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Liquidity, Collateral Quality and Interest Rate

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Abstract

This paper analyzes how collateral quality shocks affect banks' liquidity management and the risk-free rate. We develop a model where banks manage liquidity through near-cash assets and marketable securities subject to idiosyncratic and/or aggregate shocks. Collateral quality deterioration leads to non-monotonic changes in liquidity holdings: moderate declines reduce cash holdings via lower market returns, while severe declines cause precautionary hoarding and market freezes. Reduced collateral quality depresses the risk-free rate. Policy interventions, including liquidity regulation and negative interest rate policies can mitigate these effects. Our findings highlight the risks of collateral quality shocks and the importance of policy complementarities in addressing liquidity issues.

Keywords: interbank market, risk-free rate, collateral, liquidity regulation, negative interest rate, cash-hoarding.

JEL: E58, G28, G21

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1 Introduction

A recurrent feature of banking crises is that many financial institutions find themselves with insufficient collateral to cover their liquidity needs, see Bagehot (1873)'s description of the crisis of 1867, the role of the Mortgage Backed-Securities during the Great Financial Crisis of 2007-8 (Gorton 2010) and the role of sovereign debt during the European Debt Crisis (Mancini, Ranaldo, and Wrampelmeyer 2015). Also, as the Dash-for-Cash episode of March 2020 illustrates, there are shorter episodes of financial tensions triggered by issues related to abrupt variations of collateral value, and the implied need to draw on credit lines, see Vissing-Jorgensen (2021); Pelizzon and Schneider (2021). Our paper proposes a parsimonious framework to study how collateral quality shocks–or their anticipation–impacts banks' liquidity management and the risk-free rate. We ask when collateral issues impair the functioning of the secured interbank market, and its implication for the financing of the economy.

In our framework, collateral quality refers to the ability of an asset to serve, over time, as a reliable source of liquidity—immediate access to cash—when a financial institution needs it. Our model features banks which can finance either their cash holding, marketable securities and long-term investment projects. As in the episodes mentioned above, their liquidity position - cash and securities - can be weakened by either idiosyncratic or aggregate shocks to the value of the securities they hold, which may undermine their ability to continue financing long-term projects. *Individually,* banks handle this type of asset liquidity risk by adjusting *ex ante* the amount of cash they hold to match the expected distribution of this risk. We characterize the aggregate implication of those individual strategies, and describe equilibria that feature cash hoarding or cash-in-the-market pricing.

We develop a simple three date model of banks subject to liquidity shocks, in the spirit of Holmström and Tirole (1998) and Allen and Gale (2004a). More specifically banks are endowed with long-term non-tradable projects—which we think of as commercial loan with a credit line component—that may require additional cash to successfully come to maturity. Hence, the liquidity needs comes from banks' asset side (corporate liquidity demand) rather than from their liability side (depositors' liquidity demand).

As projects long-term return cannot be pledged, banks insure against those shocks by holding cash or near-cash assets and marketable securities. We think of near-cash assets as level 1 high quality liquid assets (HQLA), such as central bank reserves and sovereign bonds. Examples of marketable securities in our model would be corporate bonds or equities. Marketable securities have a higher expected return than cash but cannot be used directly to cover the liquidity shock as their value will be only realized in the long-term. Instead, they can be exchanged against cash in a market. The market may be interpreted as a secured money market in which banks borrow cash by using their securities as collateral (*e.g.* in a repo transaction), and the interest rate as the risk-free short-term rate in this model economy.[1]

A key feature of our model is that the value of securities as collateral at the interim date is uncertain. Specifically, their long-term return can take two values, high or low. The dispersion between these values measures the collateral quality, a wider dispersion being associated with a lower collateral quality. When choosing at date 0 how much cash to hold, banks must take into account the fact that at date 1 the securities they hold can turn out to have a low collateral quality, in which case they may be cash constrained for the refinancing of their long term project. When the dispersion is small, banks are less concerned about this risk. We analyze how a variation in collateral quality has an impact on the choice of cash holding by banks.

To help build the intuition and emphasize the role of collateral quality, we first consider a setup in which there is only idiosyncratic risk on collateral quality. To be specific, the variation in collateral quality is captured by a mean-preserving spread on assets long-term payoffs. In a second step, we add aggregate uncertainty on assets long-term payoffs.

In the case without aggregate uncertainty, we obtain two sets of results. In a first step, we demonstrate that in the laissez-faire equilibrium banks may choose to hold too much or too little cash compared to the first best, depending on the dispersion of the value of securities. When collateral quality is high enough, banks hold too little cash because they do not internalize the effect of their cash-holding decision on the equilibrium price of securities. In the cash-in-the-market pricing equilibrium, low cash holding implies that the

[1] Alternatively, it can be interpreted as a market in which the securities are sold outright, as is standard in this type of models Allen, Carletti, and Gale (2009). Indeed, since the return on marketable securities is publicly known when the market opens in our model, there is no difference between selling marketable securities and using them as collateral in secured loans. Holmström (2015) provides reasons for why it might be different in practice.

value of securities is low. This, in turns, reduces the extent to which banks hit by a liquidity shock can continue their non-marketable project. In contrast, if the dispersion of the value of the marketable securities is sufficiently high, banks hoard cash; that is, banks choose to hold enough cash to self-insure and continue their long-term project without accessing the interbank market. This leads to a market "freeze".

Interestingly, our results show that an increase in dispersion has a non-monotonous impact on aggregate liquidity. When the dispersion is small or moderate enough, an increase in dispersion diminishes the cash holdings of banks. However, when dispersion exceeds a certain threshold, banks switch their behavior on cash holdings, triggering a shift from a regime with low aggregate liquidity to a regime with liquidity hoarding.

It might seem counter-intuitive that, for moderate dispersion, banks would decrease their cash holdings as collateral quality deteriorates, given that this situation should naturally amplify the banks' precautionary motive for holding more cash. The underlying reason is that an opposing indirect effect counterbalances the rise in the precautionary motive. When dispersion remains relatively low, its increase prompts a decrease in the market return on cash in equilibrium. This is attributed to the declining demand for cash backed by securities due to the diminishing value of securities. Consequently, the motivation to hold cash diminishes. This indirect effect is dominant until the net market return on cash reaches 0 and can no longer adjust. Therefore, starting from this threshold level of dispersion, the precautionary motive becomes the sole driver of the decision on cash holding. This, in turn, results in a substantial surge in cash holdings.

The model also shows that the reduction in the collateral quality of securities leads to a reduction of the market return on cash, even without any change in their expected returns This carries a macroeconomic implication, as the market return on cash is the risk-free rate, i.e. the rate at which banks borrow secured.

We then study policies that can improve the equilibrium allocation. First, we show that when the economy is in an equilibrium with cash-in-the-market pricing, a liquidity regulation similar to the Basel Liquidity Coverage Ratio (LCR) improves welfare. The intuition for this result is that, by forcing banks to hold more (near-)cash assets, the regulation helps correct the fact that banks don't internalize the effect of their choice of cash holding on the price of securities at the interim date. This is the standard fire sale externality argument.

Second, we investigate the type of policies needed when the economy is in an equilibrium with cash hoarding. In such a situation, a policy of negative interest rate (NIR) reduces the incentive to hold cash, but always reduces welfare by inducing an excessive downward adjustments in banks' cash holdings and thus to too many project liquidations. However, a combination of a NIR policy and a liquidity regulation can be effective because liquidity requirement counters the negative effect of NIR on cash holding choice. This provides an additional rationale for liquidity regulation, different from the standard fire sale externality argument.

In a second step, the analysis of the case of aggregate uncertainty confirms the previous results that collateral quality can lead to cash-hoarding and cash-in-the-market pricing. It also allows getting more insights on the effect of collateral quality on the risk free rate. We model aggregate uncertainty by the possibility that the economy can be in one of two states at the interim date, a normal or a crisis state. In the normal state, all securities have the same high value while in the crisis state, a fraction becomes low type. In line with the previous case, a decrease in collateral quality has a non-monotonous impact on aggregate cash holdings. Since there is aggregate uncertainty, the risk-free rate is state-contingent. The risk free rate in the crisis state declines continuously with the decrease in collateral quality, as in the case without aggregate uncertainty. Collateral quality also affects the risk free rate in the normal state, through the initial cash holding decision of banks and their anticipation of future collateral shocks. In the normal state, the risk-free rate is non monotonous as it first increases with the decrease of the collateral quality up to a critical value after which the increasing discount on collateral quality leads to the market rate converging to the zero lower bound.

Related literature.—Our paper relates to three strands of literature.

First, our work relates to a strand of literature analyzing liquidity management by banks and its relation with the interbank market and asset prices, pioneered by Bhattacharya and Gale (1987).[2] In particular, following Allen and Gale (2004a; 2005), a number of papers have analyzed how *cash-in-the-market-pricing* affects aggregate liquidity and the scope for intervention. One important difference with extant studies is that there can be either insufficient

[2]Important contributions to this literature include Allen and Gale (1994), Freixas, Parigi, and Rochet (2000), Allen and Gale (2004a;b), Allen, Carletti, and Gale (2009), Diamond and Rajan (2011), and Freixas, Martin, and Skeie (2011), among others.

liquidity or liquidity hoarding in our setup.[3] On the one hand, the result that liquidity can be too low under laissez-faire is in line with early contributions where cash-in-the-market-pricing generates pecuniary externalities when combined with limited risk-sharing due *e.g.* to market incompleteness (Allen and Gale 2004b), asymmetric information (Bhattacharya and Gale 1987), or hidden trades (Farhi, Golosov, and Tsyvinski 2009). Recent applications of this general theme include Kara and Ozsoy (2020), Lutz and Pichler (2021) and Calomiris et al. (2024). Building on a standard rationale for a minimum liquidity requirements,[4] our results suggest that the optimal level of the requirement depends (in a non trivial way) on the collateral quality of non-HQLA securities. On the other hand, motivated by the great financial crisis, several studies have explained inefficient liquidity hoarding as arising from the expectation of fire sales (Acharya, Shin, and Yorulmazer 2011; Diamond and Rajan 2011; Gale and Yorulmazer 2013).[5] In our model cash hoarding arises from a precautionary—rather than a strategic—motive when the collateral quality of securities is depressed and the market return of cash at its lowest.[6] Interestingly, our contribution suggests that a liquidity *ceiling* is not a proper policy response in a cash hoarding regime; rather a combination between a liquidity *floor* and NIR can be effective. More recently, this literature has started to analyze the interaction between liquidity requirements and capital requirement (Walther 2016; Kara and Ozsoy 2020; Kashyap, Tsomocos, and Vardoulakis 2024; Carletti, Goldstein, and Leonello 2020) or liquidity injections by the central bank (Santos and Suarez 2019; Robatto 2023). We add to these papers by emphasizing a complementarity between liquidity requirement and a NIR policy.

Second, our results also contribute to the growing literature on the *impact* of the negative interest rate policy enacted in some jurisdictions as part of the unconventional monetary policy toolkit in the aftermath of the great financial crisis. This literature is essentially

[3] Arseneau, Rappoport, and Vardoulakis (2020) provide an over-the-counter market setup where liquidity can be inefficiently low or high due to a congestion externality.

[4] In Diamond and Kashyap (2016), liquidity regulation helps reducing the probability of run rather than correcting a pecuniary externality. A separate strand of literature analyzes liquidity requirements as a way to limit public bailouts; see Farhi and Tirole (2012) and Keister (2016), and more recently Tirole and Dewatripont (2018).

[5] The strategic motive for cash hoarding is also analyzed in Acharya, Gromb, and Yorulmazer (2012), because of market power in the interbank market.

[6] Another view is that liquidity hoarding can be explained by an exogenous increase in counterparty risk (Heider, Hoerova, and Holthausen 2015). We analyze a secured interbank market where counterparty risk is not an issue. Acharya and Skeie (2011) also propose an analysis of liquidity hoarding based on agency problem associated with high-leverage. There is no agency issue in our paper.

empirical, with no one-sided conclusion so far regarding the efficacy of a NIR policy (NIRP): a number of studies report results suggesting an expansionary effect on bank lending, while others instead point to a contractionary impact.[7] In our setup, a NIRP is contractionary if introduced alone in a configuration of low rates (high dispersion), but this negative effect can be mitigated by a regulation requiring banks to hold highly liquid assets (*e.g.* LCR). This suggests that the effect (and desirability) of NIRP is not independent of other measures, and in particular should be assessed by conditioning on policy tools targeting the level of liquidity in the banking sector. On the theoretical side, Ulate (2021), Abadi, Brunnermeier, and Koby (2023) and Eggerston et al. (2023) provide macro models where interest rate cuts in negative territory can become counterproductive beyond some point, by eroding bank profitability. We focus on a different mechanism, by showing that a NIR policy can be useful to address liquidity hoarding (if complemented by a liquidity regulation). This provides a different rationale for a NIR policy than the usual argument that going negative removes an upward bias in expectations about future policy rates (Draghi (2014), see de Groot and Haas (2023) for a macroeconomic modeling).

Finally, even though ours is a banking setup, a connection can also be made with the well-developed macro literature investigating the sources of low interest rates. In particular, a series of papers emphasize a safe asset shortage as a key factor exerting downward pressure on real interest rates (Caballero 2006; Caballero, Farhi, and Gourinchas 2008; Caballero and Farhi 2018).[8] In our model economy, there is no shortage of safe assets (HQLA), but a decrease in the capacity of (non-HQLA) assets to serve as a source of liquidity—caused *e.g.* by an increase in the dispersion of their return—leads to a lower risk-free interest rate. Interestingly, our results suggests that even though the underlying source of low rates is outside of public authorities' reach, the required mitigating policies implied by a decrease

[7] See Eisenschmidt and Smets (2018) and Heider, Saidi, and Schepens (2021) for surveys. Expansionary effect on bank lending are reported in Grandi and Guille (2023) for France, Demiralp, Eisenschmidt, and Vlassopoulos (2021) for the Eurozone, Bottero et al. (2022) for Italy, Schelling and Towbin (2022) for Switzerland, and Hong and Kandrac (2022) for Japan. Contractionary effects can be found in Heider, Saidi, and Schepens (2019) for the Eurozone, Eggerston et al. (2023) for Sweden, and Bittner et al. (2023) for Germany. Note that since these studies generally use a difference-in-difference approach for identification purposes, they cannot say much about the *aggregate* effect of NIRP.

[8] This macro literature is not surveyed here. Other determinants that have been discussed include demographic trends, productivity slowdown or globalization. See Eggerston et al. (2023), Lukasz and Summers (2019) and Marx, Mojon, and Velde (2021) for recent contributions attempting to quantify the importance of the various factors. On the specific role of safe assets, see the surveys by Caballero, Farhi, and Gourinchas (2017) and Gorton (2017).

in collateral quality may lead to further reduction in the safe interest rate.

2 Environment

The economy has 3 dates, $t \in \{0, 1, 2\}$, and is populated by a continuum of *ex ante* identical agents—labeled banks—who are risk neutral and maximize expected final consumption. The economy also includes a central bank. There is no aggregate uncertainty.[9]

At $t = 0$, each bank is endowed with one unit of a money-like asset we call "cash". Cash can be invested in marketable securities, henceforth "securities," such as bonds, equities, or asset-backed securities, or it can be stored. We denote by $s \in [0, 1]$ the share of cash that a (representative) bank chooses to hold. Securities generate a stochastic return at date 2, as described below. Cash in storage earns a net return determined by the central bank.[10]

In addition, each bank has a pre-existing non-marketable project, henceforth, "project", which we think of as an industrial loan or a commercial real-estate deal, for example. Projects yield output at date 2, and may need additional infusion of cash at date 1, as explained below.

What matters for our results is the relative return on cash, securities, and the non-marketable project, so we normalize the net return on cash to be zero. In section 5 we consider what happens if the central bank changes the net return on cash at date 1.

Securities: We think of securities as a long-term asset that generate a risky payoff $\tilde{R}$ at date 2. $\tilde{R}$ takes value R_H with probability μ and R_L with probability $1-\mu$, with $R_H \geq R_L$.[11]

As of date 0, all securities are perceived as identical, with an expected date 2 payoff $\bar{R} \equiv \mu R_H + (1-\mu) R_L$. We refer to $\theta \in \{H, L\}$ as the type of the securities and call type-H securities the 'high' type and type-L securities the 'low' type. We assume some form of limited diversification, so that all securities in a given bank portfolio turn out to be of either the low or the high type.[12] At date 1, each bank learns the actual date 2 payoff

[9] See section 6 for the case with aggregate uncertainty.

[10] This corresponds to setting the interest on reserves. We consider negative interest rate policies in section 5.

[11] There is an infinite supply of such assets. Empirically we think of these assets as any type of security with an International Security Identification Number (ISIN) or, in the U.S. context, a CUSIP, allowing its trade on an organized market. We do not model the process leading to the creation of this type of security.

[12] Limited diversification is a standard assumption in the banking literature (Holmström and Tirole 1997). Limited diversification could arise because of geographical bias in financial institutions' securities

of the securities it holds.

