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Working Papers / Documents de travail

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WP 2025 - Nr 10

Appraising the central tendency of distributions of a cardinal and an ordinal variable[*]

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April 14th 2025

Abstract

This paper provides a simple unified axiomatic framework for appraising the central tendency of distributions of a single attribute (pie) among a collection of individuals depending upon the available measurement of the attribute. Two types of measurement are considered: cardinal and ordinal. For each of them, three properties are posited on an ordering of distributions of numbers among individuals. The two first properties are the anonymity requirement that permutations of the same list of numbers be equivalent and the weak Pareto requirement that a strict increase in the value of the variable for all individuals be favorably appraised. The third property requires that inverting the numerical measurement of the variable leads to an inversion of the ranking of the any two distributions to which the inversion is applied. The mean of a distribution is shown to be the only ordering of distributions consistent with cardinal measurability that satisfies those three requirements in the cardinal context while the median is the only such ranking consistent with ordinal measurability of the variable that satisfies those same requirement if the number of individuals is odd. If the number of individuals is even, then those three requirements applied to the ordinal context are shown to be inconsistent.

JEL classification: D71, D72

Keywords:

mean, median, cardinal, ordinal, measurement, consistency

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[*]We are indebted, with the usual disclaiming qualification, to Brice Magdalou for useful discussions on the subject matter of this paper. We also acknowledge financial support from the "France 2030" investment plan managed by the French National Research Agency (ANR) Grant ANR-17-EURE-0020 and by the Excellence Initiative of Aix-Marseille University - A*MIDEX as well as from the ANR research grant MADIMIN (ANR-24-CE26-3823-21).

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1 Introduction

When can we say that there is "more to distribute" in one distribution than in another? If the distributions concern a variable such as income, consumption, or wealth that is believed to be *cardinally* measured, a (very) common answer to this question is: when the mean is larger! The mean of a distribution is, indeed, the most widely used indicator of its "central tendency". It is so important and natural that almost all concerns for "reducing inequalities" in the distribution of a cardinally measurable variable are expressed in a way that are mean-preserving. One of the most important transfer principle satisfied by a vast majority of inequality indices states indeed that a transfer of a given quantity of the variable from a relatively well-endowed to a relatively less well-endowed person should always be seen as inequality reducing. Transfers of these kinds are called Pigou-Dalton transfers. They incorporate, in their very definition, the requirement that they preserve the mean of the distribution[1].

The answer to the above question is less clear when the distributions concern a variable, such as health or declared happiness, whose meaningful measurement is only believed to be *ordinal*. As very well-argued by Allison and Foster (2004) in the case of health status, the mean of a distribution can not meaningfully serve as an indicator of the central tendency of an ordinal variable. The reason for this is clear. The ranking of two distributions on the basis of their mean is *not* invariant with respect to changes in the units of measurement of the variable that are consistent with its ordinal character. If there are three increasingly ordered levels of declared happiness - "low", "normal" and "high" say - then any numbering of those levels consistent with this ordering can serve as an ordinal scale. Consider two distributions of five individuals into these three happiness levels. In one distribution, three individuals are in the low category and the two others are in the high category. In the other distribution, all five individuals are in the middle category. The ranking of these two distributions in terms of their mean crucially depends upon the particular scale used to assign happiness levels to the categories. If one uses the (natural ?) scale which assigns numbers 1, 2 and 3 to the respective three categories, the mean will be larger in the second distribution (2) than in the first (9/5). However if one uses the square of those three numbers as indicators of happiness levels, one would obtain the reverse conclusion that the mean is larger in the first distribution (21/5) than in the second (4).

An indicator of the "size of the pie" that may seem more appropriate when applied to distributions of ordinal variables is the *median*, defended by Allison and Foster (2004), Apouey (2007) and Cowell and Flachaire (2017)

[1] A Pigou-Dalton transfer produces indeed the converse of a "mean-preserving-spread" (see e.g. Rotshchild and Stiglitz (1970)) on the distribution of the attribute.

