mathbf{I} + \mathbf{\Delta G})$ is a P-matrix, we can guarantee that a player's contribution will increase when his needs increase:

Theorem 1. *Let* $(\mathbf{I}+\mathbf{\Delta G})$ *be a* P*-matrix, and let* $\mathbf{x}$ *be the unique Nash equilibrium of the game with parameters* $(\mathbf{q}, \mathbf{\Delta}, \mathbf{G})$*. Consider* $\mathbf{q}' = (q_1+\beta, q_2, ..., q_n)^T$*, the vector of needs where the needs of player* 1 *are increased by any amount* $\beta > 0$*, and let* $\mathbf{x}'$ *be the unique Nash equilibrium of the game with parameters* $(\mathbf{q}', \mathbf{\Delta}, \mathbf{G})$*. Then*

$$x_1' \geq x_1 \text{ if } x_1 = 0$$
$$x_1' > x_1 \text{ if } x_1 > 0$$

If in addition $(\mathbf{I}+\mathbf{\Delta G})$ *is symmetric, then*

$$x_1' - x_1 > \beta \text{ when } x_1 > 0$$

We detail here the main steps of the proof of Theorem 1, since several ideas from the paper are used to prove it. Obviously, if $x_1 = 0$ then $x_1' \geq x_1$. Now assume that $x_1 > 0$. The easy case is the following: Assume the two equilibria, before and after the increase in needs, are such that everyone is active. Then the equilibria are given by the solutions to:

$$(\mathbf{I}+\mathbf{\Delta G})\mathbf{x} = \mathbf{q} \text{ and } (\mathbf{I}+\mathbf{\Delta G})\mathbf{x}' = \mathbf{q}'$$

Letting $\mathbf{M}$ denote $(\mathbf{I}+\mathbf{\Delta G})^{-1}$, we have

$$\mathbf{x} = \mathbf{M}\mathbf{q} \text{ and } \mathbf{x}' = \mathbf{M}\mathbf{q}'$$

and since $q_j = q_j'$ for all $j \neq 1$ and $q_1' = q_1 + \beta$, we get

$$\mathbf{x}' - \mathbf{x} = \mathbf{M}(\mathbf{q}' - \mathbf{q}) = \mathbf{M}(\beta, 0, ..., 0)^T = \beta(m_{11}, ..., m_{n1})^T$$

Hence $x_1' - x_1 = \beta m_{11}$. Notice that $\mathbf{M}$ is a P-matrix, because the inverse of a P-matrix is also a P-matrix. Since all principal minors of a P-matrix are strictly positive, it follows that $m_{11} > 0$. Therefore $x_1' > x_1$.[14] For the second part of the theorem, notice that if $(\mathbf{I}+\mathbf{\Delta G})$ is symmetric then it is positive definite. This implies that $m_{ii} > 1$ for all i.[15] Thus, we have $x_1' > x_1 + \beta$, and any increase in a player's needs will be amplified through the network structure and will result in an even larger increase in action.

This specific case is easy to deal with, for two reasons: there are no SI players in $\mathbf{x}$, and sets A and SI remain unchanged between $\mathbf{x}$ and $\mathbf{x}'$. In the general case, $SI(\mathbf{x})$ could be non-empty, and $SI(\mathbf{x}')$ could be different from $SI(\mathbf{x})$.

Here we illustrate why these situations are complex to deal with. Assume $a(\mathbf{x}, \mathbf{x}') = \{2\}$ and $si(\mathbf{x}, \mathbf{x}') = \emptyset$, i.e. player 2 becomes strictly inactive after the needs of player

[14] This simple case illustrates why the interaction matrix needs to be a P-matrix for this monotony result to hold. Otherwise the term m_{11} could be negative, in which case $x_1' < x_1$, as in Figure 2.

[15] See for instance Fiedler (1964), where it is shown that the product of a diagonal term of a positive definite matrix and the diagonal term of its inverse is greater than 1.

1 increase. Then $\mathbf{x} = \mathbf{Mq}$ is still true, however, $\mathbf{x}' \neq \mathbf{Mq}'$ since, by equation (7), $\mathbf{x}' = (\mathbf{I} + \boldsymbol{\Delta}\mathbf{G}_{N\setminus\{2\}})^{-1}\mathbf{q}_{N\setminus\{2\}}$, and thus operations like the above with only active players can no longer be performed.

However, we can write $\mathbf{G}_{N\setminus\{2\}} = \mathbf{G} - \mathbf{0_{-2}}$, where $\mathbf{0_{-2}}$ is a matrix of 0's except for row 2. Then, $(\mathbf{I} + \boldsymbol{\Delta}\mathbf{G}_{N\setminus\{2\}})\mathbf{x}' = \mathbf{q}_{N\setminus\{2\}} \implies (\mathbf{I} + \boldsymbol{\Delta}\mathbf{G})\mathbf{x}' = \mathbf{q}_{N\setminus\{2\}} + \delta_2\mathbf{0_{-2}}\mathbf{x}'$ and therefore $\mathbf{x}' = \mathbf{Mq}_{N\setminus\{2\}} + \delta_2\mathbf{M}\mathbf{0_{-2}}\mathbf{x}'$. By developing, we finally get $x_1' - x_1 = \beta m_{11} + (\delta_2 \sum_{i\in N} g_{2i}x_i' - q_2)m_{12}$.

It can be seen that the appearance of a new strictly inactive player adds the term $(\delta_2 \sum_{i\in N} g_{2i}x_i' - q_2)m_{12}$ to the previous situation where everyone is active. We know that $(\delta_2 \sum_{i\in N} g_{2i}x_i' - q_2) > 0$ since player 2 is strictly inactive in $\mathbf{x}'$, but the sign of m_{12} depends on the specific structure of the network and cannot be predicted by simple network statistics.[16]

In the same way, the appearance of new active players will add other terms to the difference $x_1' - x_1$. It is not possible, in general, to sign each of these terms, even less possible to sign the sum of these terms. However, when the interaction matrix is a P-matrix, we can show that the difference $x_1' - x_1$ is always positive (see the details in the proof).

Remark 2. *Usually in the literature on comparative statics in this game, the set of active players is held fixed before and after. Theorem 1 holds regardless of possible changes in the set of active and strictly inactive players at equilibrium.*

3.2 Shocks and Total Contributions

Here we analyze how the change in needs of one player affects total contributions. We first present the general case and then look into the special case of identical players on an undirected network as it lends itself to a nice interpretation of the results.

To start, let us assume that the set of active agents remains the same after the increase, except for player 1 who could become active or strictly inactive depending on his initial status, i.e. $a(\mathbf{x}, \mathbf{x}') \setminus \{1\} = si(\mathbf{x}, \mathbf{x}') \setminus \{1\} = \emptyset$. Then we have the following:

Proposition 4. *Let* $\mathbf{x}$ *be an equilibrium of the game with parameters* $(\mathbf{q}, \boldsymbol{\Delta}, \mathbf{G})$*. Let* $\mathbf{q}' = (q_1 + \beta, q_2, ..., q_n)^T$ *with* $\beta > 0$ *and* $\mathbf{x}'$ *be an equilibrium of the game with parameters* $(\mathbf{q}', \boldsymbol{\Delta}, \mathbf{G})$*, such that* $a(\mathbf{x}, \mathbf{x}') \setminus \{1\} = si(\mathbf{x}, \mathbf{x}') \setminus \{1\} = \emptyset$*. Let* $\mathbf{x}^{unc} = \mathbf{x}^{unc}((\boldsymbol{\Delta}\mathbf{G}_{A(\mathbf{x})})^T, \mathbf{1})$*. Then,*

- *If* $x_1 = x_1' = 0$*, then* $X' = X$
- *Otherwise,* $Sign(X' - X) = Sign(x_1^{unc})$

[16] The only networks for which the signs of the terms of the inverse of the interaction matrix are predetermined are the tree networks (see Roy and Xue (2021)).

Proposition 4 provides a simple way to check whether contributions will increase or decrease: what matters is whether player 1 has a positive or a negative coordinate in the associated problem's unconstrained solution. One noteworthy point is that although there is no immediate relation between $\mathbf{x}^{unc}((\mathbf{\Delta G}_{A(\mathbf{x})})^T, \mathbf{1})$ and $\mathbf{x}$ or $\mathbf{x}'$, it is $\mathbf{x}^{unc}((\mathbf{\Delta G}_{A(\mathbf{x})})^T, \mathbf{1})$ which determines aggregate effects of a shock. Surprisingly, increasing needs of some players (through an increase in wealth or a decrease in costs for instance) will decrease total contributions. This is counter-intuitive since usual comparative statics on the complete network conclude that an increase in needs of one player will always result in an increase in total contributions. This can be seen by combining Proposition 4 with Proposition 3, where it is established that $x_i^{unc} \geq 0$ for all i, hence $X' \geq X$. This, again, illustrates how the complete network is very specific.

Remark 3. *In fact, when* $1 \in A(\mathbf{x}) \cap A(\mathbf{x}')$*, i.e. when player* 1 *is active in both equilibria, then* $X' - X = \beta x_1^{unc}$*. Thus,* x_1^{unc} *can be interpreted as the marginal increase of total contributions resulting from an increase in the needs of player 1. This interpretation, however, only holds in the context of Proposition 4, where the set of active players does not change after the shock on player* 1*.*

Another surprising implication is that some players have no impact on total contributions: changing the needs of a player who plays 0 in the unconstrained solution of the associated game played with homogeneous needs, will change the equilibrium contributions of potentially all players, including himself, but will leave the sum of contributions unchanged. We call these players *neutral players.*

Definition 2. *Let* $\mathbf{x}$ *be an equilibrium of the game with parameters* $(\mathbf{q}, \mathbf{\Delta}, \mathbf{G})$*. Then, if* $x_i^{unc}((\mathbf{\Delta G}_{A(\mathbf{x})})^T, \mathbf{1}) = 0$*, player* i *is called a neutral player at* $\mathbf{x}$*.*

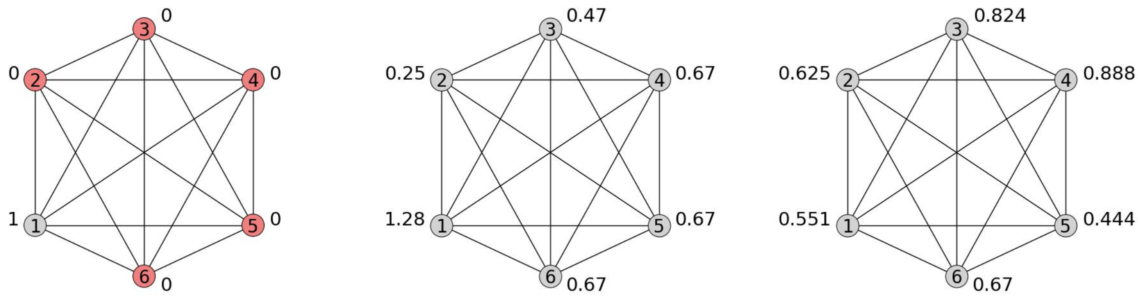
Remark 4. *By Proposition 3, in the complete network with* $\delta_1 = 1$*, every player except player* 1 *is neutral, as illustrated in Figure 3. In particular, unless the shock hits player* 1*, individual shocks on needs leave total contributions unchanged.*

We now turn to the general case where the set of active players may change after the shock.

Theorem 2. *Let* $\mathbf{x}$ *be an equilibrium of the game with parameters* $(\mathbf{q}, \mathbf{\Delta}, \mathbf{G})$*. Let* $\mathbf{q}' = (q_1 + \beta, q_2, ..., q_n)^T$ *with* $\beta > 0$ *and* $\mathbf{x}'$ *be any equilibrium of the game with parameters* $(\mathbf{q}', \mathbf{\Delta}, \mathbf{G})$*. Let* $\mathbf{x}^{unc} = \mathbf{x}^{unc}((\mathbf{\Delta G}_{A(\mathbf{x})})^T, \mathbf{1})$*. Then,*

$$\begin{cases} x_1^{unc} \geq 0, x_i^{unc} \geq 0 \text{ for all } i \in a(\mathbf{x}, \mathbf{x}') \cup si(\mathbf{x}, \mathbf{x}') & \Longrightarrow X' \geq X \\ x_1^{unc} \leq 0, x_i^{unc} \leq 0 \text{ for all } i \in a(\mathbf{x}, \mathbf{x}') \cup si(\mathbf{x}, \mathbf{x}') & \Longrightarrow X' \leq X \end{cases}$$

In addition to checking the sign of the coordinate of $\mathbf{x}^{unc}$ for the player whose needs have changed, we must now also consider the sign of coordinates for players whose status



(a) Unconstrained solution **(b)** $\mathbf{q} = (4, 1, 1, 1, 1, 1)^T$ **(c)** $\mathbf{q} = (4, 1.3, 1.3, 1.2, 0.8, 1)^T$

Figure 3: Let $\boldsymbol{\delta} = (1, 0.2, 0.15, 0.1, 0.1, 0.1)$. Panel (a) is the solution of the unconstrained problem. We see that $x_i^{unc} = 0$ for all $i \neq 1$, which implies that all players except player 1 are neutral. We then start with needs $\mathbf{q}$ and find the initial equilibrium in Panel (b), where the total contribution is 4. Then in Panel (c), needs of player 2 to 5 are modified, and although the equilibrium profile is completely changed, the total contribution remains constant.

changes after the shock. The intuition behind this is the following: when the set of active players changes after the shock, the equilibrium interaction matrix changes as well, from $(\mathbf{I} + \mathbf{\Delta G}_{A(\mathbf{x})})$ to $(\mathbf{I} + \mathbf{\Delta G}_{A(\mathbf{x}')})$. Therefore, $\mathbf{x}'$ is the (unique) solution of $(\mathbf{I} + \mathbf{\Delta G}_{A(\mathbf{x}')})\mathbf{x}' = \mathbf{q}'_{A(\mathbf{x}')}$. However, as we show in the proof, we can find a suitable needs vector, $\tilde{\mathbf{q}}$, such that $\mathbf{x}'$ is also the (unique) solution of $(\mathbf{I} + \mathbf{\Delta G}_{A(\mathbf{x})})\mathbf{x}' = \tilde{\mathbf{q}}$, where the needs of players who have changed status between $\mathbf{x}$ and $\mathbf{x}'$ are increased, but the interaction matrix remains the same as the initial equilibrium $\mathbf{x}$. Since the effect sign of increasing needs of player i depends on the sign of x_i^{unc}, as established in Proposition 4, we get the result.