Our analysis focuses on the dispersion of the value of securities. To facilitate the comparison between economies with different degree of dispersion, we introduce a parameter ε that indexes dispersion for a given expected value of the asset, $\bar{R}$ (*i.e.*, mean-preserving spread). Specifically, given ε, the value of type-L securities is expressed by

$$R_L^{\varepsilon} = (1 - \varepsilon)\,\bar{R} \tag{1}$$

Accordingly the value of type-H securities under constant $\bar{R}$ can be expressed by

$$R_H^{\varepsilon} = \left(1 + \frac{1-\mu}{\mu}\varepsilon\right)\bar{R} \tag{2}$$

Expression (1) and (2) show that a higher value for ε implies a lower value of type-L securities, and a higher value of type-H securities, which can be translated as a higher dispersion of the value of securities. When ε is 0, $R_L^0 = \bar{R} = R_H^0$ and there is no dispersion while, when ε is 1, $R_L^1 = 0 < \bar{R} < R_H^1 = \frac{1}{\mu}\bar{R}$ so that dispersion is maximal. In the remainder of the paper, we simply refer to the parameter ε as the dispersion of asset value.

Projects: Each bank is endowed with a project that yields y at $t = 2$ if it matures. At $t = 1$, with probability λ, the project is hit by a liquidity shock that requires the injection of an additional amount of cash x.[13] We do not consider aggregate uncertainty and assume that the value of λ is known at date 0. In addition, the liquidity shock is uncorrelated with the quality of a bank's securities. To simplify notation, we normalize that amount to 1. We show in appendix C that our qualitative results generalize to the case where $x < 1$. If no cash is injected, the project is liquidated at $t = 1$ and yields nothing. The project is divisible: If a bank does not have enough cash at $t = 1$ to continue the project in its entirety, it can inject $i < 1$ of cash and the project returns $i \cdot y$ at date $t = 2$. Liquidity needs are private information to the banks.

We assume that :

$$y > \bar{R} \tag{A1}$$

portfolio.

[13]We assume a law of large number holds so that the share of projects affected by the liquidity shock is also λ.

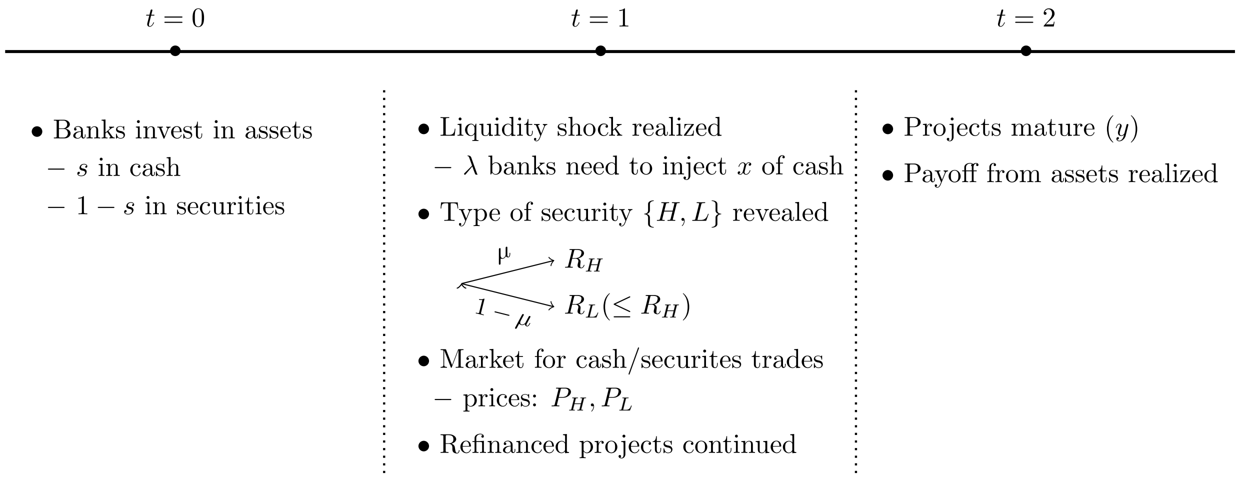


Figure 1: Timing

This condition implies that, from the perspective of date 0, the return y on the extra amount of cash that needs to be injected to continue the project in case of a liquidity shock is greater than $\bar{R}$, the expected return on securities. Finally, let $\bar{\varepsilon}$ denote the level of dispersion such that the expected value of the low type is equal to the extra cash needed to continue the project full scale, *i.e.* $R_L^{\bar{\varepsilon}} = 1$.

Timing: At date $t = 0$, banks choose the share of cash s they want to hold, with $1 - s$ representing the share of securities the bank holds. At $t = 1$, each bank learns the type of the securities it holds and whether it suffer liquidity shock. A market opens in which banks can rebalance their portfolio of cash and securities. Banks with liquidity needs can sell their securities to banks with extra cash, to continue their project. In our model, under the assumption of no counterparty risk, acquiring cash through the sale of securities is equivalent to acquiring cash through repo by pledging securities as “collateral”. At $t = 2$, projects and securities mature and their returns are realized. Consumption takes place.[14] Figure 1 summarizes the sequence of the events. Banks profits are realized.

Benchmark allocations: We consider two benchmarks, the first-best and autarky. Since $y > \bar{R}$, the first best requires that the non-marketable projects be continued in their entirely, which can be achieved by keeping an amount of cash greater or equal to λ. In addition, since $\bar{R} > 1$, the amount of cash should be as small as possible, conditional on continuing

[14] In this model, obtaining liquidity to continue the project is the only reason to sell assets.

the non-marketable projects. Hence, first-best welfare is equal to

$$\mathcal{W}^{FB} = y + (1-\lambda)\,\bar{R}. \tag{3}$$

There are two cases to consider regarding the autarky allocation. A bank could choose to hold only cash to make sure that it can continue its non-marketable project in its entirely if it is subject to a liquidity shock. In this case, expected welfare would be $y + (1-\lambda)$. Alternatively, the bank could choose to only hold securities in which case the entire non-marketable project is lost when a liquidity shock occurs and expected welfare is $(1-\lambda)y+\bar{R}$. To focus on the more relevant case in which liquidity shock is not trivial, we assume that λ is not too small, so that in autarky banks prefer to hold cash rather than securities. Specifically, we impose

$$\bar{R} - 1 < \lambda(y-1), \tag{A2}$$

so that welfare in autarky is

$$\mathcal{W}^{0} = y + (1-\lambda)\,. \tag{4}$$

3 Laissez-faire equilibrium

In this section we characterize equilibria in the absence of official sector intervention. We solve the model backward, looking first at the date 1 market for liquid securities, and then at banks' date 0 portfolio decision. Since banks are *ex ante* identical, we focus on symmetric equilibria where all banks choose the same portfolio at $t = 0$.[15] Proofs are relegated to the online appendix.

3.1 Market for liquid securities

At date 1, a market opens in which banks can trade securities for cash or, equivalently, borrow using their securities as collateral. At that date, the type of each security, $\theta \in \{H, L\}$ is publicly known. Hence, arbitrage requires that the return from purchasing either security must be the same in equilibrium. Let r denote the net market return on cash, which we

[15] This is without loss of generality. Non-symmetric equilibria exist and are equivalent in terms of welfare, since banks are risk neutral.

interpret as the risk-free interest rate. The no-arbitrage condition is $\frac{R_H^\varepsilon}{P_H^\varepsilon} = \frac{R_L^\varepsilon}{P_L^\varepsilon} = 1 + r$, with P_θ^ε, the price of a type θ security, defined as $P_\theta^\varepsilon \equiv \frac{R_\theta^\varepsilon}{1+r}$. Note that assets are priced at their fundamental values whenever $r = 0$. Lemma 1 restricts the range of market returns on cash that can be part of an equilibrium.

Lemma 1. *In any laissez-faire equilibrium, the net market return on cash satisfies (i)* $0 \leq r$ *and (ii)* $R_L^\varepsilon \leq 1 + r \leq \bar{R}$.

The proof of Lemma 1 is straightforward. The underlying intuition is as follows: Part (i) follows from observing that if $r < 0$, there is no supply for cash against assets in the date 1 market, but then $P_\theta^\varepsilon = 0$, implying that $r = \infty$. Part (ii) follows from a standard dominance argument. If $1 + r > \bar{R}$, the long term asset is dominated from an *ex ante* perspective since one can always get a higher return by investing cash in the market at date 1; but then $r = 0$ since there are no assets to buy at distressed prices. In turn, if $R_L^\varepsilon > 1+r$ banks hit by a liquidity need can always continue their project at full scale by selling their asset at $t = 1$, so that the long term asset dominates cash both in terms of return and of liquidity; but then $P_\theta^\varepsilon = 0$, meaning that $r = \infty$ as there is no cash to buy assets at $t = 1$.

Lemma 1 implies that the equilibrium market return on cash is always strictly lower than the return on using cash to continue the long term project (which we can think of as the internal return on cash),

$$1 + r < y. \tag{5}$$

Therefore, banks that are hit by a liquidity shock strictly prefer using cash to continue their non-marketable project rather than buying securities in the market. Lemma 1 also implies that equilibrium prices of securities satisfy

$$R_L^\varepsilon \leq 1 + r \leq R_H^\varepsilon. \tag{6}$$

Now consider the demand for cash—supply of securities—by banks with liquidity needs.[16] These banks need an amount $x \equiv 1$ of cash to continue their non-marketable project in its entirety and, thus, need to borrow $(1 - s)$ to complement their cash holdings s. By (6), banks with type-H collateral are unconstrained and supply just enough securities to raise

[16]Banks without liquidity needs never find it strictly beneficial to sell securities since there is symmetric information about future returns.

the cash necessary to continue their non-marketable project. This amount, $a_H \in (0, 1-s]$, satisfies $s + a_H P_H^\varepsilon = 1$. By contrast, banks with type-L collateral can be constrained if the price P_L is too low. Indeed, when $s + (1-s)P_L^\varepsilon < 1$, these banks are unable to continue their non-marketable project in its entirety, even after selling all their assets. The aggregate demand for cash is thus

$$D(s,r) = \lambda (1-s) \left(\mu + (1-\mu) \frac{R_L^\varepsilon}{1+r} \right). \tag{7}$$

Since cash can only be obtained by selling or pledging securities, $D(s,r)$ falls short of aggregate liquidity needs, $\lambda(1-s)$, whenever $R_L^\varepsilon < 1 + r$.

Banks without liquidity needs can either use their cash to obtain securities, earning the market return $1+r$, or store cash from date 1 to date 2, earning a return of 1. The supply of cash is thus

$$S(s,p) \begin{cases} = (1-\lambda)s & \text{if} \quad r > 0, \\ \in [0, (1-\lambda)s] & \text{if} \quad r = 0. \end{cases} \tag{8}$$

From (7) and (8), the market clearing condition $S(s,r) = D(s,r)$ implies pricing of assets at their fundamental value, $r = 0$, when

$$(1-\lambda)s > \lambda(1-s)(\mu + (1-\mu) R_L^\varepsilon), \tag{9}$$

and cash-in-the-market pricing (CIMP) otherwise. In this case, $r > 0$ is determined by

$$(1-\lambda)s = \lambda(1-s) \left(\mu + (1-\mu) \frac{R_L^\varepsilon}{1+r} \right). \tag{10}$$

To sum up, net market return on cash is given by

$$r(s) = \max \left\{ 0, \frac{R_L^\varepsilon}{\frac{1}{1-\mu} \left(\frac{(1-\lambda)s}{\lambda(1-s)} - \mu \right)} - 1 \right\}. \tag{11}$$

Expression (11) has important implications for our analysis. First, one can see that for a given level of s the type of pricing depends on the value R_L^ε of type-L securities, but not on that of type-H securities. For a given level of s, a low value of R_L^ε makes it more likely that *all* securities are priced at their fundamental value, because it reduces the amount of cash that can be raised by selling or pledging type-L securities.

Second, and relatedly, the effect of CIMP is different compared to the literature because we have two types of securities. To see this, compute the prices for both types of securities, using equation (11):

$$P_L^\varepsilon = \min\left\{R_L^\varepsilon, \frac{1}{1-\mu}\left(\frac{(1-\lambda)\,s}{\lambda\,(1-s)} - \mu\right)\right\}, \tag{12}$$

$$P_H^\varepsilon = \min\left\{R_H^\varepsilon, \frac{R_H^\varepsilon}{R_L^\varepsilon}\frac{1}{1-\mu}\left(\frac{(1-\lambda)\,s}{\lambda\,(1-s)} - \mu\right)\right\}. \tag{13}$$

By comparison, the standard formula from the canonical CIMP with one type of assets—see Allen and Gale (2005, p. 539)—is $P = \min\left\{R, \frac{(1-\lambda)s}{\lambda(1-s)}\right\}$. Inspection of (12) and (13) shows that the price P_L^ε depends on the cash supply but not on the fundamental value R_L^ε when there is CIMP, as in the canonical model. By contrast, the price P_H^ε depends on R_H^ε even if there is CIMP. Formally,

$$\frac{\partial P_L^\varepsilon}{\partial R_L^\varepsilon} = \begin{cases} 1, & \text{if} \quad r\,(s) = 0, \\ 0, & \text{if} \quad r\,(s) > 0, \end{cases} \qquad \frac{\partial P_H^\varepsilon}{\partial R_H^\varepsilon} = \begin{cases} 1, & \text{if} \quad r\,(s) = 0, \\ \dfrac{1}{1+r\,(s)}, & \text{if} \quad r\,(s) > 0. \end{cases}$$

3.2 Individual portfolio choice

We now consider banks' date 0 choice between cash and securities. Given the optimal reinvestment behavior conditional on date 1 shocks, a bank's expected profit is

$$\begin{aligned} \pi\,(s,r) = {} & (1-\lambda)\left(y + s\,(1+r) + (1-s)\,\bar{R}\right) \\ & + \lambda\mu\,(y + (1-s-a_H)\,R_H^\varepsilon) + \lambda\,(1-\mu)\,y\left(s + (1-s)\,\frac{R_L^\varepsilon}{1+r}\right), \end{aligned} \tag{14}$$

where $a_H = \frac{(1-s)(1+r)}{R_H^\varepsilon}$. The first term represent the bank's profits if it does not suffer from a liquidity shock. The second and third terms correspond to the profits when the bank suffers a liquidity shock and holds type-H and type-L securities, respectively.

Taking the derivative with respect to s, one obtains the following expression for the marginal return on cash (MRC) at date 0:

$$MRC\left(r\right) \equiv \frac{\partial \pi}{\partial s} = \underbrace{\left(1-\lambda\right)\left(1+r-\bar{R}\right)+\lambda\mu\left(1+r-R_{H}^{\varepsilon}\right)}_{\text{(opposite of) opportunity cost of cash}} + \underbrace{\lambda\left(1-\mu\right)y\left(1-\frac{R_{L}^{\varepsilon}}{1+r}\right)}_{\text{liquidity value of cash}}. \quad (15)$$

In words, the marginal expected return on cash is the sum of two terms, the opportunity cost of cash and its liquidity value. If the bank does not suffer from a liquidity shock, its expected date 1 marginal return on cash is $1+r-\bar{R}$, which corresponds to the market return of cash net of the expected foregone return on securities. If the bank does suffer from a liquidity shock, its expected date 1 marginal return on cash is more complicated. Conditional of holding a type-H security, the bank is unconstrained: It has enough cash to continue its entire non-marketable project and can invest additional units of cash into the market, for a marginal return of $1+r-R_{H}^{\varepsilon}$. By contrast, if the bank holds a type-L security, it invests all its cash into the non-marketable project, for a marginal return of y, compared to a marginal return of $\frac{R_{L}^{\varepsilon}}{1+r}y$ for selling type-L securities to obtain cash.

One can see from (15) that the marginal return of cash is a decreasing function of the interest rate r. Indeed, a decrease in r leads to an increase in the opportunity cost of cash. Moreover, a decrease in r decreases the liquidity value of cash compared to type-L securities, since the market price $\frac{R_{L}^{\varepsilon}}{1+r}$ increases. Rearranging (15), the marginal return of cash can also be expressed as

$$MRC\left(r\right) = \left(1+r-\bar{R}\right)+\lambda\left(1-\mu\right)\left(y-\left(1+r\right)\right)\left(1-\frac{R_{L}^{\varepsilon}}{1+r}\right). \quad (16)$$

The optimality requirement for the choice between cash and securities can thus be expressed as

$$s(r) = \begin{cases} 0, & \text{if} \quad MRC\left(r\right) < 0, \\ \in \left[0,1\right], & \text{if} \quad MRC\left(r\right) = 0, \\ 1, & \text{if} \quad MRC\left(r\right) > 0. \end{cases} \quad (17)$$

3.3 Equilibrium

Definition 1. *A laissez-faire equilibrium is a risk-free interest rate r^* and an investment choice s^* satisfying the market clearing condition (11) and the first order condition (17).*

We are interested in how the dispersion of the return on securities impacts the aggregate level of liquidity in the economy. The following proposition shows that there are two types of equilibria with different properties in terms of liquidity and pricing, depending on the level of the dispersion.