among many other contributions interested in comparing distributions of ordinal variables. It is clear indeed that the ranking of distributions provided by their median does *not* depend upon the particular numerical representation of the variable of interest. In the example above, the second distribution, where the "normal" category is the median, would be ranked above the first, where the "low" category is the median, irrespective of the particular numerical scale used to measure happiness. But the median is not the only possible indicator of the "central tendency" of a distribution that produces a ranking independent from the numerical scale used to represent the ordinal variable. Other possible candidates are the *mode*, as well as *positional dictatorial* rules (see e.g. d'Aspremont and Gevers (1977), Gevers (1979) or Blackorby, Donaldson, and Weymark (1984)) and criteria based on some prespecified quantile of the distribution (the median being only one of them). Identifying the proper criterion for appraising the central tendency of a distribution of an ordinal variable is all the more important as this criterion - if identified - would be the natural reference point with respect to which the extent of inequality in the distribution of an ordinal variable could be appraised. Allison and Foster (2004) have proposed for example a notion of "spread away from the median" as a natural definition of increase in inequality in the distribution of an ordinal variable. A "spread away from the median" is any change in the distribution that either transfer probability mass away from the median at the right or at the left of it (or both) while preserving the median. They have therefore strongly endorsed the median as the natural reference. Cowell and Flachaire (2017), who use a different approach consisting in measuring the difference between a given distribution and a specific equal distribution in which the probability mass is concentrated in one reference point, have considered in turns different such reference point, including the median and the maximal value of the variable.

Along the lines of Hammond (1976), Gravel, Magdalou, and Moyes (2021) (see also Bennia, Gravel, Magdalou, and Moyes (2022)) have proposed the notion of Hammond transfer as the analogue, for distributions of an ordinal variable, to Pigou-Dalton transfers as an elementary undisputable notion of "inequality reduction". A Hammond transfer is like a Pigou-Dalton transfer, but without the requirement - meaningless in for an ordinal variable - that what is taken away from the giver be equal to what is given to the receiver. Gravel, Magdalou, and Moyes (2021) have proposed an operational dominance criterion that is equivalent to the fact of going from the dominated to the dominating distribution by a finite sequence of such Hammond transfers.[2] While a Hammond transfer provides a somewhat plausible definition of "elementary inequality reduction" applicable to distributions of an ordinal variable, it does *not* preserve any known notion of central ten-

[2] They have not however provided a direct constructive proof of this equivalence. Such a proof has been recently provided by Gargani (2025).

dency. Gravel, Magdalou, and Moyes (2019) have actually shown that the fact of going from a distribution of a continuous ordinal variable to another by a finite sequence of Hammond transfers is equivalent to the dominance of the first distribution by the second by *both* the lexicographic extension of the maximin criterion *and* the lexicographic extension of the minimax criterion. As a consequence of this result, for any position of an individual in a distribution of the ordinal variable, one can find a distribution in which a Hammond transfer that changes the quantity of the ordinal variable received by the individual in that position can be performed. There are thus no position that will be preserved by all Hammond transfers. However, one could think of specific Hammond transfers that would preserve the median or any other positions in the distribution other than the max or the min. After all, Pigou-Dalton transfers are nothing else than specific Hammond transfers that preserve the mean of the distribution to which they are applied. The question thus arises: what is the indicator of central tendency that Hammond transfers should seek preserve when applied to distributions of an ordinal variable?

In this note, we provide an axiomatic answer to this question within an integrated framework that covers the cardinal and the ordinal case. We propose three properties that an ordering of distributions, among a given population of individuals, of an indicator of well-being (pie) should satisfy to serve as a plausible notion of the "size of the pie". The first property is the *weak Pareto* requirement that a strict increase in the amount of the pie received by everyone should be recorded as a clear instance of an increase in pie's size. The second property is the *anonymity* requirement that individuals's names don't matter, and that all permutations of the same list of values of the indicator are equivalent. The third property is the somewhat less noticed *reversal consistency* requirement that the ranking of the size of the pie of two distributions be reversed in the case where the numerical scale used to measure the pie's size is reversed. Imagine indeed that one measures the pie positively (the more the better) and that the used criterion of the size of the pie considers that there is more pie to distribute in one distribution than the other. Then, it seems quite natural that the ranking of the two distributions be reversed if pie size was measured negatively (in which case the distribution with a smaller pie should be ranked below that with a larger pie).