Finally, if we focus on the standard case of identical players (i.e. with identical needs and substitution rates) interacting on an undirected and unweighted network, by combining Proposition 2 and Theorem 2 we have the following easy to interpret result:

Proposition 5. *Assume the network* $\mathbf{G}$ *is undirected. Let* $\mathbf{x}$ *be an equilibrium with parameters* $(\mathbf{q}, \mathbf{\Delta}, \mathbf{G})$ *where* $q_i = q$ *and* $\delta_i = \delta$ *for all* i*. Let* $\mathbf{x}'$ *be an equilibrium with parameters* $(\mathbf{q}', \mathbf{\Delta}, \mathbf{G})$ *where* $\mathbf{q}' = (q + \beta, q, ..., q)^T$ *with* $\beta > 0$*. Then,*

- $1 \in A(\mathbf{x}) \setminus Z(\mathbf{x})$ *and* $si(\mathbf{x}, \mathbf{x}') = \emptyset \implies X' > X$
- $1 \in Z(\mathbf{x})$ *and* $si(\mathbf{x}, \mathbf{x}') = a(\mathbf{x}, \mathbf{x}') = \emptyset \implies X' = X$
- $1 \in SI(\mathbf{x})$ *and* $a(\mathbf{x}, \mathbf{x}') = \emptyset \implies X' < X$ *if* $x'_1 > 0$ *and* $X' \leq X$ *if* $x'_1 = 0$

All three points are partial consequences of Proposition 2 and Theorem 2. What is missing, and that we establish with this proposition, is that when players are identical and the network is undirected, players who become strictly inactive after the shock necessarily have a positive coordinate in their unconstrained solution, while players who become active after the shock necessarily have a negative one. Notice that this is true even if player 1 himself changes status.

While the first point is in line with intuition, and while the second point is in line with earlier comments about neutral players, the third point may seem counter-intuitive. Although intuition suggests that free-riders are those driving contributions down and that these are the players that should be incited to contribute, Proposition 5 tells us precisely the opposite. Increasing the needs of a strictly inactive player until he becomes active will have a negative effect on total contributions, despite the fact that this player is now contributing a positive amount.

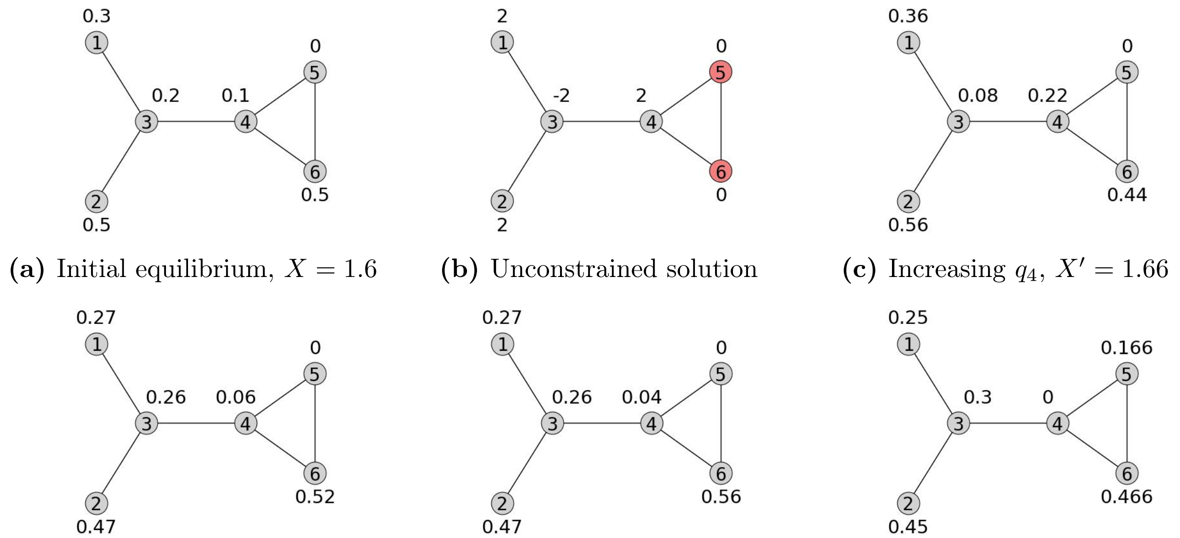
To understand why this can happen, note that a player is active whenever the contribution of his neighbors is small, while he is strictly inactive whenever it is large. Additionally, as Proposition 2 illustrates, with a symmetric interaction matrix, the contribution of each player is proportional to the unconstrained solution whenever a player is active and needs are homogeneous. This suggests that a strictly inactive player is surrounded by neighbors who exert a large outgoing effect on the entire network. Consequently, an increase in the contribution of a strictly inactive player negatively affects the needs of their neighbors, who in turn have a substantial impact on the entire network. We show in the proof that the negative effect stemming from the impact of neighbors dominates the positive effect of the increase in contribution of the strictly inactive player itself, and the net effect is proportional to $(q - \delta\overline{x}_1)$ which is negative since $1 \in SI(\mathbf{x})$.

We summarize and illustrate results of this section in the following example.

Example 1. Consider the network of Figure 4, with $\delta_i = 0.5$ for all i. Since the matrix $(\mathbf{I}+\mathbf{\Delta G})$ is symmetric and positive definite, and thus a P-matrix, there is a unique Nash equilibrium for any vector of needs. Take needs $\mathbf{q} = (0.4, 0.6, 0.65, 0.45, 0.25, 0.55)^T$. The unique equilibrium $\mathbf{x}$ is $(0.3, 0.5, 0.2, 0.1, 0, 0.5)^T$, where $SI(\mathbf{x}) = \{5\}$. To know whether increasing the needs of some players will increase or decrease total contributions, we need to construct the interaction matrix $(\mathbf{I} + \mathbf{\Delta G}_{A(\mathbf{x})})^T$ from $(\mathbf{I} + \mathbf{\Delta G})$ by taking out the incoming links of player 5 and transposing it:

$$(\mathbf{I}+\mathbf{\Delta G}) = \begin{pmatrix} 1 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 1 & 0.5 & 0 & 0 & 0 \\ 0.5 & 0.5 & 1 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 1 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0.5 & 1 & 0.5 \\ 0 & 0 & 0 & 0.5 & 0.5 & 1 \end{pmatrix} \quad (\mathbf{I}+\mathbf{\Delta G}_{A(\mathbf{x})})^T = \begin{pmatrix} 1 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 1 & 0.5 & 0 & 0 & 0 \\ 0.5 & 0.5 & 1 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 1 & 0 & 0.5 \\ 0 & 0 & 0 & 0.5 & 1 & 0.5 \\ 0 & 0 & 0 & 0.5 & 0 & 1 \end{pmatrix}$$

We now solve for $(\mathbf{I} + \mathbf{\Delta G}_{A(\mathbf{x})})^T\mathbf{x} = \mathbf{1}$ and find $\mathbf{x}^{unc} = (2, 2, -2, 2, 0, 0)^T$. According to what precedes, this implies that increasing the needs of players $1, 2$ or 4 will increase the total contributions (since $x_1^{unc} = x_2^{unc} = x_4^{unc} = +2$), increasing the needs of player 3 will decrease it (since $x_3^{unc} = -2$), and increasing the needs of player 6 will leave the total contributions unchanged since player 6 is neutral, as long as the set of strictly inactive players remains the same. Finally, an increase of needs of player 5 changes the set of actives, player 5 becoming active while player 4 becomes strictly inactive. According to Theorem 2, this will increase total contributions, since $x_5^{unc} \geq 0$ and $x_4^{unc} \geq 0$. Figure 4 illustrates these different situations.



(a) Initial equilibrium, $X = 1.6$ **(b)** Unconstrained solution **(c)** Increasing q_4, $X' = 1.66$

(d) Increasing q_3, $X' = 1.58$ **(e)** Increasing q_6, $X' = 1.6$ **(f)** Increasing q_5, $X' = 1.633$

Figure 4: In the initial equilibrium needs are $\mathbf{q} = (0.4, 0.6, 0.65, 0.45, 0.25, 0.55)^T$. In Panel (c) the needs of player 4 are increased to 0.48 and total contributions increase. In Panel (d) needs of player 3 are increased to 0.66 and total contributions decrease. In Panel (e) needs of player 6 are increased to 0.58 and total contributions are unchanged. In Panel (f) needs of player 5 are increased to 0.4, player 4 becomes strictly inactive, and total contributions increase. These effects can be inferred from the the unconstrained solution in Panel (b).

Putting together the different results from this section, and using Propositions 1 and 3, we retrieve the standard results for the complete network, which we extend to the non-symmetric case:

Corollary 2. *If* $\mathbf{G}$ *is a complete network, then for any* $\mathbf{q}$ *and* $\boldsymbol{\Delta}$*, increasing a player's needs results in an increase in his contribution. Furthermore, this increase induces an increase in total contribution.*

We finish this section with the following remark:

Remark 5. *As illustrated in Figure 1, increasing the needs of player* 1 *can either decrease or increase the number of equilibria. Theorem 2 and Propositions 4 and 5 hold true for every equilibrium.*

4 Planner's Interventions

In this section we analyze the effects of two public policies that have been discussed in the literature with complete networks. First, we consider transfers of wealth among players and we show that, contrary to the complete network case, aggregate neutrality never holds except in very specific situations. Second, we look at the effects of the state publicly providing some baseline level of public goods, either from external resources or from taxes collected from agents, and we show that, surprisingly, providing more public goods to players can actually increase the total provision of the players, again contradicting the results on complete networks. For ease of exposition we will use the Cobb-Douglas payoff function (5) in this section.

Before proceeding further, we highlight the following: in the previous section, we examined shocks that affected only one player. However, in this section, we are considering shocks that simultaneously impact the needs of multiple players. However, as the system solved by an equilibrium is linear, increasing the needs of multiple players results in simply aggregating their individual effects, as long as the set of actives remains unchanged. Thus, increasing needs of all players from $\mathbf{q}$ to $\mathbf{q} + \boldsymbol{\beta}$ where $\boldsymbol{\beta} = (\beta_i)_{i \in N}$ results in a change of total contributions of

$$X' - X = \sum_{i \in N} \beta_i x_i^{unc}$$

4.1 Transfers and Neutrality

The neutrality result of Bergstrom et al. (1986) states that, if the public good is pure and the interactions take place on a complete network, a small income redistribution among active players changes their contribution by exactly the amount of the transfer received. Thus, total contributions do not change.

Andreoni (1990) considers possibly impure public goods (i.e. possibly $\gamma_i \leq 1$ and $\gamma_i \neq \gamma_j$), still on complete networks, and shows that the previous neutrality result holds if and only if $\gamma_i = 1\ \forall i$. However, he also shows that if the ratio of the marginal demand for the public good with respect to wealth and neighbors' contributions[17] is equal between the two players involved in the transfer, then we have an *aggregate neutrality* result, according to which the sum of contributions will remain constant after a transfer. Of course, aggregate neutrality is weaker than neutrality. Nevertheless, it is still a strong result.

Allouch (2015) takes Bergstrom et al. (1986) to networks to check whether neutrality still holds on non-complete networks,[18] and proves that it only holds on specific networks

[17] Andreoni (1990) calls this coefficient, which is equal to $\frac{\lambda_i}{1-\delta_i}$ in our framework, the altruism coefficient.

[18] Transposed into our setting, Allouch (2015) restricts the analysis to networks such that the interaction matrix is positive definite, thereby guaranteeing uniqueness of the solution, in a class of games that

where all active players are linked together and where strictly inactive players are either linked to every active player or to none. Thus, neutrality tends to fail once some heterogeneity is introduced into the pattern of interactions. Here, we aim to determine the conditions under which aggregate neutrality holds. In order to compare with previous literature, we also restrict to transfers that leave the set of active players unchanged.

As mentioned earlier, a change in players' wealth translates into a change in needs. By taking the Cobb-Douglas payoff (5) with heterogeneous public good's preference parameters λ_i, needs are given by $q_i = \lambda_i w_i$, and the substitution degree δ_i is given by $(1 - \lambda_i)\gamma_i$. Thus, considering a vector of transfers of wealth $\mathbf{t} = (+\beta, -\beta, 0, ..., 0)^T$ from player 2 to player 1, is equivalent to modifying needs from $\mathbf{q} = (q_1, q_2, ..., q_n)^T$ to $\mathbf{q'} = (q_1 + \lambda_1\beta, q_2 - \lambda_2\beta, ...q_n)^T$. We use the additivity of the effects of changing needs to obtain the following:

Proposition 6 (Transfers)**.** *Let* $\mathbf{x}$ *be an equilibrium of the game played on network* $\mathbf{G}$ *with substitution rates* $\mathbf{\Delta}$ *and with wealth vector* $\mathbf{w} = (w_1, \cdots, w_n)^T$*, and* $\mathbf{x}'$ *be an equilibrium of the same game with wealth vector* $\mathbf{w} + \mathbf{t} = (w_1 + \beta, w_2 - \beta, \cdots, w_n)^T$ *where* $\beta > 0$*. Assume that* $A(\mathbf{x}) = A(\mathbf{x}')$ *and let* $\mathbf{x}^{unc} = \mathbf{x}^{unc}((\mathbf{\Delta G}_{A(\mathbf{x})})^T, \mathbf{1})$*. Then,*

$$X' - X = \beta(\lambda_1 x_1^{unc} - \lambda_2 x_2^{unc})$$

This formula contains several messages. First, a transfer between individuals sharing identical substitution rates and interacting with the same set of neighbors in the network (and thus having the same unconstrained component) will result in an increase of total contributions only if the preference for the public good of the player receiving the transfer is larger than that of the other player.