Proposition 1. *There is a threshold $\hat{\varepsilon} > 0$, defined by $MRC(0) = 0$, such that:*

1. *If $\varepsilon < \hat{\varepsilon}$, the equilibrium is unique and features CIMP, with $r^* \, (> 0)$ and $s^* \, (\leq \lambda)$ satisfying*

$$\bar{R} - (1 + r^*) = (1 - \mu)\,\lambda\,(y - (1 + r^*))\left(1 - \frac{R_L^\varepsilon}{1 + r^*}\right), \tag{18}$$

$$s^* = \frac{\lambda\left(\mu + (1 - \mu)\,\dfrac{R_L^\varepsilon}{1 + r^*}\right)}{\lambda\left(\mu + (1 - \mu)\,\dfrac{R_L^\varepsilon}{1 + r^*}\right) + (1 - \lambda)}. \tag{19}$$

2. *If $\varepsilon > \hat{\varepsilon}$, the equilibrium is unique, securities are priced at their fundamental value, and banks hoard liquidity, so that $r^* = 0$ and $s^* = 1$.*[17]

3. *If $\varepsilon = \hat{\varepsilon}$, there is a continuum of equilibria with $r^* = 0$ and $s^* \in [\hat{s}, 1]$ where $\hat{s}$ is given by (19). All these equilibria deliver the same level of welfare.*

Proof. See Appendix A. ■

Proposition 1 states that fundamental pricing and cash hoarding arise when the dispersion of returns of securities is high ($\varepsilon > \hat{\varepsilon}$), while there is CIMP when dispersion is low ($\varepsilon < \hat{\varepsilon}$). The simplest way to get the intuition underlying this result is to understand when fundamental pricing can occur in equilibrium. One requirement is that, conditional on fundamental pricing, cash is not dominated by securities in banks' initial portfolio choice.

[17] Condition (A2) ensures that $\hat{\varepsilon} < 1$, allowing us to focus on the interesting case where both cash-in-the-market or fundamental pricing configurations arise when varying the level of dispersion.

Formally, $MRC(0) \geq 0$ must hold. Using expression (16) for the marginal return of cash we get the condition

$$\lambda (1-\mu)(y-1)(1-R_L^\varepsilon) \geq \bar{R} - 1. \tag{20}$$

The right side is the marginal opportunity cost of cash in a world with no liquidity concerns; the left side represents the marginal benefit of cash when a liquidity shock occurs and the bank holds low type securities. More cash allows a bank hit by a shock to continue its non-marketable project at a scale greater than it could with a low type security.[18] Condition (20) holds for small value of R_L^ε, corresponding to high dispersion of returns ($\varepsilon \geq \hat{\varepsilon}$). When (20) holds strictly, securities are a very poor source of liquidity, so that banks choose to hold enough cash to continue their non-marketable projects at full scale without relying on the secured interbank market, $s^* = 1 > \lambda$. Aggregate cash then exceeds aggregate cash needs at date 1, validating fundamental pricing, and there is unused liquidity in the system (cash hoarding).

When the dispersion of returns of securities is low ($\varepsilon < \hat{\varepsilon}$), R_L^ε is high enough that (20) is violated or, formally, $MRC(0) < 0$. In that case fundamental pricing cannot arise since it would imply that banks choose not to hold any cash initially, and no cash would be available to buy securities at date 1. There is CIMP, with a net market return on cash $r^* > 0$ and a price for the low type security $\dfrac{R_L^\varepsilon}{1+r^*} \leq 1$ such that banks are indifferent between cash and securities at date 0. The portfolio choice is such that the aggregate amount of liquidity in the market is insufficient to continue all projects that suffered from a liquidity shock, $s^* < \lambda$, except for the limit case of no dispersion.

Note that the two polar cases in proposition 1 exhibit very different properties in terms of aggregate liquidity and efficiency loss compared to the first best. In the fundamental pricing/cash hoarding regime, banks are fully (self-)insured so that all non-marketable projects arrive at completion. However, there is an efficiency loss that comes from eschewing the return on securities. In this regime, aggregate surplus is

$$\mathcal{W}^* = y + (1-\lambda) = \mathcal{W}^0 = \mathcal{W}^{FB} - (1-\lambda)\left(\bar{R} - 1\right). \tag{21}$$

In the CIMP regime, the interbank market transfers all unused cash from banks with

[18] Observe that (20) implies that $R_L^\varepsilon < 1$.

liquidity surplus to those with liquidity needs, so that all cash is effectively used to continue non-marketable projects. However, there is insufficient cash in aggregate to continue all projects at full scale—except for the limit case of no dispersion.[19] In this pricing regime, aggregate surplus is

$$\mathcal{W}^{*} = (1 - \lambda + s^{*})\, y + (1 - s^{*})\, \bar{R} = \mathcal{W}^{FB} - (\lambda - s^{*}) \left(y - \bar{R}\right). \tag{22}$$

4 Dispersion of returns and collateral quality

In this section, we study more closely the capacity of assets to serve as a source of liquidity when needed. We term this the "collateral quality" of securities.[20] We are interested in how a variation—and in particular a decrease—in this collateral quality affect equilibrium outcomes. Two dimensions matter for the value of securities as a source of liquidity, reflecting the widespread intuition that the collateral quality of an asset relates to *adverse* shocks to the value of this asset. One dimension, captured by our dispersion parameter ε, relates to how much value the security can loose. The other dimension relates to the probability $(1 - \mu)$ that the securities looses value. For brevity we conduct our formal analysis with a focus on ε, but the results in this section will make clear that the two dimensions have qualitatively similar implications.[21]

Consider first the impact of an increase in dispersion on aggregate liquidity. From the equilibrium characterization in proposition 1, we have :

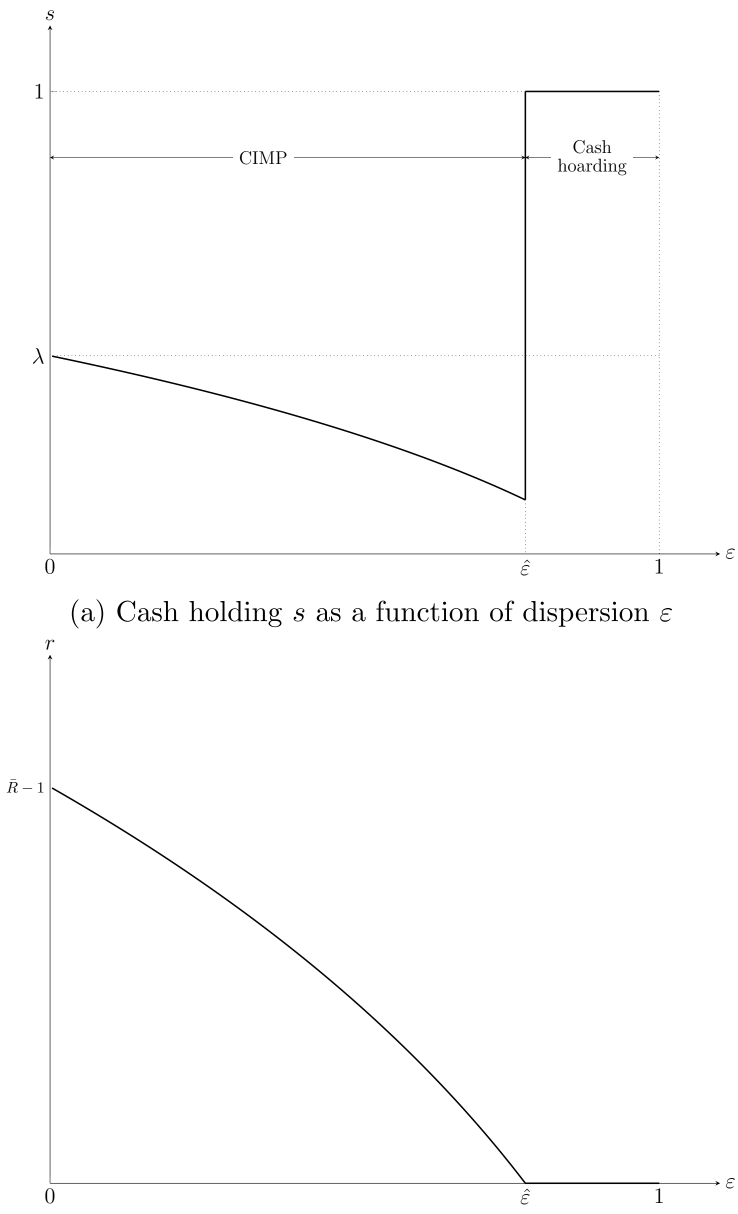
Proposition 2. *An increase in dispersion, from $\varepsilon_0 \geq 0$ to $\varepsilon_1 > \varepsilon_0$,*

1. *leads to a decrease in the equilibrium level of cash holding if $\varepsilon_0 < \varepsilon_1 < \hat{\varepsilon}$;*
2. *leads to a (large) increase in equilibrium cash holdings if $\varepsilon_0 < \hat{\varepsilon} < \varepsilon_1$.*
3. *has no impact if $\hat{\varepsilon} < \varepsilon_0$.*

[19] In the limit case with no dispersion ($\varepsilon = 0$), proposition 1 implies that $\bar{R} = R_L^{\varepsilon} = 1 + r^{*}$ and $s^{*} = \lambda = s^{FB}$. The result that $\bar{R} = 1 + r^{*}$ is standard in economies à la Diamond and Dybvig with one homogeneous long term asset, see von Thadden (1998). The fact that the first best obtains follows from our risk neutrality cum linear technology assumption.

[20] Recall that transactions in the date 1 interbank market can be interpreted either as outright asset purchases or as repos.

[21] One can observe that the two dimensions show up in the left side of (20), through R_L^{ε} and $(1 - \mu)$.



(a) Cash holding s as a function of dispersion ε

(b) Risk-free rate r as a function of dispersion ε

Figure 2: Effect of collateral quality on cash holdings and the risk-free rate

Proof. See Appendix A. ■

Proposition 2 states that an increase in dispersion—akin to a decrease in the collateral quality of securities—has a non monotonous impact on aggregate liquidity (see figure 2, panel (a)). For low level of dispersion, that is in the CIMP regime, there is a negative relationship between dispersion and cash holdings. Said differently, banks collectively increase their investment in securities as their capacity to serve as a source of liquidity decreases. This comes from two opposite effects. First, for a given pricing factor, the reduction in the liquidity that can be raised with a type-L security increases the precautionary motive for holding cash. In the absence of price adjustment, this direct effect would lead banks to invest only in cash. The second effect is indirect: in equilibrium the market return of cash decreases as the value of the collateral needed to back liquidity demand by constrained banks falls, reducing the incentives to hold cash. Proposition 2 shows that the second effect dominates, resulting in lower aggregate liquidity and—in line with intuition—a lower price for type-L collateral. The second effect disappears once dispersion reaches $\hat{\varepsilon}$, as the market return cannot adjust further. When dispersion is close to but below $\hat{\varepsilon}$, a small increase in dispersion thus leads to a switch to the cash hoarding regime and a large jump in cash holdings.

Proposition 3. *An increase in the dispersion is associated with a decrease in the risk-free rate r, a decrease in the price of low-type securities P_L^{ε}, and a decrease in aggregate surplus.*

Proof. See Appendix A. ■

Panel (b) of figure 2 illustrates the above relationship concerning the market return on cash and the price of low-type securities. Given that in this economy the market return of cash can be interpreted as the risk-free rate, we have the following implication. A decrease in collateral quality of securities leads to a decrease in the risk-free rate, even though their expected return $\bar{R}$ is unchanged. This connects our banking model to discussions of falling safe rates and stable return on capital (Caballero and Farhi 2018; Marx, Mojon, and Velde 2021). We view our work as potentially complementing explanations of low rates developed in macrodynamics models (*e.g.* Eggertsson, Mehrotra, and Robbins 2019).

We close this section with two propositions showing that similar comparative static results hold when varying $(1-\mu)$ and when considering a mean preserving spread in the distribution of returns of securities.

Proposition 4. *An increase in the probability of ending up with a low type security (holding expected return $\bar{R}$ and dispersion ε constant) has a similar impact as an increase in dispersion. Specifically, there exists $\hat{\mu}$ such that there is under-insurance in liquidity for $\hat{\mu} < \mu$ and liquidity hoarding if $\mu < \hat{\mu}$. In addition, a decrease in μ is associated with a decrease in the market return of cash and a decrease in welfare.*

Proof. See Appendix A. ■

Proposition 5. *Consider two economies $\mathcal{E}, \mathcal{E}'$ that differ only by the parameters for the binary distribution of return of securities, and assume that the distribution in $\mathcal{E}'$ is a mean preserving spread of that in $\mathcal{E}$. Then $\mathcal{E}'$ is characterized by a lower market return on cash, and is more likely to feature cash hoarding than $\mathcal{E}$.*

Proof. See Appendix A. ■

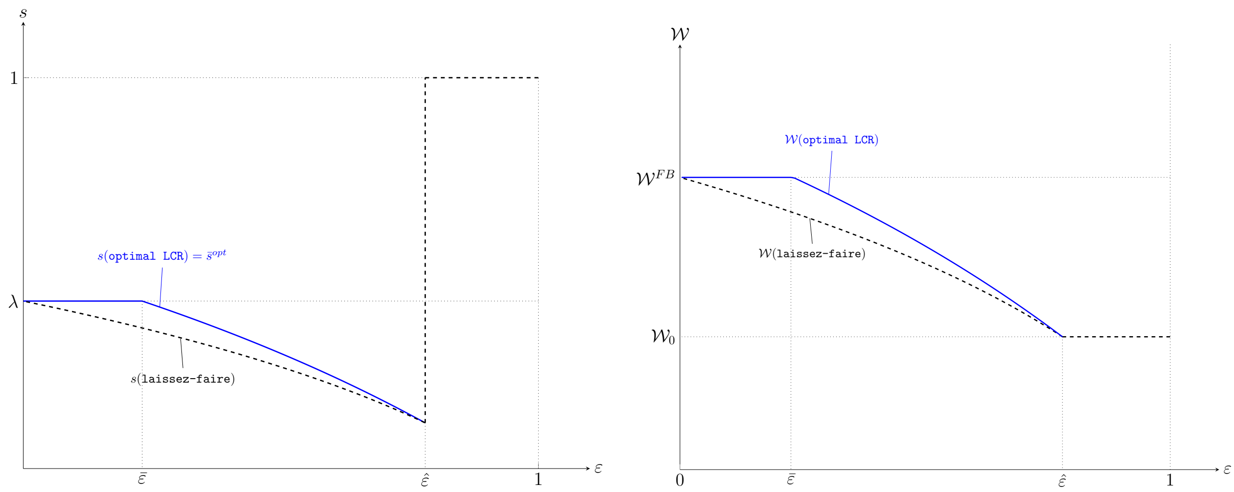
Propositions 2 to 5 show that a decrease in the collateral quality of securities—captured either through an increase in ε, an increase in $(1-\mu)$, or a mean preserving spread—leads to a lower level of the interest rate, and can trigger a shift from a regime with "low" aggregate liquidity to a regime where banks hoard liquidity.

To ease the exposition, the remainder of the formal analysis will focus on the dispersion parameter ε, having in mind that given the results above it can be interpreted more broadly as capturing the collateral quality of securities.

5 Policies

The laissez-faire equilibrium derived above can then be interpreted as the outcome obtained with a passive central bank that manages the wholesale payment system but does not seek to influence banks' liquidity management. In this section, we use our framework to discuss policy interventions.

We analyze liquidity regulation such as Basel III Liquidity Coverage Ratio (LCR) as a policy targeting insufficient liquidity on the one hand, and negative interest rates (NIR) to deal with liquidity hoarding on the other hand.



(b) Welfare $\mathcal{W}$

Figure 3: Cash holding and welfare under optimal Liquidity Coverage Ratio (LCR)

5.1 Liquidity regulation

When the economy features CIMP, aggregate liquidity under laissez-faire is lower than the first best level ($s^* < \lambda$). This makes liquidity requirement a natural candidate for welfare-enhancing policy.

In our model, a LCR-type regulation would require banks to hold at least $\bar{s}$ in cash at the initial date. The banks' optimal portfolio choice under such a constraints is given by

$$s(r) = \begin{cases} \bar{s}, & \text{if} \quad MRC(r) < 0, \\ \in [0,1], & \text{if} \quad MRC(r) = 0, \\ 1, & \text{if} \quad MRC(r) > 0. \end{cases} \tag{23}$$

The following proposition confirms that such a policy, if well designed, enhances welfare.

Proposition 6. *Let parameters be such that the economy is in a CIMP regime ($\varepsilon < \hat{\varepsilon}$). Then there exists $\bar{s}$ such that imposing a minimum level of cash $s \geq \bar{s}$ improves welfare. The optimal (surplus maximising) LCR is*

$$\bar{s}^{opt} = \min\left\{\lambda \frac{\mu + (1-\mu) R_L^{\varepsilon}}{1 - \lambda + \lambda\left(\mu + (1-\mu) R_L^{\varepsilon}\right)}, \lambda\right\}. \tag{24}$$

Proof. See Appendix A. ■

Figure 3 illustrates optimal LCR ($\bar{s}^{opt}$) in panel (a) and corresponding welfare in panel (b) for each level of dispersion (ε) respectively. Forcing banks to hold more liquidity increases surplus because it leads a reduction in the market return of cash—the economy's risk-free rate—which raises the date 1 price of securities. This allows constrained banks to raise more cash when needed and continue their long-term project at a larger scale.[22] The regulation helps because banks do not internalize the effect of their initial portfolio decision on the market price of securities. This result is reminiscent of other studies providing a rationale for liquidity regulation based on a pecuniary externality arising from fire sales (Kara and Ozsoy 2020; Lutz and Pichler 2021).[23] The optimal LCR is the level at which securities are priced at their fundamental, unless banks' liquidity needs are already satiated at a lower price. Imposing a more stringent LCR requirement would lead to unused liquidity and a reduction in surplus, since the return on cash is lower than the return on securites *ex ante*.[24]

An interesting and novel implication of our framework is that the appropriate level of the requirement depends on the collateral properties of securities, which we think of as non-HQLA assets (see figure 3, panel (a)). More precisely, the optimal liquidity requirement becomes lower as the quality of securities decreases, either on the intensive margin (decrease in R_L^{ε}, which means that less cash can be raised with securities when they have a low return) or on the extensive margin (decrease in μ, the probability that the securities have a high return). So, banks should be required to hold less cash when the quality securities assets is lower. This is because liquidity regulation can only improve allocation inefficiency *ex ante*, at date 0. At date 1, the redistribution of cash between banks with extra cash and those in need of cash relies on the market. Since the central bank cannot force banks to pay a higher price than fundamental value of collateral, requiring the banks to hold more cash than λ in aggregate result in cash unused, and a welfare loss. Therefore, if the quality of security decreases, so that the amount of cash that banks with type-L securities decreases,

[22] The effect of imposing a liquidity floor in our model is consistent with empirical studies pointing to an increase in bank lending following the implementation of liquidity requirements (Hachem and Song 2021; Ananou et al. 2021).