As it turns out, the only ordering of lists of cardinally meaningful well-being levels that satisfies these three requirements is the ordering resulting from the comparison of their mean or, equivalently in our fixed population setting, their sum. The proof of this is quite straightforward, and results mainly from a classical result by Milnor (1954) exploited by d'Aspremont and Gevers (1977). When these three requirements are imposed on an ordering of lists of ordinally meaningful well-being levels, they characterize the ranking of those lists based on their median element when the number of individuals is odd. However, they lead to an impossibility when the number of individuals is even. Hence, at least when the number of recipients of the pie is odd, it happens that the median is the analogue, when the pie quantities are only ordinally significant, to the mean in the case where those quantities are measured cardinally. From this point of view at least, the median is "the" natural criterion for assessing central tendency in the distribution of an ordinally measurable variable as is the mean when the variable is cardinally measurable. To the very best of our knowledge, this characterization of the median is novel, even though rankings of finite sets of objects that are equivalent to their median element have been characterized by Nitzan and Pattanaik (1984).

The remaining of this note is as follows. The next section describes the model and the properties imposed on the ordering of distributions. Section 3 states, proves and discusses the results and section 4 concludes.

2 The model

We are interested in comparing distributions of an *indicator* of well-being such as income, health, education, self-declared happiness, etc. between a given number, n say, of individuals.[3] Any such distribution is depicted as a list $\mathbf{y} = (y_1, ..., y_n)$ of n real numbers - interpreted as a column vector - where, for $i = 1, ..., n$, y_i denoted the value of the indicator for individual i in distribution $\mathbf{y}$. We are interested in the properties of an ordering[4] $\succsim$ on $\mathbb{R}^n$ - with asymmetric and symmetric factors $\succ$ and $\sim$ respectively - that compares any two distributions $\mathbf{y}$ and $\mathbf{z}$ in $\mathbb{R}^n$ on the basis of the "size of the pie". The very definition of this notion depends upon the information that the considered indicator of well-being is conveying. We consider here two possible such informations: ordinal, and cardinal. We first define as follows the requirement for the ranking $\succsim$ to use ordinal information on well-being.

Definition 1 *The ordering $\succsim$ on $\mathbb{R}^n$ uses ordinal information on individual well-being if for any $(y_1, ..., y_n)$ and $(z_1, ..., z_n) \in \mathbb{R}^n$ and any increasing function $f : A \longrightarrow B$ for some subsets A and B of real numbers such that $y_i \in A$ and $z_i \in A$ for $i = 1, ..., n$, one has $(y_1, ..., y_n) \succsim (z_1, ..., z_n) \Longleftrightarrow (f(y_1), ..., f(y_n)) \succsim (f(z_1), ..., f(z_n))$*

In plain English, a ranking of distributions of an indicator of individual well-being uses ordinal information about the indicator if it is unaffected by any change in the unit of measurement of the indicator that preserves the ordering of the values achieved by the indicator for the different individuals.

[3] All results of this paper can be extended to distributions involving a variable collection of individuals if the Dalton (1920) replication axiom is added to the other fixed population properties discussed herein.

[4] An ordering is a reflexive, complete and transitive binary relation.

Changes in units of measurement that preserve the ranking of the values of the indicator for the different people are precisely those changes that result from transforming the initial units of measurement by means of an increasing function.

Well-known examples of orderings on $\mathbb{R}^n$ that use ordinal information on the indicator of well-being are *positional dictatorship* rankings discussed in classical social choice theory by Gevers (1979), Roberts (1980) and Blackorby, Donaldson, and Weymark (1984) among others. To define those rankings, denote, for any distribution $\mathbf{y} \in \mathbb{R}^n$, its ordered permutation $\mathbf{y}_{(.)}$ defined by:

$$\mathbf{y}_{(.)} = \boldsymbol{\pi}.\mathbf{y}$$

for some $n \times n$ permutation matrix $\boldsymbol{\pi}$ such that $y_{(i)} \leq y_{(i+1)}$ for all $i = 1, ..., n-1$. An ordering $\succsim$ on $\mathbb{R}^n$ is called a positional dictatorship ranking if there exists a position $i \in \{1, ..., n\}$ such that for every $\mathbf{y}$ and $\mathbf{z} \in \mathbb{R}^n$, $y_{(i)} > z_{(i)} \implies \mathbf{y} \succ \mathbf{z}$. Hence, a positional dictatorship ranking of distributions is based on the well-being achievement of the individual who is in some specific position in the ranking of well-being levels. Well-known examples of positional dictatorship rankings are the Maxi-Min and the Lexi-min rules, discussed notably by Hammond (1976) and d'Aspremont and Gevers (1977), where the worst position "dictates" its preference upon the society. Observe that positional dictatorships only restrict the social ranking in cases where the well-being level of the individual in the relevant position is different in the two considered distributions (the distribution providing the position with the higher well-being being considered the better). However, positional dictatorship does not restrict in any way the social ranking of distributions that provide the individual in the relevant position with the same level of well-being. For example both the Maxi-min and the Lexi-min rankings are positional dictatorships based on the worst position.