Second, transfers between players with identical preferences for the public good will not be neutral in general, since players occupy different positions in the network, resulting in $x_1^{unc} \neq x_2^{unc}$. So, for aggregate neutrality to hold, it is necessary that both types of heterogeneity (in individual characteristics and in the interaction patterns) compensate each other.

Third, the result of Andreoni (1990) can be recovered thanks to the symmetry of positions of players in a complete network:

Corollary 3 (Andreoni)**.** *Let* $\mathbf{G}$ *be a complete network and let* $\alpha_i = \frac{\lambda_i}{1-\delta_i}$*. Then*

$$Sign(X' - X) = Sign(\alpha_1 - \alpha_2)$$

Note that α_i is the altruism degree defined in Andreoni (1990). When $\lambda_1 = \lambda_2$ and $\gamma_1 = \gamma_2$, then $\alpha_1 = \alpha_2$ and $X' = X$, so that aggregate neutrality holds on the complete network.

includes some linear best-response games. Our analysis does not restrict to situations where equilibrium is unique, it also applies to all games with linear best-responses, but to those alone.

To get some intuition on these three messages, let us focus on the case where the equilibrium interaction matrix is of spectral radius smaller than 1. In that case, we can use the power series development:

$$(\mathbf{I} + \mathbf{\Delta G})^{-1} = \sum_{k=0}^{\infty} (-1)^k (\mathbf{\Delta G})^k,$$

where the term $(\mathbf{\Delta G})^k_{ij}$ is the number of paths of length k going from player i to player j, where each link between any player l and any other player is discounted by δ_l. When a transfer takes place between player 1 and 2, it is easy to show that the net effect only depends on paths leaving from player 1 and player 2, the former being discounted by δ_1, while the latter are discounted by δ_2. Since the network is complete, to each path leaving from 1 and reaching any other player i in k steps, we can associate a path leaving from 2, reaching i in k steps, and going through the same set of players.[19] The aggregate effect of the transfer will thus be captured by the different effects of all these paths in the network. Therefore, the (positive) effect stemming from player 1 getting richer will be discounted by δ_1, while the (negative) effect stemming from player 2 getting poorer will be discounted by δ_2. Hence, the change in total contributions only depends on individual characteristics of the players, captured by the coefficients α_i, and not on the the number of paths through which effects take place.

However, once the network is not complete, paths leaving from 1 and reaching any other player i in k steps cannot be associated to an equivalent path leaving from 2 and reaching i in k steps. This asymmetry in the network explains why aggregate effects cannot be captured as simply as with the complete network.

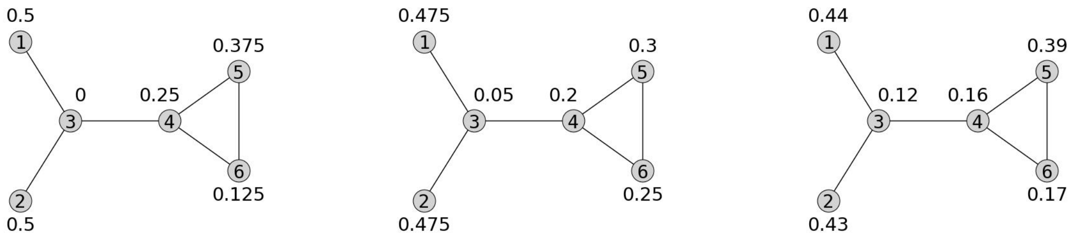
Last, by using previous observations on neutral players, we observe that transfers taking place between neutral players will leave the total contributions unchanged, despite changes in individual contributions.

Proposition 6 is illustrated in Figure 5.

4.2 State Provision and Taxes

Here, we assume that the state initially provides a public good for everyone to benefit from, either directly or through taxes $\mathbf{t} = (t_i)_{i \in N}$ collected from agents, for a total amount S. Our setting covers the case where $S = \sum_{i \in N} t_i$, which corresponds to the standard setting of Bergstrom et al. (1986) and Andreoni (1990) where the state provision is financed by taxes. It also covers the case where $S > 0$ and $t_i = 0$ for all i, which corresponds to the setting of Acemoglu and Jensen (2013) where the public good is

[19] It is in fact a bit more subtle, since player 1 can reach player 2 in one step while player 2 cannot. We leave these subtleties out since they do not change the intuition.



(a) Before transfer
$\mathbf{w} = (1, 1, 1.25, 1, 1, 1)$
$X = 1.75$

(b) Transfer from 5 to 6
$w_5 = 0.9, w_6 = 1.1$
$X = 1.75$

(c) Transfer from 2 to 3
$w_2 = 0.98, w_3 = 1.27$
$X = 1.71$

Figure 5: Take the Cobb-Douglas utility function (5) with $\lambda_i = \lambda = 0.5$ for all i, hence $q_i = 0.5w_i$. Take $\boldsymbol{\delta} = (0.5, 0.5, 0.5, 0.5, \frac{1}{3}, 0.6)$. Panel (a) shows the initial equilibrium, while the unconstrained solution of the associated game is given in Panel (b) of Figure 4, where players 5 and 6 are neutral. In Panel (b), a transfer happens from players 5 to 6, and the total contribution remains unchanged because both are neutral. In Panel (c), a transfer happens from players 2 to 3, and the sum of contributions has decreased because $x_3^{unc} < x_2^{unc}$.

directly provided by the state without taxes.

By taking the Cobb-Douglas payoff as in (5), the best-response functions with state provision S and tax vector $\mathbf{t}$ are given by

$$\forall i \in N, \;\; BR_i(\mathbf{x}_{-i}) = \max\{\lambda_i(w_i - t_i) - (1-\lambda_i)\gamma_i(\overline{x}_i + S), 0\}$$

We consider that the state increases its provision from S to S', and increases taxes from $\mathbf{t} = (t_i)_{i\in N}$ to $\mathbf{t}' = (t_i')_{i\in N}$, and we analyze what happens with total contributions. Consider for instance the case where $S > 0$ and $t_i = 0$ for all i. This is equivalent to assuming that the needs of every player are reduced by the same amount. A natural intuition thus leads us to the conclusion that potentially every player should decrease its contribution, or at least that total contributions will decrease.

Surprisingly again, this intuition works on the complete network but not necessarily otherwise, since what matters in determining whether total contributions will increase or decrease is the unconstrained solution to the associated problem. Therefore, a planner willing to increase total contributions might have to increase or decrease the state provision, depending on the interaction patterns that he is facing.

We now analyze the general case. Let $\beta_S = S' - S$ and $\beta_{t_i} = t_i' - t_i$.

Proposition 7 (State baseline provision)**.** *Assume the payoff function is a Cobb-Douglas as in (5), and the state provision increases by β_S and the tax of each player increases by β_{t_i} for each i. Let $\mathbf{x}$ be an equilibrium before the increase and $\mathbf{x}'$ be an equilibrium after* *the increase. Let* $\mathbf{x}^{unc} = \mathbf{x}^{unc}((\mathbf{\Delta G}_{A(\mathbf{x})})^T, \mathbf{1})$. *Then,*

$$X' - X = -\sum_{i \in A(\mathbf{x}) \cap A(\mathbf{x}')} (\delta_i \beta_S + \lambda_i \beta_{t_i})\, x_i^{unc} + \sum_{i \in si(\mathbf{x},\mathbf{x}')} x_i' x_i^{unc} + \sum_{i \in a(\mathbf{x},\mathbf{x}')} (\delta_i \overline{x}_i' - q_i)\, x_i^{unc} \quad (9)$$

As we can see, the aggregate effect of the state collecting taxes and providing some public good again depends on the signs of the coordinates of $\mathbf{x}^{unc}$. Therefore, if players with negative coordinates are impacted by the change, the intuitive effects could be reversed.

This is the case for instance in the example of Figure 6, where we assume that no taxes are collected, but the state increases its provision from $S = 0$ to $S' = 0.05$. Every player is in $A(\mathbf{x}) \cap A(\mathbf{x}')$ and $\mathbf{x}^{unc} = (-6, -6, 11.66, -9.33, 4.66, 3.33)$, so that $-\sum_{i \in A(\mathbf{x}) \cap A(\mathbf{x}')} \delta_i \beta_S x_i^{unc} = +0.0166$, illustrating how an increase in state provision will, in this case, actually increase total contributions to the public good.

(a) State baseline $S = 0$, $X = 6$ **(b)** State baseline $S' = 0.05$, $X' = 6.0166$

Figure 6: Panel (a) represents the unique Nash equilibrium with $\boldsymbol{\delta} = (0.5, 0.5, 0.6, 0.5, 0.5, 0.3)$ and $\mathbf{q} = (1.5, 1.5, 2.8, 2.5, 2, 1.6)^T$. When the state provides an amount $S' = 0.05$ of public good for every player, total contributions increase as shown in Panel (b).

However, on the complete network, total contributions decrease as expected:

Corollary 4. *Assume the network* $\mathbf{G}$ *is complete. Then*

- *Private provision decreases as the state provision* S *increases (Proposition 1 in Acemoglu and Jensen (2013));*
- *If* $\beta_S > \sum_{i \in A(\mathbf{x})} \beta_{t_i}$, *i.e. the increase in the state provision is larger than the increase in total tax collected from actives, the sum of private and state provisions increases. In particular, if* $\beta_S = \sum_{i \in N} \beta_{t_i}$ *and* $\beta_{t_i} > 0$ *for at least one strictly inactive player, the sum of private and state provisions increases. (Theorem 6 in Bergstrom et al. (1986));*
- *If the set of active players does not change, if* $\beta_S = \sum_i \beta_{t_i}$, *i.e. the increase in state provision is financed by the increase in tax, and if* $\beta_{t_i} = 0$ *for all strictly inactive players, then the sum of private and state provisions increases if and only if there is a player such that* $\beta_{t_i} > 0$ *and* $\gamma_i < 1$ *(Proposition 3 in Andreoni (1990)).*

To understand the second point, consider the effects of the state taxing a strictly inactive player to provide some public good. First, active players adjust their private provisions (first term in equation (9)). Second, the taxed player might start contributing (second term in equation (9)). In a complete network, the first term is negative but offset by T, and the second term is positive. Thus, both effects lead to an overall increase in total contributions.

5 Shocks on Interaction Patterns

We consider three possible shocks on the way individuals interact. Interactions are mediated through the structure of the network and through substitution rates. We thus look into changing the network either by adding or removing links between existing players, or by adding a new player to the network, while we look into the impact of substitution rates by changing the diagonal matrix $\mathbf{\Delta}$. We illustrate each time how results from the literature that hold on complete networks fail to hold on arbitrary networks.

In fact, modifying interaction patterns is in some sense equivalent to changing needs of players.

Let $\mathbf{G}^\epsilon = \mathbf{G} + \mathbf{E}$ be a network resulting from a modification of $\mathbf{G}$, where $\mathbf{E} = (\epsilon_{ij})_{i,j \in N}$ is a matrix of individual shocks on links such that $0 \leq g^\epsilon_{ij} \leq 1$ for all i, j. We have the following equivalence result:

Proposition 8. *Let* $\mathbf{x}^\epsilon$ *be an equilibrium of the game with parameters* $(\mathbf{q}, \mathbf{\Delta}, \mathbf{G}^\epsilon)$. *Then* $\mathbf{x}^\epsilon$ *is also an equilibrium of the game with parameters* $(\mathbf{q} + \boldsymbol{\psi}^\epsilon, \mathbf{\Delta}, \mathbf{G})$, *where* $\psi^\epsilon_i = -\delta_i \sum_{j \in N} \epsilon_{ij} x^\epsilon_j$.

Thus, any modification of the network $\mathbf{G}$ can also be modeled as an appropriate modification of needs of players. Also, since these modifications affect several players at once, we can use slight adaptations of Equation (9) in what follows.

Equation (9) contains three terms. Those concerning players directly affected by the change and who do not change status, those who were active before and strictly inactive after, and those who were strictly inactive before and active after. The two later terms can always have positive or negative signs, depending on the signs of elements of $\mathbf{x}^{unc}$. Therefore, variations in total contributions can go in both directions, due to the change in the status of some players. However, we argue that the inability to generalize results from complete networks to arbitrary networks does not hinge on players changing status (as in complete networks, players may also change status depending on parameters), but rather on the distinct nature of complete networks as a specific interaction structure. Therefore, in what follows, we only restrict attention to situations where the set of active players does not change before and after the shock, i.e. $a(\mathbf{x}, \mathbf{x}') = si(\mathbf{x}, \mathbf{x}') = \emptyset$. Details of what happens when this set changes are relegated to the appendix.

