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# Hammond Transfers and Ordinal Inequality Measurement

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**Abstract**

This article establishes a direct proof of the equivalence between two incomplete rankings of distributions of an ordinal attribute. The first ranking is the possibility of going from one distribution to another by a finite sequence of Hammond transfers. The second ranking is the intersection of two dominance criteria introduced by Gravel et al.(*Economic Theory*, 71 (2021), 33-80). The proof constructs an algorithm that provides a series of Hammond transfers, between any two distributions related by the intersection of the two dominances.

JEL classification: D3, D63

Keywords:

Hammond Transfers, Inequality, Algorithm

# Introduction

A Pigou-Dalton transfer (Dalton(1912), Pigou(1920)) is a mean-preserving transfer of resources from a relatively well-endowed individual to a less well-endowed one, that preserves the rank between the donator and the recipient. A Pigou-Dalton transfer is commonly considered as a natural definition of inequality reduction when applied to a cardinal variable. A cardinal variable, is in effect, measured uniquely up to a positive affine transformation. Because of this, the preservation of the mean by a Pigou-Dalton transfer is a meaningful operation for a cardinal variable because the ranking of two distributions according to their mean remains unchanged if an increasing affine transformation

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is applied to the unit of measurement. If, however, the metric of the attribute is defined up to a strictly increasing transformation, as is the case when the distributed attribute is ordinal, then the mean is no longer a "reliable" magnitude that needs to be preserved by the transfer. This observation is at the core of recent criticism of the Pigou-Dalton transfer when applied to an ordinal attribute (Abul-Naga and Yalcin (2008), Allison and Foster (2004), Apouey (2007), Cowell and Flachaire (2017), Gravel et al. (2021)).

How can we define inequality reduction when the attribute studied is ordinal? To answer this question, consider the following example of distributions of self-reported health status (SRHS) categorized into: poor, fair, good, very good and excellent. Suppose a simple distribution is composed of Chloé, who is in poor health, and Tom, who is in excellent health. If we measure health using a metric assigning the values "1, 2, 3, 4, 5" to the respective categories of "poor, fair, good, very good, and excellent", then the distribution in which both Tom and Chloé are in good health can be obtained by a Pigou-Dalton transfer. However, with the alternative metric "1, 2, 3, 4, 6", we cannot reach the distribution where Tom and Chloé are in good health by a Pigou-Dalton transfer. Indeed, what is "taken" from Tom in terms of health units is not equal to what is "given" to Chloé with this alternative metric.

To provide a more appropriate definition of inequality reduction for an ordinal variable, we need to loosen the requirement that what is taken must equal what is given. If we simply demand that Tom's health level decreases and Chloé's health level increases, then the distribution where both are in good health will always be viewed as less unequal, regardless of the metric employed. This concept of inequality reduction was introduced by Hammond (1976) in his *minimal equity principle*.

According to Hammond's principle, reducing the gap between the attributes' endowments of two agents reduces inequality irrespective of whether or not what is taken from the highly endowed agent is equal to what is given to the less endowed agent. The present paper is interested in the empirical verification of the possibility of going from a distribution to another by a finite sequence of such Hammond transfers.

The recent literature (Gravel et al.(2019), Gravel et al.(2021)) has provided two different answers to this problem. The first answer applies to comparisons of distributions of an ordinal attribute that can take continuously many values (e.g. PISA scores). In that case we know from Gravel et al.(2019) that the possibility of going from a distribution to another by a finite sequence of Hammond transfers is equivalent to ranking distributions according to the intersection of the Leximin[1] and the anti-Leximax[2] criteria. Since the two criteria are easily verifiable empirically, the problem of identifying an empirical test for the possibility of going from a distribution to another by a finite sequence of Ham-

[1]The Leximin is the lexicographic extension of the well-known Maximin.

[2]The anti-Leximax is the lexicographic extension of the Minimax, the Minimax prefers the distribution where the better-off agent has the lower utility.

mond transfers is solved for distributions of a continuous ordinal variable. The second answer applies to distributions of an ordinal variable that can take only finitely many values, like the health categories considered in the example above. In order to identify when a distribution is obtained from another by a finite sequence of Hammond transfers, Gravel et al.(2021) introduced two tests constructed from two curves, namely the $H$-curve which is a weighted sum of the cumulative density function, and the $\bar{H}$-curve which is a weighted sum of the survival function[3]. The first test checks whether the $H$-curves of two distributions cross each other. Gravel et al.(2021) show that the $H$-curves don't cross (what is referred to as the $H$-dominance) if and only if the distribution for which the $H$-curve lies below can be obtained from the other distribution by a finite sequence of Hammond transfers and/or *increments*. In Gravel et al.(2021), an increment defines an elementary rise in efficiency in a distribution, described as an improvement in the endowment of the attribute of one agent everything else remaining the same for all other agents. The second test makes an analogous analysis using the $\bar{H}$-curve. Gravel et al (2021) prove that the $\bar{H}$-curve of one distribution lies nowhere above the $\bar{H}$-curve of a second distribution (what is referred to as the $\bar{H}$-dominance) if and only if it is possible to reach the first distribution from the second by a finite sequence of Hammond transfers and/or decrements, a decrement being the opposite of an increment (i.e. the deterioration of an agent attribute).

Since increments and decrements are transformations that work in opposite directions and Hammond transfers are recognized as worth doing transformations by both types of dominance, it seems natural to ask the following question: Is the simultaneous occurrence of the two dominance an empirical test for the possibility of going from the dominated to the dominating distribution by a finite sequence of Hammond transfers only? The main contribution of this paper is to provide a positive answer to this question.

Gravel et al. (2021) have provided a somewhat indirect proof that such a positive answer can be provided. Their proof is built on the fact that Hammond transfers have the structure of a discrete cone (see Magdalou (2019)). Following this approach, they show that when there is double dominance, it is always possible to write the difference between the two distributions as a combination of Hammond transfers. However, because this method focuses on the *difference* between distributions, it allows the addition of *phantom agents*[4] to both distributions. Hence, what Gravel et al. (2021) have established is that if there is both $H$ and $\bar{H}$-dominance between two distributions, then it is possible to add non-existing (dummy) agents with specific values of the attribute to both distributions in such a way that one can go from the phantom-augmented dominated distribution to the phantom-augmented dominating distribution by a finite sequence of Hammond transfers.

[3] If we note F() the cumulative, 1-F() is the survival function

[4] Here, I reuse a terminology employed by Gravel and Moyes(2012)

In the current paper, I prove that we can go from the dominated to the dominating distribution by a finite sequence of Hammond transfers without adding dummy agents. The proof consists of an algorithm[5] that always provides a series of Hammond transfers to reach the actual dominant distribution from the actual dominated one without resorting to any other modifications of the distributions.

I will also use this algorithm to investigate whether one could identify an empirical test that corresponds precisely to the possibility of going from one distribution to another by Median-Preserving Hammond transfers.

A Hammond transfer, by definition, does not preserve any measure of the "size of the cake" (such as the mean or median). However, as recalled above, inequality reduction typically preserves the mean in the literature on inequality measurement for cardinal attributes. Several recent approaches focusing on defining inequality for ordinal attributes consider that inequality reduction should preserve the median (see Allison and Foster(2004), Abul-Naga and Yalcin(2008)). Recently, Gargani and Gravel (2025) axiomatically demonstrated that the median is the only measure of the size of the cake that is ordinally consistent. Therefore, if we adopt the perspectives of both the ordinal and cardinal literature, arguing that inequality reduction should preserve a measure of the size of the cake, median-preserving transfers can be viewed as the ordinal equivalent to mean-preserving transfers.

It would be nice to identify an implementable test that enables one to verify if one distribution has been obtained from another by means of a finite sequence of median-preserving Hammond transfers. While the current paper does not provide such a test, it explores a first question in that direction that is of some interest. The question is based on the famous Hardy et al. (1952) theorem[6], which says that when comparing two distributions with the same mean, the Lorenz curve of distribution $d$ lies nowhere below the Lorenz curve of distribution $d'$ if and only if one can go from $d'$ to $d$ by a finite sequence of mean-preserving transfers. A similar equivalence for median-preserving transfers could be the following: *We observe the double dominance between two distributions, with the same median, if and only if it is possible to reach the dominant from the dominated distribution by a finite sequence of median-preserving transfer.* Is this theorem true? The if part of this theorem is true simply because a median-preserving transfer is a specific type of Hammond transfer. However, the only if part of the theorem is false because there are some cases of double dominance between distributions with the same median for which it is impossible to reach the dominant distribution by a series of median-preserving transfers. This thus leaves open the important question: Which restriction of the double dominance test -in addition to the requirement that it applies to two distributions with the same

[5]The reader can use the algorithm at https://algorithmhammond.streamlit.app

[6]See Kolm (1969), Atkinson (1970) and Dasgupta et al. (1973) for the discovery of this theorem by economists.

median- would capture the possibility of going from the dominated to the dominating distribution by median-preserving Hammond transfers only?

The remainder of this paper is organised as follows. Section 1 introduces the basic notations, section 2 states and proves the theorem and explains the algorithm, section 3 discusses the notion of median-preserving transfer, and finally, section 4 concludes.

# 1 Main Notation

Throughout this article, I denote by $L \geq 3$ the number of different possible values of the attribute that I refer to as categories. The attribute is taken to be ordinally measured. This means that the categories are all strictly ordered (from worst to best) but that no other meaning than this ordering can be assigned to the numbers that can be applied without loss of generality to those categories. I consider a population distribution $d \in \mathbb{N}^L$ among different categories: $d = (n_1^d, ..., n_L^d)$ with $n_i^d$ the number of individuals in distribution $d$ who are in category $h \in \{1, ..., L\}$ and satisfying $\sum_{h=1}^{L} n_h^d = n$ for some population size[7] $n$. The formal definition of a Hammond transfer, underlying Hammond's equity principle, is as follows.

**Definition 1** (*Hammond transfer*) Distribution $d \in \mathbb{N}^L$ is obtained from distribution $d' \in \mathbb{N}^L$ by means of a Hammond transfer, if there exist categories $1 \leq g < i \leq j < l \leq L$ such that:

$$\text{If } i < j: \quad \begin{aligned} & n_h^d = n_h^{d'}, \forall h \neq g, i, j, l \\ & n_g^d = n_g^{d'} - 1, n_i^d = n_i^{d'} + 1 \\ & n_j^d = n_j^{d'} + 1, n_l^d = n_l^{d'} - 1 \end{aligned} \qquad \text{If } i = j: \quad \begin{aligned} & n_h^d = n_h^{d'}, \forall h \neq g, i, l \\ & n_g^d = n_g^{d'} - 1, n_i^d = n_i^{d'} + 2 \\ & n_l^d = n_l^{d'} - 1 \end{aligned}$$

In plain English, a Hammond transfer is a reduction in the gap between two individuals' attribute endowments, all other individuals' situations remaining the same.

**Example 1** (*Two distributions of education degrees*)

| Degree | $d' = (3, 5, 1, 1)$ | $d = (2, 7, 0, 1)$ |
|---|---|---|
| No degree | 3 | 2 |
| Bachelor | 5 | 7 |
| Master | 1 | 0 |
| PhD | 1 | 1 |

[7] Comparisons of distributions with differing numbers of individuals can be made thanks to the Dalton (1920) replication axiom.

In example 1, $d$ is obtained from $d'$ by a Hammond transfer. I aim to demonstrate the equivalence between performing such Hammond transfers a finite number of times and achieving a distribution that is considered better by two dominance criteria. These two dominance criteria are constructed from the cumulative distribution function and the survival function associated to a distribution $d \in \mathbb{N}^L$.

**Definition 2** *(Cumulative and Survival Functions)*

$$\forall h \in \{1, ..., L\},\ F(d,h) = \sum_{i=1}^{h} \frac{n_i^d}{n} \quad \text{and} \quad \bar{F}(d,h) = 1 - F(d,h)$$

The $H$-dominance is constructed as follows from the cumulative function:

**Definition 3** *(H-dominance)* For any two distributions $d$ and $d'$ in $\mathbb{N}^L$, we say that $d$ $H$-dominates $d'$, if the following holds:

$$\forall h \in \{1, ..., L-1\},\ H(d,h) \leq H(d',h),$$

$$\text{where,}\ \ H(d,1) = F(d,1) \text{ and } \forall h \geq 2,\ H(d,h) = \sum_{i=1}^{h-1} 2^{h-1-i} F(d,i) + F(d,h)$$

Gravel et al. (2021) have shown that $H$-dominance between two distributions is equivalent to the possibility of going from the dominated to the dominating distribution by a finite sequence of Hammond transfers and/or increments. An increment is defined as follows:

**Definition 4** (*Increment*) Distribution $d \in \mathbb{N}^L$ is obtained from distribution $d' \in \mathbb{N}^L$ through an increment, if there exists a category $i \in \{1, ..., L-1\}$ such as:

$$n_h^d = n_h^{d'}, \forall h \neq i, i+1$$
$$n_i^d = n_i^{d'} - 1, n_{i+1}^d = n_{i+1}^{d'} + 1$$

By calling a decrement the opposite of an increment[8], i.e. the deterioration of an agent's outcome. Gravel et al. (2021) showed that we observe $\bar{H}$-dominance between two distributions if and only if the dominant distribution can be obtained from the dominated one by a series of Hammond transfers and/or decrements. The $\bar{H}$-dominance is constructed from the survival function as follows:

[8]Formally, a distribution $d$ is obtained from $d'$ by a decrement if and only if $d'$ is obtained from $d$ by an increment.

**Definition 5** ($\bar{H}$*-dominance*) For any two distributions $d$ and $d'$ in $\mathbb{N}^L$, we say that $d$ $\bar{H}$-dominates $d'$, if the following holds:

$$\forall h \in \{1, ..., L-1\},\ \bar{H}(d, h) \leq \bar{H}(d', h),$$

where, $\bar{H}(d, L-1) = \bar{F}(d, L-1)$ and $\forall h \leq L-2$, $\bar{H}(d, h) = \sum_{i=h+1}^{L-1} 2^{i-h-1}\bar{F}(d, i) + \bar{F}(d, h)$

These two dominance criteria are therefore interested in comparing the values of the $H$ and $\bar{H}$-curves between different distributions, which is why I will use the following notation:

$$\begin{gathered}
\Delta_{(d',d)}H(h) = H(d', h) - H(d, h) \\
\Delta_{(d',d)}\bar{H}(h) = \bar{H}(d', h) - \bar{H}(d, h) \\
\Delta_{(d',d)}F(h) = F(d', h) - F(d, h) \\
\Delta_{(d',d)}\bar{F}(h) = \bar{F}(d', h) - \bar{F}(d, h) \\
(\text{Note that, } \Delta_{(d',d)}F(h) = -\Delta_{(d',d)}\bar{F}(h))
\end{gathered}$$

Moreover, for the rest of the article, I will keep denoting $d'$, the dominated distribution, and $d$, the dominant one.

# 2 Main Result

In their main result, Gravel et al. (2021) showed indirectly that if we want to single out Hammond transfers, we should look at the intersection of both dominance criteria. In this paper, the following theorem is proved directly.

**Theorem 1** Let $d$ and $d'$ be two distinct distributions in $\mathbb{N}^L$. The two following statements are equivalent:

$(i)$ $d$ can be obtained from $d'$ by a finite series of Hammond transfers.
$(ii)$ For all $h \in \{1, ..., L-1\}$ $\Delta_{(d',d)}H(h) \geq 0$ and $\Delta_{(d',d)}\bar{H}(h) \geq 0$

The tricky part of the proof consists of establishing that $(ii)$ implies $(i)$. To prove this implication, I build an algorithm that provides a list of Hammond transfers that allow us to reach the dominant distribution from the dominated one. The Hammond transfers performed by the algorithm are such that the resulting distribution remains *weakly* dominated by the initial dominant distribution. In other words, the algorithm aims to return a Hammond transfer that *preserves the double dominance.*

Between the curves of $d'$ and the curves of $d$, there are some gaps for some categories $h \in \{1, ..., L\}$, denote these gaps $m_h^{d'} > 0$ and $m_h^{d'} > 0$, these gaps could be defined as follows:

$$m_h^{d'} = H(d', h) - H(d, h)$$
$$\bar{m}_h^{d'} = \bar{H}(d', h) - \bar{H}(d, h)$$

When a transfer preserves the double dominance, it reduces the gaps. Hence if $d''$ is a distribution obtained from $d'$ by means of a Hammond transfer preserving the double dominance, then for all categories $h \in \{1, ..., L\}$, $m_h^{d''} \leq m_h^{d'}$ and $\bar{m}_h^{d''} \leq \bar{m}_h^{d'}$, with the inequalities being strict for at least one category. Since it is a discrete framework with a finite number of individuals, the gaps will be null after a finite number of iterations[9] of the algorithm, meaning that the dominant distribution has been reached by the algorithm.

The first part of this algorithm consists in defining a set of categories from which some Hammond transfers preserving the double dominance can be done.

## 2.1 An Important Set of Categories.

I use the cumulative distribution function to define the set of ordered categories $\{h_0, ..., h_K\} \subset \{1, ..., L\}$ as follows.

$h_0 = min(\{h \in \{1, ..., L-1\} \mid \Delta_{(d',d)}F(h) > 0\})$ and,
$\{h_1, ..., h_K\} = \{h \in \{h_0 + 1, ..., L\} \mid \Delta_{(d',d)}F(h) \geq 0 \text{ and } \Delta_{(d',d)}F(h-1) < 0\}$

The category $h_0$ is the first category at which the value of the cumulative of $d'$ is greater than the value of the cumulative of $d$. The cumulative value of $d'$ is higher in $h_0$, but this cannot be the case for all categories after $h_0$; otherwise, $d$ would not $\bar{H}$-dominates $d'$. Therefore, there must be a crossing point between the cumulative values, i.e., at which the cumulative value of $d'$ becomes strictly lower than that of $d$. After this crossing point, there must be a category for which the value of the cumulative $d'$ will once again be greater than or equal to that of $d$. Let say informally that in such an occurrence, the cumulative of both distributions "meet again". Therefore, using this informal expression, the elements of the set $\{h_1, ..., h_K\}$ are the categories for which the cumulative "meet again" after a crossing point.

Five properties of the set $\{h_0, ..., h_K\}$ will play a key role in establishing the possibility of performing a Hammond transfer from $d'$ to either $d$ or to another distribution that remains dominated by $d$ (the proof of these properties, as well as of all formal results of

[9]See Faure and Gravel(2021) for additional difficulties of building an algorithm when the variable is continuous.

this paper, are provided in the Appendix):

**Property 1** $\forall k \in \{0, ..., K\},\ n^{d'}_{h_k} \geq 1$
**Property 2** $K + 1 \geq 2$
**Property 3** $\forall k \in \{0, ..., K-1\},\ h_k \leq h_{k+1} - 2$
**Property 4** $\forall k \in \{0, ..., K-1\}$ and $\forall h \in \{h_k, ..., h_{k+1} - 1\}$, if $\Delta_{(d',d)}F(h) \geq 0$ then $\forall h' \in \{h_k, ..., h\},\ \Delta_{(d',d)}F(h') \geq 0$

From property 1, all categories in the set $\{h_0, ..., h_K\}$ contain at least one individual in the distribution $d'$. Moreover, this set always contains at least two elements (property 2) and there is always for all $k \in \{0, ..., K-1\}$, at least one category between $h_k$ and $h_{k+1}$ (property 3). Property 4 states that for categories in the set $\{h_k, ..., h_{k+1} - 1\}$ there is a category $h$ such that for categories in the set $\{h_k, ..., h_{k+1} - 1\}$ and strictly below $h$, the differences in the cumulative functions are positive whereas for categories above or in $h$, the differences of the cumulative functions are strictly negative.

Following the first three properties, it is possible to do a Hammond transfer of an individual from the category $h_k$ to the just above category $h_k + 1$ combined with the transfer of an individual from the category $h_{k+1}$ to the category just below $h_{k+1} - 1$. Since it consists in going up one category at the bottom in exchange of going down one category at the top, I will refer to it as a "Pigou-Dalton transfer". The proof will use this kind of transfer - that is also a Hammond transfer - as a diagnostic tool for the possibility of making a Hammond transfer preserving the double dominance. The detail of the algorithmic procedure is provided in the next section.

## 2.2 The Algorithm

When a Pigou-Dalton transfer described above preserves the double dominance, the algorithm does this Pigou-Dalton transfer in order to reach the dominant distribution from the dominated. Hence, the question that must be faced is: What do we do when none of the Pigou-Dalton transfers described above preserve the double dominance?

To answer this question, I consider the following two mutually exclusive possibilities in which none of the Pigou-Dalton transfers preserve the double dominance. Denote $d_k$ (with $k \in \{0, ..., K-1\}$), a distribution obtained from $d'$ by a Pigou-Dalton transfer described in the previous section.

