école d'économie d'aix-marseille
aix-marseille school of economics

Working Papers / Documents de travail

Endogenous Growth, Spatial Dynamics and Convergence: A Refinement

Raouf Boucekkine
Carmen Camacho
Weihua Ruan

WP 2025 - Nr 33

Faculté d'économie et de gestion Aix Marseille Université
Aix Marseille Université

ENDOGENOUS GROWTH, SPATIAL DYNAMICS AND CONVERGENCE: A REFINEMENT

RAOUF BOUCEKKINE[1], CARMEN CAMACHO[2], AND WEIHUA RUAN[3]

Abstract. The dynamics of capital distribution across space are an important topic in economic geography and, more recently, in growth theory. In particular, the spatial AK model has been intensively studied in the latter stream. It turns out that the positivity of optimal capital stocks over time and space for any initial capital spatial distribution has not been entirely settled even in the simple linear AK case. We use Ekeland's variational principle together with Pontryagin's maximum principle to solve an optimal spatiotemporal AK model with a state constraint (non-negative capital stock), where the capital law of motion follows a diffusion equation. We derive the necessary optimality conditions to ensure the solution satisfies the state constraints for all times and locations. The maximum principle enables the reduction of the infinite-horizon optimal control problem to a finite-horizon problem, ultimately proving the uniqueness of the optimal solution with positive capital and the non-existence of such a solution when the time discount rate is either too large or too small.

Keywords: Diffusion and growth, convergence, Optimal Control, State constraint, Ekeland's variational principle

Journal of Economic Literature: C61, O44, Q15, Q56, R11.

This paper was written while Raouf Boucekkine was fellow of the Institute of Advanced Study at Durham University, United Kingdom. We thank Alessandro Calvia and Fausto Gozzi for technical comments on an initial version of this paper. Financial support from the French National Agency is gratefully acknowledged by Boucekkine under grant ANR-17-EURE-0020 and from Excellence Initiative of Aix-Marseille University - A*MIDEX. Financial support is also acknowledged by Camacho, grant ANR-17-EURE-0001.

[1]Corresponding author. Aix Marseille Univ, CNRS, AMSE, Marseille, France.
[2]Paris School of Economics and CNRS, France.
[3] Department of mathematics, Purdue University Northwest, USA.
E-mail addresses: raouf.boucekkine@univ-amu.fr, carmen.camacho@psemail.eu, Wruan@pnw.edu .

1. Introduction

The optimal allocation of economic activities across space is a fundamental topic in economic theory and is expected to become increasingly crucial as environmental and sustainable development concerns are more thoroughly integrated (see Arnott et al. (2008) for an early contribution to this line of research). While the New Economic Geography (see Fujita and Thisse (2002)) has significantly enriched the theoretical and methodological tools available for spatial economic analysis, most studies remain static, especially in continuous space settings.[1] To address sustainability concerns, intertemporal dynamics must be incorporated, necessitating the development of spatiotemporal frameworks and inherent analytical methodologies.

The first attempt to integrate spatiotemporal optimality into a seemingly spatiotemporal optimal control problem was by Brito (2004), followed by similar studies, including Camacho et al. (2008) and Brock and Xepapadeas (2008). These works typically use transport or diffusion equations, namely parabolic partial differential equations. More recently, the so-called *Spatial AK* (SAK) model has emerged as an active research area, following Boucekkine et al. (2013). The SAK model is the canonical spatial growth model: production and consumption activities occur along a circular spatial framework with a given time-independent population density, while capital goods are distributed across locations via transport-like equations, serving production at all locations through a linear (in capital) production function. Since 2013, the SAK model has inspired numerous extensions and discussions. Fabbri (2016) generalized the analysis to arbitrary spatial Riemannian manifolds; Ballestra (2016) presented an alternative optimization method resolving a puzzling result in the 2013 publication; Boucekkine et al. (2019) introduced exogenous spatiotemporal paths for population and productivity and solved analytically a generalized AK model; Gozzi and Leocata (2022) investigated a stochastic version of the SAK; and Ricci (2025) offered preliminary insights into the existence of non-negative optimal capital paths in this model. This paper addresses a significant and fundamentally challenging question that has been largely overlooked in the existing literature.

A key question in the SAK model involves the convergence of optimal spatiotemporal paths to stationary spatial distributions and their characterization. Boucekkine et al. (2013) showed that, under certain parametric conditions involving capital productivity, time discount rates, and spatial diffusion speeds, optimal capital paths converge to a uniform distribution from any initial distribution. This result contrasts sharply with non-spatial AK models, which predict that regions or countries with lower initial capital stocks remain lagged. Interestingly, Boucekkine et al. (2013) found that the time discount rate must fall within specific bounds for convergence: a lower bound ensures the boundedness of the value function, while an upper bound prevents non-convergence or non-existence of solutions. However, they did not thoroughly analyze the existence of optimal solutions. The solutions they identified hold only if capital remains strictly positive. This limitation means the existence of optimal solutions for the original problem remains unsettled.

[1]There are some exceptions though, Mossay's work being probably the most technically convincing, see Mossay (2013). See more below.

In this paper, we address this gap building on a few novel ideas. When the solutions in Boucekkine et al. (2013) lead to negative capital, our reformulation reduces the infinite-horizon optimization problem to a two-stage problem: a finite-horizon optimal control problem followed by an infinite-horizon problem where the solution aligns with Boucekkine et al. (2013). Boucekkine et al. (2013, 2019) used a dynamic programming method adapted to infinite-dimensional problems, effective for the SAK model without state constraints. Their method assumes strictly positive capital over time and space, excluding cases where capital reaches zero at specific locations or times. In contrast, we employ Ekeland's variational principle and Pontryagin's maximum principle to handle these scenarios. The underlying problem involves an infinite-dimensional optimal control problem over an infinite horizon with state constraints-a challenging area extensively studied in classical texts (Lions (1971); Fattorini (1999); Bensoussan et al. (2007)). Pontryagin's maximum principle, in particular, has been widely used for state constraints (e.g., Li and Yong (1991); Hu and Yong (1995); Raymond and Zidani (1998, 1999); Casas and Kunisch (2022, 2023)).

Although most studies address finite-time horizons, only special cases consider infinite horizons, typically when welfare functionals are quadratic (Bensoussan et al. (2007); Casas and Kunisch (2022, 2023)). The SAK model does not fall within this category, as its forward-backward coupled diffusion equations are often ill-posed in infinite horizons (Boucekkine et al. (2009)). In this paper, we analyze the SAK model over an infinite horizon using Pontryagin's maximum principle for finite-horizon problems. This is feasible because the linearity of the SAK model's parabolic equation gives the adjoint equation a structure that limits possible solutions. We derive necessary conditions for finite-horizon solutions to satisfy state constraints (capital remaining non-negative over time and space). These conditions help eliminate most possible adjoint equation solutions when the state constraint is not binding (capital remains positive). Consequently, we prove the uniqueness of the optimal solution with positive capital and the non-existence of such solutions under some meaningful and nontrival parametric conditions. We also highlight the importance of the initial conditions regarding these curcial issues.

To give an idea of the economic mechanisms involved, let us elaborate briefly on the proved non-existence of optimal solutions do occur when the time discount rate is too large or too small. Whether such a non-existence property may emerge if time discounting is too small is a classical result in growth theory, it also occurs in the non-spatial Ramsey model as the value-function may not be bounded in this case. Nonetheless, large time discount rates are not an issue in the latter benchmark theory. Indeed, non-existence of optimal solution when time discount rates are too large is a specific outcome of the spatiotemporal nature of the SAK problem. As in any optimal control problem, both the existence of optimal solutions and their shapes depend on the eigenvalues of the underlying differential operators, in particular the principal (or dominant) eigenvalue. In our linear case we show in this paper, using the related mathematical literature (in particular Smoller (2012)), that this principal engenvalue depends in the general case on productivity (the "A" of the AK production function, which is space-dependent in the general case), and the spatial diffusion speed parameter. It's independent of the time discount rate which only shows up in the objective function of the problem. In the case where productivity is identical across locations, say equal to a constant A, then the principal eigenvalue is exactly equal to A. If productivity is low, the spatially flowing capital stock cannot be regenerated enough locally to compensate for too large time discount rates leading (classically) to excessive consumption and eventually reducing capital to zero. As a result, such "dead zones" may arise. In the case where productivity is space-dependent, the principal eigenvalue is additionally decreasing in the spatial diffusion parameter. In such a case, a large diffusion speed combined with low (local) productivity will even favor more the emergence of the "dead zones" mentioned before as the capital stock will fail even more to regenerate under large time discount rates.