[23] See also Farhi, Golosov, and Tsyvinski (2009) and Allen and Gale (2004b).

[24] As can be seen from the right side of (24) the threshold for dispersion, which we denote $\bar{\varepsilon}$, is given by $R_L^{\bar{\varepsilon}} = 1$. So for $\varepsilon \leq \bar{\varepsilon}$, the first best can be implemented, with a price $P_L^{\varepsilon} = \dfrac{R_L^{\varepsilon}}{1+r} = 1 < R_L^{\varepsilon}$ for L-type securities. For $\bar{\varepsilon} < \varepsilon < \hat{\varepsilon}$, the price at the optimal LCR is the fundamental value, $P_L^{\varepsilon} = R_L^{\varepsilon}$.

banks should be required to hold less cash.

The next proposition establishes that liquidity regulation alone cannot help when there is cash hoarding.

Proposition 7. *Let parameters be such that the economy is in a cash hoarding regime ($\varepsilon > \hat{\varepsilon}$). Then imposing a minimum level of cash (LCR) is irrelevant. Imposing a maximum level (ceiling) of cash is welfare reducing.*

Proof. See Appendix A. ■

Clearly, setting a minimum level of cash has no effect since banks already hold as much cash as they can. Forcing banks to hold less cash than they prefer would be welfare reducing because R_L^{ε} is too low for banks to raise enough cash against their low quality collateral to continue their project. The issue comes from how cash is distributed among banks and is caused by the impediments to cash redistribution (reshuffling) created by the variation of the collateral value.

5.2 Negative interest rate policy

Since mandatory minimum liquidity requirements alone cannot improve on laissez-faire when the dispersion of the value of assets used as collateral is high (when $\varepsilon \geq \hat{\varepsilon}$), in this section we consider whether implementing a negative rate can limit cash hoarding. Surprisingly, we find that negative rates alone cannot increase welfare, but a combination of negative rate and minimum liquidity requirements can.

NIR policies are often described as a tax on reserves (Hannoun 2015; Waller 2016). We assume that the central bank can implement negative rates by imposing a taxes $\rho > 0$ on each unit of cash held by banks at the end of date 1. The central bank uses the proceeds to (stand ready to) buy assets from banks that suffer from a liquidity shock.[25]

We start by analyzing the impact of a NIR policy alone. Compared to laissez-faire, a negative deposit rate merely reduces the outside option of holding reserves at the end of date 1. Specifically, banks with excess liquidity are eager to supply their cash as long as

[25] In practice, ρ cannot be too large as some economic actors could choose to hold currency. In addition, negative rates may reduce bank profitability.

$r \geq -\rho$. The supply of reserves can be written

$$S^{NIR}(s,r) \begin{cases} = (1-\lambda)s & \text{if} \quad r > -\rho, \\ \in [0,(1-\lambda)s] & \text{if} \quad r = -\rho, \end{cases} \tag{25}$$

and the equilibrium in the liquidity market now gives the risk-free interest rate:

$$r(s) = \max\left\{ -\rho, \frac{R_L^{\varepsilon}}{\frac{1}{1-\mu}\left(\frac{(1-\lambda)s}{\lambda(1-s)} - \mu\right)} - 1 \right\}. \tag{26}$$

Since the rest of the analysis is unaltered, we have the following straightforward extension of proposition 1

Proposition 8. *Let the CB follows a NIR policy with $-\rho < 0$. There is a strictly increasing function $\hat{\varepsilon}(\rho) > \hat{\varepsilon}$, with $\hat{\varepsilon}(0) = \hat{\varepsilon}$, such that*

1. *If $\varepsilon < \hat{\varepsilon}(\rho)$, the equilibrium is unique and features CIMP, with $r^* > -\rho$ and $s^* \leq \lambda$ satisfying (18) and (19).*

2. *If $\hat{\varepsilon}(\rho) < \varepsilon$, the equilibrium is unique and features liquidity hoarding, with $s^* = 1$ and $r^* = -\rho < 0$.*

3. *If $\varepsilon = \hat{\varepsilon}(\rho)$, there is a continuum of equilibria, with $r^* = -\rho$ and $s^* \in [\bar{s}^*, 1]$ where $\bar{s}^*$ is given by (19).*

Proof. See Appendix A. ■

Negative deposit rate reduces the outside option associated with cash, so that banks with liquidity surplus have incentives to buy an asset delivering R_θ in date 2 at a price up to $\frac{1}{1-\rho}R_\theta$. Is this sufficient for a NIR policy to address the inefficiency associated with a cash hoarding regime? Proposition 9 shows that a policy of negative deposit rate is successful in eliminating cash hoarding and in supporting asset prices if the rate can be sufficiently negative, but reduces welfare if *introduced alone*.

Proposition 9. ***(NIR alone can do no good)*** *Let parameters be such that the economy is in a cash hoarding regime ($\varepsilon > \hat{\varepsilon}$). There is a threshold $\hat{\rho}(\varepsilon) > 0$ such that under a negative rate $\rho > \hat{\rho}(\varepsilon)$ the configuration with cash hoarding is no longer an equilibrium, and* *there is a (unique) equilibrium where $-\rho < r^* < 0$ and $s^* < \lambda$. However, welfare is lower than in the initial equilibrium with cash hoarding. If $\rho \leq \hat{\rho}(\varepsilon)$, a NIR policy has no impact.*

Proof. See Appendix A. ■

The reason NIR alone is welfare reducing is that the lower return on cash, which is instrumental in incentivising banks with liquidity surplus to lend their cash and in supporting security prices, backfires as banks adjust their portfolio choice. This adjustment is such that equilibrium aggregate liquidity is very low (see figure 4) and that banks affected by a liquidity shock end up being more constrained, leading to the liquidation of too many projects. In a sense, a NIR policy is too powerful.[26]

Since NIR is not effective alone, we investigate whether it can be helpful when combined with other policies. Proposition 10 shows that for a given level of dispersion $\varepsilon > \hat{\varepsilon}$, a joint policy of negative interest rate and LCR can increase welfare provided that the central bank rate (deposit facility) can be lowered sufficiently (see red line in figure 4).

Proposition 10. *Let parameters be such that the economy is in a cash-hoarding regime ($\varepsilon > \hat{\varepsilon}$), and let*

$$\bar{\rho}(\varepsilon) \equiv \frac{R_L^{\hat{\varepsilon}} - R_L^{\varepsilon}}{R_L^{\hat{\varepsilon}}} \quad (> 0). \tag{27}$$

Then a NIR policy with $\rho > \bar{\rho}(\varepsilon)$ combined with a LCR can raise surplus.

Proof. See Appendix A. ■

The combination of policies is helpful as a (well designed) floor on liquidity holdings counters the negative effect of NIR on liquidity choices. Note that implementing this joint policy requires that the central bank is able to set a low-enough negative interest rate for this combination of policies to be effective,[27] i.e. the NIR must be lower than $-\bar{\rho}(\varepsilon)$.

So far, some of the empirical literature have shown expansionary impact of NIRP on credit lending by banks in the Euro area, France, Italy, Switzerland or Japan (Demiralp, Eisenschmidt, and Vlassopoulos 2021; Girotti, Horny, and Sahuc 2022; Grandi and Guille 2023; Bottero et al. 2022; Schelling and Towbin 2022; Hong and Kandrac 2022). Other

[26] It is noteworthy that when the policy has an impact (e.g. for $\varepsilon \in (\hat{\varepsilon}, \hat{\varepsilon}(\rho)))$ in equilibrium the tax is never implemented, since all the cash is used to continue projects and banks end up with zero reserves at the end of date 1.

[27] Below which currency or competing safe assets becomes a credible outside option, or below which financial stability issues become too important. See Andolfatto (2019) for a discussion of the extent to which the zero lower bound reflects a legal, rather than an economic, constraint.

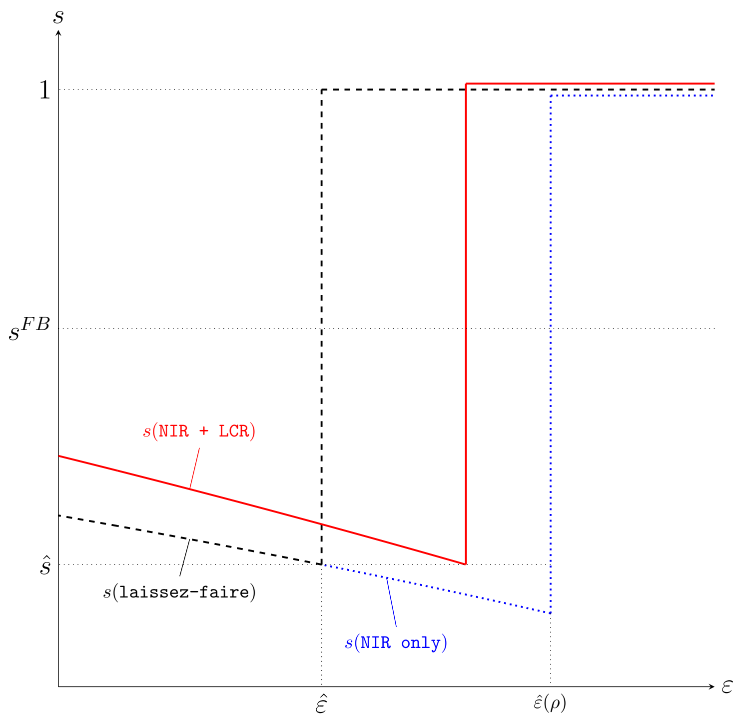


Figure 4: Impact of Negative Interest Rate (NIR) policy and NIR with LCR policy on cash holding s, depending on the dispersion ε of security values

papers have however shown a negative effect of NIRP on bank lending in the Euro area (Heider, Saidi, and Schepens 2019), Germany (Bittner et al. 2023) or (Eggerston et al. 2023), something that is also compatible with our model (Proposition 9). By modeling explicitly the interplay between credit (project) and the interbank market, our paper points to the importance of taking into account both the dispersion of collateralized asset values and the liquidity regulation and its impact on liquid asset holdings in any empirical setting testing the impact of NIRP on bank lending volume.[28]

6 The case with aggregate uncertainty

We now study the robustness of our analysis to the inclusion of an aggregate uncertainty shock on the quality of the collateral in the spirit of Allen and Gale (2005). More precisely, at date 1, there are two possible states of the world $\omega \in \{n, c\}$ where n stands for "normal" and c for "crisis". In state n, it is publicly known that all securities will return R_H at date

[28]In a similar vein, the recent empirical literature has shown the importance of the interplay between quantitative easing policy and the effectiveness of NIRP on bank lending (Bittner et al. 2023; Demiralp, Eisenschmidt, and Vlassopoulos 2021). One common feature of quantitative easing and liquidity regulation is that both policies influence the level of liquidity in the banking system.

2, while in state c the long-term return of the securities has the same structure as before, with a fraction $1-\mu$ turning into low type securities. Let the probability of state c be $\gamma \in [0,1]$, and denote the risk-free interest rate in state ω by r^{ω}.

Proposition 11 derives the conditions under which equilibria feature cash-in-the-market pricing and/or cash hoarding. See figure 5 for a graphical representation.

Proposition 11. *There exists continuous functions $\underline{\varepsilon}(\gamma)$ and $\bar{\varepsilon}(\gamma)$, with $\underline{\varepsilon}(\gamma)$ strictly increasing, $\bar{\varepsilon}(\gamma)$ strictly decreasing, $\underline{\varepsilon}(1) = \hat{\varepsilon} = \bar{\varepsilon}(1)$ such that:*

1. *If $\varepsilon < \underline{\varepsilon}(\gamma)$, the (unique) equilibrium features CIMP in both states, with $r^{n*} > r^{c*} > 0$ and $s^* < \lambda$.*
2. *If $\underline{\varepsilon}(\gamma) \leq \varepsilon < \bar{\varepsilon}(\gamma)$, the (unique) equilibrium is "hybrid" : it features CIMP in state n and fundamental pricing in state c, $r^{n*} > r^{c*} = 0$, and*
   (a) $s^* < \lambda$ if $\varepsilon < \hat{\varepsilon}$.
   (b) $s^* = \lambda$ if $\hat{\varepsilon} \leq \varepsilon$.
3. *If $\bar{\varepsilon}(\gamma) < \varepsilon$, the (unique) equilibrium features cash hoarding $s^* = 1$ cum fundamental pricing in both states $r^{n*} = r^{c*} = 0$.*

Proof. See Appendix B. ■

The introduction of aggregate uncertainty does not change qualitatively our results on the impact of the dispersion of the value of securities. Specifically, when dispersion is low enough ($\varepsilon < \underline{\varepsilon}(\gamma)$) there is cash-in-the-market pricing in both states and aggregate liquidity below aggregate liquidity needs, irrespective of the probability of the crisis state. Banks with liquidity needs are constrained in both states, and the risk-free interest rate is lower in the crisis state ($r^{n*} > r^{c*} > 0$) since there is less collateral in that state chasing the same amount of aggregate liquidity. When dispersion is high enough ($\varepsilon > \bar{\varepsilon}(\gamma)$) and the probability of crisis high enough, there is cash hoarding and a low interest rate in both states ($r^{n*} = r^{c*} = 0$). Cash hoarding arises when the precautionary motive is strong enough, that is when the probability of crisis and the potential loss in collateral value are high enough. These two types of equilibrium configurations mirror those obtained in the benchmark case without aggregate uncertainty, see Proposition 1.

For intermediate values of the dispersion of collateral values ($\underline{\varepsilon}(\gamma) < \varepsilon < \bar{\varepsilon}(\gamma)$) a new configuration emerges, with CIMP in the normal state and fundamental pricing together with the risk-free rate at its lower bound in the crisis state ($r^{n*} > r^{c*} = 0$). In this "hybrid" equilibrium, the aggregate level of liquidity is below aggregate liquidity needs when the dispersion of collateral values is not too high ($\underline{\varepsilon}(\gamma) < \varepsilon < \hat{\varepsilon}$) ; in that case banks with liquidity needs are constrained in both crisis and normal states. For higher levels of dispersion ($\hat{\varepsilon} < \varepsilon < \bar{\varepsilon}(\gamma)$), aggregate liquidity is at the first best level. However, the equilibrium allocation is not first best : while banks are unconstrained in the normal state, in the crisis state banks with liquidity needs but type L asset are constrained by its low price and cannot continue their project at full scale.

Figure 5 replicates figure 2 for the case with aggregate uncertainty and a high enough probability of the crisis state, and shows how the dispersion of collateral values impacts the level of cash holdings at date 0 and the level of interest rates in both states at date 1. From panel (a), one can see that as before there is a non-monotonous—and richer—relationship between collateral quality and aggregate liquidity.[29] For low level of dispersion ($\varepsilon < \underline{\varepsilon}(\gamma)$), an increase in dispersion leads to a decrease in cash holdings because the decrease in the value of the collateral available in the crisis state to back liquidity demand leads to a decrease in the interest rate in that state, thereby increasing the ex ante opportunity cost of cash. This is the same price effect underlying case 1 of proposition 2. In turn, the decrease in cash holdings leads to an increase the interest rate in the normal state. For higher level of dispersion ($\varepsilon > \underline{\varepsilon}(\gamma)$), the price effect of deteriorating collateral values in the crisis state ceases to operate as the interest rate in that state cannot go down further, and the impact on aggregate cash holding is driven by the precautionary motive to hold cash. When dispersion is not too high ($\underline{\varepsilon}(\gamma) < \varepsilon < \bar{\varepsilon}(\gamma)$), the precautionary motive is not strong enough to lead to cash hoarding : the interest rate in the normal must decrease to compensate for the increase in the liquidity value of cash (see panel (b)). As dispersion increases above $\underline{\varepsilon}(\gamma)$, aggregate cash holdings initially increases and then reaches a constant "plateau" at the first best liquidity level until dispersion reaches $\bar{\varepsilon}(\gamma)$. The intuition for this is that above $\hat{\varepsilon}$, banks are not constrained in the normal state so increasing cash holding further has no liquidity benefits but only a cost in that state. A sufficient increase in the

[29] In other words, the qualitative result in proposition 2 extends to the case with aggregate uncertainty.