Proposition 9 (Shocks on Interaction Patterns)**.** *Let* $\mathbf{x}$ *be an equilibrium of the game with parameters* $(\mathbf{q}, \mathbf{\Delta}, \mathbf{G})$ *and assume* $1, 2 \in A(\mathbf{x})$ *and* $g_{12} = g_{21} = 0$*. Let* $\mathbf{x}^{unc} = \mathbf{x}^{unc}((\mathbf{\Delta}\mathbf{G}_{A(\mathbf{x})})^T, \mathbf{1})$*.*

- **Adding a link:** *Let* $\mathbf{G}'$ *be such that* $g'_{12} = g'_{21} = 1$*, and* $g_{ij} = g'_{ij}$ *otherwise, and let* $\mathbf{x}'$ *be an equilibrium with parameters* $(\mathbf{q}, \mathbf{\Delta}, \mathbf{G}')$*. Then,*

$$X' - X = -\,\delta_1 x'_2 x_1^{unc} - \delta_2 x'_1 x_2^{unc}$$

- **Changing substitution rates:** *Let* $\mathbf{\Delta}'$ *be such that* $\delta'_1 = \delta_1 + \epsilon$ *(with* $\epsilon > 0$*),* $\delta'_i = \delta_i$ *otherwise, and let* $\mathbf{x}'$ *be an equilibrium with parameters* $(\mathbf{q}, \mathbf{\Delta}', \mathbf{G})$*. Then,*

$$X' - X = -\,\epsilon \overline{x}'_1 x_1^{unc}$$

- **Entry of a player:** *Let* $N^+ = N \cup \{n+1\}$*,* $\mathbf{q}^+ = (q_1, \cdots, q_{n+1})$ *and* $\mathbf{\Delta}^+ = diag(\delta_1, \cdots, \delta_{n+1})$*, and let* $\mathbf{G}^+$ *be such that* $g^+_{ij} = g_{ij}$ *for* $i, j \in N$*. Assume* $\mathbf{x}^+$ *is an equilibrium of the game with parameters* $(\mathbf{q}^+, \mathbf{\Delta}^+, \mathbf{G}^+)$*. Then,*

$$X^+_{|n} - X = -x^+_{n+1} \sum_{i \in A(\mathbf{x})} g^+_{i,n+1} \delta_i x_i^{unc}$$

where $X^+_{|n}$ *is the sum of contributions of all players except the newcomer.*

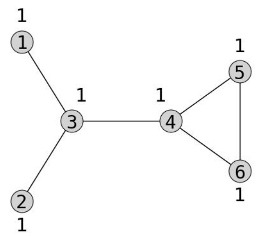
All three results in this proposition show that total contributions can decrease, like with the complete network, but they can also increase if the players who are affected by the shock have negative components in the unconstrained solution of the associated problem.

In Bramoullé et al. (2014), the authors consider players with identical needs and substitution rates interacting on an undirected network. Under these assumptions they show (Proposition 8) that adding a link or increasing substitution rates reduces total contributions in the highest equilibrium. The first two points of our proposition show that this is no longer true as soon as heterogeneity is present, since players could have negative coordinates in $\mathbf{x}^{unc}$.

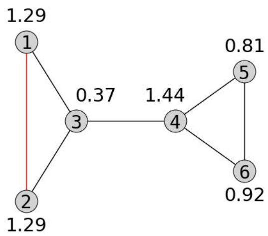
Finally, regarding the third point, it is known (Acemoglu and Jensen (2013)) that in the complete network, entry does not necessarily lead to an increase in total contributions, inclusive of the contribution of the new player, but it does decrease them, exclusive of the contribution of the new player. Here again, our proposition shows that this is not true once arbitrary networks are considered, if the new player is linked to players with $x_i^{unc} < 0$.

Figure 7 describes an example where adding a link, increasing substitution rates or entry of a new player into the game actually increases total contributions, contrary to what happens in the complete network.

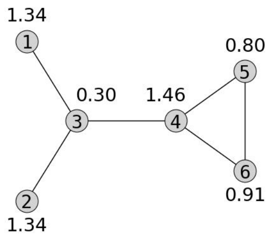
Noticing that in a complete network, $x_i^{unc} \geq 0$ for every player, we recover the standard results:



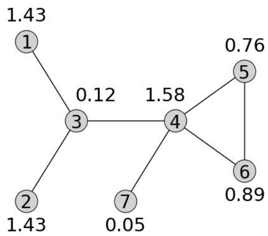
(a) Initial equilibrium: $X = 6$



(b) Adding a link $g'_{12} = g'_{21} = 0.02$: $X' = 6.15$



(c) Changing substitution rate, $\delta'_4 = 0.51$: $X' = 6.19$



(d) Entry of player 7 ($q_7 = 1$ and $\delta_7 = 0.6$): $X^+_{|6} = 6.239$

Figure 7: Panel (a) shows the unique Nash equilibrium with $\boldsymbol{\delta} = (0.5, 0.5, 0.6, 0.5, 0.5, 0.3)$ and $\mathbf{q} = (1.5, 1.5, 2.8, 2.5, 2, 1.6)^T$. In Panel (b), a link is added between 1 and 2. In Panel (c), δ_4 is increased, and in Panel (d), player 7 enters the game. In all three situations, total contributions have increased. This is because $\mathbf{x}^{unc} = (-6, -6, 11.66, -9.33, 4.66, 3.33)^T$, and the players involved in these changes have a negative component in $\mathbf{x}^{unc}$.

Corollary 5. *If $\mathbf{G}$ is the complete network with eventually heterogeneous players, total contributions decrease after*

i) increasing the intensity of a link

ii) increasing substitution rates

iii) entry of a new player

We end up with a proposition about neutral players. As we noted before, changes affecting neutral players do not affect total contributions. Moreover, we show that shocks on non-neutral players always have the same effect, irrespective of changes affecting neutral players:

Proposition 10. *Let $\mathbf{x}$ be an equilibrium of the game with parameters $(\mathbf{q}, \boldsymbol{\Delta}, \mathbf{G})$, such that player 1 is neutral. Let $\mathbf{x}'$ be an equilibrium of the game with parameters $(\mathbf{q}', \boldsymbol{\Delta}', \mathbf{G}')$, where $\mathbf{q}'$ is arbitrary, where $\boldsymbol{\Delta}'$ is such that substitution rate of player 1 is changed and where $\mathbf{G}'$ is such that incoming links of player 1 are changed. Assume also that $A(\mathbf{x}) \setminus \{1\} = A(\mathbf{x}') \setminus \{1\}$. Then,*

$$\mathbf{x}^{unc}((\boldsymbol{\Delta}\mathbf{G})^T, \mathbf{1}) = \mathbf{x}^{unc}((\boldsymbol{\Delta}'\mathbf{G}')^T, \mathbf{1})$$

In particular, player 1 remains neutral after these changes.

Thus, any shock on a neutral player will not only leave this player neutral, it will also leave every other player's component of $\mathbf{x}^{unc}$ unchanged. Since effects of shocks depend on these values, these effects are unchanged.

This result seems surprising, because modifying substitution rate, needs or incoming links of a neutral player does not affect his status of neutral player. This is because being neutral only depends on the player's outgoing effects. And these outgoing effects, which are zero in the original network, are not affected by such modifications.

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Appendix: Proofs

Proof of Proposition 1: A matrix is a P-matrix if and only if it reverses the sign of no vector except $\mathbf{0}$ (see Theorem 2 in Gale and Nikaido (1965)). That is, we cannot find a vector $\mathbf{x}$ such that $x_i((\mathbf{I}+\mathbf{\Delta G})\mathbf{x})_i \leq 0$ for all i. Let us assume otherwise. If $\mathbf{G}$ is complete, then for all i,

$$x_i(x_i + \delta_i X - \delta_i x_i) \leq 0$$

If $x_i \geq 0$ for all i (with some $x_i > 0$), then $X > 0$ and $(1-\delta_i)x_i + \delta_i X > 0$, a contradiction. Also, it is impossible that $x_i \leq 0$ for all i (with some $x_i < 0$) otherwise $x_i + \delta_i X - \delta_i x_i < 0$ and $x_i((I+\Delta G)x)_i \leq 0$ for all i. Thus, there is at least a pair (i, j) such that $x_i > 0$ and $x_j < 0$.
From that we get

$$(1-\delta_i)x_i + \delta_i X \leq 0$$

$$(1-\delta_j)x_j + \delta_j X \geq 0$$

From the first inequality we get $X < 0$, while from the second we get $X > 0$, which is a contradiction. $\square$

Proof of Proposition 2: By definition of the interaction matrix, we have

$$(\mathbf{I}+\mathbf{\Delta G}_{A(\mathbf{x})})\mathbf{x} = \mathbf{q}_{A(\mathbf{x})}$$

Let $\mathbf{M} = (\mathbf{I}+\mathbf{\Delta G}_{A(\mathbf{x})})^{-1}$. Then we have $\mathbf{x} = \mathbf{M}\mathbf{q}_{A(\mathbf{x})}$, so that $x_i = q\sum_{j\in A(\mathbf{x})} m_{ij}$. Moreover, the unconstrained solution $\mathbf{x}^{unc} = \mathbf{x}^{unc}((\mathbf{\Delta G}_{A(\mathbf{x})})^T, \mathbf{1})$ is defined as follows.

$$\mathbf{x}^{unc} = \left[(\mathbf{I}+\mathbf{\Delta G}_{A(\mathbf{x})})^T\right]^{-1}\mathbf{1}$$

Therefore, $x_i^{unc} = \sum_{j\in N} m_{ji}$, since $\left[(\mathbf{I}+\mathbf{\Delta G}_{A(\mathbf{x})})^T\right]^{-1} = \left[(\mathbf{I}+\mathbf{\Delta G}_{A(\mathbf{x})})^{-1}\right]^T = \mathbf{M}^T$.

Besides, let $\mathbf{A}[S,T]$ denote the submatrix of $\mathbf{A}$ with rows and columns indexed by the elements of sets S and T each, and let $\mathbf{A}[S]$ be the principal submatrix with rows and columns indexed by the elements of set S. By rewriting $(\mathbf{I}+\mathbf{\Delta G}_{A(\mathbf{x})})$ in blocks, separating active and strictly inactive players, we have

$$(\mathbf{I}+\mathbf{\Delta G}_{A(\mathbf{x})}) = \left(\begin{array}{c|c} (\mathbf{I}+\mathbf{\Delta G})[A(\mathbf{x})] & (\mathbf{I}+\mathbf{\Delta G})[A(\mathbf{x}), SI(\mathbf{x})] \\ \hline \mathbf{0} & \mathbf{I} \end{array}\right)$$

Thus, using the fact that $(\mathbf{I}+\mathbf{\Delta G}_{A(\mathbf{x})})\mathbf{M} = \mathbf{I}$ and by decomposing $\mathbf{M}$ into blocks, we obtain

$$(\mathbf{I}+\mathbf{\Delta G}_{A(\mathbf{x})})\mathbf{M} = \left(\begin{array}{c|c} (\mathbf{I}+\mathbf{\Delta G})[A(\mathbf{x})] & (\mathbf{I}+\mathbf{\Delta G})[A(\mathbf{x}), SI(\mathbf{x})] \\ \hline \mathbf{0} & \mathbf{I} \end{array}\right)\left(\begin{array}{c|c} \mathbf{M}_1 & \mathbf{M}_2 \\ \hline \mathbf{M}_3 & \mathbf{M}_4 \end{array}\right) = \mathbf{I}$$

where $\mathbf{M}_1$ is $|A(\mathbf{x})| \times |A(\mathbf{x})|$, $\mathbf{M}_2$ is $|A(\mathbf{x})| \times |SI(\mathbf{x})|$, $\mathbf{M}_3$ is $|SI(\mathbf{x})| \times |A(\mathbf{x})|$, $\mathbf{M}_4$ is $|SI(\mathbf{x})| \times |SI(\mathbf{x})|$ matrix. Hence, we obtain $\mathbf{M}_1 = [(\mathbf{I} + \mathbf{\Delta G})[A(\mathbf{x})]]^{-1}$, $\mathbf{M}_3 = \mathbf{0}$, and $\mathbf{M}_4 = \mathbf{I}$, and then we have

$$x_i^{unc} = \sum_{j \in N} m_{ji} = \sum_{j \in A(\mathbf{x})} m_{ji} = \sum_{j \in A(\mathbf{x})} m_{ij} = \frac{1}{q} x_i \geq 0 \tag{10}$$

for all $i \in A(\mathbf{x})$. The second equality comes from the fact that $\mathbf{M}_3 = \mathbf{0}$, and the third equality comes from the symmetry of $\mathbf{M}_1$. Especially, $x_i^{unc} = x_i = 0$ for $i \in Z(\mathbf{x})$. The first 2 points are proved.

We prove the last point. By computing the (i, j) entry of the product $\mathbf{M}(\mathbf{I} + \mathbf{\Delta G}_{A(\mathbf{x})}) = \mathbf{I}$ with $i \in A(\mathbf{x})$ and $j \in SI(\mathbf{x})$, we obtain $m_{ij} + \sum_{k \in A(\mathbf{x})} m_{ik} \delta g_{kj} = 0$. Summing up over all $i \in A(\mathbf{x})$, we get:

$$\begin{aligned} & \sum_{i \in A(\mathbf{x})} m_{ij} + \delta \sum_{i \in A(\mathbf{x})} \left[\sum_{k \in A(\mathbf{x})} m_{ik} g_{kj} \right] = 0 \\ \Leftrightarrow & \sum_{i \in A(\mathbf{x})} m_{ij} + \delta \sum_{i \in A(\mathbf{x})} \left[g_{ij} \sum_{k \in A(\mathbf{x})} m_{ki} \right] = 0 \end{aligned} \tag{11}$$

We know from (10) that $\sum_{j \in A(\mathbf{x})} m_{ji} = \sum_{j \in A(\mathbf{x})} m_{ij} = \frac{1}{q} x_i$ for all $i \in A(\mathbf{x})$. Therefore, (11) will be

$$\sum_{i \in A(\mathbf{x})} m_{ij} + \delta \sum_{i \in A(\mathbf{x})} \left[g_{ij} \sum_{k \in A(\mathbf{x})} m_{ki} \right] = \sum_{i \in A(\mathbf{x})} m_{ij} + \frac{\delta}{q} \sum_{i \in A(\mathbf{x})} g_{ij} x_i = 0.$$

Since $j \in SI(\mathbf{x})$, we know that $\delta \sum_{i \in A(\mathbf{x})} g_{ji} x_i > q_j = q \Leftrightarrow \frac{\delta}{q} \sum_{i \in A(\mathbf{x})} g_{ji} x_i > 1$. Therefore, $\sum_{i \in A(\mathbf{x})} m_{ij} + 1 < 0$. Moreover, since $\mathbf{M}_4 = \mathbf{I}$, $m_{jj} = 1$ and $m_{ij} = 0$ for $i \in SI(\mathbf{x})$. Hence, we have

$$x_j^{unc} = \sum_{i \in N} m_{ij} = m_{jj} + \sum_{i \in A(\mathbf{x})} m_{ij} = 1 + \sum_{i \in A(\mathbf{x})} m_{ij} < 0$$

for all $j \in SI(\mathbf{x})$. The third point is proved. □

Proof of Proposition 3: We first prove the following lemma.