Other examples of positional dictatorship rankings - examined more closely in this paper - are those based on the median position. The median position is defined to be $(n+1)/2$ if n is odd and either $n/2$ or $n/2+1$ if n is even. Hence the median position is not uniquely defined if the number of individuals is even and, as it turns out, this will create some difficulty. An ordering $\succsim$ on $\mathbb{R}^n$ is called a median ranking if it is a positional dictatorship for a median position (unique if n is odd). When n is odd, we denote the (unique) median ordering of $\mathbb{R}^n$ by $\succsim^{med}$

A good example of a ranking of distributions of an indicator of well-being that does not use ordinal information about that indicator is the ordering $\succsim^{sum}$ defined by:

$$\mathbf{y} \succsim^{sum} \mathbf{z} \iff \sum_{i=1}^{n} y_i \geq \sum_{i=1}^{n} z_i$$

which corresponds to the usual ranking of distributions based on the per capita value of the indicator (when the number of individuals is the same in the two distributions). This ranking does not use ordinal information about the indicator because, for $n = 2$ for example, it would consider that $(3, 3) = (9^{1/2}, 9^{1/2}) \sim (2, 4) = (4^{1/2}, 16^{1/2})$ but would not consider that $(9, 9) \sim (4, 16)$ even thought the function $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ defined by $f(x) = x^{1/2}$ is increasing.

If $\succsim^{sum}$does not use ordinal information on the indicator used to measure individual well-being, it does however use cardinal information about it. The formal definition of cardinal information used in this paper, which corresponds to what classical social theory (see e.g. Blackorby, Donaldson, and Weymark (1984)) calls "cardinal-unit comparability", is the following.

Definition 2 *The ordering $\succsim$ on $\mathbb{R}^n$ uses cardinal information on individual well-being if for any $(y_1, ..., y_n)$ and $(z_1, ..., z_n) \in \mathbb{R}^n$ and any list of $n+1$ real numbers $a_1, ..., a_n$ and b with $b > 0$ one has $(y_1, ..., y_n) \succsim (z_1, ..., z_n) \iff (a_1 + by_1, ..., a_n + by_n) \succsim (a_1 + bz_1, ..., a_n + bz_n)$*

Functions f from some subset A of real number into some (possibly different) subset of real numbers B defined, for every $x \in A$, by $f(x) = a + bx$ for some real numbers a and b are called affine functions. An affine function is increasing if and only if the real number b is strictly positive. Hence, a ranking of distributions of indicators of well-being uses a cardinal information about this indicator if it is invariant with respect to any change in the measurement of the indicator obtained by means of an increasing affine function. Observe that the requirement that an ordering uses cardinal information on individual well-being does not require that the same increasing affine transformation be applied to all individuals. Individuals' well-being may be assigned different a_i. However, they must be assigned the same strictly positive b. The multiplication of all individual values of the well-being indicator by the same b implies that the gains and losses of well-being in moving from one distribution to another are comparable across individuals and meaningful, but that their levels of well-being need not be.

We are interested in identifying the property of an ordering $\succsim$ of distributions based on the "size of their pie" that uses either ordinal or cardinal information about the measurement of the indicator as defined above. We will be interested in an ordering that satisfies the following anonymity principle, requiring that the "individuals' names don't matter".

Axiom 1 *Anonymity. For any $(y_1, ..., y_n) \in \mathbb{R}^n$, $(y_1, ..., y_n) \sim (y_{i(1)}, ..., y_{i(n)})$.*

The other requirement that we require an ordering of distributions based on the "size of their cake " to satisfy is the weak Pareto principle, which we define formally as follows.

Axiom 2 *Weak Pareto. For any $(y_1, ..., y_n)$, $(z_1, ..., z_n) \in \mathbb{R}^n$ such that $y_i \geq z_i$ for all $i \in \{1, ..., n\}$, $(y_1, ..., y_n) \succsim (z_1, ..., z_n)$ and for any $(y_1, ..., y_n)$,*

$(z_1, ..., z_n) \in \mathbb{R}^n$ such that $y_i > z_i$ for all $i \in \{1, ..., n\}$, $(y_1, ..., y_n) \succ (z_1, ..., z_n)$.