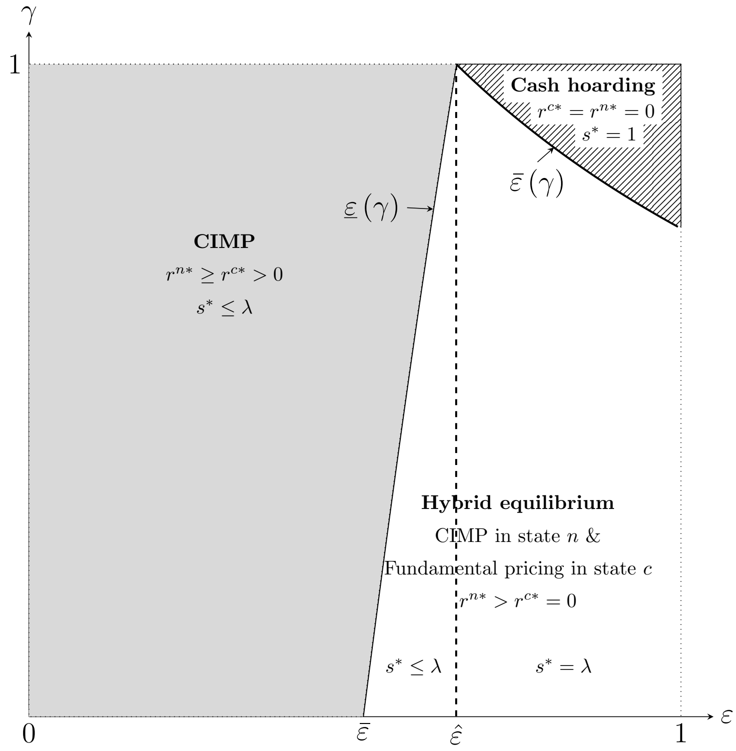


Figure 5: Aggregate uncertainty — equilibria in (ε, γ) space

value of cash in the crisis state is then needed for banks to accumulate more cash. Above $\bar{\varepsilon}(\gamma)$, the precautionary motive is so strong that banks hoard cash to be fully self-insured, as in the benchmark case without aggregate uncertainty on collateral quality.

One interesting takeaway from proposition 11 and figure 6 is that regardless of any lowering of the policy rate, the quality of collateral is an important factor in the setting of the risk-free interest rate by market participants. As a result, in crisis time, the low interest rate is explained mostly by the dispersion of collateral values. For low enough value of ε, the decrease is continuous with collateral value dispersion until the market rate reach the zero lower bound. This is also true in normal times when the probability of a crisis or the perception of collateral depreciation increases. In normal times, when the probability of a depreciation of the collateral value is sufficiently low ($\varepsilon < \bar{\varepsilon}$), an increase in the perception of a depreciation translates into an increase in the interest rate. In this region of the parameters, banks choose in period 0 to hold less cash, which increase the market risk-free rate. When ε is above the $\bar{\varepsilon}$ threshold, the interest rate starts decreasing until it reaches the zero lower bound. For high-enough value of $\bar{\varepsilon}$, at a given level of ε, an increase of the probability of crisis γ makes the market rate reduction steeper. The model proposes a market mechanism to explain those declines. The fall in the risk-free rate is the result of both an increase in crisis risk and an increase in the discount that can affect the value of collateral.[30]

7 Conclusion

Our paper proposes a simple theoretical framework to analyze how variations in collateral quality affect banks liquidity management and the risk-free rate. We show that lower collateral quality can lead to distinct liquidity regimes: cash-in-the-market pricing, where aggregate liquidity is insufficient, or liquidity hoarding, where banks self-insure. When collateral quality is high enough, banks always hold too little cash, as they do not internalize the effect of their individual cash holding decisions on the price of securities. By contrast, when collateral quality is low enough, banks choose to hold too much cash.

Policy interventions can mitigate these inefficiencies if tailored to the prevailing liquidity regime. Liquidity regulations are effective in addressing under-provision of liquidity but are insufficient in cases of liquidity hoarding. Negative interest rate policies, while potentially reducing cash hoarding, may lead to suboptimal outcomes if implemented without complementary measures. A combination of targeted liquidity requirements and negative rates can enhance welfare by balancing the liquidity allocation across banks and states of the economy. Our findings underscore the critical role of collateral quality in shaping macro-financial dynamics and offer guidance for policy design in addressing liquidity challenges.

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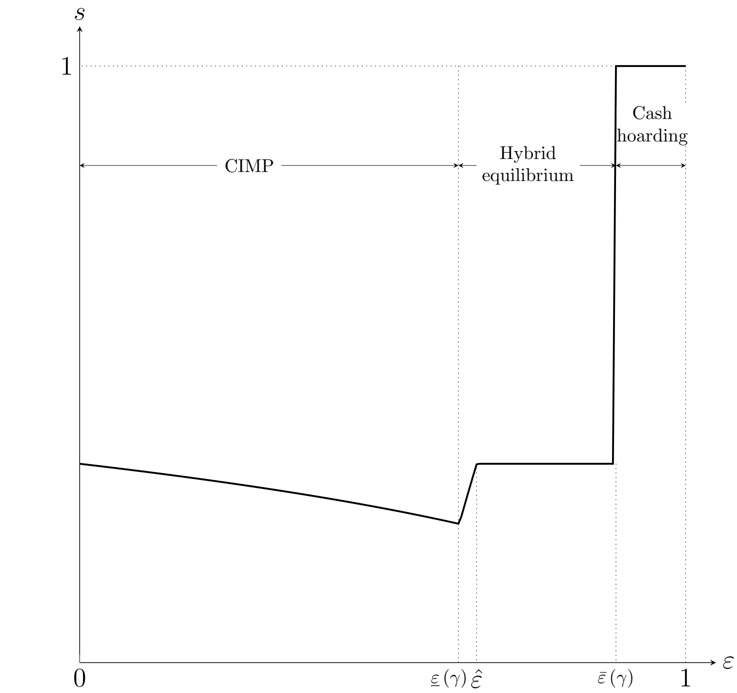
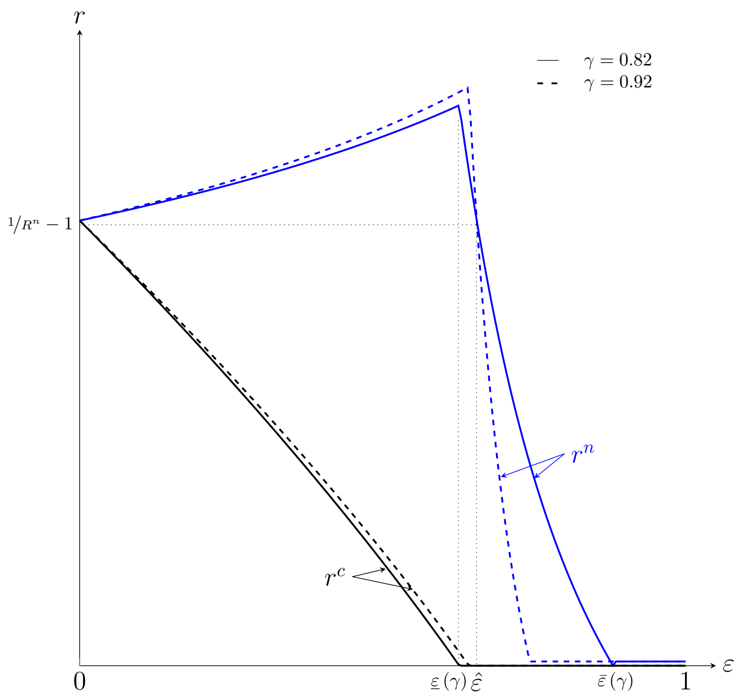
[30] The sensitivity of the risk-free interest rate to these two risks is consistent with what happened in 2007. As shown by the inspection of the US T-Bill interest rate in the Spring and the Summer of 2007, i.e. when market participants thought they were still in a normal state, the risk-free interest rate fell with the spread of the perception that the values of MBS could have been difficult to assess.



(a) Equilibrium cash holdings $s^\star$ ($\gamma = 0.82$)



(b) Risk-free interest rate : crisis state (r^c), normal state (r^n)

Figure 6: Equilibrium characterization upon ε, given a high enough γ

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Appendix A. Proofs.

We start with some preliminary computations. To make explicit the dependence on ε, write (15) as $MRC(r) = F(r,\varepsilon)$, with $F : \left[0, \bar{R}-1\right] \times [0,1] \to \mathbb{R}$ given by:

$$F(r,\varepsilon) = (1-\lambda)\left(1+r-\bar{R}\right) + \lambda\mu\left(1+r-R_H^\varepsilon\right) + \lambda(1-\mu)\,y\left(1-\frac{R_L^\varepsilon}{1+r}\right). \tag{28}$$

F is continuous and differentiable over its domain, with

$$\frac{\partial F(r,\varepsilon)}{\partial r} > 0, \qquad \frac{\partial F(r,\varepsilon)}{\partial \varepsilon} > 0. \tag{29}$$

The first part is obvious from (28). For the second part, compute

$$\frac{\partial F(r,\varepsilon)}{\partial \varepsilon} = -\lambda\left[\mu\frac{\partial R_H^\varepsilon}{\partial \varepsilon} + (1-\mu)\left(\frac{y}{1+r}\right)\frac{\partial R_L^\varepsilon}{\partial \varepsilon}\right] = -\lambda(1-\mu)\left[\frac{y}{1+r}-1\right]\frac{\partial R_L^\varepsilon}{\partial \varepsilon} > 0,$$

where the second step follows from $\frac{\partial R_H^\varepsilon}{\partial \varepsilon} = -\frac{1-\mu}{\mu}\frac{\partial R_L^\varepsilon}{\partial \varepsilon}$ (see (1)-(2)), and the final step from the restriction to values $r \leq \bar{R}-1 (< y-1)$ and $\frac{\partial R_L^\varepsilon}{\partial \varepsilon} < 0$.

We also introduce the function $M : [0,1] \times [0,1] \to \mathbb{R}$ given by

$$M(s,\varepsilon) = \frac{R_L^\varepsilon}{\frac{1}{1-\mu}\left(\frac{(1-\lambda)s}{\lambda(1-s)} - \mu\right)} - 1. \tag{30}$$

Equation (11) can thus be written $r(s) = \max\{0, M(s,\varepsilon)\}$. M is continuous and differentiable except when $s = 1$ or $\varepsilon = 1$, with

$$\frac{\partial M(s,\varepsilon)}{\partial s} < 0, \qquad \frac{\partial M(s,\varepsilon)}{\partial \varepsilon} < 0. \tag{31}$$

Proof of proposition 1 An equilibrium is a pair (r,s) satisfying (11)—or, equivalently $r(s) = \max\{0, M(s,\varepsilon)\}$—and (17), with $MRC(r) = F(r,\varepsilon)$. Note that by lemma 1 a situation where banks only invest in securities ($s^* = 0$) cannot be an equilibrium. We also know that $r \geq 0$, with $r = 0$ corresponding to fundamental pricing, and that in an equilibrium where banks are indifferent between reserves and securities $F(r,\varepsilon) = 0$. It is useful to distinguish cases depending on the sign of $F(0,\varepsilon)$.

1. Consider first the case where $F(0,\varepsilon) < 0$. An equilibrium with fundamental pricing can be excluded, since $F(0,\varepsilon) < 0$ implies $s^* = 0$. Using (28) we have

$$F\left(R_L^\varepsilon - 1, \varepsilon\right) < 0, \qquad F\left(\bar{R} - 1, \varepsilon\right) = \lambda\left(1-\mu\right)\left(y - \bar{R}\right)\varepsilon > 0,$$

   where the last step uses (A1). Together with $F(0,\varepsilon) < 0$ this implies that there exists a unique r^* such that $F(r^*,\varepsilon) = 0$, with $\max\{0, R_L^\varepsilon - 1\} < r^* < \bar{R} - 1$ (since $F(r,\varepsilon)$ is continuous and strictly increasing w.r.t. r). Using (11), which is continuous and decreasing in s, one can find a unique s^* such that $r^* = r(s^*)$ $(= M(s^*,\varepsilon))$. We thus have a unique equilibrium (r^*, s^*), featuring CIMP. With some algebra, we get expression (19) for s^*.

2. Consider next the case $F(0,\varepsilon) > 0$. Then $F(r,\varepsilon) > 0$ for all $r \geq 0$, implying $s^* = 1$ in any candidate equilibrium. In turn, the pricing formula (11) implies $r^* = 0$. We thus have a unique equilibrium, featuring liquidity hoarding and fundamental pricing.

3. The last case is $F(0,\varepsilon) = 0$. Then $r^* = 0$ (any $r < 0$ would imply $s^* = 1$ and thus $r = 0$, a contradiction). From (11), any s^* above a threshold $\hat{s}$ (defined by $M(\hat{s},\varepsilon) = 1$) is compatible with equilibrium. It is obvious to check that $\hat{s}$ is given by (19) with $r^* = 0$.

To terminate the proof, it suffices to note that $F(0,0) < 0$ and that $F(0,\varepsilon)$ is continuous and increasing in ε (see (29)). Hence there exists $\hat{\varepsilon} > 0$ (possibly ≥ 1) such that $F(0,\varepsilon) <$, $=$, or > 0 depending on $\varepsilon <$, $=$, or $> \hat{\varepsilon}$. Cases 1-3 in the statement of proposition 1 thus map with cases 1-3 above. ■

Proof of proposition 2 For $\varepsilon < \hat{\varepsilon}$, $r^* > 0$ solves $F(r^*,\varepsilon) = 0$. Given the properties of F (see (29)), applying the inverse function theorem we get $\frac{dr^*}{d\varepsilon} < 0$. Therefore, an increase in ε by $\Delta\varepsilon > 0$ from ε_0 to $\varepsilon_1 \equiv \varepsilon_0 + \Delta\varepsilon < \hat{\varepsilon}$ leads to a decrease in r^*. Now, $\frac{R_L^\varepsilon}{1+r^*}$ must decrease. To see this, use (28) to rewrite $F(r^*,\varepsilon) = 0$ as

$$(1-\mu)\,\lambda\left(\frac{R_L^\varepsilon}{1+r} - 1\right) = \frac{y - \bar{R}}{y - (1+r)} - 1.$$

The right side is an increasing function of r, implying that $\frac{R_L^\varepsilon}{1+r^*}$ decreases when ε increases. From expression (19), s^* decreases. The rest of the proposition is straightforward from proposition 1. ■

Proof of proposition 3 The negative relationship between r^* and ε, as well as between P_L^ε and ε, are established in the previous proof. The result for welfare $\mathcal{W}$ follows from the observation that in a CIMP equilibrium ($\varepsilon < \hat{\varepsilon}$) banks are indifferent as to their liquidity choice, so that

$$\mathcal{W}^* = \begin{cases} y + (1-\lambda)(1+r^*), & \text{if} \quad \varepsilon < \hat{\varepsilon}, \\ y + (1-\lambda), & \text{if} \quad \varepsilon \geq \hat{\varepsilon}. \end{cases} \tag{32}$$

■

Proof of proposition 4 The proofs of propositions 2 to 3 are based on the properties of $F(r,\varepsilon)$ and $M(s,\varepsilon)$. To show that similar results hold when varying $(1-\mu)$, we simply need to show that the properties of these two functions w.r.t. ε are also valid when considering (28) and (30) as functions of $(1-\mu)$ rather than of ε. Specifically (with a slight abuse of notation), we claim that

$$\frac{\partial F(r,\mu)}{\partial \mu} < 0, \qquad \frac{\partial M(s,\mu)}{\partial \mu} > 0. \tag{33}$$

From (16) and (30) compute

$$\frac{\partial F(r,\mu)}{\partial \mu} = -\lambda(1-\mu)(y-(1+r))\left(1-\frac{R_L^\varepsilon}{1+r}\right), \tag{34}$$

$$\frac{\partial M(s,\mu)}{\partial \mu} = \frac{1}{1-\mu}M(s,\mu) - \frac{1}{R_L^\varepsilon}\frac{1}{1-\mu} = \frac{1}{1-\mu}\left(M(s,\mu)-\frac{1}{R_L^\varepsilon}\right). \tag{35}$$

Now, from lemma 1, $r^* \geq R_L^\varepsilon - 1$. In addition, we can exclude the case $r^* = R_L^\varepsilon - 1$ except for the special case of a degenerate distribution. To see this, observe that for a non degenerate distribution $F(R_L^\varepsilon - 1, \mu) = (1-\lambda)\left(R_L^\varepsilon - \bar{R}\right) + \lambda\mu\left(R_L^\varepsilon - R_H^\varepsilon\right) > 0$. Hence, (33) holds in any configuration with CIMP. Comparing with (29) and (31), we can conclude that the formal analysis w.r.t. ε can be extended w.r.t. $(1-\mu)$. ■

Proof of proposition 5 Let $\mathcal{R}$ and $\mathcal{R}'$ denote the (binary) distributions of returns for $\mathcal{E}$ and $\mathcal{E}'$, with parameters such that $(1-\mu) R_L + \mu R_H = \bar{R} = (1-\mu') R'_L + \mu' R'_H$.[31] We will show that if $\mathcal{R}'$ differs from $\mathcal{R}$ by a mean preserving spread, then $R'_L \leq R_L$ and $1-\mu' \geq 1-\mu$ with at least one strict inequality. The result will thus follows from propositions 1 to 4.