Lemma 1. *Let* $\mathbf{G}$ *be the complete network,* $\mathbf{\Delta} = diag(\delta_i)_{i=1,\dots,n}$ *and* $\mathbf{M} = (\mathbf{I} + \mathbf{\Delta G})^{-1}$*. If* $\delta_i \in (0, 1)$ *for all* $i \in N$*,* $x_i^{unc} = \sum_j m_{ji} > 0$*. If* $\delta_i = 1$ *and* $\delta_j \in (0, 1)$ *for all* $j \neq i$*, then* $x_i^{unc} = 1$ *and* $x_j^{unc} = 0$ *for all* $j \neq i$*.*

Proof. Assume first that $\delta_i \in (0, 1)$ for all i. By taking the (i, j) and (i, i) entry of the product $\mathbf{M}(\mathbf{I}+\mathbf{\Delta G})=\mathbf{I}$, for all $i \in N$ and $j \neq i$, we have

$$m_{ij} + \sum_{l\neq j} \delta_l m_{il} = 0 \tag{12}$$

$$m_{ii} + \sum_{l\neq i} \delta_l m_{il} = 1 \tag{13}$$

For a given k and $k' \neq k$, take (i,k) and (i,k') entry of the product $\mathbf{M}(\mathbf{I}+\mathbf{\Delta G})=\mathbf{I}$, and sum up for all $i \in N$ for k and k'. We obtain

$$\sum_{i\in N} m_{ik} + \sum_{l\neq k} \delta_l \sum_{i\in N} m_{il} = 1 \tag{14}$$

$$\sum_{i\in N} m_{ik'} + \sum_{l\neq k'} \delta_l \sum_{i\in N} m_{il} = 1 \tag{15}$$

By subtracting (14) from (15), we get $\sum_{i\in N} m_{ik}+\delta_{k'}\sum_{i\in N} m_{ik'} = \sum_{i\in N} m_{ik'}+\delta_k\sum_{i\in N} m_{ik}$, resulting in

$$(1-\delta_k)\sum_{i\in N} m_{ik} = (1-\delta_{k'})\sum_{i\in N} m_{ik'} \tag{16}$$

from which $Sign(\sum_{i\in N} m_{ik}) = Sign(\sum_{i\in N} m_{ik'})$, and this is true for any k and $k' \neq k$. By (14), necessarily $\sum_{i\in N} m_{ik} > 0$ or $\sum_{i\in N} m_{il} > 0$ for some l. But, they all have the same sign. Therefore, $\sum_{i\in N} m_{ij} > 0$ for all j. Thus $x_j^{unc} > 0$ for all j.

Now assume w.l.o.g. that $\delta_1 = 1$ and $\delta_i \in (0,1)$ for all $i \neq 1$. Then, by taking (12) and (13) with $i = 1$, subtracting the one from the other, we get

$$m_{1j} = -\frac{1}{1-\delta_j}, \quad \forall j \neq 1 \tag{17}$$

Plugging it back into (13) with $i = 1$, we also get

$$m_{11} = 1 + \sum_{i\neq 1} \frac{\delta_i}{1-\delta_i} \tag{18}$$

By taking (12) and (13), each for $i \neq 1$ and $j = 1$, subtracting the one from the other, we get

$$m_{ii} = \frac{1}{1-\delta_i}, \quad i \neq 1 \tag{19}$$

Next, take (12) for 2 pairs $(i,1)$ and (i,j) such that $i \neq 1$, $j \neq 1$ and $j \neq i$, and by subtracting the one from the other, we get $m_{ij}(1-\delta_j) = 0$ for all $i \neq 1$, $j \neq 1$ and $j \neq i$. Thus

$$m_{ij} = 0, \quad i \neq 1, j \neq 1, i \neq j \tag{20}$$

Together with (17) and (19), for all $i \neq 1$ we have $x_i^{unc} = \sum_{j\in N} m_{ji} = 0$.

Finally, by plugging (20) into (13), we get for $i \neq 1$

$$m_{ii} + \delta_i m_{i1} = 1 \Leftrightarrow m_{i1} = -\frac{\delta_i}{1-\delta_i}$$

Therefore, using (18), we have $x_1^{unc} = \sum_{i \in N} m_{i1} = 1$, and the statement is proved. $\square$

Let $\mathbf{M} := (\mathbf{I} + \mathbf{\Delta G}_K)^{-1}$ for some $K \subseteq N$. By using the matrix decomposition in the proof of Proposition 2, we have

$$(\mathbf{I} + \mathbf{\Delta G}_K)\mathbf{M} = \left(\begin{array}{c|c} (\mathbf{I} + \mathbf{\Delta G})[K] & (\mathbf{I} + \mathbf{\Delta G})[K, N \setminus K] \\ \hline \mathbf{0} & \mathbf{I} \end{array}\right) \left(\begin{array}{c|c} \mathbf{M}_1 & \mathbf{M}_2 \\ \hline \mathbf{M}_3 & \mathbf{M}_4 \end{array}\right) = \mathbf{I}$$

We know from the proof of Proposition 2, that $\mathbf{M}_1 = [(\mathbf{I} + \mathbf{\Delta G})[K]]^{-1}$, $\mathbf{M}_3 = \mathbf{0}$, and $\mathbf{M}_4 = \mathbf{I}$. With $\mathbf{G}$ being complete, Lemma 1 tells us that for all $i \in K$, $x_i^{unc}((\mathbf{\Delta G}_K)^T, \mathbf{1}) = x_i^{unc}((\mathbf{\Delta G}[K])^T, \mathbf{1}) \geq 0$ since $\mathbf{G}[K]$ is complete.
Therefore, we need to prove that for all $i \in N \setminus K$, $x_i^{unc}((\mathbf{\Delta G}_K)^T, \mathbf{1}) \geq 0$.
Notice that from the structure of $\mathbf{M}$, for $j \in N \setminus K$, we have

$$x_j^{unc}((\mathbf{\Delta G}_K)^T, \mathbf{1}) = m_{jj} + \sum_{k \in K} m_{kj}$$

By taking the (i,j)-th element of $(\mathbf{I} + \mathbf{\Delta G}_K)\mathbf{M}$ for $i \in K$ and $j \in N \setminus K$, we have

$$\delta_i m_{jj} + m_{ij} + \delta_i \sum_{k \in K \setminus \{i\}} m_{kj} = 0$$

$$\Leftrightarrow \delta_i \left(m_{jj} + \sum_{k \in K} m_{kj} \right) + (1-\delta_i) m_{ij} = 0$$

Since m_{jj} is a diagonal element of $\mathbf{M}_4$, $m_{jj} = 1$. Hence, we have

$$\delta_i \left(1 + \sum_{k \in K} m_{kj} \right) + (1-\delta_i) m_{ij} = 0 \tag{21}$$

Suppose that for all $k \in K$, $m_{kj} \geq 0$. Then, $m_{ij} \geq 0$ since $i \in K$, so that (21) could not hold. Therefore, for at least one $i \in K$, $m_{ij} < 0$. Take such i, and we have

$$\delta_i \left(m_{jj} + \sum_{k \in K} m_{kj} \right) = -(1-\delta_i) m_{ij}$$

If $\delta_i \in (0,1)$, then we have $x_j^{unc} = m_{jj} + \sum_{k \in K} m_{kj} > 0$. If $\delta_i = 1$, then $x_j^{unc} = 0$. This is true for all $j \in N \setminus K$. The statement is proved. $\square$

Proof of Theorem 1: It is straightforward for the case where $x_1 = 0$. If agent 1 is inactive, then he either remains inactive or become active in $\mathbf{x}'$. Thus $x_1' \geq x_1$ if $x_1 = 0$.

Now assume $x_1 > 0$. By the best-responses, we have $(\mathbf{I} + \mathbf{\Delta G}_{A(\mathbf{x})})\mathbf{x} = \mathbf{q}_{A(\mathbf{x})}$ and $(\mathbf{I} + \mathbf{\Delta G}_{A(\mathbf{x}')})\mathbf{x}' = \mathbf{q}'_{A(\mathbf{x}')}$. From these 2 equations, we obtain

$$(\mathbf{I} + \mathbf{\Delta G}_{A(\mathbf{x})})\mathbf{x} = \left[\mathbf{I} + \mathbf{\Delta}(\mathbf{G}_{A(\mathbf{x})\cup A(\mathbf{x}')} - \mathbf{G}_{si(\mathbf{x},\mathbf{x}')})\right]\mathbf{x} = \mathbf{q}_{A(\mathbf{x})}$$
$$(\mathbf{I} + \mathbf{\Delta G}_{A(\mathbf{x}')})\mathbf{x}' = \left[\mathbf{I} + \mathbf{\Delta}(\mathbf{G}_{A(\mathbf{x})\cup A(\mathbf{x}')} - \mathbf{G}_{a(\mathbf{x},\mathbf{x}')})\right]\mathbf{x}' = \mathbf{q}'_{A(\mathbf{x}')}$$

Hence, we have

$$\left(\mathbf{I} + \mathbf{\Delta G}_{A(\mathbf{x})\cup A(\mathbf{x}')}\right)\mathbf{x} = \mathbf{q}_{A(\mathbf{x})} + \mathbf{\Delta}\overline{\mathbf{x}}_{si(\mathbf{x},\mathbf{x}')}$$
$$\left(\mathbf{I} + \mathbf{\Delta G}_{A(\mathbf{x})\cup A(\mathbf{x}')}\right)\mathbf{x}' = \mathbf{q}'_{A(\mathbf{x}')} + \mathbf{\Delta}\overline{\mathbf{x}}'_{a(\mathbf{x},\mathbf{x}')}$$

where $\overline{\mathbf{x}} = (\overline{x}_1, \cdots, \overline{x}_n)^T$ and $\overline{\mathbf{x}}' = (\overline{x}'_1, \cdots, \overline{x}'_n)^T$. Let $\mathbf{M} = (\mathbf{I} + \mathbf{\Delta G}_{A(\mathbf{x})\cup A(\mathbf{x}')})^{-1}$. Then, we obtain

$$\mathbf{x}' - \mathbf{x} = \mathbf{M}(\mathbf{q}'_{A(\mathbf{x}')} - \mathbf{q}_{A(\mathbf{x})} + \mathbf{\Delta}\overline{\mathbf{x}}'_{a(\mathbf{x},\mathbf{x}')} - \mathbf{\Delta}\overline{\mathbf{x}}_{si(\mathbf{x},\mathbf{x}')}).$$

First, assume that $1 \in A(\mathbf{x}) \cup A(\mathbf{x}')$. From the equation above, we obtain

$$x'_i - x_i = \beta m_{i1} + \sum_{j\in si(\mathbf{x},\mathbf{x}')} \left[(q_j - \delta_j\overline{x}_j)\, m_{ij}\right] + \sum_{j\in a(\mathbf{x},\mathbf{x}')} \left[\left(\delta_j\overline{x}'_j - q_j\right) m_{ij}\right]$$

Let $\beta_j = q_j - \delta_j\overline{x}_j$ for $j \in si(\mathbf{x},\mathbf{x}')$ and $\beta_j = \delta_j\overline{x}'_j - q_j$ for $j \in a(\mathbf{x},\mathbf{x}')$, so that

$$x'_i - x_i = \beta m_{i1} + \sum_{j\in si(\mathbf{x},\mathbf{x}')} \beta_j m_{ij} + \sum_{j\in a(\mathbf{x},\mathbf{x}')} \beta_j m_{ij} \begin{cases} \geq 0, & \text{for } i \in si(\mathbf{x},\mathbf{x}') \\ \leq 0, & \text{for } i \in a(\mathbf{x},\mathbf{x}') \end{cases}$$

For $i \in si(\mathbf{x},\mathbf{x}')$, $\beta_i < 0$ since he is active in $\mathbf{x}'$. For $i \in a(\mathbf{x},\mathbf{x}')$, $\beta_i > 0$ since he is active in $\mathbf{x}$. Therefore, for all $i \in si(\mathbf{x},\mathbf{x}') \cup a(\mathbf{x},\mathbf{x}')$, $\beta_i(x'_i - x_i) \leq 0$.

Let $U = \{1\} \cup si(\mathbf{x},\mathbf{x}') \cup a(\mathbf{x},\mathbf{x}')$, and $\boldsymbol{\beta}^* = (\beta, (\beta_i)_{i\in U\setminus\{1\}})^T$. Note that $U \subseteq A(\mathbf{x}) \cup A(\mathbf{x}')$. Then, we can rewrite the above equations as the following linear system.

$$\mathbf{M}[U]\boldsymbol{\beta}^* = (\mathbf{x}' - \mathbf{x})[U] \tag{22}$$

where $\mathbf{M}[U]$ is of the same definition as in the proof of Proposition 2, and $(\mathbf{x}' - \mathbf{x})[U]$ be the subvector of $(\mathbf{x}' - \mathbf{x})$ with entries indexed by the elements of set U. By the matrix decomposition in the proof of Proposition 2, we have $\mathbf{M}[A(\mathbf{x}) \cup A(\mathbf{x}')] = [(\mathbf{I} + \mathbf{\Delta G})[A(\mathbf{x}) \cup A(\mathbf{x}')]]^{-1}$. Since $(\mathbf{I} + \mathbf{\Delta G})[A(\mathbf{x}) \cup A(\mathbf{x}')]$ is a P-matrix and since the inverse of a P-matrix is a P-matrix, $\mathbf{M}[A(\mathbf{x}) \cup A(\mathbf{x}')]$ is also a P-matrix. Hence, its principal submatrix $\mathbf{M}[U]$ is also a P-matrix.