**Case 1.** There is a distribution $d_k$ obtained from $d'$ by means of a Pigou-Dalton that doesn't preserve the $\bar{H}$-dominance.

**Case 2.** All distributions $d_k$ obtained from $d'$ by means of a Pigou-Dalton preserve $\bar{H}$-dominance.

In case 1, the algorithm goes through the $d_k$ distributions in ascending order, i.e. from $d_0$ to $d_{K-1}$. It stops at the first $d_k$ distribution that does not preserve the $\bar{H}$-dominance. I call this distribution $d_t$, with $t = min(\{k \in \{0, ..., K-1\} \mid \exists h \in \{1, ..., L-1\},\ \bar{H}(d_k, h) < \bar{H}(d, h)\})$. Note that, when $t \geq 1$, $d_{t-1}$ doesn't preserve the $H$-dominance, otherwise $d_{t-1}$ would preserve the double dominance, as it already preserves the $\bar{H}$-dominance by definition of $t$.

In the algorithm, case 1 is divided in three mutually exclusive sub-cases that are treated in propositions 1 to 3 (see section 2.3). In order to explain how I divide case 1 in three sub-cases, I need to introduce a particular category. Suppose that $d'' \in \mathbb{N}^L$ is a distribution obtained from $d'$ by a Hammond transfer. Then I define $h_{max}(d'')$ as follows:

$$h_{max}(d'') = max(\{h \in \{1, ..., L-1\} \mid \Delta_{(d',d'')}\bar{H}(h) > \Delta_{(d',d)}\bar{H}(h)\})$$

In plain English, $h_{max}(d'')$ is the first category from the top, at which the preservation of $\bar{H}$-dominance of $d''$ over $d$ is broken. In case 1 and by definition of $t$, the category $h_{max}(d_t)$ exists and satisfies the two following properties:

**Property 5** For all $k \in \{1, ..., K-1\},\ \Delta_{(d',d)}F(h_{max}(d_k)) \geq \frac{1}{n}$
**Property 6** For all $k \in \{1, ..., K-1\},\ h_{max}(d_k) \in \{h_0, ..., h_{k+1} - 2\}$

I use the category $h_{max}(d_t)$ to distinguish between the following three sub-cases of case 1:

**Case 1a.** $h_{max}(d_t) \leq h_t$

**Case 1b.** $h_{max}(d_t) > h_t$ and there is a category $h \in \{h_t, ..., h_{max}(d_t)\}$ such as $\Delta_{(d',d)}F(h) \geq \frac{2}{n}$

**Case 1c.** $h_{max}(d_t) > h_t$ and for all categories $h \in \{h_t, ..., h_{max}(d_t)\}$, $\Delta_{(d',d)}F(h) \leq \frac{1}{n}$

For case 1a I provide a Hammond transfer that preserves the double dominance in Proposition 1, while cases 1b and 1c are treated in propositions 2 and 3 respectively. The intuition behind these propositions is the following. The Pigou-Dalton leading to the distribution $d_t$ generates variations of the $\bar{H}$-curve that are too large. Therefore in order for a transfer to preserve the double dominance, it needs to generate smaller variations of the $\bar{H}$-curve than the Pigou-Dalton transfer. For example, in appendix, Table 4 describes the variations generated by the Hammond transfer performed in Proposition 1, as it can be seen in Table 4, most of the variations of the $\bar{H}$-curve are null. In cases 1b and 1c the Hammond transfer defined in Proposition 1 does not necessarily preserve the $H$-dominance, therefore the Hammond transfers defined in propositions 2 and 3 are built to generate lower variations of the $H$-curve than the transfer defined in Proposition 1.

Note that in order to encounter case 1, the algorithm go through the distributions $d_k$ in ascending order. An other possible approach would have been to define the algorithm to go through distributions in descending order, i.e from $d_{K-1}$ to $d_0$. This descending approach is symmetric to the ascending approach. The descending approach corresponds in fact to the approach used in case 2. Indeed, in case 2, the algorithm goes through the $d_k$ distributions in descending order. It stops at the first $d_k$ distribution that does not preserve the $H$-dominance. Because in case 2, all distributions $d_k$ preserve $\bar{H}$-dominance and it is assumed that none of the Pigou-Dalton transfers preserve double dominance, the distribution $d_{K-1}$ can't preserve the $H$-dominance. Hence, in case 2, the algorithm will stop at the distribution $d_{K-1}$.

Case 2 is divided in three mutually exclusive sub-cases that are treated in propositions 4 to 6 (see section 2.4). In order to explain how I divide case 2 in three sub-cases, I parallel the approach of case 1 by introducing the following category $h_{min}(d'')$ defined for any distribution $d'' \in \mathbb{N}^L$ obtained from $d'$ by a Hammond transfer:

$$h_{min}(d'') = min(\{h \in \{1, ..., L-1\} \mid \Delta_{(d',d'')}H(h) > \Delta_{(d',d)}H(h)\})$$

$h_{min}(d'')$ is the first category from the bottom, at which the preservation of $H$-dominance of $d''$ over $d$ is broken. It is clear that the category $h_{min}(d_{K-1})$ exists in case 2. The following two properties of this category are worth noticing.

**Property 7** For all $k \in \{0, ..., K-1\}$, $\Delta_{(d',d)}\bar{F}(h_{min}(d_k)) \geq \frac{1}{n}$
**Property 8** For all $k \in \{0, ..., K-1\}$, $h_{min}(d_k) \in \{h_k + 1, ..., h_K - 1\}$

The three sub-cases of case 2 are distinguished as follows:

**Case 2a.** $h_{min}(d_{K-1}) = h_K - 1$
**Case 2b.** $h_{min}(d_{K-1}) < h_K - 1$ and there is a category $h \in \{h_{min}(d_{K-1}), ..., h_K - 1\}$ such as $\Delta_{(d',d)}\bar{F}(h) \geq \frac{2}{n}$
**Case 2c.** $h_{min}(d_{K-1}) < h_K - 1$ and for all categories $h \in \{h_{min}(d_{K-1}), ..., h_K - 1\}$, $\Delta_{(d',d)}\bar{F}(h) = \frac{1}{n}$.

For case 2a, I provide a Hammond transfer that preserves the double dominance in Proposition 4, cases 2b and 2c are treated in propositions 5 and 6 respectively. The transfers performed in propositions 4 to 6 follow a logic somewhat symmetrical to the case 1. In case 2, the Pigou-Dalton transfer leading to the distribution $d_{K-1}$ generates variations of the $H$-curve that are too large. Table 9 of the Appendix shows the variations generated by the Hammond transfer performed in Proposition 4. As can be seen in this table, most of the variations of the $H$-curve are null. In cases 2b and 2c the Hammond transfer defined in Proposition 4 does not necessarily preserve the $\bar{H}$-dominance, therefore the Hammond transfers defined in propositions 5 and 6 are built to generate lower variations of the $\bar{H}$-curve than the transfer defined in Proposition 4.

I present the propositions covering case 1 and case 2 in the two next sections.

## 2.3 Case 1

The Hammond transfer preserving the double dominance in the case 1a is provided in the following proposition.

**Proposition 1** Suppose $t \in \{0, ..., K-1\}$. Additionally, suppose that if $t \geq 1$, then $d_{t-1}$ doesn't preserve the $H$-dominance. Under these conditions, if $h_{max}(d_t) \leq h_t$ and $d'' \in \mathbb{N}^L$, is obtained from $d'$, by the following Hammond transfer:

$$\begin{aligned}
&n_h^{d''} = n_h^{d'},\ \forall h \neq h_t,\ h_{t+1}-1,\ h_{t+1} \\
&n_{h_t}^{d''} = n_{h_t}^{d'} - 1,\ n_{h_{t+1}-1}^{d''} = n_{h_{t+1}-1}^{d'} + 2 \\
&n_{h_{t+1}}^{d''} = n_{h_{t+1}}^{d'} - 1
\end{aligned}$$

Then $d''$ preserves the double dominance.

The distribution obtained from the Hammond transfer defined in Proposition 1 preserves the double dominance when $h_{max}(d_t) \leq h_t$ but not necessarily when $h_t < h_{max}(d_t)$.

When $h_t < h_{max}(d_t)$, the second Hammond transfer, presented in Proposition 2, applies when there is at least one category in the set $\{h_t, ..., h_{max}(d_t)\}$, such that the value of the difference between the cumulative functions is greater or equal to $\frac{2}{n}$.

**Proposition 2** Suppose $t \in \{0, ..., K-1\}$, and $h_t < h_{max}(d_t)$. Moreover suppose that there is a category $h \in \{h_t, ..., h_{max}(d_t)\}$ such as $\Delta_{(d',d)}F(h) \geq \frac{2}{n}$. Additionally, suppose that if $t \geq 1$, then $d_{t-1}$ doesn't preserve the $H$-dominance. Under these conditions, by defining the category $e = min(\{\{h \in \{h_t, ..., h_{max}(d_t)\} \mid \Delta_{(d',d)}F(h) \geq \frac{2}{n}\})$, if $d'' \in \mathbb{N}^L$, is obtained from $d'$, by the following Hammond transfer:

$$\begin{aligned}
&n_h^{d''} = n_h^{d'},\ \forall h \neq\ e,\ h_{t+1}-1,\ h_{t+1} \\
&n_e^{d''} = n_e^{d'} - 1,\ n_{h_{t+1}-1}^{d''} = n_{h_{t+1}-1}^{d'} + 2 \\
&n_{h_{t+1}}^{d''} = n_{h_{t+1}}^{d'} - 1
\end{aligned}$$

Then $d''$ preserves the double dominance.

The Hammond transfer performed in Proposition 2 is quite similar to the transfer performed in Proposition 1. If the category $e$, defined in Proposition 2, is equal to $h_t$, then the transfers performed in the two propositions are the same. When $e \neq h_t$, the transfer in Proposition 1 preserves the $\bar{H}$-dominance, but it is too efficient to preserve the $H$-dominance. Hence the transfer proposed in Proposition 2 is relatively less efficient. Indeed, the distribution obtained from the Hammond transfer defined in Proposition 2, can be reached by adding some decrements to the distribution obtained by the Hammond transfer defined in Proposition 1. However the transfer in Proposition 2 can't be done when there isn't any category for which the value of the difference in the cumulative functions is greater or equal to $\frac{2}{n}$.

The last sub-case of case 1, covered in Proposition 3, is when, for all categories $h \in \{h_t, ..., h_{max}(d_t)\}$, $\Delta_{(d',d)}F(h) \leq \frac{1}{n}$.

**Proposition 3** Suppose $t \in \{0, ..., K-1\}$, and $h_t < h_{max}(d_t)$. Moreover suppose that for all categories $h \in \{h_t, ..., h_{max}(d_t)\}$, $\Delta_{(d',d)}F(h) \leq \frac{1}{n}$. Additionally, suppose that if $t \geq 1$, then $d_{t-1}$ doesn't preserve the $H$-dominance. Under these conditions, by defining the categories $e = min(\{h \in \{h_t+1, ..., h_{max}(d_t)+1\} \mid \Delta_{(d',d)}F(h-1) > \Delta_{(d',d)}F(h)\})$ and $j = min(\{h \in \{e+1, ..., h_{t+1}\} \mid \Delta_{(d',d)}F(h-1) < \Delta_{(d',d)}F(h)\})$, if $d'' \in \mathbb{N}^L$, is obtained from $d'$ by the following Hammond transfer:

$$\begin{aligned}
\text{If } e < j-1: \quad & n_h^{d''} = n_h^{d'},\ \forall h \neq h_t,\ e,\ j-1,\ j \\
& n_{h_t}^{d''} = n_{h_t}^{d'} - 1,\ n_e^{d''} = n_e^{d'} + 1 \\
& n_{j-1}^{d''} = n_{j-1}^{d'} + 1,\ n_j^{d''} = n_j^{d'} - 1 \\
\text{If } e = j-1: \quad & n_h^{d''} = n_h^{d'},\ \forall h \neq h_t,\ e,\ j \\
& n_{h_t}^{d''} = n_{h_t}^{d'} - 1,\ n_e^{d''} = n_e^{d'} + 2 \\
& n_j^{d''} = n_j^{d'} - 1
\end{aligned}$$

Then $d''$ preserves the double dominance.

The transfer of Proposition 3 is also quite similar to the transfer performed in Proposition 1. If the category $e$, defined in Proposition 3, is equal to $h_{t+1}-1$ then the transfers performed in the two propositions are the same. When $e < h_{t+1}-1$ and the category $j$, defined in Proposition 3, is equal to $h_{t+1}$, the transfer in Proposition 1 is too efficient to preserve the $H$-dominance. Consequently the transfer proposed in Proposition 3 is relatively less efficient. Indeed, when $e < h_{t+1}-1$ and $j = h_{t+1}$, the distribution obtained from the Hammond transfer defined in Proposition 3, can be reached by applying a series of decrements to the distribution obtained from the Hammond transfer defined in Proposition 1. Finally when $e \neq h_{t+1}-1$ and $j < h_{t+1}$ the transfer in Proposition 1 still doesn't necessarily preserve the $H$-dominance but not necessarily because it is too efficient. Here, the distribution obtained from the Hammond transfer defined in Proposition 1 can be reached by applying a Hammond transfer and an increment to the distribution obtained from the Hammond transfer defined in Proposition 3.

To end the demonstration, I must consider the case 2, discussed above, where all the Pigou-Dalton transfers preserve the $\bar{H}$-dominance.

### 2.4 Case 2

The first subcase of case 2 is handled with the following Proposition 4.

**Proposition 4** Suppose that $d_{K-1}$ doesn't preserve the $H$-dominance. Additionally suppose that $h_{min}(d_{K-1}) = h_K - 1$. Under these conditions, if $d'' \in \mathbb{N}^L$, is obtained from $d'$ by the following Hammond transfer:

$$\begin{aligned}
&n_h^d = n_h^{d'},\ \forall h \neq h_{K-1},\ h_{K-1}+1,\ h_K\\
&n_{h_{K-1}}^d = n_{h_{K-1}}^{d'} - 1,\ n_{h_{K-1}+1}^d = n_{h_{K-1}+1}^{d'} + 2\\
&n_{h_K}^d = n_{h_K}^{d'} - 1
\end{aligned}$$

Then $d''$ preserves the double dominance.

The distribution obtained from the Hammond transfer defined in Proposition 4 preserves the double dominance when $h_{min}(d_{K-1}) = h_K - 1$ but not necessarily when $h_K - 1 > h_{min}(d_{K-1})$.

When $h_K - 1 > h_{min}(d_{K-1})$, the transfer presented in Proposition 5 applies when there is at least one category in the set $\{h_{min}(d_{K-1}), ..., h_K - 1\}$, such that the value of the difference between the survival functions is greater or equal to $\frac{2}{n}$.

**Proposition 5** Suppose that $d_{K-1}$ doesn't preserve the $H$-dominance but does preserve the $\bar{H}$-dominance. Moreover suppose that $h_K - 1 > h_{min}(d_{K-1})$. Additionally, suppose there is a category $h \in \{h_{min}(d_{K-1}), ..., h_K - 1\}$ such that $\Delta_{(d',d)}\bar{F}(h) \geq \frac{2}{n}$. Under these conditions, by defining $e = max(\{h \in \{h_{min}(d_{K-1}), ..., h_K - 1\} | \Delta_{(d',d)}\bar{F}(h) \geq \frac{2}{n}\})$, if $d'' \in \mathbb{N}^L$, is obtained from $d'$ by the following Hammond transfer:

$$\begin{aligned}
&n_h^{d''} = n_h^{d'},\ \forall h \neq h_{K-1},\ h_{K-1}+1,\ e+1\\
&n_{h_{K-1}}^{d''} = n_{h_{K-1}}^{d'} - 1,\ n_{h_{K-1}+1}^{d''} = n_{h_{K-1}+1}^{d'} + 2\\
&n_{e+1}^{d''} = n_{e+1}^{d'} - 1
\end{aligned}$$

Then $d''$ preserves the double dominance.

As was the case for case 1, it can be observed that the Hammond transfer performed in Proposition 5 is quite similar to that performed in Proposition 4. In fact if the category $e$

of Proposition 5 is equal to $h_K - 1$, then the two transfers are the same. When $e < h_K - 1$ the transfer in Proposition 4 preserves the $H$-dominance but is too inefficient to preserve the $\bar{H}$-dominance. Consequently the transfer proposed in Proposition 5 is relatively more efficient. Indeed, the distribution obtained from the Hammond transfer defined in Proposition 5, can be reached by adding a series of increments to the distribution obtained by the Hammond transfer defined in Proposition 4. However the transfer in Proposition 5 can't be done when there isn't any category for which the value of the difference of the survival functions is greater or equal to two.

Hence the last case to be treated by the following Proposition 6 is when, for all categories $h \in \{h_{min}(d_{K-1}), ..., h_{K-1}\}, \Delta_{(d',d)}\bar{F}(h) = \frac{1}{n}$.

**Proposition 6** Suppose that $d_{K-1}$ doesn't preserve the $H$-dominance. Moreover suppose that $h_K - 1 > h_{min}(d_{K-1})$. Additionally, suppose that for all categories $h \in \{h_{min}(d_{K-1}), ..., h_K - 1\}$, $\Delta_{(d',d)}\bar{F}(h) = \frac{1}{n}$. Under these conditions, if $d'' \in \mathbb{N}^L$, is obtained from $d'$ by the following Hammond transfer:

$$\begin{aligned}
\text{If } h_{K-1} + 1 < h_{min}(d_{K-1}): \quad n_h^{d''} &= n_h^{d'},\ \forall h \neq h_{K-1},\ h_{K-1}+1,\ h_{min}(d_{K-1}),\ h_K \\
n_{h_{K-1}}^{d''} &= n_{h_{K-1}}^{d'} - 1,\ n_{h_{K-1}+1}^{d''} = n_{h_{K-1}+1}^{d'} + 1 \\
n_{h_{min}(d_{K-1})}^{d''} &= n_{h_{min}(d_{K-1})}^{d'} + 1,\ n_{h_K}^{d''} = n_{h_K}^{d'} - 1 \\
\text{If } h_{K-1} + 1 = h_{min}(d_{K-1}): \quad n_h^{d''} &= n_h^{d'},\ \forall h \neq h_{K-1},\ h_{K-1}+1,\ h_K \\
n_{h_{K-1}}^{d''} &= n_{h_{K-1}}^{d'} - 1,\ n_{h_{K-1}+1}^{d''} = n_{h_{K-1}+1}^{d'} + 2 \\
n_{h_K}^{d''} &= n_{h_K}^{d'} - 1
\end{aligned}$$

Then $d''$ preserves the double dominance.

The transfer of Proposition 6 is also quite similar to the transfer performed in Proposition 4. If the category $h_{min}(d_{K-1})$, is equal to $h_{K-1} + 1$ then the transfers performed in the two propositions are the same. When $h_{min}(d_{K-1}) > h_{K-1} + 1$ the transfer in Proposition 4 is too inefficient to preserve the $\bar{H}$-dominance. Consequently the transfer proposed in Proposition 6 is relatively more efficient. Indeed, the distribution obtained from the Hammond transfer defined in Proposition 6 can be reached by applying a series of increments to the distribution obtained from the Hammond transfer defined in Proposition 4.

# 3 Discussion on Median-Preserving Transfers

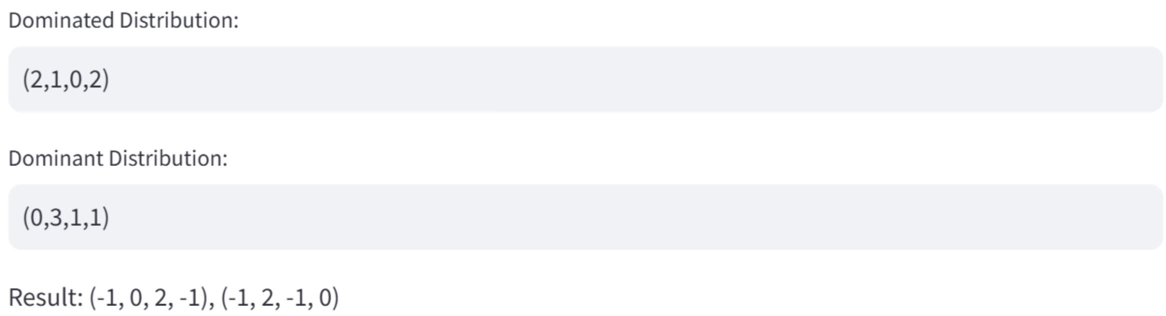
In this section, I am interested in whether the algorithm can be used to introduce a new dominance criterion to detect if a distribution can be obtained from another by a finite sequence of median-preserving transfers. The definition of a median-preserving transfer that I use is straightforward: *A distribution $d$ is obtained from a distribution $d'$ by a median-preserving transfer if and only if $d$ is obtained from $d'$ by a Hammond transfer and both distributions have the same median.* In the context of this paper, a plausible conjecture would consist of comparing the $H$-curve and the $\bar{H}$-curve of distributions with the same median.