Note that $\mathcal{R}'$ cannot be degenerate. Using the characterization of a mean preserving spread for the discrete distribution case in Rothschild and Stiglitz (1970), there exist $a_1 < a_2 < a_3 < a_4$ such that

$$\Pr_{\mathcal{R}'}(a_1) - \Pr_{\mathcal{R}}(a_1) = -[\Pr_{\mathcal{R}'}(a_2) - \Pr_{\mathcal{R}}(a_2)] \geq 0 \tag{36}$$

$$\Pr_{\mathcal{R}'}(a_4) - \Pr_{\mathcal{R}}(a_4) = -[\Pr_{\mathcal{R}'}(a_3) - \Pr_{\mathcal{R}}(a_3)] \geq 0 \tag{37}$$

Obviously, $\{a_1, a_2\} = \{R_L, R'_L\}$ and $\{a_3, a_4\} = \{R_H, R'_H\}$ (since $R_L, R'_L \leq \bar{R} \leq R_H, R'_H$).

We first show that $R'_L \leq R_L$. Assume the contrary, that is $R_L < R'_L$. Then $a_1 = R_L$ and $\Pr_{\mathcal{R}'}(a_1) = 0$. Using the positivity of the left side of (36), we then have $\Pr_{\mathcal{R}}(a_1) = 0$, that is $\mu = 1$, and $R_H = \bar{R}$. Now (since $\mathcal{R}'$ cannot be degenerate) $a_2 = R'_L < \bar{R}$ and $\Pr_{\mathcal{R}'}(a_2) = 1 - \mu' > 0$. However (36) implies $\Pr_{\mathcal{R}'}(a_2) = \Pr_{\mathcal{R}}(a_2) = 0$. A contradiction. Hence $R'_L \leq R_L$.

We now show that $1-\mu' \geq 1-\mu$. Assume the contrary, that is $1-\mu' < 1-\mu$ ($\Leftrightarrow \mu < \mu'$). We argue that, given $R'_L \leq R_L$, one cannot find combinations for $\{R_H, R'_H\}$ and $\{a_3, a_4\}$ satisfying (37). There are two cases to consider. If $a_3 = R_H < R'_H = a_4$, then (37) gives $\mu' - 0 = -(0 - \mu)$, contradicting the assumption $\mu' > \mu$. Similarly, if $a_3 = R'_H < R_H = a_4$ then (37) gives $0 - \mu = -(\mu' - 0)$, again contradicting $\mu' > \mu$. Hence, $1 - \mu' \geq 1 - \mu$.

To conclude, we must have both $R'_L \leq R_L$ and $1 - \mu' \geq 1 - \mu$. And for $\mathcal{R}$ and $\mathcal{R}'$ to differ, at least one inequality must be strict. ■

Proof of proposition 6 Note that in any candidate equilibrium with a LCR constraint, the pricing factor is still given by eq. (11). Let $\varepsilon < \hat{\varepsilon}$. Obviously imposing $s \geq s^*$ yields the laissez-faire equilibrium, where $r(s^*) > 1$ and $s^* < \bar{s}^{opt}$. From (11), taking some $\bar{s}$ slightly above s^* yields to a slight decrease in the risk-free rate such that $0 < r(\bar{s}) < r(s^*)$ and

$$\frac{\partial \pi(s,r)}{\partial s} = F(r(\bar{s}), \varepsilon) < F(r(s^*), \varepsilon) = 0.$$

[31] We omit the superscript ε in this proof.

Hence the constraint $s \geq \bar{s}$ does binds at the individual level, and the constrained equilibrium is given by $s = \bar{s}$ and $r = r(\bar{s})$. In this equilibrium, since $r(\bar{s}) > 0$ there is no unused cash at $t = 1$ and surplus is given by

$$\mathcal{W} = (1 - \lambda + \bar{s})\, y + (1 - \bar{s})\, \bar{R} > (1 - \lambda + s^*)\, y + (1 - s^*)\, \bar{R}.$$

This proves the first part of the proposition. The second part follows from noting that the same reasoning applies when further increasing $\bar{s}$ till r reaches $\max\{0, R_L^\varepsilon - 1\}$. In the former case, increasing $\bar{s}$ further has no impact on the market return on cash ($r = 0$), and since $\frac{\partial \pi(s,0)}{\partial s} = F(0, \varepsilon) < 0$ increasing $\bar{s}$ further leads to a reduction in surplus. From (19), $r = 0$ gives the first threshold in the optimal LCR in the proposition. In the latter case, $P_L^\varepsilon = 1$ and (using (19)) $\bar{s} = \lambda$ so the first best obtains. ■

Proof of proposition 7 The first part is obvious. Now consider that banks' initial portfolio choice is constrained by $s \leq \bar{s}$ where $\bar{s} < 1$. It is easy to check that $s^* = \bar{s}$ is the only equilibrium (since $\varepsilon > \hat{\varepsilon}$ implies $F(r, \varepsilon) > 0 \;\; \forall r \leq 1$). To show that surplus is inferior to $y + (1 - \lambda)$, we distinguish two cases.

Consider first the case where $\bar{s} < \hat{s}(\varepsilon)$, where $\hat{s}(\varepsilon)$ is defined by $M(\hat{s}(\varepsilon), \varepsilon) = 0$. Note that $\hat{s}(\hat{\varepsilon}) = \hat{s}$ (defined in proposition 1 and its proof) and thus (from (31)) $\hat{s}(\varepsilon) < \hat{s}$. Given (31), $M(\bar{s}, \varepsilon) > M(\hat{s}(\varepsilon), \varepsilon) = 0$ so that there is no unused cash at $t = 1$ and the surplus is given by

$$\mathcal{W} = (1 - \lambda + \bar{s})\, y + (1 - \bar{s})\, \bar{R} < (1 - \lambda + \hat{s})\, y + (1 - \hat{s})\, \bar{R} = y + (1 - \lambda),$$

where the first step follows from $\bar{s} < \hat{s}(\varepsilon) < \hat{s}$, and the last step from the definition of $\hat{s}$ and the continuity of (32).

Consider next the case where $\bar{s} \geq \hat{s}(\varepsilon)$. Given (31), $M(\bar{s}, \varepsilon) > M(\hat{s}(\varepsilon), \varepsilon) = 0$ and there is unused cash at $t = 1$, with an amount

$$S(\bar{s}, 0) - D(\bar{s}, 0) = (1 - \lambda)\bar{s} - \lambda(1 - \bar{s})\left(\mu + (1 - \mu) R_L^\varepsilon\right) \tag{38}$$

$$= \bar{s} - \lambda - \lambda(1 - \bar{s})(1 - \mu)(1 - R_L^\varepsilon). \tag{39}$$

In particular, banks with liquidity needs holding a low type security are constrained and continue their project at a reduced scale $(1-(1-\bar{s})(1-R_L^{\varepsilon}))$. The surplus is therefore

$$\mathcal{W} = y - \lambda(1-\mu)(1-\bar{s})(1-R_L^{\varepsilon})y + (1-\bar{s})\bar{R} + S(\bar{s},0) - D(\bar{s},0). \tag{40}$$

Using (39) and rearranging (40) yields

$$\mathcal{W} = y + (1-\bar{s})\bar{R} - \lambda(1-\mu)(1-\bar{s})(1-R_L^{\varepsilon})(y-1) + \bar{s} - \lambda \tag{41}$$

$$< y + (1-\bar{s})\bar{R} - (1-\bar{s})(\bar{R}-1) + \bar{s} - \lambda = y + (1-\lambda), \tag{42}$$

where the second step follows from $F(1,\varepsilon) > 0$ (that is (20) with strict inequality). We thus have $\mathcal{W} < y + (1-\lambda) = \mathcal{W}^0$ for any $\bar{s} < 1$. ■

Proof of proposition 8 The proof is the same as for proposition 1, with the difference that expression (26) for the pricing factor replaces (11). In particular, the threshold $\hat{\varepsilon}(\rho)$ is given by $F(-\rho, \hat{\varepsilon}(\rho)) = 0$. Applying the implicit function theorem, $\hat{\varepsilon}(\rho)$ is strictly increasing. ■

Proof of proposition 9 Consider some $\varepsilon > \hat{\varepsilon}$. Using proposition 8, define $\hat{\rho}(\varepsilon)$ to be the value such that $\hat{\varepsilon}(\hat{\rho}(\varepsilon)) = \varepsilon$. Then, for $\rho > \hat{\rho}(\varepsilon)$ we have $\varepsilon < \hat{\varepsilon}(\rho)$ and case 1 of proposition 8 applies. Using the fact that (as in the proof of proposition 3) in such an equilibrium banks are indifferent as to their level of liquidity the surplus is given by

$$\mathcal{W}^* = y + (1-\lambda)(1+r) = y + (1-\lambda)(1-\rho) < \mathcal{W}^0,$$

showing that welfare is reduced compared to the initial configuration with cash-hoarding. For $\rho < \hat{\rho}(\varepsilon)$, case 2 applies and the equilibrium is unaffected. ■

Proof of proposition 10 The proof uses proposition 3, adapted with the risk-free rate (26). Let $\varepsilon > \hat{\varepsilon}$, and $\rho > \hat{\rho}(\varepsilon)$. From proposition 9 and its proof, the equilibrium is characterized by a risk-free rate $-\rho < r^* < 0$ that solves $F(r^*, \varepsilon) = 0$ and a level of liquidity given by expression (19). Using the same argument as for proposition 6, one can regulate banks' liquidity by imposing a LCR constraint with

$$\bar{s} = \frac{\lambda\left(\mu + (1-\mu)\frac{1}{1-\rho}R_L^{\varepsilon}\right)}{\lambda\left(\mu + (1-\mu)\frac{1}{1-\rho}R_L^{\varepsilon}\right) + (1-\lambda)}. \tag{43}$$

For this level of required liquidity, all liquidity get allocated to projects at date 1, and the surplus is given by

$$\mathcal{W} = (1-\lambda+\bar{s})\,y + (1-\bar{s})\,\bar{R}. \tag{44}$$

To conclude, we use the fact that the threshold $\hat{s}$ in proposition 1 is such that

$$\mathcal{W}^0 = (1-\lambda+\hat{s})\,y + (1-\hat{s})\,\bar{R} \tag{45}$$

with

$$\hat{s} = \frac{\lambda\left(\mu + (1-\mu)\,R_L^{\hat{\varepsilon}}\right)}{\lambda\left(\mu + (1-\mu)\,R_L^{\hat{\varepsilon}}\right) + (1-\lambda)}. \tag{46}$$

Hence NIR plus LCR increases welfare iff $\bar{s} > \hat{s}$, that is $\frac{1}{1-\rho}R_L^{\varepsilon} > R_L^{\hat{\varepsilon}}$. Rearranging, one gets $\rho > \hat{\rho}(\varepsilon)$. (Note that $\hat{\rho}(\varepsilon) > \bar{\rho}(\varepsilon)$). ■

Appendix B. Proof of Proposition 11

Let the long-term payoff of securities in state n be $R^n = R_H$, which is fixed. In state c, a proportion $1-\mu$ of securities will have a value of $R_L^{\varepsilon} = (1-\varepsilon)R^n$.[32] The price of a security of type θ in state ω is denoted by P_θ^ω which is $\frac{R_\theta^\omega}{1+r}$. (Note: θ is omitted in the case of state n.)

Step 1. Extending lemma 1.

Lemma 2. *(i)* $0 \leq r^\omega, \quad \forall\omega$

(ii) $\mathbb{E}\left[1+r^\omega\right] \leq \mathbb{E}\left[\tilde{R}\right] \equiv \gamma\bar{R} + (1-\gamma)\,R^n$

(iii) $\min\left\{P_L^c, P^n\right\} \leq 1$.

Part (i) follows from observing that if $r^\omega < 0$, there is no supply for cash against assets in the date 1 market in state ω, but then $P_\theta^\omega = 0$, implying that $r^\omega = \infty$, a contradiction.

[32] We consider this scenario as a natural extension, which differs slightly from our benchmark with mean-preserving case. The mean-preserving case does not alter Proposition 11.

Part (ii) and (iii) follow from dominance arguments. If $\mathbb{E}[1+r^{\omega}] > \mathbb{E}\left[\tilde{R}\right]$ (>1), the long term asset is dominated from an ex ante perspective since one can always get a higher return by investing cash in the market at date 1; but then $r^n = r^c = 0$ since there are no assets to buy at distressed prices in any state, leading to $\mathbb{E}[1+r^{\omega}] = \gamma(1+r^c) + (1-\gamma)(1+r^n) = 1$.

In turn if $P_L^c > 1$ and $P^n > 1$, banks hit by a liquidity need can always continue their project full scale by selling their asset at $t = 1$, so that the long term asset dominates cash both in terms of return and of liquidity; but then $r^n = r^c = \infty$ as there is no cash to buy assets at $t = 1$.

Step 2. Pricing at date 1. Note that lemma 2 does imply that the market return on cash is strictly lower than the internal return on cash in all states $(1 + r < y, \forall \omega)$ so that bank strictly prefer continuing their non-marketable projects (when relevant) to buying assets.

In any state ω, the supply of cash is $S(r^{\omega}) = (1-\lambda)\, s\mathbb{1}_{r^{\omega} \geq 0}$.

- State n. Banks are constrained iff $P^n < 1$. The (solvent) demand for cash is then

$$D(r^n) = \lambda(1-s)\min\left\{1, \frac{R^n}{1+r^n}\right\}. \tag{47}$$

  Observing that $s < \lambda \;\Leftrightarrow\; \lambda(1-s) < (1-\lambda)s$, equating cash supply and demand yields

$$r^n = \begin{cases} \frac{\lambda(1-s)}{(1-\lambda)s} R^n - 1 \;\; (\leq R^n - 1 > 0), & \text{if} \quad s < \lambda, \\ \in [0, R^n - 1], & \text{if} \quad s = \lambda, \\ 0, & \text{if} \quad s > \lambda. \end{cases} \tag{48}$$

- State c. The (solvent) demand for cash is

$$D(r^c) = \lambda(1-s)\left(\mu \min\left\{1, \frac{R_H}{1+r^c}\right\} + (1-\mu)\min\left\{1, \frac{R_L}{1+r^c}\right\}\right). \tag{49}$$

  Equating with the supply of cash, one can see that $\frac{R_H}{1+r^c} < 1$ requires $(1-\lambda)s <$

$\mu\lambda(1-s)$. Using this, one should be able to show that in the crisis state

$$R_L \leq 1 + r^c \leq R_H \tag{50}$$

(Details. Assume $R_H < 1 + r^c$. Then $(1-\lambda)s < \mu\lambda(1-s) < \lambda(1-s)$ which implies $s < \lambda$. From (48) this implies $R^n \leq 1 + r^n$. But then (ii) in lemma 2 cannot hold. A contradiction.) In equilibrium we thus have the same net return on cash in state c as in our benchmark case (Eq. (11)).

Step 3. Date 1 individual optimization. Based on the discussion above, the marginal return on cash in the crisis state is the same as in our benchmark case (Eq. (15)).

In state n, one must distinguish cases depending on the value of s:

- if $s < \lambda$, banks with liquidity shock are constrained

$$MRC^n(r^n) = (1-\lambda)(1 + r^n - R^n) + \lambda y \left(1 - \frac{R^n}{1+r^n}\right) \tag{51}$$

$$= (1 + r^n - R^n) + \lambda(y - (1 + r^n))\left(1 - \frac{R^n}{1+r^n}\right) \tag{52}$$

(with $r^n > R^n - 1 > 0$ given by the first case in (48)).

- if $s > \lambda$,

$$MRC^n(r^n) = 1 - R^n \tag{53}$$

Putting the two together, the FOC is given by

$$s = \begin{cases} 0, & < 0, \\ \in [0,1], \quad \text{if} \quad \gamma MRC^c(r^c) + (1-\gamma) MRC^n(r^n) = 0, \\ 1, & > 0. \end{cases} \tag{54}$$

Denote MRC by

$$MRC(r^c, r^n) = \gamma MRC^c(r^c) + (1-\gamma) MRC^n(r^n). \tag{55}$$

Denote $\tilde{r}^n(r^c)$ by r^n such that the (solvent) demand for cash in both state c and n is equal for a given r^c. From Equation (47) and (49), it is provided by

$$\tilde{r}^n\left(r^c\right) = \frac{R^n}{\mu + (1-\mu)\frac{R_L}{1+r^c}} - 1. \tag{56}$$

Step 4: Additional notations and computations To make explicit the dependence on γ and ϵ, write (55) as

$$H\left(r^c, r^n, \gamma, \varepsilon\right) = \gamma H^c\left(r^c, \varepsilon\right) + (1-\gamma) H^n\left(r^n\right) \tag{57}$$

where H^ω is MRC^ω. By similar reasoning in our benchmark case, we have

$$\frac{\partial H^c\left(r^c, \varepsilon\right)}{\partial r^c} > 0, \qquad \frac{\partial H^n\left(r^n\right)}{\partial r^n} > 0, \qquad \frac{\partial H^c\left(r^c, \varepsilon\right)}{\partial \epsilon} > 0. \tag{58}$$

We also introduce the function $G : \left[0, \bar{R}-1\right] \to \mathbb{R}$ given by

$$G\left(r^c, \gamma, \epsilon\right) = H\left(r^c, \tilde{r}^n\left(r^c\right), \gamma, \epsilon\right) \tag{59}$$

$$= \gamma H^c\left(r^c, \epsilon\right) + (1-\gamma) H^n\left(\tilde{r}^n\left(r^c\right)\right) \tag{60}$$

G is continuous and differentiable over $r^c \in \left[0, \bar{R}-1\right]$ with

$$\frac{\partial G\left(r^c, \gamma, \varepsilon\right)}{\partial r^c} > 0 \tag{61}$$

To show this, compute

$$\frac{\partial G\left(r^c, \gamma, \varepsilon\right)}{\partial r^c} = \gamma \frac{\partial H^c\left(r^c, \varepsilon\right)}{\partial r^c} + (1-\gamma) \frac{\partial H^n\left(\tilde{r}^n\right)}{\partial \tilde{r}^n} \frac{\partial \tilde{r}^n\left(r^c\right)}{\partial r^c} > 0 \tag{62}$$

Define a few threshold values: First, denote r_0 by r^c such that $H^c\left(r^c, \epsilon\right) = 0$. Second, we denote two threshold values of r^n, $\tilde{r}_0 = \tilde{r}(r_0)$ and $\tilde{r}_1 = \tilde{r}(0)$. Third, we reuse the two threshold values of ε in our benchmark model, $\bar{\varepsilon}$ such that $R_L^{\varepsilon} = 1$ and $\hat{\varepsilon}$ such that $H^c\left(0, \hat{\varepsilon}\right) = 0$.