Since for all $i \in si(\mathbf{x},\mathbf{x}') \cup a(\mathbf{x},\mathbf{x}')$, $\beta_i(x'_i - x_i) \leq 0$, by applying Theorem 2 of Gale and Nikaido (1965)[20] to (22), we obtain that $\beta(x'_1 - x_1) > 0$ since $\boldsymbol{\beta}^* \neq 0$. Because $\beta > 0$,

[20] It states that a matrix $\mathbf{H}$ is a P-matrix if and only if it reverses the sign of no vector except $\mathbf{0}$, i.e. [$\mathbf{Hx} = \mathbf{y}$ with $x_i y_i \leq 0$ for all i] is true only for $\mathbf{x} = \mathbf{y} = \mathbf{0}$.

necessarily $x'_1 - x_1 > 0$.

Assume now that $1 \in a(\mathbf{x}, \mathbf{x}')$. Then, we have

$$x'_i - x_i = \sum_{j \in si(\mathbf{x},\mathbf{x}')} [(q_j - \delta_j \overline{x}_j)\, m_{ij}] + \sum_{j \in a(\mathbf{x},\mathbf{x}')} \left[\left(\delta_j \overline{x}'_j - q_j\right) m_{ij}\right]$$

Note that $\delta_1 \overline{x}'_1 - q_1 > \delta_1 \overline{x}'_1 - (q_1 + \beta) > 0$ since $1 \in SI(\mathbf{x}')$. Therefore, with the same argument of the previous case, we can prove that this is a contradiction by the fact that $\mathbf{M}[U]$ is a P-matrix. $\square$

Proof of Proposition 4: We first prove the following lemma.

Lemma 2. *Let* $\mathbf{x}$ *be an equilibrium with* $(\mathbf{q}, \mathbf{\Delta}, \mathbf{G})$ *and* $\mathbf{x}'$ *be one with* $(\mathbf{q}', \mathbf{\Delta}, \mathbf{G})$. *Then, we have*

$$X' - X = \sum_{A(\mathbf{x}') \cap A(\mathbf{x}')} (q'_i - q_i) x_i^{unc} + \sum_{i \in si(\mathbf{x},\mathbf{x}')} (q'_i - \delta_i \overline{x}'_i) x_i^{unc} + \sum_{i \in a(\mathbf{x},\mathbf{x}')} (\delta_i \overline{x}'_i - q_i) x_i^{unc}$$

where $\mathbf{M} = (\mathbf{I} + \mathbf{\Delta}\mathbf{G}_{A(\mathbf{x})})^{-1}$ *and* $x_i^{unc} = \sum_{j \in N} m_{ij}$.

Proof. By definition of $\mathbf{x}$ and $\mathbf{x}'$, we have $(\mathbf{I} + \mathbf{\Delta}\mathbf{G}_{A(\mathbf{x})})\mathbf{x} = \mathbf{q}_{A(\mathbf{x})}$ and $(\mathbf{I} + \mathbf{\Delta}\mathbf{G}_{A(\mathbf{x}')})\mathbf{x}' = \mathbf{q}'_{A(\mathbf{x}')}$. Let $\overline{\mathbf{x}} = (\overline{x}_1, \cdots, \overline{x}_n)^T$. Since $\mathbf{G}_{A(\mathbf{x}')} = \mathbf{G}_{A(\mathbf{x})} + \mathbf{G}_{si(\mathbf{x},\mathbf{x}')} - \mathbf{G}_{a(\mathbf{x},\mathbf{x}')}$, we have

$$\begin{aligned}(\mathbf{I} + \mathbf{\Delta}\mathbf{G}_{A(\mathbf{x}')})\mathbf{x}' = \mathbf{q}'_{A(\mathbf{x}')} &\Leftrightarrow (\mathbf{I} + \mathbf{\Delta}(\mathbf{G}_{A(\mathbf{x})} + \mathbf{G}_{si(\mathbf{x},\mathbf{x}')} - \mathbf{G}_{a(\mathbf{x},\mathbf{x}')}))\mathbf{x}' = \mathbf{q}'_{A(\mathbf{x}')} \\ &\Leftrightarrow (\mathbf{I} + \mathbf{\Delta}\mathbf{G}_{A(\mathbf{x})})\mathbf{x}' = \mathbf{q}'_{A(\mathbf{x}')} - \mathbf{\Delta}\overline{\mathbf{x}}'_{si(\mathbf{x},\mathbf{x}')} + \mathbf{\Delta}\overline{\mathbf{x}}'_{a(\mathbf{x},\mathbf{x}')}\end{aligned}$$

Let $\mathbf{M} = (\mathbf{I} + \mathbf{\Delta}\mathbf{G}_{A(\mathbf{x})})^{-1}$, then we have

$$\mathbf{x}' - \mathbf{x} = \mathbf{M}\left(\mathbf{q}'_{A(\mathbf{x}')} - \mathbf{\Delta}\overline{\mathbf{x}}'_{si(\mathbf{x},\mathbf{x}')} + \mathbf{\Delta}\overline{\mathbf{x}}'_{a(\mathbf{x},\mathbf{x}')} - \mathbf{q}_{A(\mathbf{x})}\right)$$

where

$$\left(\mathbf{q}'_{A(\mathbf{x}')} - \mathbf{\Delta}\overline{\mathbf{x}}'_{si(\mathbf{x},\mathbf{x}')} + \mathbf{\Delta}\overline{\mathbf{x}}'_{a(\mathbf{x},\mathbf{x}')} - \mathbf{q}_{A(\mathbf{x})}\right)_i = \begin{cases} q'_i - q_i, & \text{for } i \in A(\mathbf{x}) \cap A(\mathbf{x}') \\ q'_i - \delta_i \overline{x}'_i, & \text{for } i \in si(\mathbf{x}, \mathbf{x}') \\ \delta_i \overline{x}'_i - q_i, & \text{for } i \in a(\mathbf{x}, \mathbf{x}') \end{cases}$$

Therefore, by arranging the terms and computing the sum, we obtain

$$X' - X = \sum_{i \in A(\mathbf{x}) \cap A(\mathbf{x}')} (q'_i - q_i) x_i^{unc} + \sum_{i \in si(\mathbf{x},\mathbf{x}')} (q'_i - \delta_i \overline{x}'_i) x_i^{unc} + \sum_{i \in a(\mathbf{x},\mathbf{x}')} (\delta_i \overline{x}'_i - q_i) x_i^{unc}$$

where $x_i^{unc} = \sum_{j \in N} m_{ji}$. $\square$

In Proposition 4, $q_1' - q_1 = \beta$ and $q_i' - q_i = 0$ for all $i \neq 1$. Moreover, we have $a(\mathbf{x}, \mathbf{x}') \setminus \{1\} = si(\mathbf{x}, \mathbf{x}') \setminus \{1\} = \emptyset$. By applying Lemma 2, we obtain

$$X' - X = \begin{cases} \beta x_1^{unc}, & \text{if } 1 \in A(\mathbf{x}) \cap A(\mathbf{x}') \\ (q_1 + \beta - \delta_1 \overline{x}_1') x_1^{unc}, & \text{if } 1 \in si(\mathbf{x}, \mathbf{x}') \\ (\delta_1 \overline{x}_1' - q_1) x_1^{unc}, & \text{if } 1 \in a(\mathbf{x}, \mathbf{x}') \end{cases}$$

Since $\beta > 0$, $q_1 + \beta - \delta_1 \overline{x}_1' \geq 0$ if $1 \in si(\mathbf{x}, \mathbf{x}')$ and $\delta_1 \overline{x}_1' - q_1 > 0$ if $1 \in a(\mathbf{x}, \mathbf{x}')$, the statement is proved. □

Proof of Theorem 2: We have $q_1' - q_1 = \beta$ and $q_i' - q_i = 0$ for all $i \neq 1$. Assume first that $1 \in A(\mathbf{x}) \cap A(\mathbf{x}')$. By applying Lemma 2, we obtain

$$X' - X = \beta x_1^{unc} + \sum_{i \in si(\mathbf{x}, \mathbf{x}')} [(q_i - \delta_i \overline{x}_i')\, x_i^{unc}] + \sum_{i \in a(\mathbf{x}, \mathbf{x}')} [(\delta_i \overline{x}_i' - q_i)\, x_i^{unc}]$$

For $i \in si(\mathbf{x}, \mathbf{x}')$, since he is active in $\mathbf{x}'$, $q_i - \delta_i \overline{x}_i' \geq 0$. For $i \in a(\mathbf{x}, \mathbf{x}')$, since he is SI in $\mathbf{x}'$, $\delta_i \overline{x}_i' - q_i \geq 0$.

Next, assume that $1 \in si(\mathbf{x}, \mathbf{x}')$. Then by Lemma 2, we obtain

$$X' - X = (q_1 + \beta - \delta_1 \overline{x}_1') x_1^{unc} + \sum_{i \in si(\mathbf{x}, \mathbf{x}') \setminus \{1\}} [(q_i - \delta_i \overline{x}_i')\, x_i^{unc}] + \sum_{i \in a(\mathbf{x}, \mathbf{x}')} [(\delta_i \overline{x}_i' - q_i)\, x_i^{unc}]$$

Note that $q_1 + \beta - \delta_1 \overline{x}_1' \geq 0$ since player 1 is active in $\mathbf{x}'$.

Finally, assume that $1 \in a(\mathbf{x}, \mathbf{x}')$. Then, by Lemma2,

$$X' - X = (\delta_1 \overline{x}_1' - q_1) x_1^{unc} + \sum_{i \in si(\mathbf{x}, \mathbf{x}')} [(q_i - \delta_i \overline{x}_i')\, x_i^{unc}] + \sum_{i \in a(\mathbf{x}, \mathbf{x}') \setminus \{1\}} [(\delta_i \overline{x}_i' - q_i)\, x_i^{unc}]$$

Note that $\delta_1 \overline{x}_1' - q_1 > \delta_1 \overline{x}_1' - (q_1 + \beta) > 0$ because player 1 is SI in $\mathbf{x}'$. The statement is proved. □

Proof of Proposition 5: For the first point, by Lemma 2, we have

$$X' - X = \begin{cases} \beta x_1^{unc} + \sum_{i \in a(\mathbf{x}, \mathbf{x}')} (\delta \overline{x}_i' - q) x_i^{unc}, & \text{if } 1 \in A(\mathbf{x}) \cap A(\mathbf{x}') \\ (\delta \overline{x}_1' - q) x_1^{unc} + \sum_{i \in a(\mathbf{x}, \mathbf{x}') \setminus \{1\}} [(\delta \overline{x}_i' - q)\, x_i^{unc}], & \text{if } 1 \in a(\mathbf{x}, \mathbf{x}') \end{cases}$$

Note that $\delta \overline{x}_i' - q > 0$ for $i \in a(\mathbf{x}, \mathbf{x}') \subseteq SI(\mathbf{x}')$. Moreover, by Proposition 2, for all $i \in a(\mathbf{x}, \mathbf{x}') \subseteq A(\mathbf{x})$, $x_i^{unc} \geq 0$. If $1 \in A(\mathbf{x}) \setminus Z(\mathbf{x})$, then $x_1^{unc} = x_1 > 0$. The statement is proved.

For the second point, by Lemma 2, we have

$$X' - X = \begin{cases} \beta x_1^{unc}, & \text{if } 1 \in A(\mathbf{x}) \cap A(\mathbf{x}') \\ (\delta \overline{x}_1' - q) x_1^{unc}, & \text{if } 1 \in a(\mathbf{x}, \mathbf{x}') \end{cases}$$

In any of the 2 cases, since $1 \in Z(\mathbf{x})$, $x_1^{unc} = x_1 = 0$. The statement is proved.

For the third point, by Lemma 2, we have

$$X' - X = \begin{cases} (q + \beta - \delta \overline{x}_1') x_1^{unc} + \sum_{i \in si(\mathbf{x}, \mathbf{x}')} (q - \delta \overline{x}_i') x_i^{unc}, & \text{if } 1 \in si(\mathbf{x}, \mathbf{x}') \\ \sum_{i \in si(\mathbf{x}, \mathbf{x}')} (q - \delta \overline{x}_i') x_i^{unc}, & \text{if } 1 \in SI(\mathbf{x}) \cap SI(\mathbf{x}') \end{cases}$$

Note that $q + \beta - \delta \overline{x}_1' = x_1'$, and for all $i \in si(\mathbf{x}, \mathbf{x}')$, $q - \delta \overline{x}_i' = x_i' \geq 0$. Moreover, By Proposition 2, for all $i \in si(\mathbf{x}, \mathbf{x}') \subseteq SI(\mathbf{x})$, $x_i^{unc} < 0$. Hence, the statement is proved. □

Proof of Corollary 2: By the direct application of Proposition 3 to Theorem 2, the statement is proved. □

Proof of Proposition 6: Since the set of active players does not change before and after the transfer, without loss of generality, we can assume that both equilibria are such that everyone is active.