**Conjecture** Let $d$ and $d'$ be two distinct distributions in $\mathbb{N}^L$. The two following statements are equivalent:

$(i)$ $d$ can be obtained from $d'$ by a finite series of median-preserving transfers.

$(ii)$ $d$ and $d'$ have the same median. And, for all $h \in \{1, ..., L-1\}$ $\Delta_{(d',d)}H(h) \geq 0$ and $\Delta_{(d',d)}\bar{H}(h) \geq 0$

I show that this conjecture is false by providing the following counter-example: take $d' = (2,1,0,2)$ and $d = (0,3,1,1)$. $d'$ and $d$ have the same median, the category 2. Moreover, $d$ can be obtained from $d'$ by a finite series of Hammond transfers. However, in this series, there is a Hammond transfer that doesn't preserve the median. See below the entire series provided by the algorithm.




From the algorithm, the series is composed of two transfers. The first one consists of transferring one individual from category 1 to category 3 and another individual from category 4 to category 3, which leads to the distribution $d'' = (1,1,2,1)$, for which the median is not category 2. Because the number of categories and individuals is relatively small in this example, it is easy to see that we must go through the distribution $d''$ to reach the distribution $d$ by a series of Hammond transfers. Hence, the median must change at some point in the series of Hammond transfers.

# 4 Conclusion

Although Gravel et al. (2021) showed that the $H$ and $\bar{H}$-dominance of one distribution over another is equivalent to the possibility of going from the dominated distribution to the dominant distribution by a finite sequence of Hammond transfers, these authors could not directly establish the equivalence and had to resort to indirect theorems from convex analysis that allow for the possibility for adding phantom agents to both distributions. The present article is the first to prove directly the possibility of reaching the dominant from the dominated distribution by a series of Hammond transfers only.

This article developed new insights into the relationship between the dominance criterion introduced by Gravel et al. (2021) and Hammond's notion of transfer. I hope these insights will serve for future research. One direction for this research has been discussed at the end of the last section. It concerns median preserving Hammond transfers. Hammond transfers preserving the median are particularly interesting. Several approaches to ordinal inequality measurement (see, e.g. Alison and Foster (2004)) have argued that reducing inequality should preserve the median. However, none of them define inequality reduction as a median preserving transfer. A more in-depth study of median preserving transfer could lay the foundations for a median-based approach to measure inequality and provide new interesting tools to compare with traditional mean-based tools.

# Appendix A

*Technical Appendix*

In this part of the appendix, I provide several preliminary results that are essential in demonstrating the main result.

The Pigou-Dalton transfers discussed in section 2.1 are not formally defined. I define them now:

**Definition 6** ($d_k$ *distributions*) For $k \in \{0, ..., K-1\}$, a distribution $d_k \in \mathbb{N}^L$ is defined as follows:

$$\begin{aligned}
\text{If } h_k + 1 < h_{k+1} - 1: \quad n_h^{d_k} &= n_h^{d'}, \forall h \neq h_k, h_k+1, h_{k+1}-1, h_{k+1} \\
n_{h_k}^{d_k} &= n_{h_k}^{d'} - 1, n_{h_k+1}^{d_k} = n_{h_k+1}^{d'} + 1 \\
n_{h_{k+1}-1}^{d} &= n_{h_{k+1}-1}^{d'} + 1, n_{h_{k+1}}^{d} = n_{h_{k+1}}^{d'} - 1 \\
\text{If } h_k + 1 = h_{k+1} - 1: \quad n_h^{d_k} &= n_h^{d'}, \forall h \neq h_k, h_k+1, h_{k+1} \\
n_{h_k}^{d_k} &= n_{h_k}^{d'} - 1, n_{h_k+1}^{d_k} = n_{h_k+1}^{d'} + 2 \\
n_{h_{k+1}}^{d} &= n_{h_{k+1}}^{d'} - 1
\end{aligned}$$

From properties 1,2 and 3 these Pigou-Dalton transfers are always feasible.

The following Lemmas are extensively used to prove Propositions 1 to 6.

**Lemma 1**

$\forall h \in \{2; ...; L\}$ :

$$\Delta_{(d',d)} H(h) = \sum_{i=1}^{h-1} \Delta_{(d',d)} H(i) + \Delta_{(d',d)} F(h)$$

and

$\forall h \in \{1; ...; L-1\}$ :

$$\Delta_{(d',d)} \bar{H}(h) = \sum_{i=h+1}^{L} \Delta_{(d',d)} \bar{H}(i) + \Delta_{(d',d)} \bar{F}(h)$$

**Proof** I prove the first equality by induction.

I start with the category 2:

$$\begin{aligned}
\Delta_{(d',d)} H(2) &= H(d', 2) - H(d, 2) \\
&= F(d', 1) + F(d', 2) - F(d, 1) - F(d, 2) \\
&= \Delta_{(d',d)} H(1) + \Delta_{(d',d)} F(2)
\end{aligned}$$

Suppose by induction that for a category $h \in \{2, ..., L-1\}$ the equality of Lemma 1 is verified, then in $h+1$:

$$\begin{aligned}
\Delta_{(d',d)} H(h+1) &= \sum_{i=1}^{h} 2^{h-i} F(d', i) + F(d', h+1) - \sum_{i=1}^{h} 2^{h-i} F(d, i) - F(d, h+1) \\
&= 2H(d', h) - F(d', h) - 2H(d, h) + F(d, h) + \Delta_{(d',d)} F(h+1) \\
&= 2 \sum_{i=1}^{h-1} \Delta_{(d',d)} H(i) + 2\Delta_{(d',d)} F(h) - \Delta_{(d',d)} F(h) + \Delta_{(d',d)} F(h+1) \\
&= \sum_{i=1}^{h-1} \Delta_{(d',d)} H(i) + \sum_{i=1}^{h-1} \Delta_{(d',d)} H(i) + \Delta_{(d',d)} F(h) + \Delta_{(d',d)} F(h+1) \\
&= \sum_{i=1}^{h} \Delta_{(d',d)} H(i) + \Delta_{(d',d)} F(h+1)
\end{aligned}$$

The proof of the second equality of Lemma 1 is similar, it is then left to the reader.

**Lemma 2.** Suppose the distribution $d_k$ obtained from $d'$ by means of a Pigou-Dalton transfer, with $k \in \{0, ..., K-1\}$, doesn't preserve the $\bar{H}$-dominance, such that $h_0 \leq h_{max}(d_k) - 1$. Then for any category $h \in \{h_{max}(d_k) + 2, ..., h_K - 1\}$, the following equality holds:

$$\Delta_{(d',d)}H(h) = 2^{h-h_{max}(d_k)} \sum_{i=h_0}^{h_{max}(d_k)-1} \Delta_{(d',d)}H(i) + \sum_{i=h_{max}(d_k)}^{h} \Delta_{(d',d)}F(i) + \sum_{i=h_{max}(d_k)}^{h-2} 2^{h-2-i} \sum_{j=h_{max}(d_k)}^{i} \Delta_{(d',d)}F(j)$$

**Proof** I proceed by induction.

I start with the category $h_{max}(d_k) + 2$, using Lemma 1:

$$\begin{aligned}
\Delta_{(d',d)}H(h_{max}(d_k)+2) &= \sum_{i=h_0}^{h_{max}(d_k)+1} \Delta_{(d',d)}H(i) + \Delta_{(d',d)}F(h_{max}(d_k)+2) \\
&= \sum_{i=h_0}^{h_{max}(d_k)} \Delta_{(d',d)}H(i) + \Delta_{(d',d)}H(h_{max}(d_k)+1) + \Delta_{(d',d)}F(h_{max}(d_k)+2) \\
&= 2\sum_{i=h_0}^{h_{max}(d_k)} \Delta_{(d',d)}H(i) + \Delta_{(d',d)}F(h_{max}(d_k)+1) + \Delta_{(d',d)}F(h_{max}(d_k)+2) \\
&= 2\sum_{i=h_0}^{h_{max}(d_k)-1} \Delta_{(d',d)}H(i) + 2\Delta_{(d',d)}H(h_{max}(d_k)) + \sum_{i=h_{max}(d_k)+1}^{h_{max}(d_k)+2} \Delta_{(d',d)}F(i) \\
&= 4\sum_{i=h_0}^{h_{max}(d_k)-1} \Delta_{(d',d)}H(i) + 2\Delta_{(d',d)}F(h_{max}(d_k)) + \sum_{i=h_{max}(d_k)+1}^{h_{max}(d_k)+2} \Delta_{(d',d)}F(i) \\
&= 4\sum_{i=h_0}^{h_{max}(d_k)-1} \Delta_{(d',d)}H(i) + \Delta_{(d',d)}F(h_{max}(d_k)) + \sum_{i=h_{max}(d_k)}^{h_{max}(d_k)+2} \Delta_{(d',d)}F(i)
\end{aligned}$$

Suppose by induction that for a category $h \in \{h_{max}(d_k) + 2, ..., h_K - 2\}$ the equality of Lemma 2 is verified, then in $h + 1$:

$$\begin{aligned}
\Delta_{(d',d)}H(h+1) &= \sum_{i=1}^{h} 2^{h-i}\Delta_{(d',d)}F(i) + \Delta_{(d',d)}F(h+1) \\
&= \sum_{i=1}^{h} 2^{h-i}\Delta_{(d',d)}F(i) + \Delta_{(d',d)}F(h+1) + \Delta_{(d',d)}F(h) - \Delta_{(d',d)}F(h) \\
&= 2\Delta_{(d',d)}H(h) + \Delta_{(d',d)}F(h+1) - \Delta_{(d',d)}F(h)
\end{aligned}$$

By replacing $\Delta_{(d',d)}H(h)$ by the equality in Lemma 2, it gives:

$$\Delta_{(d',d)}H(h+1) = 2^{h+1-h_{max}(d_k)} \sum_{i=h_0}^{h_{max}(d_k)-1} \Delta_{(d',d)}H(i) + \sum_{i=h_{max}(d_k)}^{h+1} \Delta_{(d',d)}F(i) + \sum_{i=h_{max}(d_k)}^{h-1} 2^{h-1-i} \sum_{j=h_{max}(d_k)}^{i} \Delta_{(d',d)}F(j)$$

**Lemma 3** For any category $h \geq h_0 + 2$, the following equality holds:

$$\Delta_{(d',d)} H(h) = \sum_{i=h_0}^{h} \Delta_{(d',d)} F(i) + \sum_{i=h_0}^{h-2} 2^{h-2-i} \sum_{j=h}^{i} \Delta_{(d',d)} F(j)$$

**Proof** I proceed by induction. I start with the category $h_0 + 2$:

$$\begin{aligned}
\Delta_{(d',d)} H(h_0+2) &= \sum_{i=h_0}^{h_0+1} 2^{h_0+1-i} \Delta_{(d',d)} F(i) + \Delta_{(d',d)} F(h_0+2) \\
&= 2\Delta_{(d',d)} F(h_0) + \sum_{i=h+1}^{h_0+2} \Delta_{(d',d)} F(i)
\end{aligned}$$

Suppose by induction that for a category $h \in \{h_0+2, ..., L-1\}$ the equality of Lemma 3 is verified, then in $h+1$:

$$\begin{aligned}
\Delta_{(d',d)} H(h+1) &= 2\Delta_{(d',d)} H(h) + \Delta_{(d',d)} F(h+1) - \Delta_{(d',d)} F(h) \\
&= 2\sum_{i=h_0}^{h} \Delta_{(d',d)} F(i) + \sum_{i=h_0}^{h-2} 2^{h-1-i} \sum_{j=h}^{i} \Delta_{(d',d)} F(j) + \Delta_{(d',d)} F(h+1) - \Delta_{(d',d)} F(h) \\
&= \sum_{i=h_0}^{h+1} \Delta_{(d',d)} F(i) + \sum_{i=h_0}^{h-1} 2^{h-1-i} \sum_{j=h}^{i} \Delta_{(d',d)} F(j)
\end{aligned}$$

**Lemma 4** For any category $h \leq h_K - 3$, the following equality holds:

$$\Delta_{(d',d)} \bar{H}(h) = \sum_{i=h}^{h_K-1} \Delta_{(d',d)} \bar{F}(i) + \sum_{i=h+2}^{h_K-1} 2^{i-h-2} \sum_{j=i}^{h_K-1} \Delta_{(d',d)} \bar{F}(j)$$

**Proof** I proceed by induction. I start with the category $h_K - 3$:

$$\begin{aligned}
\Delta_{(d',d)} \bar{H}(h_K-3) &= 2\Delta_{(d',d)} \bar{F}(h_K-1) + \Delta_{(d',d)} \bar{F}(h_K-2) + \Delta_{(d',d)} \bar{F}(h_K-3) \\
&= \sum_{i=h_K-3}^{h_K-1} \Delta_{(d',d)} \bar{F}(i) + \Delta_{(d',d)} \bar{F}(h_K-1)
\end{aligned}$$

Suppose by induction that for a category $h \in \{1, ..., h_K-3\}$ the equality of Lemma 4 is verified, then in $h-1$:

$$\begin{aligned}
\Delta_{(d',d)} \bar{H}(h-1) &= \sum_{i=h}^{h_K-1} 2^{i-h} \Delta_{(d',d)} \bar{F}(i) + \Delta_{(d',d)} \bar{F}(h-1) \\
&= \sum_{i=h}^{h_K-1} 2^{i-h} \Delta_{(d',d)} \bar{F}(i) + \Delta_{(d',d)} \bar{F}(h-1) + \Delta_{(d',d)} \bar{F}(h) - \Delta_{(d',d)} \bar{F}(h) \\
&= 2\Delta_{(d',d)} \bar{H}(h) + \Delta_{(d',d)} \bar{F}(h-1) - \Delta_{(d',d)} \bar{F}(h) \\
&= 2\sum_{i=h}^{h_K-1} \Delta_{(d',d)} \bar{F}(i) + \sum_{i=h+2}^{h_K-1} 2^{i-h-1} \sum_{j=i}^{h_K-1} \Delta_{(d',d)} \bar{F}(j) + \Delta_{(d',d)} \bar{F}(h-1) - \Delta_{(d',d)} \bar{F}(h) \\
&= \sum_{i=h-1}^{h_K-1} \Delta_{(d',d)} \bar{F}(i) + \sum_{i=h+1}^{h_K-1} 2^{i-h-2} \sum_{j=i}^{h_K-1} \Delta_{(d',d)} \bar{F}(j)
\end{aligned}$$

**Lemma 5** Suppose the distribution $d_k$ obtained from $d'$ by means of a Pigou-Dalton transfer, with $k \in \{0, ..., K-1\}$, doesn't preserve the $H$-dominance, such that $h_K - 1 > h_{min}(d_k)$. Then, for any category $h \in \{h_0, ..., h_{min}(d_t) - 2\}$:

$$\Delta_{(d',d)}\bar{H}(h) = 2^{h_{min}(d_{t-1})-h} \sum_{i=h_{min}(d_{t-1})+1}^{h_K-1} \Delta_{(d',d)}\bar{H}(i) + \sum_{i=h}^{h_{min}(d_{t-1})} \Delta_{(d',d)}\bar{F}(i) + \sum_{i=h+2}^{h_{min}(d_{t-1})} 2^{i-h-2} \sum_{j=i}^{h_{min}(d_{t-1})} \Delta_{(d',d)}\bar{F}(j)$$

**Proof** Similar to the four previous Lemmas.

I now prove all the properties of the set $\{h_0, ..., h_K\}$ discussed in sections (2.1) and (2.2).

**Property 1** $\forall k \in \{0, ..., K\},\ n_{h_k}^{d'} \geq 1$

**Proof** Note that for all $h < h_0,\ \Delta_{(d',d)}F(h) = 0$. Hence in $h_0$:

$$\Delta_{(d',d)}F(h_0) = n_{h_0}^{d'} - n_{h_0}^{d} > 0 \implies n_{h_0}^{d'} > 0$$

In $h_k$, with $k \in \{1, ..., K\}$, $\Delta_{(d',d)}F(h_k) \geq 0$ and $\Delta_{(d',d)}F(h_k - 1) < 0$, knowing that:

$$\Delta_{(d',d)}F(h_k) = \Delta_{(d',d)}F(h_k - 1) + n_{h_k}^{d'} - n_{h_k}^{d} \geq 0$$

Then we must have $n_{h_k}^{d'} - n_{h_k}^{d} > 0 \implies n_{h_k}^{d'} > 0$

**Property 2** $K + 1 \geq 2$

**Proof** Here I prove that the following sets are non empty:

$$A = \{h \in \{1, ..., L\} \mid \Delta_{(d',d)}F(h) > 0\}$$
$$B = \{h \in \{2, ..., L\} \mid \Delta_{(d',d)}F(h) \geq 0 \text{ and } \Delta_{(d',d)}F(h-1) < 0\}$$

Denote $a = min(\{h \in \{1, ..., L\} | \Delta_{(d',d)}H(h) > 0\})$.

Because $d$ dominates $d'$, for all categories $h \in \{1, ..., L\},\ \Delta_{(d',d)}H(h) \geq 0$. Hence by definition of $a$, for all categories $h < a,\ \Delta_{(d',d)}H(h) = 0$. Using Lemma 1:

$$\Delta_{(d',d)}H(a) = \Delta_{(d',d)}F(a)$$

Hence $\Delta_{(d',d)}F(a) > 0$, so the set $A$ is non empty.

Denote $b = max(\{h \in \{1, ..., L\} | \Delta_{(d',d)}\bar{H}(h) > 0\})$.

For all categories $h > b,\ \Delta_{(d',d)}\bar{H}(h) = 0$. Using Lemma 1, we get:

$$\Delta_{(d',d)}\bar{H}(b) = \Delta_{(d',d)}\bar{F}(b) = -\Delta_{(d',d)}F(b)$$

Hence $\Delta_{(d',d)}F(b) < 0$, so the set $B$ is non empty because by definition of $b$, $\Delta_{(d',d)}F(b + 1) \geq 0$. Indeed, if $\Delta_{(d',d)}F(b + 1) < 0$ we would have $\Delta_{(d',d)}\bar{H}(b + 1) > 0$ which is impossible by definition of $b$. To conclude note that $|A| + |B| = K + 1 \geq 2$.

**Property 3** $\forall k \in \{0, ..., K - 1\},\ h_K \leq h_{k+1} - 2$

**Proof** Suppose by contradiction that there is a $k \in \{0, ..., K-1\}$ such that $h_k \geq h_{k+1} - 1$. In this case because $h_k < h_{k+1}$ we must have $h_k = h_{k+1} - 1$. However in $h_{k+1} - 1$, $\Delta_{(d',d)}F(h_{k+1} - 1) < 0$ which is in contradiction with the definition of $h_k$.

**Property 4** $\forall k \in \{0, ..., K-1\}$ and $\forall h \in \{h_k, ..., h_{k+1} - 1\}$, if $\Delta_{(d',d)}F(h) \geq 0$ then $\forall h' \in \{h_k, ..., h\}$, $\Delta_{(d',d)}F(h') \geq 0$

**Proof** Suppose by contradiction that there is a $k \in \{0, ..., K-1\}$ such that for a category $h \in \{h_k, ..., h_{k+1}-1\}$, with $\Delta_{(d',d)}F(h) \geq 0$ and there is a category $h' \in \{h_k, ..., h\}$, such that $\Delta_{(d',d)}F(h') < 0$. It is then possible to define the following category:

$$a = max(\{h'' \in \{h_k, ..., h\} | \Delta_{(d',d)}F(h'') < 0\})$$

By definition of $a$, $\Delta_{(d',d)}F(a+1) \geq 0$, moreover by definition of the set $\{h_1, ..., h_K\}$, we must have $a + 1 = h_{k+1}$ which is impossible because $a + 1 \leq h < h_{k+1}$.

By definition of $b$, $\Delta_{(d',d)}\bar{F}(b-1) \geq \frac{1}{n} \implies \Delta_{(d',d)}F(b-1) \leq -\frac{1}{n}$ hence we must have $b = h_{k+1}$, which is impossible.

Before proving the last properties of the set $\{h_0, ..., h_K\}$, two Claims are needed, namely Claims 1 and 2. Which establish that if a Hammond transfer preserves the double dominance for all the categories in the set $\{h_0, ..., h_K - 1\}$ then the dominance is also preserved for all categories in the set $\{1, ..., L\}$. In other words, the analysis can be restricted to the set $\{h_0, ..., h_K - 1\}$.