While part of the striking results outlined just above are to a certain extent due to the linear AK structure (notably the emergence of "dead zones"), they are still interesting from the economic viewpoint as they illustrate in this benchmark case how time discount rates (a key parameter in optimal spatiotemporal decision making) interact with the state of the technology and spatial diffusion (as featured by the transport equation used). Obviously, the outcomes may not be rigorously identical if the linear production function is replaced by a strictly concave production function. As shown by Xepapadeas and Yannacopoulos (2023) in the latter Ramsey case, the optimal (long-term) shapes of capital spatial distributions may markedly vary depending on the parameters of the model, the uniform solution being only one of the possible shapes. The role of the production function is also relevant in particular for the size of agglomeration effects: as shown and numerically illustrated by these authors, agglomeration increases with diminishing returns to capital if the capital flows indicate high propensity to move to a high marginal productivity locations. It is also clear that the continuous space and the inherent use of transport equations to model capital diffusion comes with its own specific dynamic outcomes. Nonetheless, this frame is arguably more appealing to study agglomeration/dispersion across space than the metaphorical two-country or core-periphery models (Krugman (1991)). It's useful to recall here the continuous space models due to Krugman (1996) or Mossay (2013). However, only skilled labor mobility is considered in these important contributions.

The paper is structured as follows. Section 2 presents the model. Section 3 derives the necessary conditions for finite-horizon optimal control. Section 4 applies these conditions to the SAK model. Section 5 examines the case of spatially homogeneous technology and population distributions. Numerical illustrations are presented in Section 6. Section 7 concludes. A technical lemma proof is provided in the Appendix.

2. The SAK model

We consider a closed economy, where both land and households are distributed over circular spatial support on the plane, $\mathcal{S} = \{(\sin\theta, \cos\theta) \in \mathbb{R}^2 : \theta \in [0, 2\pi]\}$. This is the most popular spatial setting in economic theory (see e.g, Hutchinson and Kennedy (2008), and more recently, Boucekkine et al. (2022)): it's indeed the simplest spatial frame, not requiring boundaries, and thus bearing the advantage of preserving the global dynamics of the involved variables without absorbing or reflecting boundaries. The population is distributed over the circular support according to a given density $N(\theta)$, $\theta \in [0, 2\pi]$. Local production $Y(t, \theta)$ of the unique final good in the economy is a linear function of physical capital, $K(t, \theta)$:

$$Y(t, \theta) = A(\theta) K(t, \theta),$$

$A(\theta)$ stands for the exogenous location-dependent technological level. Capital $K(t, \theta)$ evolves according to

$$\begin{aligned} &K_t - DK_{\theta\theta} = AK - Nc, && \text{for } t > 0, \quad \theta \in \mathcal{S}, \\ &K(0, \theta) = K_0(\theta), && \text{for } \theta \in (0, 2\pi) \end{aligned} \tag{1}$$

where D is a positive constant and $c(t,\theta)$ is the per capita consumption. To simplify notations, we use the slightly incorrect notation $\theta \in \mathcal{S}$ throughout this paper to indicate both the point on the unit circle and the corresponding value of θ. Also, any function y defined on $\mathcal{S}$ is assumed to satisfy the periodic condition

$$y(t,0) = y(t,2\pi), \qquad y_\theta(t,0) = y_\theta(t,2\pi) \qquad \text{for } t > 0,$$

and therefore the boundary conditions to the initial-boundary value problems in this paper will not be explicitly stated.) In addition, the capital should be nonnegative, i.e.,

$$K(t,\theta) \geq 0 \qquad \text{for } t \geq 0, \qquad \theta \in [0,2\pi]. \tag{2}$$

The policy maker chooses $c(\cdot,\cdot)$ to maximize overall welfare, defined as

$$J(c) := \int_0^\infty e^{-\rho t} \int_0^{2\pi} \frac{c(t,\theta)^{1-\sigma}}{1-\sigma} N(\theta)\, d\theta dt \tag{3}$$

where $\rho > 0$ and $\sigma \in (0,\infty) \setminus \{1\}$ are positive constants.

It is natural to assume that consumption remains nonnegative everywhere. Additionally, optimal consumption should guarantee that the capital stock remains nonnegative at all locations and throughout all times. The central challenge, then, is to identify a solution for optimal consumption that satisfies these conditions.

3. Pontryagin's Maximum Principle

In this section, we derive Pontryagin's maximum principle for a slightly generalized parabolic control problem with state constraint. We consider the initial-boundary value problem in a bounded time interval,

$$\begin{cases} y_t - D y_{\theta\theta} = a(t,\theta)\, y + b(t,\theta)\, c & \text{for } (t,\theta) \in (0,T) \times \mathcal{S}, \\ y(0,\theta) = y_0(\theta) & \text{for } \theta \in \mathcal{S} \end{cases} \tag{4}$$

where $T > 0$ is a constant, y is the state variable, c is the control, D is a positive constant, and a, b and y_0 are given functions. The system is subject to the state constraint

$$y(t,\theta) \geq 0 \quad \text{for } (t,\theta) \in [0,T] \times \mathcal{S}.$$

In addition, the control, $c(t,\theta)$, is a nonnegative measurable function in $[0,T] \times \mathcal{S}$. Given the initial state y_0, the welfare functional to be maximized is

$$J(y_0,c) = \int_0^T \int_{\mathcal{S}} g(t,\theta,c(t,\theta))\, d\theta dt + h(T, y(T,\cdot)), \tag{5}$$

where the functional $h : \mathbb{R}_+ \times C(\mathcal{S}) \longmapsto \mathbb{R}$ has a Fréchet derivative with respect to y.

We use the notations $\mathcal{S}_T = [0,T] \times \mathcal{S}$ and

$$Q = \{ y \in C(\mathcal{S}_T) : y(t,\theta) \geq 0 \quad \text{for all } (t,\theta) \in \mathcal{S}_T \}. \tag{6}$$

We make the following assumptions.

Assumption 3.1 *(1) $a, b : \mathcal{S}_T \longmapsto \mathbb{R}$ are continuously differentiable.*

(2) $g : \mathcal{S}_T \times \mathbb{R}_+ \longmapsto \mathbb{R}_+$ is continuously differentiable and there is a constant M such that

$$|g(t,\theta,c)| \leq M \quad \textit{for } (t,\theta,c) \in \mathcal{S}_T \times \mathbb{R}_+,$$

and $h : \mathbb{R}_+ \times \mathbb{R}_+ \longmapsto \mathbb{R}_+$ is continuously differentiable.

The main result of this section is the maximum principle for the above optimal control problem given in Proposition 3.2. It is the basis of the analysis of the SAK model in the subsequent sections. Similar results to Proposition 3.2 exist in the literature, but none can be directly applied to the problem of this paper. For example, Theorem 10.3.1 in Fattorini (1999) gives the maximum principle for the general control system in an arbitrary Banach space with both a state constraint and a target condition, and Theorem 2.2 in Hu and Yong (1995) gives the maximum principle for the control system with a general semilinear parabolic equation with homogeneous Dirichlet boundary condition and without scrap value. Thus, we provide a version of the maximum principle that specifically applicable to the control system (4)–(6).

For convenience, we denote

$$f(t,\theta,y,c) = a(t,\theta)\, y + b(t,\theta)\, c$$

and define the Hamiltonian

$$H(t,\theta,y,c,\mu,\psi) = \mu g(t,\theta,c) + \psi f(t,\theta,y,c) \quad \text{in } \mathcal{S}_T \times \mathbb{R}_+ \times \mathbb{R}_+$$

where $\mu \geq 0$ is a constant. In addition, we let $d_0(y) : C(\mathcal{S}_T) \longmapsto \mathbb{R}_+$ denote the functional

$$d_0(y) = |\min\{y,0\}|_{C(\mathcal{S}_T)} \quad \text{for any } y \in C(\mathcal{S}_T). \tag{7}$$

It is clear that $d_0(y) = 0$ if y is nonegative in $\mathcal{S}_T$. In addition, d_0 is Gâteaux differentiable at every $y \in C(\mathcal{S}_T) \backslash Q$, and its Gâteaux derivative $\nabla d_0(y)$ is the same as the Clarke's generalized gradient, which is convex and weak*-compact. As a result,

$$|\nabla d_0(y)|_{\mathcal{M}(\mathcal{S}_T)} = 1 \qquad \text{if } y \notin Q \tag{8}$$

and for any $\xi \in \partial d_0(y)$,

$$\langle \xi, z - y\rangle + d_0(y) \leq d_0(z) \qquad \text{for any } z \in C(\mathcal{S}_T) \tag{9}$$

where $\mathcal{M}(\mathcal{S}_T)$ is the set of all Radon measures on $\mathcal{S}_T$.