Let $\mathbf{q} = (\lambda_1 w_1, \cdots, \lambda_n w_n)^T$ be the vector of needs before the transfer and $\mathbf{q}' = (\lambda_1(w_1 + \beta), \lambda_2(w_2 - \beta), \lambda_3 w_3, \cdots, \lambda_n w_n)^T$ be the one after the transfer.

We have $(\mathbf{I} + \mathbf{\Delta G})\mathbf{x} = \mathbf{q}$, and $(\mathbf{I} + \mathbf{\Delta G})\mathbf{x}' = \mathbf{q}'$. Therefore, by defining $\mathbf{M} = (\mathbf{I} + \mathbf{\Delta G})^{-1}$, we have $\mathbf{x}' - \mathbf{x} = \mathbf{M}(\mathbf{q}' - \mathbf{q})$ where

$$(\mathbf{q}' - \mathbf{q})_i = \begin{cases} \lambda_1 \beta, & \text{for } i = 1 \\ -\lambda_2 \beta, & \text{for } i = 2 \\ 0, & \text{otherwise} \end{cases}$$

Hence, we obtain $X' - X = \beta(\lambda_1 x_1^{unc} - \lambda_2 x_2^{unc})$, which proves the statement. □

Proof of Corollary 3: It is sufficient to prove that $\lambda_1 x_1^{unc} - \lambda_2 x_2^{unc} > 0$ if and only if $\frac{1-\delta_2}{1-\delta_1} > \frac{\lambda_2}{\lambda_1}$. Note that $\delta_i = (1 - \lambda_i)\gamma_i$.

By the definition of $\mathbf{x}^{unc}$, we have $(\mathbf{I} + \mathbf{\Delta G})\mathbf{x}^{unc} = \mathbf{1}$. $\mathbf{G}$ being a complete network, we have

$$x_1^{unc} + \delta_2 x_2^{unc} + \sum_{i \neq 1,2} \delta_i x_i^{unc} = 1$$

$$\delta_1 x_1^{unc} + x_2^{unc} + \sum_{i \neq 1,2} \delta_i x_i^{unc} = 1$$

Subtracting the one from the other, we obtain $(1-\delta_1)x_1^{unc} = (1-\delta_2)x_2^{unc} \Leftrightarrow \frac{1-\delta_2}{1-\delta_1} = \frac{x_1^{unc}}{x_2^{unc}}$. Assume that $\frac{1-\delta_2}{1-\delta_1} > \frac{\lambda_2}{\lambda_1}$. Then,

$$\frac{x_1^{unc}}{x_2^{unc}} = \frac{1-\delta_2}{1-\delta_1} > \frac{\lambda_2}{\lambda_1} \Leftrightarrow \frac{\lambda_1 x_1^{unc}}{\lambda_2 x_2^{unc}} > 1$$

and thus $\lambda_1 x_1^{unc} - \lambda_2 x_2^{unc} > 0$, as desired. The converse is also true. $\square$

Proof of Proposition 7: Assume that $\mathbf{q}$ is the need with the initial level of state provision and tax. When the state provision level and tax increase by β_S and $\boldsymbol{\beta}_{\mathbf{t}} = (\beta_{t_1}, \cdots, \beta_{t_n})$, the best-response functions give:

$$\begin{aligned} x_i + \delta_i \overline{x}_i = q_i - \delta_i \beta_S - \lambda_i \beta_{t_i} &\Rightarrow x_i = q_i - \delta_i \beta_S - \lambda_i \beta_{t_i} - \delta_i \overline{x}_i \\ \delta_i \overline{x}_i < q_i - \delta_i \beta_S - \lambda_i \beta_{t_i} &\Rightarrow x_i = 0 \end{aligned}$$

Let $\beta_i = \delta_i \beta_S + \lambda_i \beta_{t_i}$. Then, $\mathbf{x}'$ is an equilibrium with $(\mathbf{q} - \boldsymbol{\beta}, \boldsymbol{\Delta}, \mathbf{G})$. By applying Lemma 2, we obtain

$$\begin{aligned} X' - X &= -\sum_{i \in A(\mathbf{x}) \cap A(\mathbf{x}')} \beta_i x_i^{unc} + \sum_{i \in si(\mathbf{x}, \mathbf{x}')} (q_i - \beta_i - \delta_i \overline{x}'_i)\, x_i^{unc} + \sum_{i \in a(\mathbf{x}, \mathbf{x}')} (\delta_i \overline{x}'_i - q_i)\, x_i^{unc} \\ &= -\sum_{i \in A(\mathbf{x}) \cap A(\mathbf{x}')} (\delta_i \beta_S + \lambda_i \beta_{t_i})\, x_i^{unc} + \sum_{i \in si(\mathbf{x}, \mathbf{x}')} x'_i x_i^{unc} + \sum_{i \in a(\mathbf{x}, \mathbf{x}')} (\delta_i \overline{x}'_i - q_i)\, x_i^{unc} \end{aligned}$$

which proves the statement. $\square$

Proof of Corollary 4: We first prove the following lemma.

Lemma 3. *Let $\mathbf{x}$ be an equilibrium with $(\mathbf{q}, \boldsymbol{\Delta}, \mathbf{G})$ and $\mathbf{x}'$ be an equilibrium with $(\mathbf{q}', \boldsymbol{\Delta}, \mathbf{G})$, where the network $\mathbf{G}$ is complete. If there exists $i \in si(\mathbf{x}, \mathbf{x}')$ such that $q'_i - q_i \leq 0$, then $X' - X < 0$.*

Proof. Assume that $i \in si(\mathbf{x}, \mathbf{x}')$ (i.e. $i \in SI(\mathbf{x})$ and $i \in A(\mathbf{x}')$), and $q_i - q'_i = \beta \geq 0$. Since $i \in SI(\mathbf{x})$, we have $\delta_i \sum_{j \in N} g_{kj} x_j = \delta_j X > q_j$. Since $i \in A(\mathbf{x}')$, we also have

$$x'_i + \delta_i \sum_{j \in N} g_{ij} x'_j = q'_i \Leftrightarrow x'_i + \delta_i (X' - x'_i) = q_i - \beta$$

Thus, $q_i - \beta - (1-\delta_i)x'_i - \delta_i X' = 0$. Therefore,

$$\begin{aligned} q_i - \beta - (1-\delta_i)x'_i - \delta_i X' &> q_i - \delta_i X \\ \Leftrightarrow \delta_i (X' - X) &< -\beta - (1-\delta_i)x'_i < 0 \end{aligned}$$

Therefore, $X' - X < 0$. $\square$

Let β_S be the increase of the state provision and β_{t_i} be the increase of tax for player i.

First point: If $si(\mathbf{x},\mathbf{x}') \neq \emptyset$, by Lemma 3, we know that the sum of contribution decreases since the increase in β_S corresponds to the decrease in q_i for all i. Thus, we assume $si(\mathbf{x},\mathbf{x}') = \emptyset$. From Proposition 7 and by using the fact that the network is complete and $a(\mathbf{x},\mathbf{x}') \subseteq SI(\mathbf{x})$, we have

$$X' - X = -\sum_{i\in A(\mathbf{x})\cap A(\mathbf{x}')} (\delta_i\beta_S + \lambda_i\beta_{t_i})x_i^{unc} + \sum_{i\in a(\mathbf{x},\mathbf{x}')} (\delta_i X' - q_i)\, x_i^{unc}$$

Moreover, since $i \in a(\mathbf{x},\mathbf{x}') \subseteq A(\mathbf{x})$, we have $q_i = x_i + \delta_i\overline{x}_i = x_i + \delta_i(X - x_i) = (1-\delta_i)x_i + \delta_i X$. Therefore,

$$\begin{aligned} X' - X &= -\sum_{i\in A(\mathbf{x})\cap A(\mathbf{x}')} (\delta_i\beta_S + \lambda_i\beta_{t_i})x_i^{unc} + \sum_{i\in a(\mathbf{x},\mathbf{x}')} \left[\delta_i X' - (1-\delta_i)x_i - \delta_i X\right] x_i^{unc} \\ &= -\sum_{i\in A(\mathbf{x})\cap A(\mathbf{x}')} (\delta_i\beta_T + \lambda_i\beta_{t_i})x_i^{unc} + \sum_{i\in a(\mathbf{x},\mathbf{x}')} \left[\delta_i(X' - X) - (1-\delta_i)x_i\right] x_i^{unc} \end{aligned}$$

and then we obtain

$$\left(1 - \sum_{i\in a(\mathbf{x},\mathbf{x}')} \delta_i x_i^{unc}\right)(X' - X) = -\sum_{i\in A(\mathbf{x})\cap A(\mathbf{x}')} (\delta_i\beta_S + \lambda_i\beta_{t_i})x_i^{unc} - \sum_{i\in a(\mathbf{x},\mathbf{x}')} (1-\delta_i)x_i x_i^{unc} < 0$$

Thus, $1 - \sum_{i\in a(\mathbf{x},\mathbf{x}')} \delta_i x_i^{unc} > 0 \Leftrightarrow X' - X < 0$.
By definition, $\mathbf{x}^{unc}$ satisfies $(\mathbf{I} + \mathbf{\Delta}\mathbf{G}_{A(\mathbf{x})})^T \mathbf{x}^{unc} = \mathbf{1}$, so that, $\mathbf{G}$ being a complete network, we have

$$x_i^{unc} + \sum_{j\in A(\mathbf{x})} \delta_j x_j^{unc} = 1, \text{ for } i \notin A(\mathbf{x}) \tag{23}$$

$$x_i^{unc} + \sum_{j\in A(\mathbf{x})\setminus\{i\}} \delta_j x_j^{unc} = 1, \text{ for } i \in A(\mathbf{x}) \tag{24}$$

By Proposition 3, $x_i^{unc} \geq 0$ for all i if $\mathbf{G}$ is complete. Since we have $\delta_i \leq 1$, we have $\sum_{j\in A(\mathbf{x})} \delta_j x_j^{unc} \leq 1$.
Notice that $si(\mathbf{x},\mathbf{x}') = \emptyset$, thus $a(\mathbf{x},\mathbf{x}') \subsetneq A(\mathbf{x})$ (since otherwise every player would be playing 0 in $\mathbf{x}'$ which is impossible). So, there is a player k such that $x_k > 0$ and $k \notin a(\mathbf{x},\mathbf{x}')$. Therefore, $\sum_{j\in a(\mathbf{x},\mathbf{x}')} \delta_j x_j^{unc} < \sum_{j\in A(\mathbf{x})} \delta_j x_j^{unc} \leq 1$, and thus $X' - X < 0$.

Second point : Let $\beta_S > \sum_{i\in A(\mathbf{x})} \beta_{t_i}$.

$$\begin{aligned}
&X' - X + \beta_S \\
&= \beta_S - \sum_{i\in A(\mathbf{x})\cap A(\mathbf{x}')} (\delta_i \beta_S + \lambda_i \beta_{t_i}) x_i^{unc} + \sum_{i\in si(\mathbf{x},\mathbf{x}')} x_i' x_i^{unc} + \sum_{i\in a(\mathbf{x},\mathbf{x}')} (\delta_i \overline{x}_i' - q_i)\, x_i^{unc} \\
&= \beta_S - \sum_{i\in A(\mathbf{x})\cap A(\mathbf{x}')} (\delta_i \beta_S + \lambda_i \beta_{t_i}) x_i^{unc} + \sum_{i\in si(\mathbf{x},\mathbf{x}')} x_i' x_i^{unc} + \sum_{i\in a(\mathbf{x},\mathbf{x}')} [\delta_i X' - (q_i' + \lambda_i \beta_{t_i})]\, x_i^{unc} \\
&= \beta_S \left(1 - \sum_{i\in A(\mathbf{x})\cap A(\mathbf{x}')} \delta_i x_i^{unc}\right) - \sum_{i\in A(\mathbf{x})} \lambda_i \beta_{t_i} x_i^{unc} + \sum_{i\in si(\mathbf{x},\mathbf{x}')} x_i' x_i^{unc} + \sum_{i\in a(\mathbf{x},\mathbf{x}')} (\delta_i X' - q_i')\, x_i^{unc} \\
&\geq \beta_S \left(1 - \sum_{i\in A(\mathbf{x})\cap A(\mathbf{x}')} \delta_i x_i^{unc}\right) - \sum_{i\in A(\mathbf{x})} \lambda_i \beta_{t_i} x_i^{unc} \qquad (25)
\end{aligned}$$

The last inequality comes from the fact that the last two terms of the LHS of the inequality are non-negative. Hence, it is sufficient to prove that (25) is non-negative.
By definition, $\mathbf{x}^{unc}$ is the solution to the linear system $(\mathbf{I}+\mathbf{\Delta G}_{A(\mathbf{x})})^T \mathbf{x}^{unc} = \mathbf{1}$. Therefore, if $\mathbf{G}$ is complete, we have (23) and (24). From equation (23), we have $\sum_{j\in A(\mathbf{x})} \delta_j x_j^{unc} = 1 - x_i \geq \sum_{j\in A(\mathbf{x})\cap A(\mathbf{x}')} \delta_j x_j^{unc}$, since $A(\mathbf{x}) \cap A(\mathbf{x}') \subseteq A(\mathbf{x})$ and $x_i^{unc} > 0$ for all $i \in N$.
Moreover, $x_i^{unc} \geq 0$ for all $i \in N$ since $\mathbf{G}$ is complete and by taking $i \in A(\mathbf{x})$, and $j \notin A(\mathbf{x})$ for equation (23) and (24), we have

$$x_j^{unc} - x_i^{unc} + \delta_i x_i^{unc} = 0 \Leftrightarrow x_i^{unc} = \frac{x_j^{unc}}{1-\delta_i}, \text{ for any } i \in A(\mathbf{x}) \text{ and } j \notin A(\mathbf{x})$$