**Claim 1** Suppose a distribution $d''$ obtained from $d'$ by means of a Hammond transfer, such that for all categories $h < h_0$, $\Delta_{(d',d'')}F(h) = 0$ and for all categories $h > h_K - 1$, $\Delta_{(d',d'')}F(h) = 0$. If for all categories $h \in \{h_0, ..., h_K - 1\}$, $\Delta_{(d',d'')}\bar{H}(h) \leq \Delta_{(d',d)}\bar{H}(h)$ then for all categories $h \in \{1, ..., L\}$, we have $\Delta_{(d',d'')}\bar{H}(h) \leq \Delta_{(d',d)}\bar{H}(h)$.

**Proof** Note that for all categories $h \in \{h_K, ..., L\}$:

$$\Delta_{(d',d)}\bar{H}(h) = \Delta_{(d',d'')}\bar{H}(h) = 0$$

In $h_0 - 1$, knowing that $\forall h < h_0$, $\Delta_{(d',d)}F(h) = 0$ and by using Lemma 1:

$$\Delta_{(d',d)}\bar{H}(h_0 - 1) = \sum_{i=h_0}^{h_K - 1} \Delta_{(d',d)}\bar{H}(i)$$

Similarly:

$$\Delta_{(d',d'')}\bar{H}(h_0 - 1) = \sum_{i=h_0}^{h_K - 1} \Delta_{(d',d'')}\bar{H}(i)$$

Because for all categories $h \in \{h_0, ..., h_K - 1\}$,:

$$\Delta_{(d',d'')}\bar{H}(h) \leq \Delta_{(d',d)}\bar{H}(h)$$

It is clear that:

$$\Delta_{(d',d'')}\bar{H}(h_0 - 1) \leq \Delta_{(d',d)}\bar{H}(h_0 - 1)$$

Suppose by induction for a category $h \in \{2, ..., h_0 - 1\}$ that we have for all categories $h' \in \{h, ..., h_K - 1\}$:

$$\Delta_{(d',d'')}\bar{H}(h') \leq \Delta_{(d',d)}\bar{H}(h')$$

Then in $h - 1$:

$$\Delta_{(d',d'')}\bar{H}(h-1) = \sum_{i=h}^{h_K-1} \Delta_{(d',d'')}\bar{H}(i) \leq \sum_{i=h}^{h_K-1} \Delta_{(d',d)}\bar{H}(i) = \Delta_{(d',d)}\bar{H}(h-1)$$

**Claim 2** Suppose a distribution $d''$ obtained from $d'$ by means of a Hammond transfer, such that for all $h < h_0$, $\Delta_{(d',d'')}F(h) = 0$ and for all categories $h > h_K - 1$, $\Delta_{(d',d'')}F(h) = 0$. If for all categories $h \in \{h_0, ..., h_K - 1\}$, $\Delta_{(d',d'')}H(h) \leq \Delta_{(d',d)}H(h)$ then for all categories $h \in \{1, ..., L\}$, we have $\Delta_{(d',d'')}H(h) \leq \Delta_{(d',d)}H(h)$.

**Proof** Similar to Claim 1.

**Property 5** For all $k \in \{1, ..., K - 1\}$, $h_{max}(d_k) \in \{h_0, ..., h_{k+1} - 2\}$

**Proof** Note that for all $h \geq h_{k+1}$, $\bar{F}(d', h) = \bar{F}(d_k, h)$. Hence $\Delta_{(d',d_k)}\bar{H}(h) = 0$. Moreover in $h_{k+1} - 1$, $\Delta_{(d',d_k)}\bar{F}(h_{k+1} - 1) = \frac{1}{n}$. Using Lemma 1, we have:

$$\Delta_{(d',d_k)}\bar{H}(h_{k+1} - 1) = \frac{1}{n}$$

It can be shown that:

$$\Delta_{(d',d)}H(h_{k+1} - 1) \geq \frac{1}{n}$$

First if $k = K - 1$:

$$\Delta_{(d',d)}H(h_K - 1) = \Delta_{(d',d)}F(h_K - 1) \geq \frac{1}{n}$$

Finally if $k < K - 1$:

$$\begin{aligned}
\Delta_{(d',d)}\bar{H}(h_{k+1} - 1) &= \sum_{i=h_{k+1}}^{L} \Delta_{(d',d)}\bar{H}(i) + \Delta_{(d',d)}\bar{F}(h_{k+1} - 1) \\
\Delta_{(d',d)}\bar{H}(h_{k+1} - 1) &\geq \Delta_{(d',d)}\bar{H}(h_K - 1) + \Delta_{(d',d)}\bar{F}(h_{k+1} - 1) \\
\Delta_{(d',d)}\bar{H}(h_{k+1} - 1) &\geq \frac{1}{n}
\end{aligned}$$

Hence $h_{max}(d_k) < h_{k+1} - 1$. Finally, note that for all $h \leq h_k$, $\bar{F}(d', h) = \bar{F}(d_k, h)$, hence, from Claim 1, $h_{max}(d_k)$ must be greater or equal to $h_0$.

**Property 6** For all $k \in \{1, ..., K - 1\}$, $\Delta_{(d',d)}F(h_{max}(d_k)) \geq \frac{1}{n}$

Property 6 will be shown below to be implied by Claim 4 (to be stated).

**Property 7** For all $k \in \{0, ..., K - 1\}$ $h_{min}(d_k) \in \{h_k + 1, ..., h_K - 1\}$

**Proof** Note that for all $h < h_k$, $F(d', h) = F(d_k, h)$, hence $\Delta_{(d',d_k)}H(h) = 0$. Moreover in $h_k$, $\Delta_{(d',d_k)}F(h_k) = \frac{1}{n}$. Using Lemma 1:

$$\Delta_{(d',d_k)}H(h_k) = \frac{1}{n}$$

It can be shown that:

$$\Delta_{(d',d)}H(h_k) \geq \frac{1}{n}$$

First if $k = 0$:

$$\Delta_{(d',d)}H(h_0) = \Delta_{(d',d)}F(h_0) \geq \frac{1}{n}$$

Finally if $k > 0$:

$$\Delta_{(d',d)}H(h_k) = \sum_{i=1}^{h_k-1} \Delta_{(d',d)}H(i) + \Delta_{(d',d)}F(h_k)$$
$$\Delta_{(d',d)}H(h_k) \geq \Delta_{(d',d)}H(h_0) + \Delta_{(d',d)}F(h_k)$$
$$\Delta_{(d',d)}H(h_k) \geq \frac{1}{n}$$

Hence $h_{min}(d_k) > h_k$. Finally, note that for all $h \geq h_{k+1}$, $F(d',h) = F(d_k,h)$, hence, from Claim 2, $h_{min}(d_k)$ must be smaller or equal to $h_K - 1$.

**Property 8** For all $k \in \{0,...,K-1\}$ $\Delta_{(d',d)}\bar{F}(h_{min}(d_k)) \geq \frac{1}{n}$

Property 8 will be shown to be implied by Claim 3 stated and proved below.

In order to prove Claim 3, the following two tables provide the value of the variations in the $H$ and $\bar{H}$-curves that result from performing specific Pigou-Dalton transfers.

| $h$ | $\Delta_{(d',d_k)}H(h)$ |
|---|---|
| (If any) $h \leq h_k - 1$ | $0$ |
| $h_k$ | $\frac{1}{n}$ |
| $h_k + 1 \leq h \leq h_k - 2$ | $\frac{2^{h-h_{k-1}-1}}{n}$ |
| $h_k$ | $\frac{2^{h_t-h_{k-1}-2}}{n} - \frac{1}{n}$ |
| $h \geq h_{k+1}$ | $\frac{2^{h-h_k-1}}{n} - \frac{2^{h-h_{k+1}}}{n}$ |

**Table 1:** $\Delta_{(d',d_k)}H(h)$

| $h$ | $\Delta_{(d',d_k)}\bar{H}(h)$ |
|---|---|
| For $h \geq h_{k+1}$ | $0$ |
| $h_{k+1} - 1$ | $\frac{1}{n}$ |
| $h_k + 1 \leq h \leq h_{k+1} - 2$ | $\frac{2^{h_{k+1}-2-h}}{n}$ |
| $h_k$ | $\frac{2^{h_{k+1}-2-h_t}}{n} - \frac{1}{n}$ |
| (If any) $h \leq h_k - 1$ | $\frac{2^{h_{k+1}-2-h}}{n} - \frac{2^{h_k-1-h}}{n}$ |

**Table 2:** $\Delta_{(d',d_k)}\bar{H}(h)$

These two tables enable the statement and proof of the following important Claim. Claim 3 applies when a Pigou-Dalton transfer doesn't preserve the $\bar{H}$-dominance, it gives a minimal value to the differences between the cumulative functions of $d'$ and $d$.

**Claim 3** Suppose that the distribution $d_k$, with $k \in \{0,...,K-1\}$, doesn't preserve the $H$-dominance. If $h_{min}(d_k) < h_{k+1} - 1$ then:

$$\forall h \in \{h_k + 1,...,h_{min}(d_k)\}, \quad \sum_{i=h}^{h_{min}(d_k)} \Delta_{(d',d)}\bar{F}(i) \geq \frac{1}{n}$$

$$\text{And,} \quad \forall h \in \{h_0,...,h_k\}, \quad \sum_{i=h}^{h_{min}(d_k)} \Delta_{(d',d)}\bar{F}(i) \geq 0$$

Otherwise, if $h_{min}(d_k) \geq h_{k+1} - 1$ then:

$$\forall h \in \{h_k + 1, ..., h_{min}(d_k)\}, \quad \sum_{i=h}^{h_{min}(d_k)} \Delta_{(d',d)} \bar{F}(i) \geq \frac{2}{n}$$
$$\text{And,} \quad \forall h \in \{h_0, ..., h_k\}, \quad \sum_{i=h}^{h_{min}(d_k)} \Delta_{(d',d)} \bar{F}(i) \geq \frac{1}{n}$$

**Proof** Suppose $k \in \{0, ..., K-1\}$, by definition of $h_{min}(d_k)$):

$$\Delta_{(d',d)} H(h_{min}(d_k)) < \Delta_{(d',d_k)} H(h_{min}(d_k)) \tag{1}$$

Rewriting the left term of this inequality, using Lemma 1 gives:

$$\sum_{i=h_0}^{h_{min}(d_k)-1} \Delta_{(d',d)} H(i)) + \Delta_{(d',d)} F(h_{min}(d_k)) < \Delta_{(d',d_k)} H(h_{min}(d_k)) \tag{2}$$

From Table 1, for all $h < h_k$:

$$\Delta_{(d',d_k)} H(h) = 0$$

Moreover in $h_k$:

$$\Delta_{(d',d)} H(h_k) \geq \frac{1}{n} = \Delta_{(d',d_k)} H(h_k)$$

Hence $h_{min}(d_k) \in \{h_k + 1, ..., h_K - 1\}$.

First, I treat the case when $h_0 + 1 = h_{min}(d_k)$. In this case, the previous Inequality (2) can be written as follows:

$$\Delta_{(d',d)} H(h_0) + \Delta_{(d',d)} F(h_0 + 1) < \Delta_{(d',d)} H(d_k, h_0 + 1) \tag{3}$$

Note that because $h_0 + 1 = h_{min}(d_k)$, we must have $i = 0$, and $\Delta_{(d',d)} H(d_k, h_i + 1) = \frac{1}{n}$. So Inequality (3) becomes:

$$\Delta_{(d',d)} H(h_0) + \Delta_{(d',d)} F(h_{min}(d_k)) < \frac{1}{n} \tag{4}$$

Moreover, $\Delta_{(d',d)} H(h_0) = \Delta_{(d',d)} F(h_0) \geq \frac{1}{n}$, hence from Inequality (4) we get:

$$0 < \Delta_{(d',d)} \bar{F}(h_0 + 1)$$
$$\implies \frac{1}{n} \leq \Delta_{(d',d)} \bar{F}(h_0 + 1)$$

Finally, if I come back to Inequality (3), the following inequality completes the proof for the specific case when $h_0 + 1 = h_{min}(d_k)$

$$\Delta_{(d',d)} \bar{F}(h_0) + \Delta_{(d',d)} \bar{F}(h_{min}(d_k)) > -\frac{1}{n}$$
$$\implies \Delta_{(d',d)} \bar{F}(h_0) + \Delta_{(d',d)} \bar{F}(h_{min}(d_k)) \geq 0$$

I turn to the general case when $h_{min}(d_k) \in \{h_0 + 2, ..., h_K - 1\}$. Using Lemma 1 on the left term of Inequality (2), we get:

$$2 \sum_{i=h_0}^{h_{min}(d_k)-2} \Delta_{(d',d)} H(i) + \sum_{i=h_{min}(d_k)-1}^{h_{min}(d_k)} \Delta_{(d',d)} F(k) < \Delta_{(d',d)} H(d_k, h_{min}(d_k)) \tag{5}$$

It is possible to generalise the Inequality (5) for any category $u \in \{h_0, ..., h_{min}(d_k) - 2\}$:

$$\sum_{i=h_0}^{u} \Delta_{(d',d)} H(i) + \sum_{i=u}^{h_{min}(d_k)-2} \sum_{j=h_0}^{i} \Delta_{(d',d)} H(i) + \sum_{i=u+1}^{h_{min}(d_k)} \Delta_{(d',d)} F(i) < \Delta_{(d',d)} H(d_k, h_{min}(d_k)) \tag{6}$$

From Inequality (6), when $u = h_0$, I can develop the left term and obtain:

$$\sum_{i=h_0}^{h_{min}(d_k)-2} \sum_{j=h_0}^{i} \Delta_{(d',d)} H(j) + \sum_{i=h_0}^{h_{min}(d_k)} \Delta_{(d',d)} F(i) < \Delta_{(d',d)} H(d_k, h_{min}(d_k)) \tag{7}$$

From inequalities (2), (6) and (7), we write the following inequalities:

$$\sum_{i=h_0}^{h_{min}(d_k)-1} \Delta_{(d',d)} H(i)) - \Delta_{(d',d)} H(h_{min}(d_k)) < \Delta_{(d',d)} \bar{F}(h_{min}(d_k))$$

$$\sum_{i=h_0}^{u} \Delta_{(d',d)} H(i) + \sum_{i=u}^{h_{min}(d_k)-2} \sum_{j=h_0}^{i} \Delta_{(d',d)} H(j) - \Delta_{(d',d_{t-1})} H(h_{min}(d_k)) < \sum_{i=u+1}^{h_{min}(d_k)} \Delta_{(d',d)} \bar{F}(i)$$

$$\sum_{i=h_0}^{h_{min}(d_k)-2} \sum_{j=h_0}^{i} \Delta_{(d',d)} H(j) - \Delta_{(d',d_k)} H(h_{min}(d_k)) < \sum_{i=h_0}^{h_{min}(d_k)} \Delta_{(d',d)} \bar{F}(i)$$

By definition , for any category $h < h_{min}(d_k)$, $\Delta_{(d',d)} H(h) \geq \Delta_{(d',d_k)} H(h)$, hence, the previous system of inequalities become:

$$\sum_{i=h_0}^{h_{min}(d_k)-1} \Delta_{(d',d_k)} H(i) - \Delta_{(d',d_k)} H(h_{min}(d_k)) < \Delta_{(d',d)} \bar{F}(h_{min}(d_k)) \tag{8}$$

$$\sum_{i=h_0}^{u} \Delta_{(d',d_k)} H(i) + \sum_{i=u}^{h_{min}(d_k)-2} \sum_{j=h_0}^{i} \Delta_{(d',d_k)} H(j) - \Delta_{(d',d_k)} H(h_{min}(d_k)) < \sum_{i=u+1}^{h_{min}(d_k)} \Delta_{(d',d)} \bar{F}(i) \tag{9}$$

$$\sum_{i=h_0}^{h_{min}(d_k)-2} \sum_{j=h_0}^{i} \Delta_{(d',d_k)} H(j) - \Delta_{(d',d_k)} H(h_{min}(d_k)) < \sum_{i=h_0}^{h_{min}(d_k)} \Delta_{(d',d)} \bar{F}(i) \tag{10}$$

I first focus on the left hand side of Inequality (8). Applying Lemma 1:

$$\sum_{i=h_0}^{h_{min}(d_k)-1} \Delta_{(d',d_k)} H(i) - \Delta_{(d',d_k)} H(h_{min}(d_k)) = -\Delta_{(d',d_k)} F(h_{min}(d_k))$$

Moreover, for all possible $h_{min}(d_k) \in \{h_k+1, ..., h_K-1\}$, such that $h_{min}(d_k) \neq h_{k+1}-1$, $\Delta_{(d',d_k)} F(h_{min}(d_k)) = 0$, then from Inequality (8):

$$0 < \Delta_{(d',d)} \bar{F}(h_{min}(d_k))$$
$$\implies \frac{1}{n} \leq \Delta_{(d',d)} \bar{F}(h_{min}(d_k))$$

Otherwise if $h_{min}(d_k) = h_{k+1} - 1$, $\Delta_{(d',d_k)} F(h_{min}(d_k)) = -\frac{1}{n}$ and then:

$$\frac{1}{n} < \Delta_{(d',d)} \bar{F}(h_{min}(d_k))$$
$$\implies \frac{2}{n} \leq \Delta_{(d',d)} \bar{F}(h_{min}(d_k))$$

Let me now focus on the left hand side of Inequality (9). I prove the following equality by induction, for $u \in \{h_0, ..., h_{min}(d_k) - 2\}$:

$$\sum_{i=u}^{h_{min}(d_k)-2} \sum_{j=h_0}^{i} \Delta_{(d',d_k)} H(j) - \Delta_{(d',d_k)} H(h_{min}(d_k))$$
$$= -\Delta_{(d',d_k)} H(u+1) - \sum_{i=u+2}^{h_{min}(d_k)} \Delta_{(d',d_k)} F(i) \quad (11)$$

Applying (11) on the left hand side of Inequality (9):

$$\sum_{i=h_0}^{u} \Delta_{(d',d_k)} H(i) - \Delta_{(d',d_k)} H(u+1) - \sum_{i=u+2}^{h_{min}(d_k)} \Delta_{(d',d_k)} F(i) < \sum_{i=u+1}^{h_{min}(d_k)} \Delta_{(d',d)} \bar{F}(i)$$

Using Lemma 1, for any $u \in \{h_0, ..., h_{min}(d_k) - 2\}$:

$$- \sum_{i=u+1}^{h_{min}(d_k)} \Delta_{(d',d_k)} F(i) < \sum_{i=u+1}^{h_{min}(d_k)} \Delta_{(d',d)} \bar{F}(i) \quad (12)$$

Finally, replacing the left hand side of Inequality (10) with (11), I obtain:

$$-\Delta_{(d',d)} H(d_k, h_0 + 1) - \sum_{i=h_0+2}^{h_{min}(d_k)} \Delta_{(d',d_k)} F(i) < \sum_{i=h_0}^{h_{min}(d_k)} \Delta_{(d',d)} \bar{F}(i) \quad (13)$$

Using Lemma 1, I obtain from Inequality (13):

$$- \sum_{i=h_0}^{h_{min}(d_k)} \Delta_{(d',d_k)} F(i) < \sum_{i=h_0}^{h_{min}(d_k)} \Delta_{(d',d)} \bar{F}(i)$$

Claim 4 is symmetric to Claim 3. It applies when a Pigou-Dalton transfer doesn't preserve the $H$-dominance, it gives a minimal value to the differences between the survival functions of $d'$ and $d$.