From the definition of $\hat{\varepsilon}$ and (58), we can verify that $r_0 > 0$ and $\frac{R_L}{1+r_0} < 1$ when $\varepsilon < \hat{\varepsilon}$ and $r_0 \leq 0$ when $\varepsilon \geq \hat{\varepsilon}$. It is useful to distinguish cases depending on ε.

Step 5: Case with $\varepsilon < \hat{\varepsilon}$: We start by excluding the cases with no equilibrium:

Lemma 3. *The equilibrium risk-free rates under $\varepsilon < \hat{\varepsilon}$ satisfy $r_0 \geq r^{c*} \geq \max\{0, R_L - 1\}$, $r^{n*} \geq R^n - 1$, and $H(r^{c*}, r^{n*}) = 0$*

$r^{c*} \geq \max\{0, R_L - 1\}$ is derived from (11). To see the remaining part: i) If $r^c < r_0$ and $r^n < R^n - 1$, then $H^c < 0$, $H^n < 0$ from (58), yielding $F < 0$. It would imply that $s = 0$ whereas $r^n < R^n - 1$ requires $s \geq \lambda$ from (48), a contradiction; ii) If $r^c > r_0$ and $r^n > R^n - 1$, then $H^c > 0$, $H^n > 0$ from (58), yielding $H > 0$. It would imply that $s = 1$ whereas $r^n > R^n - 1$ requires $s \leq \lambda$ from (48), a contradiction; iii) $r^c > r_0$ and $r^n < R^n - 1$ cannot be an equilibrium because $\frac{R_L}{1+r^c} < \frac{R_L}{1+r_0} < 1$ implies $s < \lambda$ from (11) whereas $r^n < R^n - 1$ requires $s > \lambda$. Therefore, an equilibrium, if any, should satifsy $r_0 \geq r^{c*} \geq \max\{0, R_L - 1\}$, $r^{n*} \geq R^n - 1$. In such r^c and r^n, $H > 0$ cannnot occur in equilibrium because $H > 0$ would imply $s = 1$ whereas $r^{n*} \geq R^n - 1$ requires $s \leq \lambda$ from (48), a contradition. We can conclude that $H = 0$ in equilibrium.

Under $\varepsilon < \hat{\varepsilon}$, we can show that

$$G(r_0, \gamma, \epsilon) > 0, \qquad G(R_L - 1, \gamma, \epsilon) < 0 \tag{63}$$

For the first part, using $\frac{R_L}{1+r_0} < 1$ under $\varepsilon < \hat{\varepsilon}$, we have $\tilde{r}_0 > R^n - 1$ from (56), which yields $H^n(\tilde{r}_0) > H^n(R^n - 1) = 0$ from (58). Together with $H^c(r_0, \epsilon) = 0$, we obtain $G(r_0, \gamma, \epsilon) > 0$. The second part is obtained from $H^c(R_L - 1, \varepsilon) < 0$ and $H^n(\tilde{r}^n(R_L - 1)) = H^n(R^n - 1) = 0$.

We distinguish two cases upon ε according to the sign of $R_L^{\varepsilon} - 1$ (*i.e.*, $\varepsilon \lesseqgtr \bar{\varepsilon}$).

1. If $\varepsilon < \bar{\varepsilon}(< \hat{\varepsilon})$, there is a unique r^{c*} such that $G(r^c, \gamma, \epsilon) = 0$ with $r_0 > r^{c*} > R_L - 1\,(> 0)$. This is straightforward from the fact that G is continuous and (63). r^{n*} can be obtained from $r^{n*} = \tilde{r}^n(r^{c*})$. One can find a unique s^* such that $r^{n*} = r^n(s)$ in (48). We thus have unique equilibrium (r^{c*}, r^{n*}, s^*), featuring CIMP in both states. With some algebra, we get

$$s^* = \frac{\lambda \frac{R^n}{1+r^{n*}}}{\lambda \frac{R^n}{1+r^{n*}} + (1 - \lambda)}\,(\leq \lambda)\,. \tag{64}$$

2. If $\varepsilon \geq \bar{\varepsilon}(< \hat{\varepsilon})$, we can distinguish two cases upon the sign of $G(0, \gamma, \epsilon)$:

(a) If $G(0,\gamma,\epsilon)<0$, there is a unique r^{c*} such that $G(r^{c},\gamma,\epsilon)=0$ with $r_0>r^{c*}>0\,(\geq R_L-1)$. This is straightforward from the first part of (63) and the fact that G is continuous. r^{n*} and s^* can be obtained in the same way as in the above case.

(b) If $G(0,\gamma,\epsilon)>0$, then $H(r^c,\tilde{r}^n(r^c),\gamma,\varepsilon)>0$ for all $r_0>r^c>0\,(\geq R_L-1)$. $r^c,\tilde{r}^n(r^c)$ cannot be an equilibrium price combination from lemma 3. This implies that an equilibrium with $S=D^n=D^c$ does not exist. Given that $r^n\geq R^n-1$ requires $S=D^n$ from (48), the only possible equilibrium with $F=0$ is the case with $S=D^n>D^c$, which requires $r^{c*}=0$ and r^{n*} is r^n such that $H(0,r^n,\gamma,\epsilon)=0$ where $r^{n*}<\tilde{r}^n(0)$. s^* is determined by (48), which gives (64). Such r^{n*} satisfies $r^n\geq R^n-1$ (Since $H^c(0,\varepsilon)<0$ for all $\varepsilon<\hat{\varepsilon}$, $F=0$ thus requires $H^n>0$, implying that $r^n\geq R^n-1$.). We can also verify that s^* satisfies $S=D^n>D^c$ (from $H(0,\tilde{r}^n(0),\gamma,\varepsilon)>H(0,r^{n*},\gamma,\varepsilon)=0$ together with the second part of (58)). The equilibrium features CIMP in the state n and fundamental pricing in the state c

By some algebra, we can verify that $G(0,\gamma,\bar{\varepsilon})<0$. $G(0,\gamma,\varepsilon)$ is continuous and increasing in ε (see (58)). Hence, for $\bar{\varepsilon}\leq\varepsilon<\hat{\varepsilon}$ there exists a threshold $\underline{\varepsilon}(\gamma)$ such that $G(0,\gamma,\underline{\varepsilon}(\gamma))$ <, =, or > 0, depending on ε >, =, or < $\underline{\varepsilon}(\gamma)$ for a given γ, with $\underline{\varepsilon}(0)=\bar{\varepsilon}$ and $\underline{\varepsilon}(1)=\hat{\varepsilon}$. Cases 1 and 2.(a), as well as case 2.(b) correspond to 1 and 2.(a) in the statement of proposition 11, respectively.

Step 6: Case with $\varepsilon\geq\hat{\varepsilon}$ We start by excluding the cases with no equilibrium:

Lemma 4. *An equilibrium under $\varepsilon\geq\hat{\varepsilon}$ should satisfy $r^{c*}=0$, and $r^{n*}\leq R^n-1$.*

Note that $H^c(r^c,\varepsilon)>0$ for all $r^c\geq 0$ under $\varepsilon>\hat{\varepsilon}$. i) If $r^{n*}>R^n-1$, then $H^n(r^n)>0$ from (58). This yields $H>0$, which implies $s^*=1$ from (54), whereas $r^{n*}>R^n-1$ requires $s<\lambda$, a contradiction; ii) $r^{n*}\leq R^n-1$ and $r^{c*}>0$ cannot be an equilibrium. $r^{c*}>0$ requires $s^*<\lambda$ from (11) and $P_L^c=\frac{R_L}{1+r^c}<1$ (since $R_L^{\varepsilon}<1$ under $\varepsilon>\hat{\varepsilon}>\bar{\varepsilon}$), whereas $r^{n*}\leq R^n-1$ requires $s\geq\lambda$ from (48), a contradiction. Therefore, an equilibrium, if any, should satisfy $r^{c*}=0$, and $r^{n*}\leq R^n-1$.

Note that under $r^c = 0$, $r^{n*} \leq R^n - 1$, and $\varepsilon \geq \hat{\varepsilon}$

$$\frac{\partial H(0, r^n, \gamma, \varepsilon)}{\partial \gamma} > 0 \tag{65}$$

It is derived from $H^c(0, \varepsilon) \geq 0$ for $\varepsilon \geq \hat{\varepsilon}$ and $H^n(r^n) < 0$ (since $H^n(R^n - 1) < 0$ and (58)).

It is useful to distinguish cases depending upon the sign of $H(0, 0, \gamma, \varepsilon)$.

1. Consider first the case where $H(0, 0, \gamma, \varepsilon) < 0$. An equilibrium with fundamental pricing in both states can be excluded, since $H(0, 0, \gamma, \varepsilon) < 0$ implies $s^* = 0$. Using (58) and $H^c(0, \varepsilon) \geq 0$ we have $H(0, R^n - 1, \gamma, \varepsilon) \geq 0$. This implies that there exists a unique r^{n*} such that $H(0, r^{n*}, \gamma, \varepsilon) = 0$, with $0 < r^{n*} \leq R^n - 1$. Using (48), we obtain $s^* = \lambda$. We can verify that $r^c = 0$ and $s = \lambda$ satisfy (11) given $R_L < 1$. We thus have a unique equilibrium featuring CIMP in the state n and fundamental pricing in the state c.

2. Consider next the case $H(0, 0, \gamma, \varepsilon) > 0$. $H(0, r^n, \gamma, \varepsilon) > 0$ for all $r^n \geq 0$, implying $s^* = 1$ from the FOC (54). In turn, (48) and (11) imply $r^{n*} = r^{c*} = 0$. We thus have a unique equilibrium featuring liquidity hoarding and fundamental pricing in both states.

Note that $H(0, 0, \gamma, \hat{\varepsilon}) \leq 0$ and $H(0, 0, \gamma, \varepsilon)$ is continuous and increasing in ε (see (58)). Hence, for $\varepsilon \geq \hat{\varepsilon}$, there exists a $\bar{\varepsilon}(\gamma)$ such that $F(1, 1, \gamma, \varepsilon) <, =,$ or > 0 depending on $\varepsilon <, =,$ or $> \bar{\varepsilon}(\gamma)$ for a given γ, with $\bar{\varepsilon}(1) = \hat{\varepsilon}$. The cases 1 and 2 above thus map with cases 2.(b) and 3 in the statement of proposition 11, respectively. ■

Appendix C. Extension with liquidity shock $x \leq 1$

Until now, we have assumed that bank needs the amount of liquidity $x \equiv 1$ to continue its project in its entirety when it is affected by liquidity shock. In this appendix, we relax this assumption by considering that the amount of liquidity needed to continue project is $x \leq 1$.

Lemma 1 still apply. The aggregate supply of cash in the market is $(1 - \lambda) s$, same as the equation (8). The demand for cash depends on whether banks hit by liquidity shock is constrained by the value of their securities or not. Banks with the H-security is never constrained. The aggregate demand depends on the value of L-security. Denote $C(s,r)$ by the total cash that can be available to the banks with s of cash and $1-s$ of L-security:

$$C(s,r) = s + (1-s)\frac{R_L^{\varepsilon}}{1+r} \tag{66}$$

The demand for cash can be written by:

$$D_x(s,r) = \begin{cases} \lambda(x-s) & \text{if } \; C(s,r) \geq x \\ \lambda\left[\mu(x-s) + (1-\mu)(1-s)\frac{R_L^{\varepsilon}}{1+r}\right] & \text{if } \; C(s,r) < x \end{cases} \tag{67}$$

Now we consider individual decision on cash holding level by banks. They decide their cash holding level at $t=0$ maximizing their expected profits. Their decision depends on the expected marginal (net) return on cash conditional on the price of securities p, which is given by

$$MRC_x(s,r) = \begin{cases} 1+r-\bar{R} & \text{if } \; C(s,r) \geq x \\ \left(1+r-\bar{R}\right) + (1-\mu)\lambda\left(\frac{y}{x} - (1+r)\right)\left(1 - \frac{R_L^{\varepsilon}}{1+r}\right) & \text{if } \; C(s,r) < x \end{cases} \tag{68}$$

It is useful to distinguish the case upon the value of $C(s,r)$. Combining (67) and (68), we obtain the following lemma:

Lemma 5. *The only equilibrium with $C(s^*,r^*) > x$ is $s^* = \lambda x$ and $r^* = \bar{R} - 1$.*

The equilibrium with fundamental pricing ($r^* = 0$) with $C(s,r) > x$ cannot exist. This equilibrium requires $S > D$ so that $s^* > \lambda x$ while $MRC = 1 - \bar{R} < 0$ yielding $s^* = 0$. A contradiction.

The case with $C(s,r) \leq x$ is analogous to our benchmark model. From (67) and $S = (1-\lambda)s$, the market clearing condition yields pricing of assets at their fundamental value, $r = 0$, when

$$(1-\lambda)s > \lambda\left[\mu(x-s) + (1-\mu)(1-s)\frac{R_L^{\varepsilon}}{1+r}\right] \tag{69}$$

and CIMP with $r > 0$ otherwise. To sum up, net market return on cash is given by

$$r(s) = \max\left\{0, \frac{R_L^\varepsilon}{\frac{1}{1-\mu}\left[\frac{(1-\lambda)s}{\lambda(1-s)} - \mu\frac{x-s}{1-s}\right]} - 1\right\}. \tag{70}$$

We can characterize the equilibrium as follows:

Proposition 12. *For $x < 1$ but not too small so that liquidity shock is relevant, precisely for $\frac{y}{y+\bar{R}-1} \leq x < 1$, there are thresholds $\hat{\varepsilon}_x$, $\tilde{\varepsilon}_x$ $(\hat{\varepsilon}_x > \tilde{\varepsilon}_x > 0)$ such that the following holds:*

1. *If $\varepsilon \leq \tilde{\varepsilon}_x$, the equilibrium is unique and features first-best with $s^* = \lambda x$ and r^* satisfying*

$$r^* = \min\left\{\bar{R} - 1, \frac{1-\lambda x}{x - \lambda x}R_L^\varepsilon - 1\right\} \tag{71}$$

2. *When $\tilde{\varepsilon}_x < \varepsilon < \hat{\varepsilon}_x$, the equilibrium is unique and features CIMP, with $r^* (> 0)$ and $s^* (\leq \lambda x)$ satisfying*

$$1 + r^* = \bar{R} - (1-\mu)\lambda\left(\frac{y}{x} - (1+r^*)\right)\left(1 - \frac{R_L^\varepsilon}{1+r^*}\right) \tag{72}$$

$$s^* = \frac{\lambda\left(\mu x + (1-\mu)\frac{R_L^\varepsilon}{1+r^*}\right)}{\lambda\left(\mu + (1-\mu)\frac{R_L^\varepsilon}{1+r^*}\right) + (1-\lambda)} \tag{73}$$

3. *If $\varepsilon > \hat{\varepsilon}_x$, the equilibrium is unique and features liquidity hoarding and fundamental pricing, with $s^* = \frac{x-R_L^\varepsilon}{1-R_L^\varepsilon}$ and $r^* = 0$.*

4. *If $\varepsilon = \hat{\varepsilon}_x$, there is a continuum of equilibria, with $r^* = 0$ and $s^* \in \left[\hat{s}_x, \frac{x-R_L^\varepsilon}{1-R_L^\varepsilon}\right]$ where $\hat{s}_x$ is given by (73).*

Proof is relegated to the end of the appendix. The proposition 12 shows that our main outcome of laissez-faire equilibrium in the benchmark case ($x = 1$) is preserved in all cases with the liquidity shock lower than or equal to 1. We can verify easily that substituting $x = 1$ results in the proposition 1. We can consider that our benchmark is an extreme case.