The last property that seems plausible to impose on the ordering $\succsim$ on $\mathbb{R}^n$ is a *consistency* with respect to the way in which well-being (cake) is measured. We may indeed want to measure income positively (more affluence is better) or negatively (less poverty is better). Similarly, we may want to measure the quality of the environment positively (a cleaner environment is better) or negatively (more pollution is worse). If what we are after is a criterion for appraising the "size of the pie", it would seem natural that the ranking of a pie measured positively be the opposite of the ranking of a pie measured negatively. It seems therefore natural to impose one of the following two properties of measurement consistency whose formulation depend upon whether or not the numerical indicator is supposed to convey ordinal or cardinal information on well-being (pie).

Axiom 3 *Ordinal Reversal Consistency. For any $(y_1, ..., y_n)$, $(z_1, ..., z_n) \in \mathbb{R}^n$ and any strictly decreasing function $f : A \longrightarrow B$ for some subsets A and B of real numbers such that $y_i \in A$ and $z_i \in A$ for $i = 1, ..., n$, one has $(y_1, ..., y_n) \succsim (z_1, ..., z_n) \Longleftrightarrow (f(y_1), ..., f(y_n)) \precsim (f(z_1), ..., f(z_n))$.*

Axiom 4 *Cardinal Reversal Consistency. For any $(y_1, ..., y_n)$, $(z_1, ..., z_n) \in \mathbb{R}^n$ and any list of $n + 1$ real numbers $a_1, ..., a_n$ and b with $b < 0$ one has $(y_1, ..., y_n) \succsim (z_1, ..., z_n) \Longleftrightarrow (a_1 + by_1, ..., a_n + by_n) \precsim (a_1 + bz_1, ..., a_n + bz_n)$.*

The requirement of ordinal reversal consistency has been used under the name of *duality* in the somewhat different context of ranking finite sets of objects by Nitzan and Pattanaik (1984). Requirements that a ranking of distributions be restricted in a certain way when an inversion in the measurement of the variable is performed have also been discussed a bit in the context of inequality measurement, notably by Yalonetsky (2022) and Abul-Naga and Yalonetsky (2024). In the context of measuring inequality, it is often required that the ranking of inequality be invariant to an inversion in the measurement of the variable. This makes some sense for a notion of inequality that is concerned with the dispersion of the variable away from its central tendency. Hence some argument could be made in favor of the requirement that this "pure dispersion" be independent from whether or not the phenomenon is measured negatively or positively. However, it seems quite natural that a ranking of distributions based on their central tendency be reversed if the measurement of the phenomenon becomes negative.

A first thing to notice about those requirements of ordinal and cardinal reversal consistency is that they each entail ordinal and cardinal measurement of Definitions 1 and 2 respectively, as established in the following proposition.

Proposition 1 *If an ordering* $\succsim$ *of* $\mathbb{R}^n$ *satisfies Ordinal Reversal Consistency, then it uses ordinal information on individual well-being. Similarly, if* $\succsim$ *of* $\mathbb{R}^n$ *satisfies Cardinal Reversal Consistency, then it uses cardinal information on individual well-being.*

Proof. *Suppose that* $\succsim$ *is an ordering of* $\mathbb{R}^n$ *that satisfies Ordinal Reversal Consistency. Hence, for any two distributions* $(y_1, ..., y_n)$, $(z_1, ..., z_n) \in \mathbb{R}^n$ *and any strictly decreasing function* $f : A \longrightarrow B$ *for some subsets* A *and* B *of real numbers such that* $y_i \in A$ *and* $z_i \in A$ *for* $i = 1, ..., n$, *one has* $(y_1, ..., y_n) \succsim (z_1, ..., z_n) \iff (f(y_1), ..., f(y_n)) \precsim (f(z_1), ..., f(z_n)) \Leftrightarrow (-f(y_1), ..., -f(y_n)) \succsim (-f(z_1), ..., -f(z_n))$. *Hence* $\succsim$ *uses ordinal information on individual well-being as per Definition 1 since a function* $f : A \longrightarrow B$ *is decreasing (increasing) if and only if* $-f$ *is increasing (decreasing). Similarly, assume that hat* $\succsim$ *is an ordering of* $\mathbb{R}^n$ *that satisfies Cardinal Reversal Consistency. Then, for any* $(y_1, ..., y_n)$, $(z_1, ..., z_n) \in \mathbb{R}^n$ *and any list of* $n+1$ *real numbers* $a_1, ..., a_n$ *and* b *with* $b < 0$ *one has* $(y_1, ..., y_n) \succsim (z_1, ..., z_n) \iff (a_1 + by_1, ..., a_n + by_n) \precsim (a_1 + bz_1, ..., a_n + bz_n) \iff (a_1 - by_1, ..., a_n - by_n) \succsim (a_1 - bz_1, ..., a_n - bz_n)$. *Hence* $\succsim$ *uses cardinal information on individual well-being as per Definition 2.* ■