The following is maximum principle for system (4)–(6).

PROPOSITION 3.2 *Let Assumption 3.1 hold and let $y_0 \in C^{\alpha}(\mathcal{S})$ for some α satisfying $0 < \alpha < 1$. In addition, let $\{c^*, y^*\}$ be an optimal pair. Then there exists a constant $\mu \geq 0$, a function $\psi \in L^q\left(0,T;W^{1,q}(\mathcal{S})\right)$ $(1 < q < 3/2)$ and a Radon measure $\nu \in \{\nabla d_0(y^*)\}$ such that*

$$\mu + |\nu|_{\mathcal{M}(\mathcal{S}_T)} > 0,$$

$\langle \nu, z - y^*\rangle \leq 0$ *for any* $z \in Q$,

$$\begin{aligned} &\psi_t + D\psi_{\theta\theta} = -a(t,\theta)\,\psi - \nu|_{(0,T)\times\mathcal{S}}, \\ &\psi(T,\cdot) = \mu h_y(T, y^*(T,\cdot)) + \nu|_{\{T\}\times\mathcal{S}}, \end{aligned} \tag{10}$$

and

$$H(t,\theta,y^*(t,\theta), c^*(t,\theta), \mu, \psi(t,\theta)) = \max_{c\geq 0} H(t,\theta,y^*(t,\theta), c, \mu, \psi(t,\theta)). \tag{11}$$

A proof is in Appendix.

The following observation plays an important role in the next section. It distinguishes the optimal solution with positive state from others.

REMARK 3.3 Let $\nu \in \mathcal{M}(\mathcal{S}_T)$ be the Radon measure in Proposition 3.2. It is easy to see that

$$\text{supp } \nu \subset \{(t,\theta) \in \mathcal{S}_T : y^*(t,\theta) = 0\}. \tag{12}$$

Indeed, for any $\eta \in C(\mathcal{S}_T)$ with supp $\eta \subset \mathcal{S}_T \backslash \text{supp } y^*$, there is $\varepsilon > 0$ such that $z^{\pm} := y^* \pm \varepsilon\eta \in Q$. Hence,

$$\pm\varepsilon \langle \nu, \eta \rangle = \left\langle \nu, z^{\pm} - y^* \right\rangle \leq 0.$$

This implies that $\langle \nu, \eta \rangle = 0$.

4. Optimal solution for the SAK model

We now start the exploration of the existence conditions of the optimal solution with strictly positive capital stock disclosed in Boucekkine et al. (2013). We take the initial capital stock spatial distribution **as given**, and we essentially aim at identifying the conditions on the parameters of the model such that this solution holds. We shall see in the numerical Section 6 that initial conditions also matter for economic reasons we will expose.

In Boucekkine et al. (2013) the value function $\bar{V}(K_0)$ is derived with the control

$$\bar{c}(t,\theta) = (\lambda_0 - r)\frac{\langle K_0, \varphi_0 \rangle}{\langle f, \varphi_0 \rangle}[\varphi_0(\theta)]^{-1/\sigma} e^{rt}, \tag{13}$$

where λ_0 is the principal eigenvalue of the differential operator $\mathcal{L}$ defined by

$$\mathcal{L}u(\theta) := Du''(\theta) + A(\theta)u(\theta) \quad \text{for } \theta \in \mathcal{S} \tag{14}$$

and φ_0 is the normalized (in $L^2(\mathcal{S})$) positive eigenfunction corresponding to λ_0,

$$f(\theta) = [\varphi_0(\theta)]^{-1/\sigma} N(\theta), \tag{15}$$

and

$$r = \frac{\lambda_0 - \rho}{\sigma}. \tag{16}$$

As apparent in equation (13), r is the long-term growth rate of the AK model, both capital and consumption grow at this rate. As it stands equation (16) features a relationship between the growth rate, r, the principal eigenvalue, λ_0, the time discount rate, ρ, and the inverse of the elasticity of substitution, σ. So, while the preference parameters (ρ, σ) do not intervene in the law of motion of the capital stock, they are related to the growth rate of the economy, r, along the optimal growth paths. That is not only the preference time discount rate, ρ, shapes the optimal dynamics, the elasticity parameter, σ, does too, though we have not dwelt on it in the Introduction section to unburden the presentation. It will show up in the parametric conditions below quite frequently.

With this control, the value function is

$$\bar{V}(K_0) = \frac{\langle K_0, \varphi_0 \rangle^{1-\sigma}}{1-\sigma}\left[\frac{\langle f, \varphi_0 \rangle}{\lambda_0 - r}\right]^{\sigma}. \tag{17}$$

This value function is valid only if the associated trajectory, $\bar{K}(t,\theta)$, which satisfies (1) with c replaced by $\bar{c}$, is nonnegative for all $t \geq 0$, $\theta \in \mathcal{S}$. In the case where $\bar{K}(t,\theta)$ takes negative value for some (t,θ), we call the pair $\left\{\bar{c}, \bar{K}\right\}$ *not feasible.*

In this section, we focus on the case $0 < \sigma < 1$ and explore the feasibility of $\left\{\bar{c}, \bar{K}\right\}$ and show that it is the only possible optimal pair with positive state if it is feasible. In the case where it is not feasible, we show how an optimal solution can be obtained by solving a finite-horizon optimal problem, provided that an optimal pair with eventually positive state exists. We finally determine asymptotic behavior of such a state.

Let us first recall a well-known property of the eigenvalues of $\mathcal{L}$. For reference, see, e.g., Theorems 2.4.2 and 2.5.1 in Brown et al. (2012):

Proposition 4.1 *Suppose $D > 0$ is a constant and $A \in C(\mathcal{S}; \mathbb{R}^+)$. Then, the eigenvalues of the operator $\mathcal{L}$ on $W^{2,2}(\mathcal{S}; \mathbb{R})$ are countable and can be ordered as a decreasing sequence $\{\lambda_n\}_{n=0}^{\infty}$ such that $\lambda_n \to -\infty$. Furthermore, the multiplicity of the principal eigenvalue, λ_0, is one, and there exists a strictly positive eigenfunction φ_0 corresponding to λ_0.*

We can go a step further and say more on the dependence of the principal eigenvalue λ_0 on the parameters of our model. Using the work of Smoller (2012), we can formulate the following properties.

Proposition 4.2 *Suppose $D > 0$ is a constant and $A \in C(\mathcal{S}; \mathbb{R}^+)$. The principal eigenvalue, λ_0, is independent of ρ but is dependent of A in an increasing way and of D in a decreasing way.*

This proposition directly derives from the following characterization given by Smoller (2012):

$$\lambda_0 = \sup_{\|\varphi\|_{L^2}=1} \int_{\mathcal{S}} \left[A(\theta) |\varphi(\theta)|^2 - D \left|\varphi'(\theta)\right|^2 \right] d\theta.$$

In the special case where A is a constant function, detailed in Section 5 below, $\lambda_0 = A$, which is independent of D, as already mentioned in the Introduction.

4.1. **Small time discount.** In the case where $\rho \leq (1-\sigma)\lambda_0$,

$$\lambda_0 - r = \frac{1}{\sigma}\left[\rho - (1-\sigma)\lambda_0\right] < 0.$$

Thus $\bar{c}$ defined in (13) is nonpositive. As a result, $\{\bar{c}, \bar{K}\}$ is undefined. We prove that any optimal solution leads to the capital vanish somewhere sometime.

Proposition 4.3 *Suppose that $0 < \sigma < 1$, $\rho \leq (1-\sigma)\lambda_0$ and $K_0 \in C^{\alpha}(\mathcal{S})$ with α satisfying $0 < \alpha < 1$. If an optimal solution $\{c^*, K^*\}$ exists with $c^*(t,\theta) > 0$ for all (t,θ), then for any $T > 0$ there exists (t,θ) such that $t > T$, $\theta \in [0, 2\pi]$ and $K^*(t,\theta) = 0$.*

Proof. We first show that if $\{c^*, K^*\}$ is an optimal pair, then there is no $T > 0$ such that $K^*(t,\theta) > 0$ for all $t > T$, $\theta \in \mathcal{S}$.