**Claim 4** Suppose the distribution $d_k$ with $k \in \{0, ..., K-1\}$ doesn't preserve the $\bar{H}$-dominance. If $h_{max}(d_k) > h_k$ :

$$\forall h \in \{h_{max}(d_k), ..., h_{k+1} - 2\}, \quad \sum_{i=h_{max}(d_k)}^{h} \Delta_{(d',d)} F(i) \geq \frac{1}{n}$$
$$\text{And, } \forall h \in \{h_{k+1} - 1, ..., h_K - 1\}, \quad \sum_{i=h_{max}(d_k)}^{h} \Delta_{(d',d)} F(i) \geq 0$$

Otherwise, if $h_{max}(d_k) \leq h_k$:

$$\forall h \in \{h_{max}(d_k), ..., h_{k+1} - 2\}, \quad \sum_{i=h_{max}(d_k)}^{h} \Delta_{(d',d)} F(i) \geq \frac{2}{n}$$
$$\text{And, } \forall h \in \{h_{k+1} - 1, ..., h_K - 1\}, \quad \sum_{i=h_{max}(d_k)}^{h} \Delta_{(d',d)} F(i) \geq \frac{1}{n}$$

**Proof** Similar to Claim 3.

# Appendix B

*Proof of Propositions 1 to 6*

**Proposition 1**

Through this proof call $d''$, the distribution obtained from $d'$ by the Hammond transfer. I provide the variations generated by this transfer in Tables 3 and 4.

| $h$ | $\Delta_{(d',d'')}H(h)$ |
|---|---|
| (If any) $h \leq h_t - 1$ | $0$ |
| $h_t \leq h \leq h_{t+1} - 2$ | $\frac{2^{h-h_t}}{n}$ |
| $h = h_{t+1} - 1$ | $\frac{2^{h-h_t}}{n} - \frac{2}{n}$ |
| $h \geq h_{t+1}$ | $\frac{2^{h-h_t}}{n} - \frac{2^{h-h_{t+1}+2}}{n} + \frac{2^{h-h_{t+1}}}{n}$ |

**Table 3:** $\Delta_{(d',d'')}H(h)$

| $h$ | $\Delta_{(d',d'')}\bar{H}(h)$ |
|---|---|
| For $h \geq h_{t+1}$ | $0$ |
| $h_{t+1} - 1$ | $\frac{1}{n}$ |
| $h_t \leq h \leq h_{t+1} - 2$ | $0$ |
| (If any) $h \leq h_t - 1$ | $\frac{2^{h_t-1-h}}{n}$ |

**Table 4:** $\Delta_{(d',d'')}\bar{H}(h)$

Along the proof I suppose that $t \in \{0, ..., K-1\}$. Additionally suppose that if $t \geq 1$ then $d_{t-1}$ doesn't preserve the $\bar{H}$-dominance. Finally, recall that from Property 5 $h_{max}(d_t) \in \{h_0, ..., h_{t+1} - 2\}$, here I suppose that $h_{max}(d_t) \leq h_t$.

Under these conditions, I will first show that $d''$ preserves the $H$-dominance (**Part 1**) and then that it also preserves the $\bar{H}$-dominance (**Part 2**).

**Part 1, the H dominance.** From Table 3 and Claim 2, I only focus on categories $h \in \{h_t + 1, ..., h_K - 1\}$.

Indeed, for any categories strictly below $h_t$, $\Delta_{(d',d'')}H(h) = 0$, which is, by definition of the dominance, always smaller or equal to $\Delta_{(d',d)}H(h)$. Moreover $\Delta_{(d',d)}F(h_t) \geq 0$. Using Lemma 1, when $t > 0$:

$$\Delta_{(d',d)}H(h_t) = \sum_{i=h_0}^{h_t-1} \Delta_{(d',d)}H(i) + \Delta_{(d',d)}F(h_t) \geq \frac{1}{n} = \Delta_{(d',d'')}H(h_t)$$

And if $t = 0$:

$$\Delta_{(d',d)}F(h_0) = \Delta_{(d',d)}H(h_0) \geq \frac{1}{n} = \Delta_{(d',d'')}H(h_0)$$

Hence I will only focus on categories $h \in \{h_t + 1, .., h_K - 1\}$. Note that for these categories, $h \geq h_{max}(d_t) + 1$. I decompose this inequality into two possibilities:

$$\textit{(i)}\ h \geq h_{max}(d_t) + 2 \ \text{ or } \ \textit{(ii)}\ h = h_{max}(d_t) + 1$$

I also decompose *(i)* in two sub-cases:

$$\textit{(ia)}\ h \geq h_{max}(d_t) + 2 \text{ and } h_0 \leq h_{max}(d_t) - 1$$
$$\textit{(ib)}\ h \geq h_{max}(d_t) + 2 \text{ and } h_0 = h_{max}(d_t)$$

I start with the sub-case *(ia)*. By definition of $h_0$, $\Delta_{(d',d)}H(h_0) \geq \frac{1}{n}$. Hence for any category $h \in \{h_t+1, ..., h_K-1\}$:

$$2^{h-h_{max}(d_t)} \sum_{i=h_0}^{h_{max}(d_t)-1} \Delta_{(d',d)}H(i) \geq \frac{2^{h-h_{max}(d_t)}}{n} \tag{14}$$

Moreover from Claim 4, $\forall h \in \{h_{max}(d_t), ..., h_K-1\}$:

$$\sum_{i=h_{max}(d_t)}^{h} \Delta_{(d',d)}F(i) \geq 0 \tag{15}$$

Finally I can combine the previous inequalities (14) and (15) leading to the following inequality for any category $h \in \{h_t+1, ..., h_K-1\}$, such that $h \geq h_{max}(d_t)+2$:

$$\begin{aligned}2^{h-h_{max}(d_t)} \sum_{i=h_0}^{h_{max}(d_t)-1} \Delta_{(d',d)}H(i) + \sum_{i=h_{max}(d_t)}^{h} \Delta_{(d',d)}F(i) + \sum_{i=h_{max}(d_t)}^{h-2} 2^{h-2-i} \sum_{j=h_{max}(d_t)}^{i} \Delta_{(d',d)}F(j) \\ \geq \frac{2^{h-h_{max}(d_t)}}{n}\end{aligned}$$

Hence from Lemma 2, for any category $h \in \{h_t+1, ..., h_K-1\}$, such that $h \geq h_{max}(d_t)+2$:

$$\Delta_{(d',d)}H(h) \geq \frac{2^{h-h_{max}(d_t)}}{n} \tag{16}$$

Because in Proposition 1, the main assumption states that $h_{max}(d_t) \leq h_t$, the previous Inequality (16) leads to the following inequality, which concludes case (*ia*):

$$\Delta_{(d',d)}H(h) \geq \frac{2^{h-h_t}}{n} \geq \Delta_{(d',d'')}H(h)$$

I now turn to the case *(ib)*, where $h \geq h_{max}(d_t)+2$ and $h_0 = h_{max}(d_t)$. Because $h_0 = h_{max}(d_t) \leq h_t$, from Claim 4:

$$\forall h \in \{h_0, ..., h_{t+1}-2\}, \ \sum_{i=h_0}^{h} \Delta_{(d',d)}F(i) \geq \frac{2}{n}$$

$$\forall h \in \{h_{t+1}-1, ..., h_K-1\}, \ \sum_{i=h_0}^{h} \Delta_{(d',d)}F(i) \geq \frac{1}{n}$$

From Lemma 3, for any category $h \in \{h_t+1, ..., h_K-1\}$, such that $h \geq h_0+2$:

$$\Delta_{(d',d)}H(h) = \sum_{i=h_0}^{h} \Delta_{(d',d)}F(i) + \sum_{i=h_0}^{h-2} 2^{h-2-i} \sum_{j=h_0}^{i} \Delta_{(d',d)}F(j)$$

From Lemma 3 and Claim 4, the following inequality holds for any category $h \in \{h_t+1, ..., h_{t+1}-2\}$, such that $h \geq h_0+2$:

$$\begin{aligned}\sum_{i=h_0}^{h} \Delta_{(d',d)}F(i) + \sum_{i=h_0}^{h-2} 2^{h-2-i} \sum_{j=h_0}^{i} \Delta_{(d',d)}F(j) \geq \frac{2}{n} + \sum_{i=h_0}^{h-2} \frac{2^{h-1-i}}{n} \\ \geq \frac{2^{h-h_0}}{n}\end{aligned}$$

Moreover from Table 3, for any $h \in \{h_t+1, ..., h_{t+1}-2\}$:

$$\Delta_{(d',d'')}H(h) = \frac{2^{h-h_t}}{n} \leq \frac{2^{h-h_0}}{n}$$

For $h \in \{h_{t+1}-1, h_{t+1}\}$:

$$\sum_{i=h_0}^{h} \Delta_{(d',d)}F(i) + \sum_{i=h_0}^{h-2} 2^{h-2-i} \sum_{j=h_0}^{i} \Delta_{(d',d)}F(j) \geq \frac{1}{n} + \sum_{i=h_0}^{h-2} \frac{2^{h-1-i}}{n}$$
$$\geq \frac{2^{h-h_0}}{n} - \frac{1}{n}$$

Similarly, again from Table 3, for any $h \in \{h_{t+1}-1, h_{t+1}\}$:

$$\Delta_{(d',d'')}H(h) \leq \frac{2^{h-h_t}}{n} - \frac{1}{n}$$

Finally to conclude the case *i(b)* for $h \in \{h_{t+1}+1, ..., h_K-1\}$, Lemma 3 implies:

$$\Delta_{(d',d)}H(h) = \sum_{i=h_0}^{h} \Delta_{(d',d)}F(i) + \sum_{i=h_0}^{h_{t+1}-2} 2^{h-2-i} \sum_{j=h_0}^{i} \Delta_{(d',d)}F(j) + \sum_{i=h_{t+1}-1}^{h-2} 2^{h-2-i} \sum_{j=h_0}^{i} \Delta_{(d',d)}F(j)$$

Using Claim 4, once again one obtains:

$$\Delta_{(d',d)}H(h) \geq \frac{1}{n} + \sum_{i=h_0}^{h_{t+1}-2} \frac{2^{h-1-i}}{n} + \sum_{i=h_{t+1}-1}^{h-2} \frac{2^{h-2-i}}{n}$$
$$\geq \frac{1}{n} + \frac{2^{h-h_0}}{n} - \frac{2^{h-h_{t+1}+1}}{n} + \frac{2^{h-h_{t+1}}}{n} - \frac{1}{n}$$
$$\geq \frac{2^{h-h_0}}{n} - \frac{2^{h-h_{t+1}}}{n}$$

Moreover, from Table 3, $\Delta_{(d',d'')}H(h) \leq \frac{2^{h-h_0}}{n} - \frac{2^{h-h_{t+1}}}{n}$:

For any $h \in \{h_{t+1}+1, ..., h_K-1\}$

Which concludes case *(ib)*. I now turn to case *(ii)*, where the focus is on category $h_{max}(d_t)+1 \in \{h_t+1, ..., h_K-1\}$. I decompose case *(ii)* in two sub-cases:

*(iia)* $h_0 = h_t$

*(iib)* $h_0 < h_t$

In case *(ii)*, $h_{max}(d_t) = h_t$, otherwise $h_{max}(d_t)+1$ could not be an element of the set $\{h_t+1, ..., h_K-1\}$ and be smaller or equal to $h_t+1$ as assumed in the proposition. Also note that if $h_t+1 = h_{t+1}-1$, from row 3 on Table 3:

$$\Delta_{(d',d'')}H(h_t+1) = 0$$

Hence, to focus on the non-trivial part of case *(ii)*, I need to assume that $h_t+1 \leq h_{t+1}-2$. Because $h_{max}(d_t) = h_t$ and $h_t+1 \leq h_{t+1}-2$, from Claim 4, in $h_t+1$:

$$\Delta_{(d',d)}F(h_t) + \Delta_{(d',d)}F(h_t+1) \geq \frac{2}{n}$$

Moreover, from Table 3:

$$\Delta_{(d',d'')}H(h_t+1) \leq \frac{2}{n}$$

It concludes case *(iia)*, because when $h_0 = h_t$, in $h_t+1$:

$$\Delta_{(d',d')}H(h_t+1) = \Delta_{(d',d)}F(h_t) + \Delta_{(d',d)}F(h_t+1)$$

Finally in case *(iib)*, by using Lemma 1, I conclude that:

$$\begin{aligned}\Delta_{(d',d)}H(h_t+1) &= \sum_{i=h_0}^{h_t} \Delta_{(d',d)}H(i) + \Delta_{(d',d)}F(h_t+1) \\ &= 2\sum_{i=h_0}^{h_t-1} \Delta_{(d',d)}H(i) + \Delta_{(d',d)}F(h_t) + \Delta_{(d',d)}F(h_t+1) \geq \frac{2}{n}\end{aligned}$$

**Part 2, the $\bar{\mathbf{H}}$ dominance**. From Table 4 and Claim 1, I only focus on categories $h \in \{h_0, ..., h_t-2\}$. Indeed for categories $h \in \{h_{t+1}, ..., h_K - 1\} \cup \{h_t, ..., h_{t+1} - 2\}$:

$$\Delta_{(d',d'')}\bar{H}(h) = 0$$

Which is, by definition of the dominance, always smaller or equal to $\Delta_{(d',d)}\bar{H}(h)$. By definition of categories $h_t - 1$ and $h_{t+1} - 1$:

$$\Delta_{(d',d)}\bar{F}(h_t - 1) \geq \frac{1}{n} \text{ and } \Delta_{(d',d)}\bar{F}(h_{t+1} - 1) \geq \frac{1}{n}$$

By using Lemma 1, it gives:

$$\Delta_{(d',d)}\bar{H}(h_t - 1) = \sum_{i=h_t}^{h_K-1} \Delta_{(d',d)}\bar{H}(i) + \Delta_{(d',d)}\bar{F}(h_t - 1) \geq \frac{1}{n}$$

$$\Delta_{(d',d)}\bar{H}(h_{t+1} - 1) = \sum_{i=h_{t+1}}^{h_K} \Delta_{(d',d)}\bar{H}(i) + \Delta_{(d',d)}\bar{F}(h_{t+1} - 1) \geq \frac{1}{n}$$

From Table 4:

$$\Delta_{(d',d'')}\bar{H}(h_t - 1) = \frac{1}{n} \text{ and } \Delta_{(d',d'')}\bar{H}(h_{t+1} - 1) = \frac{1}{n}$$

Hence, I will only focus on categories $h \in \{h_0, ..., h_t - 2\}$. Note that these categories only exist when $t > 0$. I treat the two following possibilities:

$$\textit{(i) } h_{min}(d_{t-1}) = h_K - 1 \text{ or } \textit{(ii) } h_{min}(d_{t-1}) \in \{h_{t-1} + 1, ..., h_K - 2\}$$

It is important to note from Table 4 that for any $h \in \{h_0, ..., h_t - 2\}$:

$$\Delta_{(d',d'')}\bar{H}(h) = \frac{2^{h_t-1-h}}{n}$$

Hence in both cases *(i)* and *(ii)* I will show that for any $h \in \{h_0, ..., h_t - 2\}$:

$$\Delta_{(d',d')}\bar{H}(h) \geq \frac{2^{h_t-1-h}}{n}$$

I start with case *(i)*. Because in this case $h_{min}(d_{t-1}) = h_K - 1 \geq h_{t-1}$, from Claim 3:

$$\forall h \in \{h_{t-1} + 1, ..., h_K - 1\}, \sum_{i=h}^{h_K-1} \Delta_{(d',d)}\bar{F}(i) \geq \frac{2}{n}$$

$$\forall h \in \{h_0..., h_{t-1}\}, \sum_{i=h}^{h_K-1} \Delta_{(d',d)}\bar{F}(i) \geq \frac{1}{n}$$

For any category $h \in \{h_0, ..., h_t - 2\}$, from Lemma 4:

$$\Delta_{(d',d)}\bar{H}(h) = \sum_{i=h}^{h_K-1} \Delta_{(d',d)}\bar{F}(i) + \sum_{i=h+2}^{h_K-1} 2^{i-h-2} \sum_{j=i}^{h_K-1} \Delta_{(d',d)}\bar{F}(j)$$

Using Claim 3, for any categories $h \in \{h_{t-1}+1, ..., h_K - 1\}$, it gives:

$$\begin{aligned}\sum_{i=h}^{h_K-1} \Delta_{(d',d)}\bar{F}(i) + \sum_{i=h+2}^{h_K-1} 2^{i-h-2} \sum_{j=i}^{h_K-1} \Delta_{(d',d)}\bar{F}(j) &\geq \frac{2}{n} + \sum_{i=h+2}^{h_K-1} \frac{2^{i-h-1}}{n} \\ &\geq \frac{2^{h_K-h-1}}{n}\end{aligned}$$

Recalling that $h_t < h_K$, one gets:

$$\frac{2^{h_t-1-h}}{n} < \frac{2^{h_K-h-1}}{n}$$

Turning to categories $h \in \{h_{t-1}-1, h_{t-1}\}$, we have:

$$\begin{aligned}\sum_{i=h}^{h_K-1} \Delta_{(d',d)}\bar{F}(i) + \sum_{i=h+2}^{h_K-1} 2^{i-h-2} \sum_{j=i}^{h_K-1} \Delta_{(d',d)}\bar{F}(j) &\geq \frac{1}{n} + \sum_{i=h+2}^{h_K-1} \frac{2^{i-h-1}}{n} \\ &\geq \frac{2^{h_K-h-1}}{n} - \frac{1}{n}\end{aligned}$$

It can be shown that:

$$\frac{2^{h_K-h-1}}{n} - \frac{1}{n} \geq \frac{2^{h_t-1-h}}{n}$$

Indeed, because we are in a discrete setting:

$$\begin{aligned}h_t < h_K &\implies h_t \leq h_K - 1 \\ \text{So, } \frac{2^{h_t-h}}{n} \leq \frac{2^{h_K-h-1}}{n} &\implies \frac{2^{h_t-1-h}}{n} \leq \frac{2^{h_K-h-1}}{n} - \frac{1}{n}\end{aligned}$$

Finally, for categories $h \in \{h_0, ..., h_{t-1} - 2\}$, Lemma 4 can be written as follows:

$$\begin{aligned}\Delta_{(d',d)}\bar{H}(h) = \sum_{i=h}^{h_K-1} \Delta_{(d',d)}\bar{F}(i) + \sum_{i=h+2}^{h_{t-1}} 2^{i-h-2} \sum_{j=i}^{h_K-1} \Delta_{(d',d)}\bar{F}(j) \\ + \sum_{i=h_{t-1}+1}^{h_K-1} 2^{i-h-2} \sum_{j=i}^{h_K-1} \Delta_{(d',d)}\bar{F}(j)\end{aligned} \tag{17}$$

With the help of Claim 3, (17) can be decomposed into the following two inequalities:

$$\sum_{i=h}^{h_K-1} \Delta_{(d',d)}\bar{F}(i) + \sum_{i=h+2}^{h_{t-1}} 2^{i-h-2} \sum_{j=i}^{h_K-1} \Delta_{(d',d)}\bar{F}(j) \geq \frac{1}{n} + \sum_{i=h+2}^{h_{t-1}} \frac{2^{i-h-2}}{n} \tag{18}$$

$$\sum_{i=h_{t-1}+1}^{h_K-1} 2^{i-h-2} \sum_{j=i}^{h_K-1} \Delta_{(d',d)}\bar{F}(j) \geq \sum_{i=h_{t-1}+1}^{h_K-1} \frac{2^{i-h-1}}{n} \tag{19}$$

Summing (18) and (19) yields:

$$\begin{aligned}\Delta_{(d',d)}\bar{H}(h) &\geq \frac{2^{h_{t-1}-h-1}}{n} + \frac{2^{h_K-h-1}}{n} - \frac{2^{h_{t-1}-h}}{n} \\ &\geq \frac{2^{h_K-h-1}}{n} - \frac{2^{h_{t-1}-h-1}}{n}\end{aligned}$$

To conclude case *(i)*, it can be shown that:

$$\frac{2^{h_K-h-1}}{n} - \frac{2^{h_{t-1}-h-1}}{n} \geq \frac{2^{h_t-1-h}}{n}$$

Indeed, since $h_K - 2 \geq h_t$:

$$\frac{2^{h_K-h-2}}{n} \geq \frac{2^{h_t-1-h}}{n}$$
$$\implies \frac{2^{h_K-h-1}}{n} - \frac{2^{h_K-h-2}}{n} \geq \frac{2^{h_t-1-h}}{n}$$

Moreover, note that:

$$-\frac{2^{h_{t-1}-h-1}}{n} \geq -\frac{2^{h_K-h-2}}{n}$$
$$\implies \frac{2^{h_K-h-1}}{n} - \frac{2^{h_{t-1}-h-1}}{n} \geq \frac{2^{h_K-h-1}}{n} - \frac{2^{h_K-h-2}}{n}$$

Hence, we have:

$$\frac{2^{h_K-h-1}}{n} - \frac{2^{h_{t-1}-h-1}}{n} \geq \frac{2^{h_t-1-h}}{n}$$

I now turn to case *(ii)*, where $h_{min}(d_{t-1}) \in \{h_{t-1}+1, ..., h_K-2\}$. In this case, I proceed by contradiction. Suppose that there exists a category $h_{max}(d'') \in \{h_0, ..., h_t - 2\}$. Then in $h_{max}(d'')$:

$$\Delta_{(d',d)} F(h_{max}(d'')) \geq 0 \tag{20}$$

Indeed, suppose that Inequality (20) doesn't hold. Then, using Lemma 1:

$$\Delta_{(d',d)} \bar{H}(h_{max}(d'')) = \sum_{i=h_{max}(d'')+1}^{h_K-1} \Delta_{(d',d)} \bar{H}(i) + \Delta_{(d',d)} \bar{F}(h_{max}(d''))$$