In the next section, we show that the requirements of cardinal and ordinal measurement consistency imposed on anonymous orderings of $\mathbb{R}^n$ that satisfy weak Pareto characterize the ranking $\succsim^{sum}$ and $\succsim^{med}$ (the later when n is odd) respectively. We also show that there does not exist an anonymous ordering of $\mathbb{R}^n$ that satisfies weak Pareto and ordinal measurement consistency when n is even.

3 Results

We first characterize the ordering $\succsim^{sum}$ as the only ordering of $\mathbb{R}^n$ that satisfies Anonymity, Weak Pareto and Cardinal Measurement Consistency. We provide the proof for completeness even though it can be found in Milnor (1954) (Theorem 2, p. 53) and d'Aspremont and Gevers (1977) (Theorem 3, p. 203) thanks to Proposition 1.

Theorem 1 *An ordering* $\succsim$ *of* $\mathbb{R}^n$ *satisfies Anonymity, Weak Pareto and Cardinal Reversal Consistency if and only if* $\succsim = \succsim^{sum}$.

Proof. *We leave to the reader the easy task of verifying that* $\succsim^{sum}$ *is an ordering of* $\mathbb{R}^n$ *that satisfies Anonymity, Weak Pareto and Cardinal Reversal Consistency. To go in the other direction, consider an ordering* $\succsim$ *of* $\mathbb{R}^n$ *that satisfies Anonymity, Weak Pareto and Cardinal Reversal Consistency. Let us first show that any two distributions* $(y_1, ..., y_n)$, $(z_1, ..., z_n) \in \mathbb{R}^n$ *such that* $\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} z_i$ *should be equivalent. Indeed, by Anonymity* $(y_1, ..., y_n) \sim$

$(y_{(1)}, ..., y_{(n)})$ *and* $(z_1, ..., z_n) \sim (z_{(1)}, ..., z_{(n)})$. *Since by Proposition 1,* $\succsim$ *uses cardinal information on individual well-being as per Definition 2, it follows that* $(y_1, ..., y_n) \sim (y_{(1)}, ..., y_{(n)}) \succsim (z_{(1)}, ..., z_{(n)}) \sim (z_1, ..., z_n) \Leftrightarrow (y_{(1)} - \min(y_{(1)}, z_{(1)}), ..., y_{(n)} - \min(y_{(n)}, z_{(n)})) \succsim (z_{(1)} - \min(y_{(1)}, z_{(1)}), ..., z_{(n)} - \min(y_{(n)}, z_{(n)}))$. *Each of the distribution* $(y_{(1)} - \min(y_{(1)}, z_{(1)}), ..., y_{(n)} - \min(y_{(n)}, z_{(n)}))$ *and* $(z_{(1)} - \min(y_{(1)}, z_{(1)}), ..., z_{(n)} - \min(y_{(n)}, z_{(n)}))$ *is equivalent to its respective ordered permutation thanks to Anonymity again. Reapplying the same operation of subtracting from each component of the two ordered permutation their minimal value leads eventually, because the sum of the components of the two vectors is the same, to the conclusion that* $(0, ..., 0) \succsim (0, ..., 0) \Longleftrightarrow (y_1, ..., y_n) \succsim (z_1, ..., z_n)$ *thanks to the transitivity of* $\succsim$. *The statement* $(y_1, ..., y_n) \sim (z_1, ..., z_n)$ *then follows at once from the reflexivity of* $\succsim$. *Applying weak Pareto and transitivity to this conclusion that* $(y_1, ..., y_n) \sim (z_1, ..., z_n)$ *for any two* $(y_1, ..., y_n)$ *and* $(z_1, ..., z_n)$ $\mathbb{R}^n$ *such that* $\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} z_i$ *gives the required conclusion that* $(y_1, ..., y_{(n)}) \succsim (z_1, ..., z_n) \Longleftrightarrow \sum_{i=1}^{n} y_i \geq \sum_{i=1}^{n} z_i$. ■

We now show that a similar result holds for the median ordering $\succsim^{med}$when cardinal measurement consistency is replaced by ordinal measurement consistency and the number of individuals is odd.