Assume for contradiction that such a T exists. Using a time translation if necessary, we can assume that $T = 0$. Hence, in view of Remark 3.3, the Radon measure ν in Proposition 3.2 vanishes. As a result, $\mu > 0$ and we can choose $\mu = 1$. Therefore, the adjoint variable ψ and c are related by (11) which takes the form

$$e^{-\rho t}\frac{c^*(t,\theta)^{1-\sigma}}{1-\sigma} - \psi(t,\theta)\, c^*(t,\theta) = \arg\max_{c\geq 0}\left\{ e^{-\rho t}\frac{c^{1-\sigma}}{1-\sigma} - \psi(t,\theta)\, c \right\}.$$

It follows that

$$\psi(t,\theta) = e^{-\rho t} c^*(t,\theta)^{-\sigma}, \tag{18}$$

then ψ satisfies the adjoint problem (10) which takes the form

$$\begin{cases} \psi_t + D\psi_{\theta\theta} + A\psi = 0, & \text{if } 0 < t < \tau,\ \theta \in \mathcal{S}, \\ \psi(\tau,\theta) = e^{-\rho\tau} c^*(\tau,\theta)^{-\sigma}, & \text{for } \theta \in \mathcal{S}. \end{cases} \tag{19}$$

for any $\tau > 0$. Since τ is arbitrary, we see that the first equation in (19) holds for any $t > 0$, $\theta \in (0, 2\pi)$. Using Fourier expansion, we can write

$$\psi(t,\theta) = \sum_{n\geq 0} e^{-\lambda_n t} \sum_i \langle \psi(0,\cdot), \varphi_{n,i}\rangle \varphi_{n,i}(\theta)$$

where $\{\varphi_{n,i}\}$ is the set of normalized eigenfunctions of $\mathcal{L}$ corresponding to the nth eigenvalue, λ_n. Since $\lambda_n \to -\infty$ as $n \to \infty$, the above series converges only if there are only finitely many n such that $\langle \psi(0,\cdot), \varphi_{n,i}\rangle \neq 0$. In addition, since $c^*(t,\theta) > 0$ for all (t,θ), it is necessary that $\psi(t,\theta) > 0$ for all (t,θ). We show that this fact implies $\langle \psi(0,\cdot), \varphi_{n,i}\rangle = 0$ for any $n \geq 1$. To see this, assume that $\bar{n}$ is the largest of the integers n such that $\langle \psi(0,\cdot), \varphi_{n,i}\rangle \neq 0$ for some i and $\bar{n} \geq 1$. Then $\varphi_{\bar{n},1}$ and $\varphi_{\bar{n},2}$ change sign on $\mathcal{S}$. Hence, there exists $\theta^* \in \mathcal{S}$ such that

$$\sum_i \langle \psi(0,\cdot), \varphi_{\bar{n},i}\rangle \varphi_{\bar{n},i}(\theta^*) < 0.$$

Since $\lambda_{\bar{n}} < \lambda_n < \lambda_0$ for any $n = 1, \ldots, \bar{n}-1$, it follows that

$$e^{-\lambda_{\bar{n}} t} \sum_i \left|\langle \psi(0,\cdot), \varphi_{\bar{n},i}\rangle \varphi_{\bar{n},i}(\theta^*)\right| > \left| \sum_{n=0}^{\bar{n}-1} e^{-\lambda_n t} \sum_i e \langle \psi(0,\cdot), \varphi_{n,i}\rangle \varphi_{n,i}(\theta) \right|$$

if t is sufficiently large. Therefore, $\psi(t,\theta)$ cannot remain positive for all $t > 0$.

As a result, there is $a > 0$ such that $\psi(t,\theta) = a e^{-\lambda_0 t} \varphi_0(\theta)$. By (18),

$$c^*(t,\theta) = \alpha \varphi_0(\theta)^{-1/\sigma} e^{rt}, \tag{20}$$

where $\alpha = a^{-1/\sigma}$ and r is given by (16).

We show that K^* becomes negative for any $a > 0$. Let $u(t,\theta) = K^*(t,\theta) e^{-rt}$. Then

$$u_t = (\mathcal{L} - r) u - \alpha f$$

where f is given by (15). The equation is linear, and its solution can be written as

$$u(t,\theta) = \tilde{u}(t,\theta) - \alpha \hat{u}(t,\theta)$$

where $\tilde{u}$ and $\hat{u}$ are solutions to the initial-boundary value problems

$$\begin{cases} \tilde{u}_t - D\tilde{u}_{\theta\theta} = (A - r)\tilde{u} & \text{for } t > 0, \quad \theta \in \mathcal{S}, \\ \tilde{u}(0,\theta) = K_0(\theta) & \text{for } \theta \in \mathcal{S}, \end{cases} \tag{21}$$

and

$$\begin{cases} \hat{u}_t - D\hat{u}_{\theta\theta} = (A - r)\hat{u} + f & \text{for } t > 0, \quad \theta \in \mathcal{S}, \\ \hat{u}(0,\theta) = 0 & \text{for } \theta \in \mathcal{S}, \end{cases} \tag{22}$$

respectively. Clearly, $u(t,\theta)$ and $K^*(t,\theta)$ have the same sign. We show that $\alpha\hat{u}(t,\theta) > \tilde{u}(t,\theta)$ if t is large. It suffices to show that

$$\alpha \langle \hat{u}(t,\cdot), \varphi_0\rangle > \langle \tilde{u}(t,\cdot), \varphi_0\rangle$$

for large t. Note that $\rho \leq (1-\sigma)\lambda_0$ is equivalent to $\lambda_0 \leq r$. We denote

$$\hat{u}_0(t) = \langle \hat{u}(t,\cdot), \varphi_0\rangle, \qquad \tilde{u}_0(t) = \langle \tilde{u}(t,\cdot), \varphi_0\rangle.$$

Multiplying both sides of the differential equations in (21) and (22) by φ_0, and integrating over $[0, 2\pi]$, we find

$$\begin{aligned} \hat{u}_0' (t) &= (\lambda_0 - r) \hat{u}_0 (t) + \langle f, \varphi_0 \rangle, \qquad && \hat{u}_0 (0) = 0, \\ \tilde{u}_0' (t) &= (\lambda_0 - r) \tilde{u}_0 (t), && \tilde{u}_0 (0) = \langle K_0, \varphi_0 \rangle. \end{aligned}$$

The solutions are

$$\hat{u}_0 (t) = \frac{\langle f, \varphi_0 \rangle}{\lambda_0 - r} \left[e^{(\lambda_0 - r)t} - 1 \right], \qquad \tilde{u}_0 (t) = \langle K_0, \varphi_0 \rangle e^{(\lambda_0 - r)t} \tag{23}$$

if $\lambda_0 \neq r$ and

$$\hat{u}_0 (t) = \langle f, \varphi_0 \rangle t, \qquad \tilde{u}_0 (t) = \langle K_0, \varphi_0 \rangle \tag{24}$$

if $\lambda_0 = r$. Since $\langle f, \varphi_0 \rangle > 0$, in both cases $\alpha \hat{u}_0 (t) > \tilde{u}_0 (t)$ for large t. This proves that there is no $T > 0$ such that $K^* (t, \theta) > 0$ for all $t > T$ and $\theta \in \mathcal{S}$.

Recall that $\bar{c}$ is given by (13) which is in the form (20) with

$$\alpha = (\lambda_0 - r) \frac{\langle K_0, \varphi_0 \rangle}{\langle f, \varphi_0 \rangle}, \tag{25}$$

it follows that $\left\{ \bar{c}, \bar{K} \right\}$ is not feasible.

This completes the proof.

4.2. **Larger time discount.** In the case where $\rho > (1 - \sigma) \lambda_0$, the pair $\left\{ \bar{c}, \bar{K} \right\}$ may or may not be feasible. We first show that whenever $\left\{ \bar{c}, \bar{K} \right\}$ is feasible, it is the only possible optimal pair with positive state.