Moreover by definition of $h_{max}(d'')$, it leads to the following inequalities:

$$\begin{aligned}
\Delta_{(d',d)} \bar{H}(h_{max}(d'')) &\geq \sum_{i=h_{max}(d'')+1}^{h_K-1} \Delta_{(d',d'')} \bar{H}(i) + \frac{1}{n} \\
&\geq \sum_{i=h_{max}(d'')+1}^{h_t-1} \frac{2^{h_t-1-i}}{n} + \frac{1}{n} \\
&\geq \frac{2^{h_t-1-h_{max}(d'')}}{n} = \Delta_{(d',d'')} \bar{H}(h_{max}(d''))
\end{aligned}$$

Hence, Inequality (20) must hold. We can deduce from this inequality that:

$$h_{max}(d'') < h_{min}(d_{t-1}) \tag{21}$$

Indeed, recall from Property 7:

$$h_{min}(d_{t-1}) \in \{h_{t-1}+1, ..., h_K - 1\}$$

Whenever $h_{min}(d_{t-1}) > h_t - 2$ or $h_{max}(d'') < h_{t-1} + 1$ it is trivial that Inequality (21) holds. When $h_{min}(d_{t-1}) \leq h_t - 2$ and $h_{max}(d'') \geq h_{t-1} + 1$, both categories belong to the set $\{h_{t-1}+1, ..., h_t - 2\}$, on this set, from the property 4 it is impossible for a category $h'$ for which $\Delta_{(d',d)} F(h') \geq 0$ to be greater than a category $h''$ for which $\Delta_{(d',d)} F(h'') < 0$. Because, from Property 8, $\Delta_{(d',d)} F(h_{min}(d_{t-1})) < 0$ , we deduce that Inequality (21) holds. I now decompose Inequality (21) into two sub-cases:

*(iia)* $h_{max}(d'') = h_{min}(d_{t-1}) - 1$

*(iib)* $h_{max}(d'') \in \{h_0, ..., h_{min}(d_{t-1}) - 2\}$

I start with *(iia)*. Then $h_{min}(d_{t-1})-1 \in \{h_0,...,h_t-2\}$, from Lemma 1:

$$\Delta_{(d',d)}\bar{H}(h_{min}(d_{t-1})-1) = 2\sum_{i=h_{min}(d_{t-1})+1}^{h_K-1} \Delta_{(d',d)}\bar{H}(i) + \Delta_{(d',d)}\bar{F}(h_{min}(d_{t-1})) + \Delta_{(d',d)}\bar{F}(h_{min}(d_{t-1})-1) \quad (22)$$

I can rewrite (22) as:

$$\Delta_{(d',d)}\bar{H}(h_{min}(d_{t-1})-1) = 2\sum_{i=h_{min}(d_{t-1})+1}^{h_t} \Delta_{(d',d)}\bar{H}(i) + 2\sum_{i=h_t+1}^{h_K-1}\Delta_{(d',d)}\bar{H}(i) + \Delta_{(d',d)}\bar{F}(h_{min}(d_{t-1})) + \Delta_{(d',d)}\bar{F}(h_{min}(d_{t-1})-1) \quad (23)$$

I decompose the right hand side of (23) into the following three inequalities:

$$\sum_{i=h_{min}(d_{t-1})+1}^{h_t} \Delta_{(d',d)}\bar{H}(i) \geq \sum_{i=h_{min}(d_{t-1})+1}^{h_t} \Delta_{(d',d'')}\bar{H}(i), \text{ by definition of } h_{max}(d'')$$

$$\sum_{i=h_t+1}^{h_K-1} \Delta_{(d',d)}\bar{H}(i) \geq \frac{1}{n}, \text{ because } \Delta_{(d',d)}\bar{H}(h_K-1) \geq \frac{1}{n}$$

$$\text{From Claim 3, } \Delta_{(d',d)}\bar{F}(h_{min}(d_{t-1})) + \Delta_{(d',d)}\bar{F}(h_{min}(d_{t-1})-1) \geq 0$$

Hence, summing these three inequalities gives:

$$\Delta_{(d',d)}\bar{H}(h_{min}(d_{t-1})-1) \geq 2\sum_{i=h_{min}(d_{t-1})+1}^{h_t} \Delta_{(d',d'')}\bar{H}(i) + \frac{2}{n} \quad (24)$$

If $h_t = h_{min}(d_{t-1})+1$ then from Inequality (24) and Table 4:

$$\Delta_{(d',d)}\bar{H}(h_{min}(d_{t-1})-1) \geq \frac{2}{n} = \frac{2^{h_t-1-(h_{min}(d_{t-1})-1)}}{n} = \Delta_{(d',d'')}\bar{H}(h_{min}(d_{t-1})-1)$$

Finally, if $h_t > h_{min}(d_{t-1})+1$ then from Inequality (24) and Table 4:

$$\Delta_{(d',d)}\bar{H}(h_{min}(d_{t-1})-1) \geq 2\sum_{i=h_{min}(d_{t-1})+1}^{h_t-1} \frac{2^{h_t-1-i}}{n} + \frac{2}{n} = \frac{2^{h_t-1-(h_{min}(d_{t-1})-1)}}{n}$$

This concludes the case *(iia)* as it is impossible to have $h_{max}(d'') = h_{min}(d_{t-1})-1$. I now turn to the case *(iib)* where $h_{max}(d'') \in \{h_0,...,h_{min}(d_{t-1})-2\}$. I distinguish two possibilities in this case:

$$h_K - 1 > h_{min}(d_{t-1}) \text{ or } h_K - 1 = h_{min}(d_{t-1})$$

Consider first $h_K - 1 > h_{min}(d_{t-1})$. From Claim 3:

$$\sum_{i=h\_max(d'')}^{h_{min}(d_{t-1})} \Delta_{(d',d)}\bar{F}(i) \geq 0$$

In this case from Lemma 5, we have for $h_{max}(d'') \in \{h_0,...,h_{min}(d_{t-1})-2\}$:

$$\Delta_{(d',d)}\bar{H}(h_{max}(d'')) = 2^{h_{min}(d_{t-1})-h_{max}(d'')}\sum_{i=h_{min}(d_{t-1})+1}^{h_K-1}\Delta_{(d',d)}\bar{H}(i) + \sum_{i=h_{max}(d'')}^{h_{min}(d_{t-1})}\Delta_{(d',d)}\bar{F}(i) + \sum_{i=h_{max}(d'')+2}^{h_{min}(d_{t-1})} 2^{i-h_{max}(d'')-2}\sum_{j=i}^{h_{min}(d_{t-1})}\Delta_{(d',d)}\bar{F}(j)$$

Moreover:

$$\Delta_{(d',d)}\bar{H}(h_K - 1) \geq \frac{1}{n}$$

Hence, from Lemma 5 and Claim 3:

$$\Delta_{(d',d)}\bar{H}(h_{max}(d'')) \geq \frac{2^{h_{min}(d_{t-1})-h_{max}(d'')}}{n} \tag{25}$$

Hence, if $h_{min}(d_{t-1}) \geq h_t - 1$, we conclude from Inequality (25) and Table 4:

$$\Delta_{(d',d)}\bar{H}(h_{max}(d'')) \geq \frac{2^{h_t-1-h_{max}(d'')}}{n} = \Delta_{(d',d'')}\bar{H}(h_{max}(d''))$$

If however $h_{min}(d_{t-1}) < h_t - 1$, Lemma 5 needs to be rewritten as follows:

$$\begin{aligned}\Delta_{(d',d)}\bar{H}(h_{max}(d'')) = 2^{h_{min}(d_{t-1})-h_{max}(d'')}(&\sum_{i=h_{min}(d_{t-1})+1}^{h_t-1}\Delta_{(d',d)}\bar{H}(i) + \sum_{i=h_t}^{h_K-1}\Delta_{(d',d)}\bar{H}(i)) \\ &+ \sum_{i=h_{max}(d'')}^{h_{min}(d_{t-1})}\Delta_{(d',d)}\bar{F}(i) + \sum_{i=h_{max}(d'')+2}^{h_{min}(d_{t-1})} 2^{i-h_{max}(d'')-2}\sum_{j=i}^{h_{min}(d_{t-1})}\Delta_{(d',d)}\bar{F}(j)\end{aligned}$$

Once again, from Lemma 5 and Claim 3:

$$\Delta_{(d',d)}\bar{H}(h_{max}(d'')) \geq 2^{h_{min}(d_{t-1})-h_{max}(d'')}(\sum_{i=h_{min}(d_{t-1})+1}^{h_t-1}\Delta_{(d',d'')}\bar{H}(i) + \frac{1}{n})$$

Note from Table 4 that for any $h \leq h_t - 1$:

$$\Delta_{(d',d'')}\bar{H}(h) \geq \frac{2^{h_t-1-h}}{n}$$

Hence:

$$\begin{aligned}\Delta_{(d',d)}\bar{H}(h_{max}(d'')) &\geq 2^{h_{min}(d_{t-1})-h_{max}(d'')}(\sum_{i=h_{min}(d_{t-1})+1}^{h_t-1}\frac{2^{h_t-1-i}}{n} + \frac{1}{n}) \\ &\geq 2^{h_{min}(d_{t-1})-h_{max}(d'')} \times \frac{2^{h_t-1-h_{min}(d_{t-1})}}{n} \\ &\geq \frac{2^{h_t-1-h_{max}(d'')}}{n} = \Delta_{(d',d'')}\bar{H}(h_{max}(d''))\end{aligned}$$

Hence when $h_K - 1 > h_{min}(d_{t-1})$, $h_{max}(d'')$ cannot exist. Finally I turn to the case when $h_K - 1 = h_{min}(d_{t-1})$. Because $h_{min}(d_{t-1}) = h_K - 1$:

$$\forall h \in \{h_0, ..., h_{min}(d_k)\}, \sum_{i=h}^{h_{min}(d_k)}\Delta_{(d',d)}\bar{F}(i) \geq \frac{1}{n}$$

From Lemma 4:

$$\Delta_{(d',d)}\bar{H}(h_{max}(d'')) = \sum_{i=h_{max}(d'')}^{h_K-1}\Delta_{(d',d'')}\bar{F}(i) + \sum_{i=h_{max}(d'')+2}^{h_K-1} 2^{i-h_{max}(d'')-2}\sum_{j=i}^{h_K-1}\Delta_{(d',d)}\bar{F}(j)$$

Using Lemma 4 and Claim 3, it gives:

$$\begin{aligned}\Delta_{(d',d)}\bar{H}(h_{max}(d'')) &\geq \frac{1}{n} + \sum_{i=h_{max}(d'')+2}^{h_K-1}\frac{2^{i-h_{max}(d'')-2}}{n} \\ &\geq \frac{2^{h_K-h_{max}(d'')-2}}{n}\end{aligned}$$

Knowing that $h_t < h_K$, one concludes that:

$$\Delta_{(d',d)}\bar{H}(h_{max}(d'')) \geq \frac{2^{h_t - h_{max}(d'')-1}}{n} = \Delta_{(d',d'')}\bar{H}(h_{max}(d''))$$

Hence, the category $h_{max}(d'')$ cannot exist, which concludes the proof.

**Proposition 2** I provide the variations generated by the Hammond transfer in Tables 5 and 6.

| $h$ | $\Delta_{(d',d'')}H(h)$ |
|---|---|
| (If any) $h \leq e-1$ | $0$ |
| $e \leq h \leq h_{t+1}-2$ | $\frac{2^{h-e}}{n}$ |
| $h = h_{t+1}-1$ | $\frac{2^{h-e}}{n} - \frac{2}{n}$ |
| $h \geq h_{t+1}$ | $\frac{2^{h-e}}{n} - \frac{2^{h-h_{t+1}-2}}{n} + \frac{2^{h-h_{t+1}}}{n}$ |

**Table 5:** $\Delta_{(d',d'')}H(h)$

| $h$ | $\Delta_{(d',d'')}\bar{H}(h)$ |
|---|---|
| $h \geq h_{t+1}$ | $0$ |
| $h_{t+1}-1$ | $\frac{1}{n}$ |
| $e \leq h \leq h_{t+1}-2$ | $0$ |
| (If any) $h \leq e-1$ | $\frac{2^{e-1-h}}{n}$ |

**Table 6:** $\Delta_{(d',d'')}\bar{H}(h)$

Along the proof I suppose $t \in \{0, ..., K-1\}$, and $h_t < h_{max}(d_t)$. Moreover suppose that there is a category $h \in \{h_t, ..., h_{max}(d_t)\}$ such that $\Delta_{(d',d)}F(h) \geq \frac{2}{n}$. Additionally, suppose that if $t \geq 1$, then $d_{t-1}$ doesn't preserve the $H$-dominance.

**Part 1, the H dominance.** From Table 5 and Claim 2, I only need focus on categories $h \in \{e+1, ..., h_K - 1\}$. It is important to see from Table 5 that for any category $h \in \{e+1, ..., h_K - 1\}$:

$$\Delta_{(d',d'')}H(h) \leq \frac{2^{h-e}}{n}$$

Along the proof, I will systematically show that:

$$\Delta_{(d',d)}H(h) \geq \frac{2^{h-e}}{n}$$

I start with categories $h \in \{e, ..., h_{max}(d_t)\}$. I proceed by induction. Note that for category $\{h_{max}(d_t)\}$, from Property 6, we have:

$$\Delta_{(d',d)}F(h_{max}(d_t)) \geq \frac{1}{n}$$

and so for categories $h \in \{e, ..., h_{max}(d_t)\}$, from property 4, we have:

$$\Delta_{(d',d)}F(h) \geq 0$$

Using Lemma 1, in $e$:

$$\Delta_{(d',d)}H(e) \geq \sum_{i=h_0}^{e-1} \Delta_{(d',d)}H(i) \geq \frac{1}{n} = \Delta_{(d',d'')}H(e)$$

Now suppose by induction that there is a $h \in \{e, ..., h_{max}(d_t) - 1\}$ such that for all $h' \in \{e, ..., h\}$, $\Delta_{(d',d)} H(h') \geq \frac{2^{h'-e}}{n}$, then, using Lemma 1 in $h+1$:

$$\Delta_{(d',d)} H(h+1) \geq \sum_{i=h_0}^{e-1} \Delta_{(d',d)} H(i) + \sum_{i=e}^{h-1} \frac{2^{i-e}}{n} \geq \frac{2^{h-e}}{n}$$

Hence, I proved by induction that for any $h \in \{e, ..., h_{max}(d_t)\}$:

$$\Delta_{(d',d)} H(h) \geq \Delta_{(d',d'')} H(h)$$

I now turn to categories $h \in \{h_{max}(d_t)+1, ..., h_K - 1\}$ (recall from Property 8 that $h_{max}(d_t)+1 \leq h_K - 1$). For the category $h_{max}(d_t) + 1$, by Lemma 1:

$$\Delta_{(d',d)} H(h_{max}(d_t) + 1) = 2 \sum_{i=h_0}^{h_{max}(d_t)} \Delta_{(d',d)} H(i) + \sum_{i=h_{max}(d_t)}^{h_{max}(d_t)+1} \Delta_{(d',d)} F(i) \tag{26}$$

Moreover for categories $h \geq h_{max} + 2$, we obtain from Lemma 2:

$$\begin{aligned} \Delta_{(d',d)} H(h) = 2^{h-h_{max}(d_t)} \sum_{i=h_0}^{h_{max}(d_t)-1} \Delta_{(d',d)} H(i) + \sum_{i=h_{max}(d_t)}^{h} \Delta_{(d',d)} F(i) \\ + \sum_{i=h_{max}(d_t)}^{h-2} 2^{h-2-i} \sum_{j=h_{max}(d_t)}^{i} \Delta_{(d',d)} F(j) \end{aligned} \tag{27}$$

Recall that Claim 4 states that for any categories $h \in \{h_{max}(d_t), ..., h_K - 1\}$:

$$\sum_{i=h_{max}(d_t)}^{h} \Delta_{(d',d)} F(i) \geq 0$$

Using Claim 4, (26) and (27) lead to the following inequality, for any $h \in \{h_{max}(d_t) + 1, ..., h_K - 1\}$:

$$\Delta_{(d',d)} H(h) \geq 2^{h-h_{max}(d_t)} \sum_{i=h_0}^{h_{max}(d_t)-1} \Delta_{(d',d)} H(i) \tag{28}$$

In the specific case when $h_{max}(d_t) = e$, we conclude from Inequality (28) that the following holds for any categories $h \in \{h_{max}(d_t) + 1, ..., h_K - 1\}$:

$$\Delta_{(d',d)} H(h) \geq \frac{2^{h-h_{max}(d_t)}}{n} = \frac{2^{h-e}}{n}$$

Finally, I turn to the case when $h_{max}(d_t) > e$. I previously proved that for all category $h \in \{e, ..., h_{max}(d_t)\}$:

$$\Delta_{(d',d)} H(h) \geq \frac{2^{h-e}}{n}$$

Using this result, we conclude from Inequality (28) that the following holds for any category $h \in \{h_{max}(d_t) + 1, ..., h_K - 1\}$:

$$\begin{aligned} \Delta_{(d',d)} H(h) &\geq 2^{h-h_{max}(d_t)} \sum_{i=e}^{h_{max}(d_t)-1} \frac{2^{i-e}}{n} + 2^{h-h_{max}(d_t)} \sum_{i=h_0}^{e-1} \Delta_{(d',d)} H(i) \\ &\geq 2^{h-h_{max}(d_t)} \sum_{i=e}^{h_{max}(d_t)-1} \frac{2^{i-e}}{n} + \frac{2^{h-h_{max}(d_t)}}{n} \\ &\geq \frac{2^{h-e}}{n} \end{aligned}$$

Hence, $d''$ preserves the $H$-dominance.

**Part 2, the $\bar{\mathbf{H}}$ dominance.** From Table 6 and Claim 1, I only need to focus on categories $h \in \{h_0, ..., e-1\}$. Note that, by definition of $e$, for categories $h \in \{h_t, ..., e-1\}$:

$$\Delta_{(d',d)}\bar{F}(h) \geq -\frac{1}{n}$$

Moreover, by definition of the dominance, in $e$:

$$\Delta_{(d',d)}\bar{H}(e) \geq 0$$

By using Lemma 1:

$$\sum_{i=e+1}^{h_K-1} \Delta_{(d',d)}\bar{H}(i) + \Delta_{(d',d)}\bar{F}(e) \geq 0 \tag{29}$$

From Inequality (29) and the definition of $e$, we have:

$$\sum_{i=e+1}^{h_K-1} \Delta_{(d',d)}\bar{H}(i) \geq \frac{2}{n} \tag{30}$$

Given (30), we have in the category $e-1$:

$$\Delta_{(d',d)}\bar{H}(e-1) \geq \sum_{i=e}^{h_K-1} \Delta_{(d',d)}\bar{H}(i) - \frac{1}{n} \geq \frac{1}{n}$$

Thus I proved that $\Delta_{(d',d)}\bar{H}(e-1) \geq \frac{1}{n}$. Suppose now by induction that there is a category $h \in \{h_t+1, ..., e-1\}$ such that for all categories $h' \in \{h, ..., e-1\}$, $\Delta_{(d',d)}\bar{H}(h') \geq \frac{2^{e-1-h'}}{n}$. Then in $h-1$:

$$\begin{aligned}\Delta_{(d',d)}\bar{H}(h-1) &\geq \sum_{i=e}^{h_K-1} \Delta_{(d',d)}\bar{H}(i) + \sum_{i=h}^{e-1} \Delta_{(d',d)}\bar{H}(i) - \frac{1}{n} \\ \implies \Delta_{(d',d)}\bar{H}(h-1) &\geq \frac{2}{n} + \sum_{i=h}^{e-1} \frac{2^{e-1-i}}{n} - \frac{1}{n} = \frac{2^{e-h}}{n}\end{aligned}$$

Hence for any $h \in \{h_t, ..., e-1\}$, $\Delta_{(d',d)}\bar{H}(h) \geq \frac{2^{e-1-h}}{n}$. To conclude, note from Table 6 that, for categories $h \in \{h_t, ..., e-1\}$:

$$\Delta_{(d',d'')}\bar{H}(h) < \frac{2^{e-h}}{n}$$

I omit the proof for categories $h \in \{h_0, ..., h_t-1\}$ because it is almost identical to that of Proposition 1.