Theorem 2 *An ordering* $\succsim$ *of* $\mathbb{R}^n$ *with* n *odd satisfies Anonymity, Weak Pareto and Ordinal Measurement Consistency if and only if* $\succsim = \succsim^{med}$.

Proof. *We leave to the reader the task of verifying, if* n *is odd, that* $\succsim^{med}$ *satisfies Anonymity, Weak Pareto and Ordinal Measurement Consistency. In the other direction, Let* $\succsim$ *be an ordering of* $\mathbb{R}^n$ *that satisfies Anonymity, Weak Pareto and Ordinal Measurement Consistency. We observe that, thanks to Proposition 1,* $\succsim$ *uses ordinal information on individual well-being. Hence, applying Theorem 4 in Gevers (1979) (see also Roberts (1980), Theorem 4), there exists a position* $i \in \{1, ..., n\}$ *such that for any two distributions* $(y_1, ..., y_n)$ *and* $(z_1, ..., z_n)$ *in* $\mathbb{R}^n$*, one has (thanks to anonymity and transitivity)* $(y_1, ..., y_n) \sim (y_{(1)}, ..., y_{(n)}) \succ (z_{(1)}, ..., z_{(n)}) \sim (z_1, ..., z_n)$ *whenever* $y_{(i)} > z_{(i)}$. *Let us show that the dictatorship of any position* i *other than* $(n + 1)/2$ *could lead to a violation of Ordinal Measurement Consistency. To do so, consider any such position* i *distinct from* $(n + 1)/2$ *and two distributions* y *and* z *in* $\mathbb{R}^n$ *such that* $y_{(i)} > z_{(i)}$ *and* $y_{(n-i+1)} < z_{(n-i+1)}$. *It is always possible to find two such distributions whenever* $i \neq (n+1)/2$. *It would not be possible of course if* $i = (n+1)/2$ *because in this case,* $i = (n + 1)/2 = n - i + 1 = n - ((n + 1)/2) + 1 = (n + 1)/2$. *Since* i *is a positional dictator,* $\mathbf{y} \succ \mathbf{z}$. *Consider now the distributions* $(f(y_{(1)}), ..., f(y_{(n)}))$ *and* $(f(z_{(1)}), ..., f(z_{(n)}))$ *for some decreasing function* f.

Since f is decreasing, we have that $f(y_{(1)}) \geq f(y_{(2)}) \geq ..., \geq f(y_{(n)})$ and $(f(z_{(1)}) \geq f(z_{(2)}) \geq ..., \geq f(z_{(n)})$ so that the (increasingly) ordered permutations of $(f(y_{(1)}), ..., f(y_{(n)}))$ and $(f(z_{(1)}), ..., f(z_{(n)}))$ are $(f(y_{(n)}), ..., f(y_{(1)}))$ and $(f(z_{(n)}), ..., f(z_{(1)}))$. Observe now that the well-being levels associated to position i in the vectors $(f(y_{(n)}), ..., f(y_{(1)}))$ and $(f(z_{(n)}), ..., f(z_{(1)}))$ are respectively $f(y_{(n-i+1)})$ and $f(z_{(n-i+1)})$. Since $y_{(n-i+1)} < z_{(n-i+1)}$ and f is decreasing, we have $f(y_{(n-i+1)}) > f(z_{(n-i+1)})$ which, by the assumed positional dictatorial status of i, implies $(f(y_{(n)}), ..., f(y_{(1)})) \succ (f(z_{(n)}), ..., f(z_{(1)}))$ which, given the anonymity and transitivity of $\succsim$, leads to a violation of ordinal measurement consistency. The only positional dictatorship rule that satisfies ordinal measurement consistency is the position $(n+1)/2$ - which exists only if n is odd - because for any distribution $\mathbf{y}$ in $\mathbb{R}^n$ and any decreasing function f, $y_{(n+1)/2}$ is the component in the $(n+1)/2$ position in the increasingly ordered permutation of $\mathbf{y}$ if and only if $f(y_{(n+1)/2})$ is in the $(n+1)/2$ position of the increasingly ordered permutation of the distribution $f(\mathbf{y})$. Hence any strict ranking of $y_{(n+1)/2}$ against any $z_{(n+1)/2}$ would be the converse, for a decreasing f, of the ranking of $f(y_{(n+1)/2})$ against $f(z_{(n+1)/2})$. ■