PROPOSITION 4.4 *Suppose that* $0 < \sigma < 1$, $\rho > (1 - \sigma) \lambda_0$ *and that* $\left\{ \bar{c}, \bar{K} \right\}$ *is feasible. If* $\{c^*, K^*\}$ *is an optimal pair such that* $K^* (t, \theta) > 0$ *for all* $t > 0$ *and* $\theta \in \mathcal{S}$, *then* $\{c^*, K^*\} = \left\{ \bar{c}, \bar{K} \right\}$.

Proof. Let $\psi (t, \theta)$ be defined by (18). As in the proof of Proposition 4.3,

$$\psi (t, \theta) = a e^{-\lambda_0 t} \varphi_0 (\theta)$$

for some $a > 0$. As a result, $c^* (t, \theta)$ satisfies (20) with $\alpha = a^{-1/\sigma}$. It remains to show that α is given by (25).

Let $\bar{\alpha}$ denote the right-hand side of (25). We show that $\alpha \leq \bar{\alpha}$. Let $\tilde{u}$ and $\hat{u}$ be solutions of (21) and (22), respectively. Then, as in the proof of Proposition 4.3,

$$\tilde{u} (t, \theta) - \alpha \hat{u} (t, \theta) = K^* (t, \theta) e^{-rt}.$$

Since $K^* (t, \theta) > 0$ for all $t > 0$, $\theta \in \mathcal{S}$, it follows that

$$\langle \tilde{u} (t, \theta), \varphi_0 \rangle > \alpha \langle \hat{u} (t, \theta), \varphi_0 \rangle \qquad \text{for all } t > 0. \tag{26}$$

Note that $\rho > (1 - \sigma) \lambda_0$ is equivalent to $\lambda_0 > r$. It follows from (23) that

$$\langle \tilde{u} (t, \theta), \varphi_0 \rangle = \langle K_0, \varphi_0 \rangle e^{(\lambda_0 - r)t}, \qquad \langle \hat{u} (t, \theta), \varphi_0 \rangle = \frac{\langle f, \varphi_0 \rangle}{\lambda_0 - r} \left[e^{(\lambda_0 - r)t} - 1 \right].$$

Hence, (26) implies

$$\alpha < (\lambda_0 - r) \frac{\langle K_0, \varphi_0 \rangle}{\langle f, \varphi_0 \rangle} \frac{e^{(\lambda_0 - r)t}}{e^{(\lambda_0 - r)t} - 1} \qquad \text{for all } t > 0.$$

This leads to

$$\alpha \leq (\lambda_0 - r) \frac{\langle K_0, \varphi_0 \rangle}{\langle f, \varphi_0 \rangle} \equiv \bar{\alpha}_0. \tag{27}$$

Now, since $\alpha \leq \bar{\alpha}_0$ and the welfare function $J(K_0, c)$ given by (3) is increasing in c, and since $\{\bar{c}, \bar{K}\}$ is feasible, it follows that $\alpha = \bar{\alpha}_0$.

This completes the proof.

4.3. **Initial period.** In the case where $\{\bar{c}, \bar{K}\}$ is not feasible, an optimal pair $\{c^*, K^*\}$ can still exists. In this case either K^* is eventually positive in the sense that there is $T > 0$ such that $K^*(t,\theta) > 0$ for all $t > T$ and $\theta \in \mathcal{S}$, or there exist a sequence $\{(t_k, \theta_k)\}_{k=1}^{\infty}$ in $(0,\infty) \times \mathcal{S}$ such that

$$0 < t_1 < \cdots < t_k < \cdots \to \infty,$$

and $K^*(t_k, \theta_k) = 0$. In the former case, by a time-translation we see from Proposition 4.4 that $\{c^*, K^*\} = \{\bar{c}, \bar{K}\}$ for $t > T$ provided that $\{\bar{c}, \bar{K}\}$ for $t \geq T$ is feasible, where $\bar{c}$ is given by

$$\bar{c}(t,\theta) = (\lambda_0 - r) \frac{\langle K^*(T,\cdot), \varphi_0 \rangle}{\langle f, \varphi_0 \rangle} [\varphi_0(\theta)]^{-1/\sigma} e^{r(t-T)} \qquad \text{for } t \geq T, \tag{28}$$

and $\bar{K}$ is the solution to the initial-boundary value problem

$$\begin{cases} K_t - DK_{\theta\theta} = AK - Nc, & \text{for } t > T, \quad \theta \in \mathcal{S}, \\ K(T,\theta) = K^*(T,\theta), & \text{for } \theta \in \mathcal{S} \end{cases} \tag{29}$$

During the initial period, $0 \leq t < T$, we can determine $c^*(t,\theta)$ by solving a finite-horizon optimal control problem

$$J(K_0, c^*) = \sup_{c \geq 0} \int_0^T \int_0^{2\pi} e^{-\rho t} \frac{c(t,\theta)^{1-\sigma}}{1-\sigma} N(\theta)\, d\theta dt + e^{-\rho T} \bar{V}(K^*(T,\cdot)). \tag{30}$$

We prove

PROPOSITION 4.5 *Suppose that* $0 < \sigma < 1$, $\rho > (1-\sigma)\lambda_0$, *and that* $\{c^*, K^*\}$ *is an optimal pair such that* $K^*(t,\theta) > 0$ *for all* $(t,\theta) \in (T,\infty) \times \mathcal{S}$. *Suppose* $\{\bar{c}, \bar{K}\}$ *with* $\bar{c}$ *given by (28) and* $\bar{K}$ *solving (29) is also feasible for* $t \geq T$, *then, the costate* ψ *given by (18) is a solution to the terminal-boundary value problem*

$$\begin{cases} \psi_t + D\psi_{\theta\theta} + A\psi = -\left.\nu\right|_{(0,T)\times\mathcal{S}}, & \text{if } 0 < t < T, \quad \theta \in \mathcal{S}, \\ \psi(T,\theta) = \left.\nu\right|_{\{T\}\times\mathcal{S}} + e^{-\rho T} \bar{c}(T,\theta)^{-\sigma}, & \text{for } \theta \in \mathcal{S}. \end{cases} \tag{31}$$

where

$$\bar{c}(T,\theta) = (\lambda_0 - r) \frac{\langle K^*(T,\cdot), \varphi_0 \rangle}{\langle f, \varphi_0 \rangle} [\varphi_0(\theta)]^{-1/\sigma}.$$

Proof. Proposition 3.2 with

$$\begin{aligned} f(t,\theta,K,c) &= A(\theta) K - N(\theta) c, \qquad g(t,\theta,c) = e^{-\rho t} \frac{c^{1-\sigma}}{1-\sigma} N(\theta), \\ h(t,K) &= e^{-\rho t} \bar{V}(K) \end{aligned}$$

implies that there exists a constant μ, a function ψ, and a Radon measure ν such that $\mu + |\nu|_{\mathcal{M}(\mathcal{S}_T)} > 0$ and

$$\begin{cases} \psi_t + D\psi_{\theta\theta} + A(\theta)\psi = -\nu|_{(0,T)\times\mathcal{S}}, & \text{if } 0 < t < T, \quad \theta \in \mathcal{S}, \\ \psi(T,\theta) = \nu|_{\{T\}\times\mathcal{S}} + \mu e^{-\rho T}\nabla_K \bar{V}(K^*(T,\cdot)), & \text{for } \theta \in \mathcal{S}. \end{cases} \tag{32}$$

Note that the Hamiltonian is given by

$$H(t,\theta,K,c,\mu,\psi) = \mu e^{-\rho t}\frac{c^{1-\sigma}}{1-\sigma}N(\theta) + \psi(t,\theta)[A(\theta)K - N(\theta)c].$$

Thus, the optimal pair $\{c^*, K^*\}$ satisfies

$$\begin{aligned} & \mu e^{-\rho t}\frac{c^*(t,\theta)^{1-\sigma}}{1-\sigma}N(\theta) + \psi(t,\theta)[A(\theta)K^*(t,\theta) - N(\theta)c^*(t,\theta)] \\ = \ & \arg\max_{c\geq 0}\left\{\mu e^{-\rho t}\frac{c^{1-\sigma}}{1-\sigma}N(\theta) + \psi(t,\theta)[A(\theta)K^*(t,\theta) - N(\theta)c]\right\} \end{aligned}$$

for $(t,\theta) \in (0,T)\times\mathcal{S}$. It follows that

$$\mu e^{-\rho t}c^*(t,\theta)^{-\sigma} = \psi(t,\theta) \qquad \text{for } (t,\theta) \in (0,T)\times\mathcal{S}. \tag{33}$$

We first show that $\mu > 0$. If $\mu = 0$, then $|\nu|_{\mathcal{M}(\mathcal{S}_T)} > 0$. Therefore, (32) is non-homogeneous. As a result, $\psi(t,\theta)$ is nontrivial. This contradicts (33). This proves that $\mu \neq 0$. Since μ is nonnegative by Proposition (3.2), it follows that $\mu > 0$. Using scaling if necessary, we can choose $\mu = 1$.