**Proposition 3** I provide the variations generated by the Hammond transfer in Tables 7 and 8.

| $h$ | $\Delta_{(d',d'')}H(h)$ |
|---|---|
| $h < h_t$ | $0$ |
| $h_t \leq h \leq e-1$ | $\frac{2^{h-h_t}}{n}$ |
| $e \leq h \leq j-2$ | $\frac{2^{h-h_t}}{n} - \frac{2^{h-e}}{n}$ |
| $h = j-1$ | $\frac{2^{h-h_t}}{n} - \frac{2^{h-e}}{n} - \frac{1}{n}$ |
| $h \geq j$ | $\frac{2^{h-h_t}}{n} - \frac{2^{h-e}}{n} - \frac{2^{h-j}}{n}$ |

**Table 7:** $\Delta_{(d',d'')}H(h)$

| $h$ | $\Delta_{(d',d'')}\bar{H}(h)$ |
|---|---|
| $h \geq i$ | $0$ |
| $j-1$ | $\frac{1}{n}$ |
| $e \leq h \leq j-2$ | $\frac{2^{j-2-h}}{n}$ |
| $h_t \leq h \leq e-1$ | $\frac{2^{j-2-h}}{n} - \frac{2^{e-1-h}}{n}$ |
| $h \leq h_t - 1$ | $\frac{2^{j-2-h}}{n} - \frac{2^{e-1-h}}{n} + \frac{2^{h_t-1-h}}{n}$ |

**Table 8:** $\Delta_{(d',d'')}\bar{H}(h)$

Along the proof I suppose $t \in \{0, ..., K-1\}$, and $h_t < h_{max}(d_t)$. Moreover suppose that for all categories $h \in \{h_t, ..., h_{max}(d_t)\}$, $\Delta_{(d',d)}F(h) \leq \frac{1}{n}$. Additionally, suppose that if $t \geq 1$, then $d_{t-1}$ doesn't preserve the $H$-dominance.

**Part 1 the H dominance** From Table 7 and Claim 2 I only need to focus on categories $h \in \{h_t+1, ..., h_K - 1\}$. Note that for all categories $h \in \{2, ..., L\}$:

$$\Delta_{(d',d)}H(h) = 2\Delta_{(d',d)}H(h-1) + \Delta_{(d',d)}F(h) - \Delta_{(d',d)}F(h-1) \tag{31}$$

From (31) and knowing that for categories $h \in \{h_t, ..., e-1\}$, $\Delta_{(d',d)}F(h-1) \leq \Delta_{(d',d)}F(h)$, for these categories:

$$\Delta_{(d',d)}H(h) \geq 2\Delta_{(d',d)}H(h-1)$$

Which implies by induction, that for any $h \in \{h_t, ..., e-1\}$:

$$\Delta_{(d',d)}H(h) \geq 2^{h-h_t}\Delta_{(d',d)}H(h_t) \geq \frac{2^{h-h_t}}{n}$$

Turning to categories $h \in \{e, ..., h_K - 1\}$, I treat two possibilities:

$$\textit{(i)}\ e \leq h_{max}(d_t) \quad \text{or} \quad \textit{(ii)}\ e = h_{max}(d_t) + 1$$

I start with the case *(i)*. For categories $h \in \{e, ..., h_{max}(d_t)\}$:

$$\Delta_{(d',d)}F(h) \geq 0$$

By using Lemma 1, in $e$:

$$\begin{aligned}\Delta_{(d',d)}H(e) &\geq \sum_{i=h_t}^{e-1} \Delta_{(d',d)}H(i) \\ &\geq \sum_{i=h_t}^{e-1} \frac{2^{i-h_t}}{n} \\ &\geq \frac{2^{e-h_t}}{n} - \frac{1}{n}\end{aligned}$$

Suppose by induction that there is a $h \in \{e, ..., h_{max}(d_t)-1\}$ such that for all $h' \in \{e, ..., h\}$, $\Delta_{(d',d)}H(h') \geq \frac{2^{h'-h_t}}{n} - \frac{2^{h'-j}}{n}$, then in $h+1$:

$$\Delta_{(d',d)}H(h+1) \geq \sum_{i=h_t}^{e-1} \Delta_{(d',d)}H(i) + \sum_{i=e}^{h}(\frac{2^{i-h_t}}{n} - \frac{2^{i-e}}{n}) \geq \frac{2^{h+1-h_t}}{n} - \frac{2^{h+1-e}}{n}$$

Hence, I proved by induction that for any $h \in \{e, ..., h_{max}(d_t)\}$:

$$\Delta_{(d',d)}H(h) \geq \frac{2^{h-h_t}}{n} - \frac{2^{h-e}}{n}$$

Note from Table 7 that for any $h \in \{e, ..., h_{max}(d_t)\}$:

$$\Delta_{(d',d'')}H(h) \leq \frac{2^{h-h_t}}{n} - \frac{2^{h-e}}{n}$$

For the category $h_{max}(d_t)+1$, from Lemma 1:

$$\Delta_{(d',d)}H(h_{max}(d_t)+1) = 2 \sum_{i=h_t}^{h_{max}(d_t)-1} \Delta_{(d',d)}H(i) + \Delta_{(d',d)}F(h_{max}(d_t)) + \Delta_{(d',d)}F(h_{max}(d_t)+1) \quad (32)$$

From Claim 4, we obtain from Equality (32) the following inequality:

$$\Delta_{(d',d)}H(h_{max}(d_t)+1) \geq 2 \sum_{i=h_t}^{h_{max}(d_t)-1} \Delta_{(d',d)}H(i) \quad (33)$$

Moreover for categories $h \in \{h_{max}(d_t)+2, ..., h_K-1\}$, we have from Lemma 2:

$$\begin{aligned}\Delta_{(d',d)}H(h) = 2^{h-h_{max}(d_t)} &\sum_{i=h_0}^{h_{max}(d_t)-1} \Delta_{(d',d)}H(i) + \sum_{i=h_{max}(d_t)}^{h} \Delta_{(d',d)}F(i) \\ &+ \sum_{i=h_{max}(d_t)}^{h-2} 2^{h-2-i} \sum_{j=h_{max}(d_t)}^{i} \Delta_{(d',d)}F(j)\end{aligned}$$

Similarly, because of Claim 4, we have for any $h \in \{h_{max}(d_t)+2, ..., h_K-1\}$:

$$\Delta_{(d',d)}H(h) \geq 2^{h-h_{max}(d_t)} \sum_{i=h_0}^{h_{max}(d_t)-1} \Delta_{(d',d)}H(i) \quad (34)$$

If $j < h_{max}(d_t)$ it is possible to rewrite Inequalities (33) and (34) as follows, for any category $h \in \{h_{max}(d_t)+1, ..., h_K-1\}$:

$$\Delta_{(d',d)}H(h) \geq 2^{h-h_{max}(d_t)}(\sum_{i=h_t}^{e-1} \frac{2^{i-h_t}}{n} + \sum_{i=e}^{h_{max}(d_t)-1} (\frac{2^{i-h_t}}{n} - \frac{2^{i-e}}{n})) = \frac{2^{h-h_t}}{n} - \frac{2^{h-e}}{n}$$

Finally to conclude case *(i)*, if $e = h_{max}(d_t)$, from Inequalities (33) and (34), for any category $h \in \{h_{max}(d_t)+1, ..., h_K-1\}$:

$$\Delta_{(d',d)}H(h) \geq 2^{h-h_{max}(d_t)} \sum_{i=h_t}^{h_{max}(d_t)-1} \frac{2^{i-h_t}}{n} = \frac{2^{h-h_t}}{n} - \frac{2^{h-e}}{n}$$

I now turn to the case *(ii)* where $e = h_{max}(d_t)+1$. In category $e$, we know thanks to Lemma 1:

$$\Delta_{(d',d)}H(e) = 2 \sum_{i=h_t}^{h_{max}(d_t)-1} \frac{2^{i-h_t}}{n} + \Delta_{(d',d)}F(h_{max}(d_t)) + \Delta_{(d',d)}F(h_{max}(d_t)+1) \quad (35)$$

From Claim 4, if $e \leq h_{t+1} - 2$, (35) implies:

$$\begin{aligned}
\Delta_{(d',d)} H(e) &\geq 2 \sum_{i=h_t}^{h_{max}(d_t)-1} \frac{2^{i-h_t}}{n} + \frac{1}{n} \\
&\geq \frac{2^{e-h_t}}{n} - \frac{1}{n} \geq \Delta_{(d',d'')} H(e)
\end{aligned}$$

If however $e > h_{t+1} - 2$, i.e $e = h_{t+1} - 1$, one can conclude from Claim 4 and (35) that:

$$\begin{aligned}
\Delta_{(d',d)} H(e) &\geq 2 \sum_{i=h_t}^{h_{max}(d_t)-1} \frac{2^{i-h_t}}{n} \\
&\geq \frac{2^{e-h_t}}{n} - \frac{2}{n} = \Delta_{(d',d'')} H(e)
\end{aligned}$$

I then turn to categories $h \in \{e+1, ..., h_K - 1\}$, from Lemma 2:

$$\begin{aligned}
\Delta_{(d',d)} H(h) = 2^{h-h_{max}(d_t)} \sum_{i=h_0}^{h_{max}(d_t)-1} \Delta_{(d',d)} H(i) + \sum_{i=h_{max}(d_t)}^{h} \Delta_{(d',d)} F(i) \\
+ \sum_{i=h_{max}(d_t)}^{h-2} 2^{h-2-i} \sum_{j=h_{max}(d_t)}^{i} \Delta_{(d',d)} F(j) \quad (36)
\end{aligned}$$

For categories $h \in \{e+1, ..., h_{t+1} - 2\}$, we conclude from Claim 4:

$$\sum_{i=h_{max}(d_t)}^{h} \Delta_{(d',d)} F(i) \geq \frac{1}{n}$$

Hence, (36) and Claim 4 imply that, for any $h \in \{e+1, ..., h_{t+1} - 2\}$:

$$\Delta_{(d',d)} H(h) \geq 2^{h-h_{max}(d_t)} \sum_{i=h_0}^{h_{max}(d_t)-1} \Delta_{(d',d)} H(i) + \frac{1}{n} + \sum_{i=h_{max}(d_t)}^{h-2} \frac{2^{h-2-i}}{n} \quad (37)$$

Recalling that in case $(ii)$, $e = h_{max}(d_t) + 1$, one can rewrite Inequality (37) as:

$$\begin{aligned}
\Delta_{(d',d)} H(h) &\geq 2^{h-e+1} \sum_{i=h_0}^{e-2} \Delta_{(d',d)} H(i) + \frac{1}{n} + \sum_{i=e-1}^{h-2} \frac{2^{h-2-i}}{n} \\
&\geq 2^{h-j+1} \sum_{i=h_0}^{e-2} \Delta_{(d',d)} H(i) + \frac{2^{h-e}}{n} \\
&\geq 2^{h-e+1} \sum_{i=h_t}^{e-2} \frac{2^{i-h_t}}{n} + \frac{2^{h-e}}{n} \\
&\geq \frac{2^{h-h_t}}{n} - \frac{2^{h-e}}{n}
\end{aligned}$$

Note that for any category $h \in \{e+1, ..., h_{t+1} - 2\}$, from Table 7:

$$\frac{2^{h-h_t}}{n} - \frac{2^{h-e}}{n} \geq \Delta_{(d',d'')} H(h)$$

Turning now to categories $h \in \{h_{t+1} - 1, ..., h_K - 1\}$, we conclude from Claim 4 that:

$$\sum_{i=h_{max}(d_t)}^{h} \Delta_{(d',d)} F(i) \geq 0$$

Hence (36) and Claim 4, both imply, for any $h \in \{h_{t+1}-1, h_{t+1}\}$:

$$\begin{aligned}
\Delta_{(d',d)}H(h) &\geq \frac{2^{h-h_{max}(d_t)}}{n}\sum_{i=h_0}^{h_{max}(d_t)-1}\Delta_{(d',d)}H(i) + \sum_{i=h_{max}(d_t)}^{h-2}\frac{2^{h-2-i}}{n} \\
&\geq \frac{2^{h-e+1}}{n}\sum_{i=h_t}^{j-2}\frac{2^{i-h_t}}{n} + \frac{2^{h-e}}{n} - \frac{1}{n} \\
&\geq \frac{2^{h-h_t}}{n} - \frac{2^{h-e}}{n} - \frac{1}{n}
\end{aligned}$$

Note that $h_{t+1}-1 \geq j-1$. Hence from Table 7, we have for any category $h \in \{h_{t+1}-1, h_{t+1}\}$:

$$\Delta_{(d',d'')}H(h) \leq \frac{2^{h-h_t}}{n} - \frac{2^{h-e}}{n} - \frac{1}{n}$$

Finally, for categories $h \in \{h_{t+1}, ..., h_K - 1\}$, we also deduce from (36) and Claim 4:

$$\begin{aligned}
\Delta_{(d',d)}H(h) &\geq \frac{2^{h-h_{max}(d_t)}}{n}\sum_{i=h_0}^{h_{max}(d_t)-1}\Delta_{(d',d)}H(i) + \sum_{i=h_{max}(d_t)}^{h_{t+1}-2}\frac{2^{h-2-i}}{n} \\
&\geq \frac{2^{h-e+1}}{n}\sum_{i=h_t}^{e-2}\frac{2^{i-h_t}}{n} + \sum_{i=e-1}^{h_{t+1}-2}\frac{2^{h-2-i}}{n} \\
&\geq \frac{2^{h-h_t}}{n} - \frac{2^{h-e}}{n} - \frac{2^{h-h_{t+1}}}{n}
\end{aligned}$$

Hence, since $h_{t+1} \geq j$, we have for any category $h \in \{h_{t+1}, ..., h_K - 1\}$:

$$\begin{aligned}
\Delta_{(d',d'')}H(h) &= \frac{2^{h-h_t}}{n} - \frac{2^{h-e}}{n} - \frac{2^{h-j}}{n} \\
&\leq \frac{2^{h-h_t}}{n} - \frac{2^{h-e}}{n} - \frac{2^{h-h_{t+1}}}{n}
\end{aligned}$$

**Part 2 the $\bar{\mathbf{H}}$ dominance**. From Table 8 and Claim 1, I only need to focus on categories $h \in \{h_0, ..., j-1\}$. Note that for all categories $h \in \{1, ..., L-1\}$:

$$\Delta_{(d',d)}\bar{H}(h) = 2\Delta_{(d',d)}\bar{H}(h+1) + \Delta_{(d',d)}\bar{F}(h) - \Delta_{(d',d)}\bar{F}(h+1) \tag{38}$$

In $j-1$, $\Delta_{(d',d)}\bar{F}(j-1) > \Delta_{(d',d)}\bar{F}(j)$. Hence, from (38), in $j-1$:

$$\Delta_{(d',d)}\bar{H}(j-1) \geq 2\Delta_{(d',d)}\bar{H}(j) + \Delta_{(d',d)}\bar{F}(j-1) - \Delta_{(d',d)}\bar{F}(j) \geq \frac{1}{n}$$

Moreover for categories $h \in \{e, ..., j-2\}$:

$$\Delta_{(d',d)}\bar{F}(h) \geq 0$$

It thus follows from Lemma 1, that in $j-2$:

$$\Delta_{(d',d)}\bar{H}(j-2) \geq \sum_{i=j-1}^{h_K-1}\Delta_{(d',d)}\bar{H}(i) \geq \frac{1}{n}$$

Suppose by induction that there is a $h \in \{e+1, ..., j-2\}$ such that for all $h' \in \{h, ..., j-2\}$, $\Delta_{(d',d)}\bar{H}(h') \geq$

$\frac{2^{j-2-h'}}{n}$, then in $h-1$:

$$\begin{aligned}\Delta_{(d',d)}\bar{H}(h-1) &\geq \sum_{i=h}^{j-2}\Delta_{(d',d)}\bar{H}(i) + \sum_{i=j-1}^{h_K-1}\Delta_{(d',d)}\bar{H}(i)\\ &\geq \sum_{i=h}^{j-2}\frac{2^{j-2-i}}{n} + \frac{1}{n}\\ &\geq \frac{2^{j-1-h}}{n}\end{aligned}$$

Hence I proved by induction that for any $h \in \{e, ..., j-2\}$:

$$\Delta_{(d',d)}\bar{H}(h) \geq \frac{2^{j-2-h}}{n}$$

I now turn to categories $h \in \{h_t, ..., e-1\}$. Note that by definition of $e$, $\{h_t, ..., e-1\} \subset \{h_t, ..., h_{max}(d_t)\}$, moreover for categories $h \in \{h_t, ..., h_{max}(d_t)\}$, it is assumed that:

$$\Delta_{(d',d)}\bar{F}(h) \geq -\frac{1}{n} \tag{39}$$

Hence from Lemma 1, in $e-1$:

$$\begin{aligned}\Delta_{(d',d)}\bar{H}(e-1) &\geq \sum_{i=e}^{j-2}\Delta_{(d',d)}\bar{H}(i) + \sum_{i=j-1}^{h_K-1}\Delta_{(d',d)}\bar{H}(i) - \frac{1}{n}\\ &\geq \sum_{i=e}^{j-2}\frac{2^{j-2-i}}{n}\\ &\geq \frac{2^{j-1-e}}{n} - \frac{1}{n}\end{aligned}$$

For categories $h \in \{h_t, ..., e-2\}$, knowing that $\Delta_{(d',d)}\bar{F}(h) \geq \Delta_{(d',d)}\bar{F}(h+1)$, from (38):

$$\Delta_{(d',d)}\bar{H}(h) \geq 2\Delta_{(d',d)}\bar{H}(h+1) \tag{40}$$

Which implies that for any $h \in \{h_t, ..., e-1\}$:

$$\begin{aligned}\Delta_{(d',d)}\bar{H}(h) &\geq 2^{e-1-h}\Delta_{(d',d)}\bar{H}(e-1)\\ &\geq 2^{e-1-h}(\frac{2^{j-1-e}}{n} - \frac{1}{n})\\ &\geq \frac{2^{j-2-h}}{n} - \frac{2^{e-1-h}}{n}\end{aligned}$$

The proof for categories $h \in \{h_0, ..., h_t - 1\}$ is omitted because it is almost identical to that of Proposition 1.

**Proposition 4** I provide the variations generated by the Hammond transfer in Tables 9 and 10.

| $h$ | $\Delta_{(d',d'')}H(h)$ |
|---|---|
| (If any) $h \leq h_{K-1}-1$ | $0$ |
| $h_{K-1}$ | $\frac{1}{n}$ |
| $h_{K-1}+1 \leq h \leq h_K-1$ | $0$ |
| $h \geq h_K$ | $\frac{2^{h-h_K}}{n}$ |

**Table 9:** $\Delta_{(d',d'')}H(h)$

| $h$ | $\Delta_{(d',d'')}\bar{H}(h)$ |
|---|---|
| For $h \geq h_K$ | $0$ |
| $h_{K-1}+1 \leq h \leq h_K-1$ | $\frac{2^{h_K-1-h}}{n}$ |
| $h_{K-1}$ | $\frac{2^{h_K-1-h}}{n} - \frac{2}{n}$ |
| (If any) $h \leq h_{K-1}-1$ | $\frac{2^{h_K-1-h}}{n} - \frac{2^{h_{K-1}+1-h}}{n} + \frac{2^{h_{K-1}-1-h}}{n}$ |

**Table 10:** $\Delta_{(d',d'')}\bar{H}(h)$

We suppose all through the proof that $d_{K-1}$ doesn't preserve the $H$-dominance and that $h_{min}(d_{K-1}) = h_K - 1$.

**Part 1, the H dominance.** From Table 9 and Claim 2 I only need to focus on categories $\{h_{K-1}+1, ..., h_K-1\}$. Note that for all $h \in \{h_{K-1}+1, ..., h_K-1\}$:

$$\Delta_{(d',d'')}H(h) = 0$$

Hence, $d''$ preserves the $H$-dominance.

**Part 2, the $\bar{\mathbf{H}}$ dominance**. From Table 10 and Claim 1, I only need to focus on categories $h \in \{h_0, ..., h_K-2\}$. I start with the category $h_K-2$. If $h_K-2 \geq h_{K-1}+1$ we conclude from Claim 3:

$$\Delta_{(d',d'')}\bar{H}(h_K-2) = \Delta_{(d',d)}\bar{F}(h_K-2) + \Delta_{(d',d)}\bar{F}(h_K-1) \geq \frac{2}{n}$$

Otherwise, if $h_K-2 = h_{K-1}$, from Table 10:

$$\Delta_{(d',d'')}\bar{H}(h_K-2) = 0$$

For categories $h \in \{h_0, ..., h_K-3\}$, we conclude from Lemma 4:

$$\Delta_{(d',d)}\bar{H}(h) = \sum_{i=h}^{h_K-1} \Delta_{(d',d)}\bar{F}(i) + \sum_{i=h+2}^{h_K-1} 2^{i-h-2} \sum_{j=i}^{h_K-1} \Delta_{(d',d)}\bar{F}(j) \tag{41}$$

If $h \geq h_{K-1}+1$, we conclude from Claim 3 and (41):

$$\begin{aligned}\Delta_{(d',d)}\bar{H}(h) &\geq \frac{2}{n} + \sum_{i=h+2}^{h_K-1} \frac{2^{i-h-1}}{n} \\ &\geq \frac{2^{h_K-h-1}}{n} = \Delta_{(d',d'')}\bar{H}(h)\end{aligned}$$

If on the other hand $h \in \{h_{K-1}-1, h_{K-1}\}$:

$$\begin{aligned}\Delta_{(d',d)}\bar{H}(h) &\geq \frac{1}{n} + \sum_{i=h+2}^{h_K-1} \frac{2^{i-h-1}}{n} \\ &\geq \frac{2^{h_K-h-1}}{n} - \frac{1}{n} \geq \Delta_{(d',d'')}\bar{H}(h)\end{aligned}$$

Finally, if $h \leq h_{K-1} - 2$:

$$\begin{aligned}\Delta_{(d',d)}\bar{H}(h) &\geq \frac{1}{n} + \sum_{i=h_{K-1}+1}^{h_K-1} \frac{2^{i-h-1}}{n} + \sum_{i=h+2}^{h_{K-1}} \frac{2^{i-h-2}}{n} \\ &\geq \frac{2^{h_K-1-h}}{n} - \frac{2^{h_{K-1}-1-h}}{n} \geq \Delta_{(d',d'')}\bar{H}(h)\end{aligned}$$

Hence $d''$ preserves the $\bar{H}$-dominance.