It should be observed that this Theorem shows that Ordinal Reversal Consistency is stronger than the requirement that the ordering uses ordinal information on individual well-being. This latter requirement combined with Anonymity and Weak Pareto would generate the whole family of positional dictatorships. The fact of imposing Ordinal Reversal Consistency singles out the median as the only admissible one in this large class. The importance of the restriction that n is odd is significant for this result however. Indeed, as shown in the following simple theorem, there does not exist any ordering of $\mathbb{R}^n$ that satisfies Anonymity, Weak Pareto and Ordinal Reversal Consistency if n is even.

Theorem 3 *There does not exist any ordering of $\mathbb{R}^n$ that satisfies Anonymity, Weak Pareto and Ordinal Measurement Consistency if n is even..*

Proof. *Let $\succsim$ be an ordering of $\mathbb{R}^n$ that satisfies Anonymity, Weak Pareto and Ordinal Measurement Consistency. We observe that, thanks to Proposition 1, $\succsim$ uses ordinal information on individual well-being. Hence, applying Theorem 4 in Gevers (1979) (see also Roberts (1980), Theorem 4), there exists a position $i \in \{1, ..., n\}$ such that for any two distributions $(y_1, ..., y_n)$ and $(z_1, ..., z_n)$ in $\mathbb{R}^n$, one has (thanks to anonymity and transitivity) $(y_1, ..., y_n) \sim (y_{(1)}, ..., y_{(n)}) \succ (z_{(1)}, ..., z_{(n)}) \sim (z_1, ..., z_n)$ whenever $y_{(i)} > z_{(i)}$. Since there are no position i such that $i = (n+1)/2$ if n is even, one can thus apply the reasoning of Theorem 2 to show that any other positional dictatorial rule would violate Ordinal Measurement Consistency (given transitivity and anonymity).* ■

4 Conclusion

The main conclusion of this paper is that if one accepts the prerequisite that any ordering of distributions of an attribute between a given collection of individuals on the basis of their central tendency should satisfy the weak Pareto principle, be anonymous and be "reversely consistent" with the nature of the measurement of the attribute, then one is led to the conclusion that the ordering should be based on the mean in the cardinal case and on the median in the ordinal case (when the number of individuals is odd). This suggests, therefore, that the median is a highly natural criterion for appraising the central tendency - or the size of the pie - of a distribution of an ordinal variable. This immediately suggests an important avenue for future research in ordinal inequality measurement: that of obtaining an implementable criterion for verifying when a distribution has been obtained from another by a finite sequence of median preserving Hammond transfers. Gravel, Magdalou, and Moyes (2021) (and more directly Gargani (2025)) have identified an easily implementable criterion, the intersection of two dominance, that is equivalent to the fact of going from the dominated to the dominating distribution by a finite sequence of Hammond transfers. Yet many of these transfers do not preserve the median, and therefore cannot be considered to capture pure equalization in an ordinal setting. One could think of course of applying the intersection of the two dominance proposed in Gravel, Magdalou, and Moyes (2021) to distributions with the same median. However, as shown in Gargani (2025), there are examples of situations where the intersection of the two dominances of Gravel, Magdalou, and Moyes (2021) is observed between two distributions with the same median but where it is not possible to go from the doubly dominated to the doubly dominant distribution only by median preserving Hammond transfers. Some non-median preserving Hammond transfers may be required in the process. Hence, the identification of an implementable criterion that coincides with the possibility of going from a distribution to another by median preserving transfers is an important step in the research agenda.

Another step would be to alleviate a bit the invariance requirement imposed on the ordering when applied to distributions of cardinally measurable variables. This invariance requirement, albeit satisfied by comparisons of distribution on the basis of their sum (or their mean), is not terribly natural, because it allows the transformation of the value of the indicators in a way that depends upon the individuals. It would be obviously more satisfactory to use the weaker invariance requirement underlying the full cardinal comparability (using the terminology of Blackorby, Donaldson, and Weymark (1984) or Roemer (1996), chapter 4) or, at the very least, an invariance requirement akin to the almost co-cardinality property discussed in Gevers (1979). We hope to make progress in those direction in future work.

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