It remains to show that

$$\nabla_K \bar{V}(K^*(T,\cdot)) = \bar{c}(T,\theta)^{-\sigma}. \tag{34}$$

Let $T' > T$. Since $\{\bar{c}, \bar{K}\}$ with $\bar{c}$ given by (28) and $\bar{K}$ solving (29) is feasible and optimal, by dynamic programming

$$J(K^*(T,\cdot),\bar{c}) = \sup_{c\geq 0}\int_T^{T'}\int_0^{2\pi} e^{-\rho(t-T)}\frac{c(t,\theta)^{1-\sigma}}{1-\sigma}N(\theta)\,d\theta dt + e^{-\rho(T'-T)}\bar{V}\left(\bar{K}(T',\cdot)\right).$$

As a result, since $\bar{K}(t,\theta) > 0$ for $(t,\theta) \in (T,T')\times\mathcal{S}_T$, by Proposition 3.2, the costate $\bar{\psi} =: e^{-\rho(t-T)}\bar{c}(t,\theta)^{-\sigma}$ satisfies

$$\begin{cases} \bar{\psi}_t + D\bar{\psi}_{\theta\theta} + A\bar{\psi} = 0, & \text{if } T < t < T', \quad \theta \in \mathcal{S}, \\ \bar{\psi}(T',\theta) = e^{-\rho(T'-T)}\nabla_K \bar{V}\left(\bar{K}(T',\cdot)\right), & \text{for } \theta \in \mathcal{S}. \end{cases}$$

This leads to

$$e^{-\rho(T'-T)}\nabla_K \bar{V}\left(\bar{K}(T',\cdot)\right) = \bar{\psi}(T',\theta) = e^{-\rho(T'-T)}\bar{c}(T',\theta)^{-\sigma}.$$

Taking $T' \to T$ on both sides and using continuity, we obtain (34).

The proof is complete.

4.4. **Asymptotic behavior.** Finally, in the case where an optimal pair $\{c^*, K^*\}$ has an eventually positive state K^*, we can determine its asymptotic behavior by deriving the limit of the detrended capital, $K^*(t,\theta)\, e^{-gt}$.

PROPOSITION 4.6 *Suppose that $0 < \sigma < 1$, $\rho > (1-\sigma)\lambda_0$ and $\{c^*, K^*\}$ is an optimal pair. Also suppose that $K^* > 0$ for all $t > T$, $\theta \in \mathcal{S}$, and that $\{\bar{c}, \bar{K}\}$ defined by (28) and (29) is feasible for $t > T$. Let n be the smallest positive integer such that $\langle K^*(T,\cdot), \varphi_{n,i}\rangle \neq 0$ for some eigenfunction $\varphi_{n,i}$ corresponding to eigenvalue λ_n. Then,*

$$\rho \leq \lambda_0 - \sigma\lambda_n. \tag{35}$$

Furthermore,

$$\lim_{t\to\infty} K^*(t,\theta)\, e^{-rt} = e^{-rT}\left\{\langle K^*(T,\cdot), \varphi_0\rangle \varphi_0(\theta) + \sum_{k\geq n}\sum_i \frac{\bar{\alpha}_T \langle f, \varphi_{k,i}\rangle}{\lambda_k - r}\varphi_{k,i}(\theta)\right\} \tag{36}$$

if (35) holds strictly, where

$$\bar{\alpha}_T = (\lambda_0 - r)\frac{\langle K^*(T,\cdot), \varphi_0\rangle}{\langle f, \varphi_0\rangle}. \tag{37}$$

and

$$\begin{aligned}\lim_{t\to\infty} K^*(t,\theta)\, e^{-rt} = e^{-rT}&\left\{\langle K^*(T,\cdot), \varphi_0\rangle \varphi_0(\theta) + \sum_i \langle K^*(T,\cdot), \varphi_{n,i}\rangle \varphi_{n,i}(\theta)\right. \\ &\left.+ \sum_{k>n}\sum_i \frac{\bar{\alpha}_T \langle f, \varphi_{k,i}\rangle}{\lambda_k - r}\varphi_{k,i}(\theta)\right\}\end{aligned} \tag{38}$$

if (35) holds equally.

Proof. By Proposition 4.4,

$$c^*(t,\theta) = \bar{c}(t,\theta) \equiv \bar{\alpha}_T [\varphi_0(\theta)]^{-1/\sigma} e^{r(t-T)} \qquad \text{for } t \geq T \tag{39}$$

Substituting the right-hand side of (39) into (29) and using Fourier series expansion, we obtain

$$\begin{aligned}K^*(t,\theta) &= e^{r(t-T)}\left\{\langle K^*(T,\cdot), \varphi_0\rangle \varphi_0(\theta) + \sum_{k\geq n}\sum_i \frac{\bar{\alpha}_T \langle f, \varphi_{k,i}\rangle}{\lambda_k - r}\varphi_{k,i}(\theta)\right\} \\ &\quad + \sum_{k\geq n}\sum_i \left[\langle K^*(T,\cdot), \varphi_{k,i}\rangle - \frac{\bar{\alpha}_T \langle f, \varphi_{k,i}\rangle}{\lambda_k - r}\right] e^{\lambda_k(t-T)}\varphi_{k,i}(\theta).\end{aligned}$$

Since $K^*(t,\theta) > 0$ for $t > T$, and since $\varphi_{k,i}(\theta)$ changes the sign for $k \geq 1$, it follows that $r \geq \lambda_n$, which is equivalent to (35).

In the case where $r > \lambda_n$,

$$\begin{aligned}K^*(t,\theta)\, e^{-rt} &= e^{-rT}\left\{\langle K^*(T,\cdot), \varphi_0\rangle \varphi_0(\theta) + \sum_{k\geq n}\sum_i \frac{\bar{\alpha}_T \langle f, \varphi_{k,i}\rangle}{\lambda_k - r}\varphi_{k,i}(\theta)\right\} \\ &\quad + \sum_{k\geq n}\sum_i \left[\langle K^*(T,\cdot), \varphi_{k,i}\rangle - \frac{\bar{\alpha}_T \langle f, \varphi_{k,i}\rangle}{\lambda_k - r}\right] e^{(\lambda_k - r)t - \lambda_k T}\varphi_{k,i}(\theta)\end{aligned}$$

converges to the right-hand side of (36).

In the case where $r=\lambda_n$, by Proposition 4.1, $r>\lambda_k$ for $k>n$. Hence, the above relation leads to

$$\lim_{t\to\infty} K^*(t,\theta)e^{-rt} = e^{-rT}\left\{\langle K^*(T,\cdot),\varphi_0\rangle\varphi_0(\theta)+\sum_{k\geq n}\sum_i \frac{\bar{\alpha}_T\langle f,\varphi_{k,i}\rangle}{\lambda_k-r}\varphi_{k,i}(\theta) + \sum_i\left[\langle K^*(T,\cdot),\varphi_{n,i}\rangle - \frac{\bar{\alpha}_T\langle f,\varphi_{n,i}\rangle}{\lambda_n-r}\right]\varphi_{k,i}(\theta)\right\}.$$

This is the same as (38).

This completes the proof.

Clearly, in the case where $\{\bar{c},\bar{K}\}$ is feasible, the results of Proposition 4.6 hold with $T=0$. Conversely, since any eigenfunctions corresponding to eigenvalues λ_n with $n\geq 1$ change sign in $\mathcal{S}$, it is possible that the right-hand sides of (36) and (38) with $T=0$ reaches negative values in $\mathcal{S}$. In this case $\{\bar{c},\bar{K}\}$ cannot be feasible. This observation leads to the following corollary.