**Proposition 5.** I provide the variations generated by the Hammond transfer in Tables 11 and 12.

| $h$ | $\Delta_{(d',d'')}H(h)$ |
|---|---|
| (If any) $h \leq h_{K-1} - 1$ | $0$ |
| $h_{K-1}$ | $\frac{1}{n}$ |
| $h_{K-1} + 1 \leq h \leq e$ | $0$ |
| $h \geq e+1$ | $\frac{2^{h-e-1}}{n}$ |

**Table 11:** $\Delta_{(d',d'')}H(h)$

| $h$ | $\Delta_{(d',d'')}\bar{H}(h)$ |
|---|---|
| $h \geq e+1$ | $0$ |
| $h_{K-1} + 1 \leq h \leq e$ | $\frac{2^{e-h}}{n}$ |
| $h_{K-1}$ | $\frac{2^{e-h_{K-1}}}{n} - \frac{2}{n}$ |
| (If any) $h \leq h_{K-1} - 1$ | $\frac{2^{e-h}}{n} - \frac{2^{h_{K-1}+1-h}}{n} + \frac{2^{h_{K-1}-1-h}}{n}$ |

**Table 12:** $\Delta_{(d',d'')}\bar{H}(h)$

Along the proof, it is supposed that $d_{K-1}$ does not preserve the $H$-dominance and does preserve the $\bar{H}$-dominance. Moreover suppose that $h_K - 1 > h_{min}(d_{K-1})$. It is also supposed that there is a category $h \in \{h_{min}(d_{K-1}), ..., h_K - 1\}$ such that $\Delta_{(d',d)}\bar{F}(h) \geq \frac{2}{n}$.

**Part 1, the H dominance.** From Table 11 and Claim 1, I only need to focus on categories $h \in \{e+1, ..., h_K - 1\}$. By definition of $e$, we have for these categories:

$$\Delta_{(d',d)}\bar{F}(h) \leq \frac{1}{n}$$

Moreover, from property 4:

$$\Delta_{(d',d)}\bar{F}(h) \geq \frac{1}{n}$$

Hence for any category $h \in \{e+1, ..., h_K - 1\}$:

$$\Delta_{(d',d)}\bar{F}(h) = \frac{1}{n}$$

Using Lemma 1 and by definition of the dominance, we have in category $e$:

$$\Delta_{(d',d)}H(e) = \sum_{i=h_0}^{e-1} \Delta_{(d',d)}H(i) + \Delta_{(d',d)}F(e) \geq 0$$

Which implies that:

$$\sum_{i=h_0}^{e-1} \Delta_{(d',d)}H(i) \geq \Delta_{(d',d)}\bar{F}(e) \geq \frac{2}{n}$$

Hence in $e+1$:

$$\Delta_{(d',d)}H(e+1) = \sum_{i=h_0}^{e} \Delta_{(d',d)}H(i) - \frac{1}{n} \geq \frac{1}{n}$$

Knowing that for all $h \in \{e+1, ..., h_{K-1}\}$:

$$\Delta_{(d',d)}H(h) = 2^{h-e-1}\Delta_{(d',d)}H(e+1)$$

Hence, to conclude:

$$\Delta_{(d',d)}H(h) \geq \frac{2^{h-e-1}}{n}$$

Hence, $d''$ preserves the $H$-dominance.

**Part 2, the $\bar{\mathbf{H}}$ dominance.** From Table 12 and Claim 1, I only need to focus on categories $h \in \{h_0, ..., e-1\}$. Moreover, I only need to focus on the case when $e = h_K - 1$. Indeed, suppose that:

$$e \leq h_K - 2$$

In this case the reader can easily verify that the distribution $d_{K-1}$ can be obtained from $d''$ by a Hammond transfer. It implies that the $\bar{H}$-curve of $d_{K-1}$ lies nowhere above the $\bar{H}$-curve of $d''$ and, because it is assumed that $d_{K-1}$ preserves the $\bar{H}$-dominance, it implies that $d''$ also preserves the $\bar{H}$-dominance. I turn to the non-trivial case, when:

$$h_K - 1 = e$$

For categories $h \in \{h_{min}(d_{K-1}), ..., h_K - 1\}$:

$$\Delta_{(d',d)}\bar{F}(h) \geq \frac{1}{n}$$

Moreover in $h_K - 1$, by definition of $e$:

$$\Delta_{(d',d)}\bar{H}(h_K - 1) \geq \frac{2}{n}$$

Suppose by induction that there is a category $h \in \{h_{min}(d_{K-1}) + 1, ..., h_K - 1\}$ such that for any $h' \in \{h, ..., h_K - 1\}$, $\Delta_{(d',d)}\bar{H}(h') \geq \frac{2^{h_K-1-h'}}{n}$, then in $h-1$, from Lemma 1:

$$\begin{aligned}\Delta_{(d',d)}\bar{H}(h-1) &\geq \sum_{i=h}^{h_K-1} \frac{2^{h_K-1-i}}{n} + \frac{1}{n} \\ &\geq \frac{2^{h_K-h}}{n}\end{aligned}$$

Hence I just proved by induction that for any $h \in \{h_{min}(d_{K-1}), ..., h_K - 1\}$:

$$\Delta_{(d',d)}\bar{H}(h) \geq \frac{2^{h_K-1-h}}{n}$$

I now turn to the category $h_{min}(d_{K-1}) - 1$. First if $h_{min}(d_{K-1}) = h_K - 1$:

$$\Delta_{(d',d)}\bar{H}(h_{min}(d_{K-1}) - 1) = \Delta_{(d',d)}\bar{F}(h_{min}(d_{K-1})) + \Delta_{(d',d)}\bar{F}(h_{min}(d_{K-1}) - 1)$$

From Claim 3, if $h_K - 2 = h_{K-1}$ :

$$\Delta_{(d',d)}\bar{F}(h_{min}(d_{K-1})) + \Delta_{(d',d)}\bar{F}(h_{min}(d_{K-1}) - 1) \geq \frac{1}{n}$$

Moreover note that if $h_K - 2 = h_{K-1}$, from Table 12:

$$\Delta_{(d',d)}\bar{H}(h_K - 2) = \frac{1}{n}$$

Otherwise if $h_K - 2 > h_{K-1}$, from Claim 3:

$$\Delta_{(d',d)}\bar{F}(h_{min}(d_{K-1})) + \Delta_{(d',d)}\bar{F}(h_{min}(d_{K-1}) - 1) \geq \frac{2}{n} = \Delta_{(d',d)}\bar{H}(h_K - 2)$$

Second, if $h_{min}(d_{K-1}) = h_K - 2$, from Lemma 1 and Claim 3:

$$\begin{aligned}\Delta_{(d',d)}\bar{H}(h_{min}(d_{K-1}) - 1) &\geq \Delta_{(d',d)}\bar{H}(h_K - 1) + \Delta_{(d',d)}\bar{H}(h_K - 2) + \frac{1}{n} \\ &\geq \frac{4}{n}\end{aligned}$$

Finally, if $h_{min}(d_{K-1}) \leq h_K - 3$, from Lemma 1 and Claim 3:

$$\begin{aligned}\Delta_{(d',d)}\bar{H}(h_{min}(d_{K-1}) - 1) &\geq 2 \sum_{i=h_{min}(d_K-1)+1}^{h_K-1} \Delta_{(d',d)}\bar{H}(i) + \sum_{i=h_{min}(d_K-1)+1}^{h_{min}(d_{K-1})} \Delta_{(d',d)}\bar{F}(i) \\ &\geq 2 \sum_{i=h_{min}(d_K-1)+1}^{h_K-2} \frac{2^{h_K-1-i}}{n} + 2\Delta_{(d',d)}\bar{H}(h_K - 1) \\ &\geq 2(\frac{2^{h_K-1-h_{min}(d_K-1)}}{n} - \frac{2}{n}) + \frac{4}{n} \\ &\geq \frac{2^{h_K-h_{min}(d_K-1)}}{n}\end{aligned}$$

I now turn to categories $h \in \{h_0, ..., h_{min}(d_{K-1}) - 2\}$, from Lemma 5:

$$\begin{aligned}\Delta_{(d',d)}\bar{H}(h) = 2^{h_{min}(d_{K-1})-h} &\sum_{i=h_{min}(d_{K-1})+1}^{h_K-1} \Delta_{(d',d)}\bar{H}(i) \\ &+ \sum_{i=h}^{h_{min}(d_{K-1})} \Delta_{(d',d)}\bar{F}(i) + \sum_{i=h+2}^{h_{min}(d_{K-1})} 2^{i-h-2} \sum_{j=i}^{h_{min}(d_{K-1})} \Delta_{(d',d)}\bar{F}(j)\end{aligned}$$

If $h_{min}(d_{K-1}) = h_K - 2$, using Claim 3 and Lemma 5:

$$\begin{aligned}\Delta_{(d',d)}\bar{H}(h) &\geq 2^{h_{min}(d_{K-1})-h}\Delta_{(d',d)}\bar{H}(h_K - 1) \\ &\geq \frac{2^{h_{min}(d_{K-1})-h+1}}{n} \\ &\geq \frac{2^{h_K-1-h}}{n}\end{aligned}$$

Otherwise, if $h_{min}(d_{K-1}) < h_K - 2$, for categories $h \in \{h_0, ..., h_{min}(d_{K-1})\}$ from Lemma 5 and Claim 3:

$$\begin{aligned}\Delta_{(d',d)}\bar{H}(h) \geq 2^{h_{min}(d_{K-1})-h}\Delta_{(d',d)}\bar{H}(h_K - 1) + 2^{h_{min}(d_{K-1})-h} &\sum_{i=h_{min}(d_{K-1})+1}^{h_K-2} \Delta_{(d',d)}\bar{H}(i) \\ &+ \sum_{i=h_{K-1}+1}^{h_{min}(d_{K-1})} \frac{2^{i-h-2}}{n}\end{aligned}$$

which can be rewritten as:

$$\begin{aligned}\Delta_{(d',d)}\bar{H}(h) \geq \frac{2^{h_{min}(d_{K-1})-h+1}}{n} + 2^{h_{min}(d_{K-1})-h}(&\frac{2^{h_K-1-h_{min}(d_{K-1})}}{n} - \frac{2}{n}) + \frac{2^{h_{min}(d_{K-1})-h-1}}{n} - \frac{2^{h_{K-1}-1-h}}{n} \\ &\geq \frac{2^{h_K-1-h}}{n} + \frac{2^{h_{min}(d_{K-1})-h-1}}{n} - \frac{2^{h_{K-1}-1-h}}{n} \geq \frac{2^{h_K-1-h}}{n}\end{aligned}$$

which concludes the proof.

**Proposition 6.** I provide the variations generated by the Hammond transfer in Tables 13 and 14.

| $h$ | $\Delta_{(d',d'')}H(h)$ |
|---|---|
| (If any) $h \leq h_{K-1} - 1$ | $0$ |
| $h_{K-1}$ | $\frac{1}{n}$ |
| $h_{K-1} + 1 \leq h \leq h_{min}(d_{K-1}) - 1$ | $\frac{2^{h-h_{K-1}-1}}{n}$ |
| $h_{min}(d_{K-1}) \leq h \leq h_K - 1$ | $\frac{2^{h-h_{K-1}-1}}{n} - \frac{2^{h-h_{min}(d_{K-1})}}{n}$ |

**Table 13:** $\Delta_{(d',d'')}H(h)$

| $h$ | $\Delta_{(d',d'')}\bar{H}(h)$ |
|---|---|
| $h_{min}(d_{K-1}) \leq h \leq h_K - 1$ | $\frac{2^{h_K-1-h}}{n}$ |
| $h_{K-1} + 1 \leq h \leq h_{min}(d_{K-1}) - 1$ | $\frac{2^{h_K-1-h}}{n} - \frac{2^{h_{min}(d_{K-1})-1-h}}{n}$ |
| $h_{K-1}$ | $\frac{2^{h_K-1-h}}{n} - \frac{2^{h_{min}(d_{K-1})-1-h}}{n} - \frac{1}{n}$ |
| (If any) $h \leq h_{K-1} - 1$ | $\frac{2^{h_K-1-h}}{n} - \frac{2^{h_{min}(d_{K-1})-1-h}}{n} - \frac{2^{h_{K-1}-1-h}}{n}$ |

**Table 14:** $\Delta_{(d',d'')}\bar{H}(h)$

As before, I suppose here that $d_{K-1}$ does not preserve the $H$-dominance and that $h_{min}(d_{K-1}) < h_K - 1$ and for all $h \in \{h_{min}(d_{K-1}), ..., h_K - 1\}$, $\Delta_{(d',d)}\bar{F}(h) = \frac{1}{n}$.

**Part 1, the H dominance.** I only focus on the case when $h_{min}(d_{K-1}) > h_{K-1} + 1$. Indeed when $h_{min}(d_{K-1}) \leq h_{K-1} + 1$ it is trivial because for all $h \in \{h_{K-1} + 1, ..., h_K - 1\}$, $\Delta_{(d',d'')}H(h) = 0$. From Table 13, I only need to focus on categories $h \in \{h_{min}(d_{K-1}), ..., h_K - 1\}$. Indeed for categories $h \in \{h_{K-1}, ..., h_{min}(d_{K-1}) - 1\}$, $\Delta_{(d',d'')}H(h) = \Delta_{(d',d_{K-1})}H(h)$. Turning to categories $h \in \{h_{min}(d_{K-1}), ..., h_K - 1\}$ we note that:

$$\Delta_{(d',d)}F(h) = -\frac{1}{n}$$

It implies that:

$$\Delta_{(d',d)}H(h) = 2^{h-h_{min}(d_{K-1})}\Delta_{(d',d)}H(h_{min}(d_{K-1})) \tag{42}$$

From Lemma 1, in the category $h_{min}(d_{K-1})$:

$$\Delta_{(d',d)}H(h_{min}(d_{K-1})) \geq \Delta_{(d',d)}H(h_{K-1}) + \sum_{i=h_{K-1}+1}^{h_{min}(d_{K-1})-1} \Delta_{(d',d)}H(i) - \frac{1}{n} \tag{43}$$

Then, from Inequality (43):

$$\begin{aligned}\Delta_{(d',d)}H(h_{min}(d_{K-1})) &\geq \sum_{i=h_{K-1}+1}^{h_{min}(d_{K-1})-1} \Delta_{(d',d'')}H(i) \\ &\geq \sum_{i=h_{K-1}+1}^{h_{min}(d_{K-1})-1} \frac{2^{i-h_{K-1}-1}}{n} \\ &\geq \frac{2^{h_{min}(d_{K-1})-h_{K-1}-1}}{n} - \frac{1}{n}\end{aligned}$$

Given this, we conclude from (42) that, for all $h \in \{h_{min}(d_{K-1}), ..., h_K - 1\}$:

$$\Delta_{(d',d)} H(h_{min}(d_{K-1})) \geq \frac{2^{h-h_{K-1}-1}}{n} - \frac{2^{h-h_{min}(d_{K-1})}}{n} = \Delta_{(d',d'')} H(h)$$

**Part 2, the $\bar{\mathbf{H}}$-dominance.** For $h \in \{h_{min}(d_{K-1}), ..., h_K - 1\}$:

$$\Delta_{(d',d)} \bar{H}(h) = 2^{h_K-1-h} \Delta_{(d',d)} \bar{H}(h_K - 1) \geq \frac{2^{h_K-1-h}}{n}$$

In the category $h_{min}(d_{K-1}) - 1$:

$$\Delta_{(d',d)} \bar{H}(h_{min}(d_{K-1}) - 1) = 2 \sum_{i=h_{min}(d_{K-1})+1}^{h_K-1} \Delta_{(d',d)} \bar{H}(i) + \sum_{i=h_{min}(d_{K-1})-1}^{h_{min}(d_{K-1})} \Delta_{(d',d)} \bar{F}(i) \tag{44}$$

If $h_{min}(d_{K-1}) - 1 \geq h_{K-1} + 1$, then from Claim 3 and (44):

$$\begin{aligned}\Delta_{(d',d)} \bar{H}(h_{min}(d_{K-1}) - 1) &\geq 2 \sum_{i=h_{min}(d_{K-1})+1}^{h_K-1} \frac{2^{h_K-1-k}}{n} + \frac{1}{n} \\ &= \frac{2^{h_K-h_{min}(d_{K-1})}}{n} - \frac{1}{n} = \Delta_{(d',d'')} \bar{H}(h_{min}(d_{K-1}) - 1)\end{aligned}$$

Otherwise, if $h_{min}(d_{K-1}) - 1 = h_{K-1}$ we conclude again from Claim 3 and (44):

$$\begin{aligned}\Delta_{(d',d)} \bar{H}(h_{min}(d_{K-1}) - 1) &\geq 2 \sum_{i=h_{min}(d_{K-1})+1}^{h_K-1} \frac{2^{h_K-1-i}}{n} \\ &= \frac{2^{h_K-h_{min}(d_{K-1})}}{n} - \frac{2}{n} = \Delta_{(d',d'')} \bar{H}(h_{min}(d_{K-1}) - 1)\end{aligned}$$

Finally for categories $h \in \{h_0, ..., h_{min}(d_{K-1}) - 2\}$, we conclude from Lemma 5:

$$\begin{aligned}\Delta_{(d',d)} \bar{H}(h) = 2^{h_{min}(d_{K-1})-h} &\sum_{i=h_{min}(d_{K-1})+1}^{h_K-1} \Delta_{(d',d)} \bar{H}(i) \\ &+ \sum_{i=h}^{h_{min}(d_{K-1})} \Delta_{(d',d)} \bar{F}(i) + \sum_{i=h+2}^{h_{min}(d_{K-1})} \sum_{j=i}^{h_{min}(d_{K-1})} \Delta_{(d',d)} \bar{F}(j)\end{aligned}$$

If $h \geq h_{K-1} + 1$, Lemma 5 and Claim 3 both imply:

$$\begin{aligned}\Delta_{(d',d)} \bar{H}(h) \geq 2^{h_{min}(d_{K-1})-h} &\sum_{i=h_{min}(d_{K-1})+1}^{h_K-1} \frac{2^{h_K-1-i}}{n} + \frac{1}{n} + \sum_{i=h+2}^{h_{min}(d_{K-1})} \frac{2^{i-h-2}}{n} \\ &= \frac{2^{h_K-1-h}}{n} - \frac{2^{h_{min}(d_{K-1})-h-1}}{n} = \Delta_{(d',d'')} \bar{H}(h)\end{aligned}$$

In the case where $h \in \{h_{K-1} - 1, h_{K-1}\}$, we obtain from Claim 3 that $\sum_{i=h}^{h_{min}(d_{K-1})} \bar{F}(i) \geq 0$. Hence from Lemma 5 and Claim 3:

$$\Delta_{(d',d)} \bar{H}(h) \geq \frac{2^{h_K-1-h}}{n} - \frac{2^{h_{min}(d_{K-1})-h-1}}{n} - \frac{1}{n} = \Delta_{(d',d'')} \bar{H}(h)$$

Finally, for $h \in \{h_0, ..., h_{K-1} - 1\}$, from Lemma 5 and Claim 3:

$$\begin{aligned}\Delta_{(d',d)} \bar{H}(h) \geq 2^{h_{min}(d_{K-1})-h} &\sum_{i=h_{min}(d_{K-1})+1}^{h_K-1} \frac{2^{h_K-1-i}}{n} + \sum_{i=h_{K-1}+1}^{h_{min}(d_{K-1})} \frac{2^{i-h-2}}{n} \\ &= \frac{2^{h_K-1-h}}{n} - \frac{2^{h_{min}(d_{K-1})-1-h}}{n} - \frac{2^{h_{K-1}-1-h}}{n} = \Delta_{(d',d'')} \bar{H}(h)\end{aligned}$$

which concludes the proof.

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