Figure 7 illustrates the relationship between the dispersion and bank's cash holding in the case with $x < 1$. When the level of dispersion is low or intermediate (case 1 and 2),

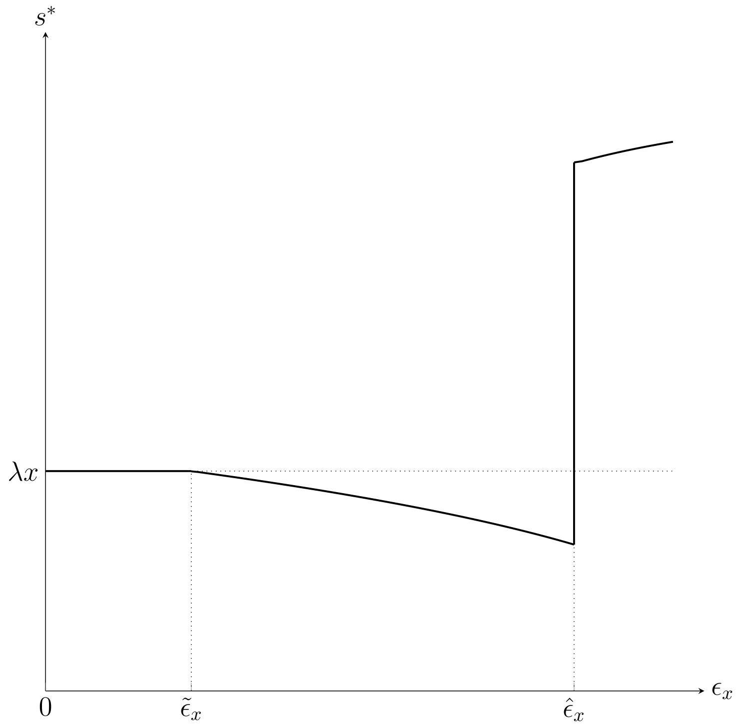

Figure 7: Effect of changes in dispersion on banks cash holdings when $x < 1$

there is CIMP. The main difference, compared to the benchmark case ($x = 1$), lies in the case where dispersion is small enough ($\varepsilon \leq \tilde{\varepsilon}_x$). In this case, we can achieve the first-best outcome, whereas in the benchmark case, achieving the first-best outcome was only possible when there was no dispersion ($\varepsilon = 0$). This is because liquidity shock is less severe ($x < 1$) so that the amount of liquidity required is lower than in benchmark case ($x = 1$). In contrast, when the level of dispersion is high, bank choose to hold enough cash, leading cash hoarding. Price is equal to the fundamental value. Note that banks hold the amount of cash lower than 1 in equilibrium while they hold only cash in the benchmark case. It is because the shock is lower so that banks choose the cash level allowing to continue the project in full scale in the worst case, in other words when they end up with low security and are hit by liquidity shock, which is $s^* + (1 - s^*) R_L^{\varepsilon} = x$.

Proof of proposition 12 To make explicit the dependence on ε, denote $E = C(s, r) - x$, the difference between the total cash that can be available to the banks with s of cash and $1 - s$ of L-security and the cash needed to continue the project in full scale, with $E : [0, 1] \times \left[0, \bar{R} - 1\right] \times [0, 1] \to \mathbb{R}$ given by

$$E(s,r,\varepsilon) = s + (1-s)\frac{R_L^{\varepsilon}}{1+r} - x \tag{74}$$

E is continuous and differentiable over its domain, with

$$\frac{\partial E(s,r,\varepsilon)}{\partial r} < 0, \qquad \frac{\partial E(s,r,\varepsilon)}{\partial \varepsilon} < 0. \tag{75}$$

and given ε, for any equilibrium candidate p given by lemma 1 that is $R_L^{\varepsilon} - 1 \leq r \leq \bar{R} - 1$,

$$\frac{\partial E(s,r,\varepsilon)}{\partial s} > 0 \tag{76}$$

Write (68) in case $E(s,r,\varepsilon) < 0$ as $MRC_x(r) = F_x(r,\varepsilon)$, with $F_x : \left[0, \bar{R}-1\right] \times [0,1] \to \mathbb{R}$ given by:

$$F_x(r,\varepsilon) = MRC_x \mid_{E<0} = \left(1 + r - \bar{R}\right) + (1-\mu)\lambda\left(\frac{y}{x} - (1+r)\right)\left(1 - \frac{R_L^{\varepsilon}}{1+r}\right) \tag{77}$$

F_x is continuous and differentiable over its domain, with

$$\frac{\partial F_x(r,\varepsilon)}{\partial r} > 0, \qquad \frac{\partial F_x(r,\varepsilon)}{\partial \varepsilon} > 0. \tag{78}$$

For the first part, compute

$$\frac{\partial F_x(r,\varepsilon)}{\partial r} = 1 - \lambda(1-\mu) + \lambda(1-\mu)\frac{y}{x}\frac{R_L^{\varepsilon}}{(1+r)^2} > 0 \tag{79}$$

For the second part, compute

$$\frac{\partial F_x(r,\varepsilon)}{\partial \varepsilon} = -\lambda(1-\mu)\left[\frac{y}{x}\left(\frac{1}{1+r}\right) - 1\right]\frac{\partial R_L^{\varepsilon}}{\partial \varepsilon} > 0,$$

where $\frac{y}{x}\left(\frac{1}{1+r}\right) - 1 > 0$ is obvious from the lemma 1 and (A1), and $\frac{\partial R_L^{\varepsilon}}{\partial \varepsilon} < 0$

We also introduce the function $M_x : [0,1] \times [0,1] \to \mathbb{R}$ given by

$$M_x(s,\varepsilon) = \frac{R_L^{\varepsilon}}{\frac{1}{1-\mu}\left[\frac{(1-\lambda)s}{\lambda(1-s)} - \mu\frac{x-s}{1-s}\right]} - 1. \tag{80}$$

Equation (70) can thus be written $r(s) = \max\{0, M_x(s,\varepsilon)\}$. M_x is continuous and differentiable except when $s = 1$ or $\varepsilon = 1$, with

$$\frac{\partial M_x(s,\varepsilon)}{\partial s} < 0, \qquad \frac{\partial M_x(s,\varepsilon)}{\partial \varepsilon} < 0. \tag{81}$$

Define a few threshold values: First, denote ε_0 by ε such that $E\left(\lambda x, \bar{R}-1, \varepsilon\right) = 0$. Second, denote $\tilde{\varepsilon}_x$ by ε such that $F_x\left(r_0^\varepsilon, \varepsilon\right) = 0$ where r_0^ε is r such that $E\left(\lambda x, r, \varepsilon\right) = 0$ for a given ε. r_0^ε is thus given by

$$r_0^\varepsilon = \frac{1-\lambda x}{x-\lambda x} R_L^\varepsilon - 1 \tag{82}$$

with

$$\frac{\partial r_0^\varepsilon}{\partial \varepsilon} < 0, \qquad \frac{\partial F_x\left(r_0^\varepsilon, \varepsilon\right)}{\partial \varepsilon} < 0 \tag{83}$$

The first part is obvious from $x < 1$, $\lambda < 1$, and $\frac{\partial R_L^\varepsilon}{\partial \varepsilon} < 0$. For the second part, compute $\frac{\partial F_x\left(r_0^\varepsilon, \varepsilon\right)}{\partial \varepsilon}$ by substituting r_0^ε for r in (77)

$$\frac{\partial F_x\left(r_0^\varepsilon, \varepsilon\right)}{\partial \varepsilon} = \left[1 - \lambda\left(1-\mu\right)\frac{1-x}{1-\lambda x}\right]\left(\frac{1-\lambda x}{x-\lambda x}\right)\frac{\partial R_L^\varepsilon}{\partial \varepsilon} < 0$$

Finally, denote $\hat{\varepsilon}_x$ by ε such that $F_x\left(0, \varepsilon\right) = 0$.

Note that $\varepsilon_0 < \tilde{\varepsilon}_x$ because at $\varepsilon = \varepsilon_0$

$$F_x\left(r_0^{\varepsilon_0}, \varepsilon\right) = \left(1-\mu\right)\lambda\left(\frac{y}{x} - \bar{R}\right)\left(1 - \frac{R_L^{\varepsilon_0}}{\bar{R}}\right) > 0$$

while $F_x\left(r_0^\varepsilon, \varepsilon\right) > 0$ for all $\varepsilon < \tilde{\varepsilon}_x$ from the second part of (83).

Note that $\tilde{\varepsilon}_x < \hat{\varepsilon}_x$ for $x < \frac{y}{y+\bar{R}-1} \equiv x_0$. To show this, it suffices to demonstrate that $F(0,\varepsilon_1) < 0$, where ε_1 is ε such that $r_0^\varepsilon = 0$. The reasoning is as follows: $F(0,\varepsilon_1) < 0$ implies that $\varepsilon_1 < \hat{\varepsilon}_x$ (since $F(0,\hat{\varepsilon}_x) = 0$ and (78)). This, in turn, implies that $r_0^{\hat{\varepsilon}_x} < 0 = r_0^{\varepsilon_1}$ (see (83). Consequently, $F(r_0^{\hat{\varepsilon}_x}, \hat{\varepsilon}_x) < 0$ (since $F(0,\hat{\varepsilon}_x) = 0$ and (78)). This means that $\tilde{\varepsilon}_x < \hat{\varepsilon}_x$ (since $F(r_0^{\tilde{\varepsilon}_x}, \tilde{\varepsilon}_x) = 0$ and (83)). We can verify that $F(0,\varepsilon_1) < 0$ at $x = x_0$ and since $F(0,\varepsilon_1)$ is decreasing in x, it follows that $F(0,\varepsilon_1) < 0$ for all $x \geq x_0$.

Lemma 6. *We can exclude two types of equilibria upon ε:*

1. *When $\varepsilon \geq \varepsilon_0$, there exists no equilibrium with $E > 0$.*

2. *When $\varepsilon \leq \tilde{\varepsilon}_x$, there exists no equilibrium with $E < 0$.*

The first part of lemma 6 results from the fact that the only equilibrium with $E > 0$ is $s^* = \lambda x$ and $r^* = \bar{R} - 1$ by lemma 5 whereas $E\left(\lambda x, \bar{R} - 1, \varepsilon\right) \leq 0$ when $\varepsilon \geq \varepsilon_0$. We can prove the second part of lemma 6 by contradiction. Suppose that there exist an equilibrium with $E < 0$ under $\varepsilon \leq \tilde{\varepsilon}_x$: i) if $r^* = 0$ (fundamental pricing), then $F_x(0, \varepsilon) < 0$, which implies $s^* = 0$. Contradiction to the fundamental pricing which require $S > D$; ii) if $r^* > 0$, then r^* is such that $F_x(r^*, \varepsilon) = 0$. Since $F_x(r_0^\varepsilon, \varepsilon) > 0$, $r^* < r_0^\varepsilon$ from (78), and s^* satisfies $M_x(s^*, \varepsilon) = r^*$. Simple algebra shows that $M_x(\lambda x, \varepsilon) = r_0^\varepsilon$. Therefore, $s^* > \lambda x$. This results in $E(s^*, r^*, \varepsilon) > E(\lambda x, r_0^\varepsilon, \varepsilon) \geq 0$ (see (75)). A contradiction.

In the following, we consider the equilibrium upon ε.

1. $\varepsilon < \varepsilon_0$: An equilibrium with $E(s, r, \varepsilon) = 0$ can be excluded. If $r^* = 0$ (fundamental pricing), $F_x(r, \varepsilon) < 0$, which implies $s^* = 0$. If $r^* > 0$, then $s = \lambda x$ since $S = D$. Thus, $r^* = r_0^\varepsilon$. However, $r_0^\varepsilon > \bar{R} - 1$ for $\varepsilon < \varepsilon_0$ from (83). A contradiction to lemma 1. If $E > 0$ in equilibrium, $s^* = \lambda x$ and $r^* = \bar{R} - 1$ by lemma 5. $E\left(\lambda x, \bar{R} - 1, \varepsilon\right) > 0$ for all $\varepsilon < \varepsilon_0$. Therefore, this is an equilibrium.

2. $\varepsilon_0 \leq \varepsilon \leq \tilde{\varepsilon}_x$: The only equilibrium, if any, should be with $E = 0$ from lemma 6: i) if $r^* = 0$, then s should satisfy $E(s^*, 0, \varepsilon) = 0$ and $s^* > \lambda x$. Since $F_x(0, \varepsilon) < 0 < F_x(r_0^\varepsilon, \varepsilon)$, $r_0^\varepsilon > 0$ from (78). This requires $s^* < \lambda x$. A contradiction; ii) if $r^* > 0$, then the only possible equilibrium is $r^* = r_0^\varepsilon$ and $s^* = \lambda x$. Given that MRC_x is not continuous at $E = 0$ from (68), the existence of such equilibrium requires both $MRC_x(s + \epsilon, r_0^\varepsilon) \leq 0$ and $MRC_x(s - \epsilon, r_0^\varepsilon) \geq 0$ where ϵ is very small close to 0 and positive. We compute

$$MRC_x(s + \epsilon, , r_0^\varepsilon) = 1 + r_0^\varepsilon - \bar{R} \leq 0 \quad (84)$$

$$MRC_x(s - \epsilon, , r_0^\varepsilon) = F_x(r_0^\varepsilon, \varepsilon) \geq 0 \quad (85)$$

(84) is from $r_0^\varepsilon \leq \bar{R} - 1$ for all $\varepsilon \geq \varepsilon_0$ based on the definition of ε_0 and (75). (85) is obvious for all $\varepsilon \leq \tilde{\varepsilon}_x$ from the definition of $\tilde{\varepsilon}_x$ and (78). Therefore this is an equilibrium.

3. $\tilde{\varepsilon}_x < \varepsilon < \hat{\varepsilon}_x$: We can exclude an equilibrium with $E = 0$. $r^* = 0$ cannot be an equilibrium for the same reason in the above. $r^* = r_0^\varepsilon > 0$ and $s^* = \lambda x$ cannot be an

equilibrium either. To show this, we compute,

$$MRC_x\left(s>\lambda x, r_0^\varepsilon\right) = 1+r_0^\varepsilon-\bar{R}<0 \tag{86}$$

$$MRC_x\left(s<\lambda x, r_0^\varepsilon\right) = F_x\left(r_0^\varepsilon, \varepsilon\right)<0 \tag{87}$$

(86) is from $r_0^\varepsilon<\bar{R}-1$ for all $\varepsilon>\varepsilon_0>\tilde{\varepsilon}_x$ based on the definition of ε_0 and (75). (87) is obvious for all $\varepsilon>\tilde{\varepsilon}_x$ from the definition of $\tilde{\varepsilon}_x$ and (78). $s=0$ would be profitable deviation. Consider now the equilibrium with $E<0$. An equilibrium with fundamental pricing can be excluded, since $F\left(0,\varepsilon\right)<0$ implies $s^*=0$. Using (77) we have

$$F_x\left(R_L^\varepsilon-1,\varepsilon\right)<0, \qquad F_x\left(\bar{R}-1,\varepsilon\right)=\lambda\left(1-\mu\right)\left(\frac{y}{x}-\bar{R}\right)\varepsilon>0,$$

where the last step uses (A1). Together with $F_x\left(0,\varepsilon\right)<0$, this implies that there exists a unique r^* such that $F_x\left(r^*,\varepsilon\right)=0$, with $\max\{0,R_L^\varepsilon-1\}<r^*<\bar{R}-1$ from (78). Using (70), which is continuous and decreasing in s, one can find a unique s^* such that $r^*=r\left(s^*\right)$. We thus have a unique equilibrium $\left(r^*,s^*\right)$, featuring CIMP. With some algebra, we get expression (73) for s^*. This equilibrium satisfies $E<0$ given that $\left(1-s^*\right)\frac{R_L^\varepsilon}{1+r^*}<x-s^*$.

4. $\varepsilon>\hat{\varepsilon}_x$: The equilibrium with $E<0$ can be excluded since $F_x\left(r,\varepsilon\right)>F_x\left(0,\varepsilon\right)>0$ for all $r>0$ from (78), which implying $s^*=1$. In turn, $E>0$. A contradiction. We now consider an equilibrium with $E=0$. Under $E=0$, $r^*=r_0^\varepsilon$ cannot be an equilibrium because $r_0^\varepsilon<0$ from $F_x(r_0^\varepsilon,\varepsilon)<0<F_x(0,\varepsilon)$ and (78). Therefore, the only equilibrium with $E=0$, if any, is $r^*=0$ and s^* should satisfy $E(s^*,0,\varepsilon)=0$ and $s^*>\lambda x$. We get $s^*=\frac{x-R_L^\varepsilon}{1-R_L^\varepsilon}$ from $E(s^*,0,\varepsilon)=0$.

5. $\varepsilon=\hat{\varepsilon}_x$: Since $F_x\left(0,\varepsilon\right)=0$, $r^*=0$ (any $r>0$ would imply that $s^*=1$ and thus $r=0$, a contradiction). s^* satisfies (69) and $E\left(s,0,\hat{\varepsilon}_x\right)\leq 0$. The lower bound of s^* is thus $\hat{s}_x$ is given by (69) with equality, which is (73), and its upper bound is given by $E\left(s,0,\hat{\varepsilon}_x\right)=0$, which is $\frac{x-R_L^\varepsilon}{1-R_L^\varepsilon}$.

Case 1 in the statement of proposition 12 map with case 1-2 above, and case 2-4 match case 3-5 above, respectively. ■