COROLLARY 4.7 *Suppose $0<\sigma<1$ and $\rho>(1-\sigma)\lambda_0$. Let $\bar{\alpha}_0$ be defined by (27) and let n be the smallest positive integer such that $\langle K_0,\varphi_{n,i}\rangle\neq 0$ for some eigenfunction $\varphi_{n,i}$ corresponding to eigenvalue λ_n. Then, $\{\bar{c},\bar{K}\}$ is not feasible in the following cases:*

(1) $\rho>\lambda_0-\sigma\lambda_n$;
(2) $\rho<\lambda_0-\sigma\lambda_n$ and the function

$$\theta\mapsto\langle K_0,\varphi_0\rangle\varphi_0(\theta)+\sum_{k\geq n}\sum_i\frac{\bar{\alpha}_0\langle f,\varphi_{k,i}\rangle}{\lambda_k-r}\varphi_{k,i}(\theta)$$

is not positive for all $\theta\in\mathcal{S}$;
(3) $\rho=\lambda_0-\sigma\lambda_n$ and the function

$$\theta\mapsto\langle K_0,\varphi_0\rangle\varphi_0(\theta)+\sum_i\langle K_0,\varphi_{n,i}\rangle\varphi_{n,i}(\theta)+\sum_{k\geq n}\sum_i\frac{\bar{\alpha}_0\langle f,\varphi_{k,i}\rangle}{\lambda_k-g}\varphi_{k,i}(\theta)$$

is not positive for all $\theta\in\mathcal{S}$.

The proof follows immediately from Proposition 4.6.

We note from Item 1 in the above corollary that for any initial capital which is not proportional to the principal eigenfunction φ_0, $\{\bar{c},\bar{K}\}$ is not feasible if ρ is sufficiently large. Therefore, high time discount rate will lead to depletion of the capital in a future time at certain locations.

5. CONSTANT A AND N

We consider the case where A and N are constants. In this case the eigenvalues of $\mathcal{L}$ are,

$$\lambda_n=A-Dn^2 \qquad \text{for } n=0,1,2,\ldots \tag{40}$$

and the corresponding normalized eigenfunctions are

$$\varphi_0(\theta) = \frac{1}{\sqrt{2\pi}}, \qquad \varphi_{n,1}(\theta) = \frac{\sin n\theta}{\sqrt{\pi}}, \qquad \varphi_{n,2}(\theta) = \frac{\cos n\theta}{\sqrt{\pi}}.$$

In addition, $\bar{c}$ is independent of θ and

$$\bar{c}(t) = \frac{A-r}{2\pi N} e^{rt} \int_0^{2\pi} K_0(\theta)\, d\theta. \tag{41}$$

We first show that $\{\bar{c}, \bar{K}\}$ is feasible if K_0 is constant.

PROPOSITION 5.1 *Suppose that A and N are constants, $0 < \sigma < 1$, $\rho > (1-\sigma) A$, and K_0 is a constant. Then $\{\bar{c}, \bar{K}\}$ given by*

$$\bar{c}(t) = \frac{K_0}{N}(A-r)e^{rt}, \qquad \bar{K}(t) = K_0 e^{rt} \qquad \text{for all } t \geq 0$$

is feasible and optimal.

Proof. Since A, N, and K_0 are constant, by (41),

$$\bar{c}(t) = \frac{K_0}{N}(A-r)e^{rt}.$$

Substituting it into (1) and noting that $\bar{K}$ depends only on t, we see that

$$\frac{d\bar{K}}{dt} = A\bar{K} - K_0(A-r)e^{rt}, \qquad \bar{K}(0) = K_0.$$

It is easy to see that the solution is $\bar{K}(t) = K_0 e^{rt}$.

The optimality of $\{\bar{c}, \bar{K}\}$ follows from Theorem 3.1 in Boucekkine et al. (2013).

This completes the proof.

In the case where $K_0(\theta)$ is not constant, we obtain from Propositions 4.4 and 4.6 that

COROLLARY 5.2 *Suppose $\rho > (1-\sigma) A$ and that $\{\bar{c}, \bar{K}\}$ is feasible. Let n be the smallest positive integer such that*

$$\int_0^{2\pi} K_0(\theta)\cos n\theta d\theta \neq 0 \text{ or } \int_0^{2\pi} K_0(\theta)\sin n\theta d\theta \neq 0. \tag{42}$$

Then

$$\rho \leq (1-\sigma) A + \sigma D n^2. \tag{43}$$

Furthermore,

$$\lim_{t\to\infty} K^*(t,\theta) e^{-rt} = \frac{1}{2\pi}\int_0^{2\pi} K_0(\theta)\, d\theta$$

if (43) holds strictly and

$$\begin{aligned}\lim_{t\to\infty} K^*(t,\theta) e^{-rt} &= \frac{1}{2\pi}\int_0^{2\pi} K_0(\theta)\, d\theta + \frac{1}{\pi}\int_0^{2\pi} K_0(\theta)\sin n\theta d\theta \sin n\theta \\ &\quad + \frac{1}{\pi}\int_0^{2\pi} K_0(\theta)\cos n\theta d\theta \cos n\theta\end{aligned}$$

if (43) holds equally.

To prove, it suffices to observe that the function f in (15) is constant. Therefore, $\langle f, \varphi_{k,i}\rangle = 0$ for any $k \geq 1$. Also, notice that since $\left\{\bar{c}, \bar{K}\right\}$ is feasible, $T = 0$.

The above result also shows that for any non-constant initial capital distribution, $K_0(\theta)$, $K(t, \theta)$ will reach zero at some future time and location if ρ is sufficiently large (equivalently, if D is sufficiently small).

Corollary 5.3 *Let $\{c^*, K^*\}$ is an optimal pair, and let n be the smallest positive integer such that (42) is true. If either*

$$\rho > (1-\sigma) A + \sigma D n^2 \tag{44}$$

or

$$\rho = (1-\sigma) A + \sigma D n^2$$

and the function

$$\begin{aligned} K_\infty(\theta) &= \frac{1}{2\pi}\int_0^{2\pi} K_0(\theta)\, d\theta + \frac{1}{\pi}\int_0^{2\pi} K_0(\theta) \sin n\theta d\theta \sin n\theta \\ &\quad + \frac{1}{\pi}\int_0^{2\pi} K_0(\theta)\cos n\theta d\theta \cos n\theta \end{aligned}$$

is not nonnegative everywhere in $\mathcal{S}$, then $\left\{\bar{c}, \bar{K}\right\}$ is not feasible. That, is, there exists $(t, \theta) \in (0, \infty) \times \mathcal{S}$ at which $K^*(t, \theta) = 0$.

This result follows immediately from Corollary 5.2. Since (43) does not hold, $\left\{\bar{c}, \bar{K}\right\}$ is not feasible. Hence, $c^* \neq \bar{c}$ for some $(t, \theta) \in (0, \infty) \times \mathcal{S}$. Therefore, the costate ψ does not satisfy the first equation in (19) for all (t, θ). As a result, $\nu \neq 0$. In view of Remark 3.3, the set $\{(t, \theta) : K^*(t, \theta) = 0\}$ is not empty.

Remark 5.4 In general, a non-constant function has non-zero inner products with eigenfunctions corresponding to λ_1. Therefore, from Corollary 5.3 we see that if

$$\rho > (1-\sigma) A + \sigma D,$$

then $\left\{\bar{c}, \bar{K}\right\}$ is generally not feasible. This means the optimal trajectory of the state, $K^*(t, \theta)$, is generally not eventually positive. I.e., there exists a sequence (t_k, θ_k) with $t_k \to \infty$ and $\theta_k \in \mathcal{S}$ such that $K^*(t_k, \theta_k) = 0$ for each k.

6. Numerical examples

In this section we illustrate results from the previous sections. In a first subsection, we reconsider the numerical examples in Boucekkine et al. (2013) in which the pair $\left\{\bar{c}, \bar{K}\right\}$ is feasible. Our results confirms theirs in the case where the time discount is low, and extend theirs when the time discount is high. In the second subsection, we examine a case where the pair $\left\{\bar{c}, \bar{K}\right\}$ is not feasible, but there exists an optimal pair $\{c^*, K^*\}$ with K^* eventually positive. Therefore, there is an initial period in which the capital trajectory along the optimal control reaches zero. In this subsection, we highlight also the role of initial conditions, considered as given in the previous sections as we have focused on the role of the structural parameters of the model instead. In the second subsection, we consider an initial capital stock distribution which is lower than the one considered in Boucekkine et al. (2013). Because of essentially the same economic mechanisms already pointed out in the Introduction, the initial capital distribution shape can also contribute to the emergence of "dead zones". Indeed, for given structural parameters, (uniformly) low enough initial capital distributions may in principle lead to zero capital locations for given structural parameters. A low enough initial capital distribution is mechanically an additional favorable condition to prevent local capital from regenerating locally in the face of spatial diffusion, greedy consumption or low productivity. This is illustrated in the second subsection here below.