By taking some $j \notin A(\mathbf{x})$, equation (25) writes

$$\begin{aligned}
\beta_S \left(1 - \sum_{i\in A(\mathbf{x})\cap A(\mathbf{x}')} \delta_i x_i^{unc}\right) - \sum_{i\in A(\mathbf{x})} \lambda_i \beta_{t_i} x_i^{unc} &\geq \beta_S x_j^{unc} - \sum_{i\in A(\mathbf{x})} \lambda_i \beta_{t_i} \frac{x_j^{unc}}{1-\delta_i} \\
&= \beta_S x_j^{unc} - x_j^{unc} \sum_{i\in A(\mathbf{x})} \beta_{t_i} \frac{\lambda_i}{1-(1-\lambda_i)\gamma_i} \\
&\geq \beta_S x_j^{unc} - x_j^{unc} \sum_{i\in A(\mathbf{x})} \beta_{t_i} \\
&> \beta_S x_j^{unc} - \beta_S x_j^{unc} = 0
\end{aligned}$$

The first equality comes from the definition of δ_i, the second inequality comes from $\frac{\lambda_i}{1-(1-\lambda_i)\gamma_i} \leq 1$ since $\gamma_i \leq 1$ for all i, and the last inequality comes from $\beta_S > \sum_{i\in A(\mathbf{x})} \beta_{t_i}$.
Third point : Since no one changes status, we have

$$X' - X + \beta_S = \beta_S - \sum_{i\in A(\mathbf{x})} (\delta_i \beta_S + \lambda_i \beta_{t_i}) x_i^{unc} = \beta_S \left(1 - \sum_{i\in A(\mathbf{x})} \delta_i x_i^{unc}\right) - \sum_{i\in A(\mathbf{x})} \lambda_i \beta_{t_i} x_i^{unc} \qquad (26)$$

With the same argument as in the proof of the second point, we have

$$\begin{aligned}\beta_S\left(1-\sum_{i\in A(\mathbf{x})}\delta_i x_i^{unc}\right)-\sum_{i\in A(\mathbf{x})}\lambda_i\beta_{t_i}x_i^{unc} &= \beta_S x_j^{unc}-\sum_{i\in A(\mathbf{x})}\lambda_i\beta_{t_i}\frac{x_j^{unc}}{1-\delta_i}\\ &= \beta_S x_j^{unc}-x_j^{unc}\sum_{i\in A(\mathbf{x})}\beta_{t_i}\frac{\lambda_i}{1-(1-\lambda_i)\gamma_i}\\ &\geq \beta_S x_j^{unc}-x_j^{unc}\sum_{i\in A(\mathbf{x})}\beta_{t_i}\\ &= \beta_S x_j^{unc}-\beta_S x_j^{unc}=0\end{aligned}$$

The inequality is strict if and only if there is at least one agent $i \in A(\mathbf{x})$ such that $\beta_{t_i} > 0$ and $\gamma_i < 1$. □

Proof of Proposition 8: By the best-responses, we have

$$\begin{aligned}q_i-\delta_i\sum_{j\in N}g_{ij}^{\epsilon}x_j^{\epsilon}\geq 0 &\Rightarrow x_i^{\epsilon}=q_i-\delta_i\sum_{j\in N}g_{ij}^{\epsilon}x_j^{\epsilon}\\ q_i-\delta_i\sum_{j\in N}g_{ij}^{\epsilon}x_j^{\epsilon}<0 &\Rightarrow x_i^{\epsilon}=0\end{aligned}$$

By definition of $\mathbf{G}^{\epsilon}$, we have

$$\begin{aligned}q_i-\delta_i\sum_{j\in N}g_{ij}^{\epsilon}x_j^{\epsilon} &= q_i-\delta_i\sum_{j\in N}(g_{ij}+\epsilon_{ij})x_j^{\epsilon}\\ &= q_i-\delta_i\sum_{j\in N}g_{ij}x_j^{\epsilon}-\delta_i\sum_{j\in N}\epsilon_{ij}x_j^{\epsilon}\end{aligned}$$

Let $\psi_i = -\delta_i \sum_{j\in N}\epsilon_{ij}x_j^{\epsilon}$, then

$$\begin{aligned}q_i+\psi_i-\delta_i\sum_{j\in N}g_{ij}x_j^{\epsilon}\geq 0 &\Rightarrow x_i^{\epsilon}=q_i+\psi_i-\delta_i\sum_{j\in N}g_{ij}x_j^{\epsilon}\\ q_i+\psi_i-\delta_i\sum_{j\in N}g_{ij}x_j^{\epsilon}<0 &\Rightarrow x_i^{\epsilon}=0\end{aligned}$$

The statement is proved. □

Proof of Proposition 9:

Adding a link

Assume first that $1, 2 \in A(\mathbf{x}) \cap A(\mathbf{x}')$. By definition, we have $(\mathbf{I}+\mathbf{\Delta G}_{A(\mathbf{x})})\mathbf{x} = \mathbf{q}_{A(\mathbf{x})}$ and $(\mathbf{I}+\mathbf{\Delta G}'_{A(\mathbf{x}')})\mathbf{x} = \mathbf{q}_{A(\mathbf{x}')}$.

By Proposition 8, x' is an equilibrium with parameter $(\mathbf{q}+\boldsymbol{\psi}, \boldsymbol{\Delta}, \mathbf{G})$ where

$$\psi_i = \begin{cases} -\delta_1 x'_2, & \text{for } i = 1 \\ -\delta_2 x'_1, & \text{for } i = 2 \\ 0, & \text{otherwise} \end{cases}$$

By applying Lemma 2, we obtain

$$X' - X = -\delta_1 x'_2 x_1^{unc} - \delta_2 x'_1 x_2^{unc} + \sum_{i \in si(\mathbf{x},\mathbf{x}')} (q_i - \delta_i \overline{x}'_i) x_i^{unc} + \sum_{i \in a(\mathbf{x},\mathbf{x}')} (\delta_i \overline{x}'_i - q_i) x_i^{unc}$$

If $si(\mathbf{x},\mathbf{x}') = a(\mathbf{x},\mathbf{x}') = \emptyset$, we obtain the statement of the proposition.

Changing substitution rates

By Proposition 8, $\mathbf{x}'$ is the equilibrium with parameters $(\mathbf{q}+\boldsymbol{\psi}, \boldsymbol{\Delta}, \mathbf{G})$ with

$$\psi_i = \begin{cases} -\epsilon \sum_{i \in N} g_{1i} x'_i = -\epsilon \overline{x}'_1, & \text{for } i = 1 \\ 0, & \text{otherwise} \end{cases}$$

By applying Lemma 2, we obtain

$$X' - X = -\epsilon \overline{x}'_1 x_1^{unc} + \sum_{i \in si(\mathbf{x},\mathbf{x}')} (q_i - \delta_i \overline{x}'_i) x_i^{unc} + \sum_{i \in a(\mathbf{x},\mathbf{x}')} (\delta_i \overline{x}'_i - q_i) x_i^{unc}$$

If $si(\mathbf{x},\mathbf{x}') = a(\mathbf{x},\mathbf{x}') = \emptyset$, we obtain the statement of the proposition.

Entry of a player

Since it does not change anything if $x^+_{n+1} = 0$, assume that $x^+_{n+1} > 0$. By definition, we have $(\mathbf{I} + \boldsymbol{\Delta}\mathbf{G}_{A(\mathbf{x})})\mathbf{x} = \mathbf{q}_{A(\mathbf{x})}$ and by defining $\mathbf{M} = (\mathbf{I} + \boldsymbol{\Delta}\mathbf{G}_{A(\mathbf{x})})^{-1}$, we have $\mathbf{x} = \mathbf{M}\mathbf{q}_{A(\mathbf{x})}$. Let $\overline{\mathbf{G}}^+ = (\overline{g}^+_{ij})_{i,j \in N^+}$ be such that

$$\overline{g}^+_{ij} = \begin{cases} 0, & \text{if } i = n+1 \text{ or } j = n+1 \\ g^+_{ij} = g_{ij}, & \text{otherwise} \end{cases}$$

This is the matrix whose $n+1$-th row and column are all zero. Thus, we have $\mathbf{G}^+ = \overline{\mathbf{G}}^+ + \mathbf{E}$, with $\mathbf{E} = (\epsilon_{ij})_{i,j \in N^+}$ where

$$\epsilon_{ij} = \begin{cases} g^+_{ij}, & \text{if } i = n+1 \text{ or } j = n+1 \\ 0, & \text{otherwise} \end{cases}$$

Therefore, by Proposition 8, $\mathbf{x}^+$ is the equilibrium with parameters $(\mathbf{q}^+ + \boldsymbol{\psi}^+, \boldsymbol{\Delta}^+, \overline{\mathbf{G}}^+)$ where

$$\psi^+_i = \begin{cases} -\delta_i g^+_{i,n+1} x^+_{n+1}, & \text{for } i \in N \\ -\delta_{n+1} \overline{x}^+_{n+1}, & \text{for } i = n+1 \end{cases}$$

Therefore, for all $i \in \{1, \cdots, n\}$, $\mathbf{x}^+$ satisfies

$$q_i + \psi_i^+ - \delta_i \sum_{j \in N} g_{ij} x_j^+ \geq 0 \Rightarrow x_i^+ = q_i + \psi_i^+ - \delta_i \sum_{j \in N} g_{ij} x_j^+ \tag{27}$$

$$q_i + \psi_i^+ - \delta_i \sum_{j \in N} g_{ij} x_j^+ < 0 \Rightarrow x_i^+ = 0 \tag{28}$$

Let $\boldsymbol{\psi} = (\psi_1^+, \cdots, \psi_n^+)^T$, and $\mathbf{x}' = (x_1^+, \cdots, x_n^+)^T$. Since $\mathbf{x}'$ satisfies (27) and (28), $\mathbf{x}'$ is an equilibrium with $(\mathbf{q}[N] + \boldsymbol{\psi}, \boldsymbol{\Delta}, \mathbf{G})$. By applying Lemma 2, we obtain

$$X'_{|n} - X = \sum_{i \in A(\mathbf{x}) \cap A(\mathbf{x}')} \psi_i^+ x_i^{unc} + \sum_{i \in si(\mathbf{x}, \mathbf{x}')} (q_i + \psi_i - \delta_i \overline{x}_i^+) x_i^{unc} + \sum_{i \in a(\mathbf{x}, \mathbf{x}')} (\delta_i \overline{x}_i^+ - q_i) x_i^{unc}$$

where $\overline{x}_i^+ = \sum_{j \in N} g_{ij} x_j^+$. If $si(\mathbf{x}, \mathbf{x}') = a(\mathbf{x}, \mathbf{x}') = \emptyset$, we obtain the statement. $\square$

Proof of Corollary 5: Since any of the 3 cases can be considered as a decrease in needs of some players, we can directly use the argument of the proof of Corollary 4, and the statement is proved. $\square$

Proof of Proposition 10: To ease notations, we write $\mathbf{A} = (\mathbf{I} + \boldsymbol{\Delta}\mathbf{G})$ and $\mathbf{B} = \mathbf{A}^{-1}$. We want to show that if player 1 is neutral with $\boldsymbol{\Delta}\mathbf{G}$ he will also be neutral with $\boldsymbol{\Delta}'\mathbf{G}$ where $\boldsymbol{\Delta}' = diag(\delta_i')_{i \in N}$ such that $\delta_i' = \delta_i$ for all $i \neq 1$ and $\delta_1' \neq \delta_1$. To do that, we change element a_{12} of matrix $\mathbf{A}$ to $a_{12} + \epsilon$ and show that $\mathbf{x}^{unc}((\boldsymbol{\Delta}\mathbf{G})^T, \mathbf{1}) = \mathbf{x}^{unc}((\boldsymbol{\Delta}'\mathbf{G})^T, \mathbf{1})$.

Using Sherman Morrison formula with $\mathbf{u} = (1, 0, ..., 0)^T$ and $\mathbf{v}^T = (0, 1, 0, ..., 0)$ we get

$$\begin{aligned}
(\mathbf{A} + \mathbf{u}\mathbf{v}^T)^{-1} - \mathbf{A}^{-1} &= -\frac{\mathbf{B}\mathbf{u}\mathbf{v}^T\mathbf{B}}{1 + \mathbf{v}^T\mathbf{B}\mathbf{u}} \\
&= -\frac{\epsilon}{1 + b_{21}} \begin{pmatrix} b_{12}b_{11} & \cdots & b_{n2}b_{11} \\ b_{12}b_{21} & \cdots & b_{n2}b_{21} \\ \vdots & \ddots & \vdots \\ b_{12}b_{n1} & \cdots & b_{n2}b_{n1} \end{pmatrix}
\end{aligned}$$

Therefore, the column sum vector is

$$\begin{aligned}
\mathbf{1}^T \left[(\mathbf{A} + \mathbf{u}\mathbf{v}^T)^{-1} - \mathbf{A}^{-1} \right] &= -\frac{\epsilon}{1 + b_{21}} \left(b_{12} \sum_j b_{j1}, ..., b_{n2} \sum_j b_{j1} \right) \\
&= -x_1^{unc}((\boldsymbol{\Delta}\mathbf{G})^T, \mathbf{1}) \frac{\epsilon}{1 + b_{21}} (b_{12}, ..., b_{n2})
\end{aligned}$$

Since $x_1^{unc}((\boldsymbol{\Delta}\mathbf{G})^T, \mathbf{1}) = 0$ because player 1 is neutral, we have

$$\mathbf{1}^T \left[(\mathbf{A} + \mathbf{u}\mathbf{v}^T)^{-1} - \mathbf{A}^{-1} \right] = 0$$

Hence, the statement is proved. $\square$

Declaration of generative AI and AI-assisted technologies in the writing process

During the preparation of this work the authors used ChatGPT in order to improve language and readability. After using this tool, the authors reviewed and edited the content as needed and take full responsibility for the content of the publication.

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