In both subsections we use the same parameter values as in Boucekkine et al. (2013),

$$A = 1/3, \qquad D = 1, \qquad N = 1, \qquad \sigma = 0.8$$

6.1. **Cases with $\{\bar{c}, \bar{K}\}$ feasible.** We use the same initial capital as in Boucekkine et al. (2013),

$$K_0(\theta) = \begin{cases} 20, & \text{if } 0 \leq \theta < \pi, \\ 10, & \text{if } \pi \leq \theta < 2\pi. \end{cases} \tag{45}$$

In this case, $\lambda_0 = 1/3$ is the only positive eigenvalue of the operator $\mathcal{L}$, $(1-\sigma)\lambda_0 = 0.0667$, and

$$\langle K_0, \varphi_0 \rangle = 15\sqrt{2\pi}, \qquad \langle f, \varphi_0 \rangle = (2\pi)^{(1+\sigma)/2\sigma}.$$

As in the baseline paper, we run two exercises. One with $\rho = 0.07$ which is slightly greater than the lower bound $(1-\sigma)A$. The other with ρ at the upper bound $(1-\sigma)A + \sigma D = 0.8667$.

Case 1: Low discount rate. With $\rho = 0.07$, we find $r = 0.3292$ and $\bar{\alpha} = 0.0198$. Computations show that $\{\bar{c}, \bar{K}\}$ is feasible detrended capital $K(t,\theta)e^{-rt}$ converges to

$$\frac{1}{2\pi}\int_0^{2\pi} K_0(\theta) = 15 \qquad \text{for } \theta \in \mathcal{S}$$

as time tends to infinite. Figure 1 shows the evolution of detrended capital. Note that very fast, initial disparities vanish and detrended capital reaches a constant value.

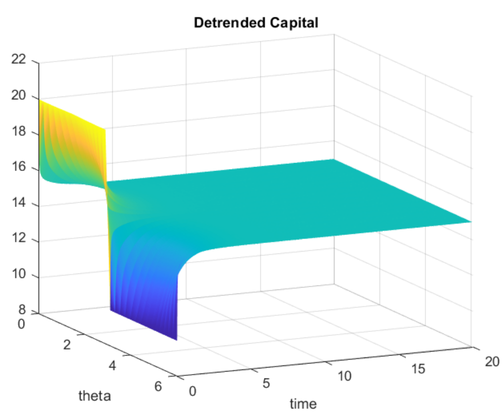

FIGURE 1. Low discount rate, $\rho = 0.07$.

Case 2: High discount rate. With $\rho = 0.8667$, we find $r = -2/3$ and $\bar{\alpha} = 4.7559$. Computations shows that $\{\bar{c}, \bar{K}\}$ is again feasible. The detrended capital converges to

$$K_D(\theta) = 15 + \frac{20}{\pi}\sin\theta$$

as predicted by Corollary 5.2. The evolution of the detrended capital is showing in Fig. 2

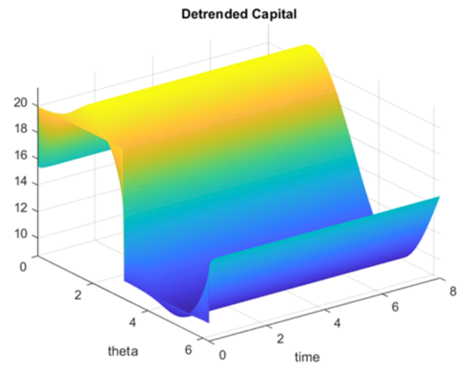

FIGURE 2. High discount rate $\rho = 0.8667$.

6.2. **A case with $\{\bar{c}, \bar{K}\}$ not feasible.** We use the initial capital

$$K_0(\theta) = \begin{cases} 10, & \text{if } 0 \leq \theta < \pi, \\ 0, & \text{if } \pi \leq \theta < 2\pi, \end{cases}$$

and choose $\rho = 0.3$. Computation shows that $\{\bar{c}, \bar{K}\}$ is not feasible with $\bar{K}$ taking negative values in the period $0 \leq t \leq 0.3$. An optimal pair $\{c^*, K^*\}$ satisfies

$$\begin{cases} K_t^* - DK_{\theta\theta}^* = AK^* - Nc^*, & \text{for } 0 < t < T, \quad \theta \in \mathcal{S}, \\ K^*(0, \theta) = K_0(\theta) & \text{for } \theta \in \mathcal{S} \end{cases} \tag{46}$$

where $T > 0$ such that $K^*(t, \theta) > 0$ for $t \geq T$. The costate

$$\psi^*(t, \theta) = c^*(t, \theta)^{-\sigma} e^{-\rho t} \tag{47}$$

is a solution to the terminal-value problem

$$\begin{cases} \psi_t^* + D\psi_{\theta\theta}^* + A\psi^* = -\nu|_{(0,T)\times\mathcal{S}}, & \text{if } 0 < t < T, \quad \theta \in \mathcal{S}, \\ \psi^*(T, \theta) = \psi_T(\theta), & \text{for } \theta \in \mathcal{S}. \end{cases} \tag{48}$$

where, by Proposition 4.5,

$$\psi_T(\theta) = e^{-\rho T} \left[(\lambda_0 - r) \int_0^{2\pi} K^*(T, \theta) \, d\theta \right]^{-\sigma} \tag{49}$$

and ν is the Gâteaux derivative of the functional

$$d_0(K^*) = |\min\{K^*, 0\}|_{C(\mathcal{S}_T)}.$$

(Note that ν does not show up in the terminal condition because of the relation (12) and $K^*(T, \theta) > 0$ in $\mathcal{S}$.) System (46)–(49) form a couple system that can be can be represented by operators

$$K^* = \mathcal{P}[\psi^*], \qquad \psi^* = \mathcal{Q}[K^*] \tag{50}$$

where $\mathcal{P}[\psi]$ is the solution of (46) with $c^* = \left[\psi e^{\rho t}\right]^{-1/\sigma}$, and $\mathcal{Q}[K]$ is the solution of (48) with ν the Gâteaux derivative of $d_0(K)$ and ψ_T given by (49) with K^* replaced by K. Thus, (K^*, ψ^*) is a fixed point of the operator $(K, \psi) \mapsto (\mathcal{P}[\psi], \mathcal{Q}[K])$.

We obtain a numerical approximation of the solution (K^*, ψ^*). Computation shows that $K^*(t, \theta) > 0$ for $t \geq T \approx 0.26$. Graphs of K^* and c^* for $0 \leq t \leq 0.4$ are showing in Fig.3.

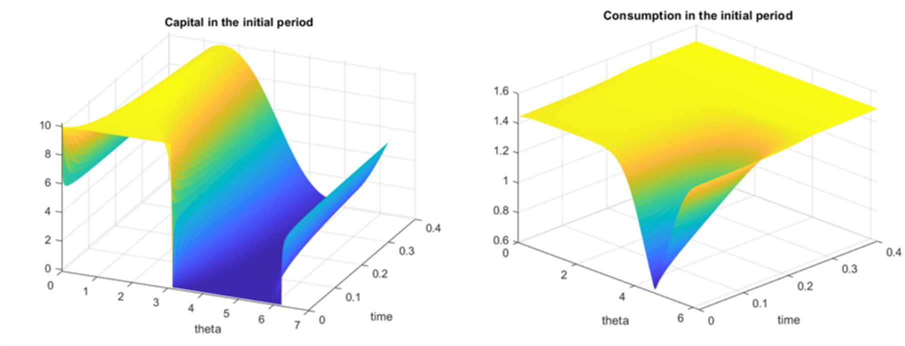

FIGURE 3. A case with $\left\{\bar{c}, \bar{K}\right\}$ not feasible. Initial period is roughly $0 \leq t \leq 0.26$. Left: the capital in the initial stage. Right: the consumption in the initial stage.

As can be seen, unlike the case where $\left\{\bar{c}, \bar{K}\right\}$ is feasible, the consumption c^* in the initial period is not spatially uniform. Specifically c^* is lower near locations where K^* is near zero.

7. Concluding remarks

This paper refines several results obtained for the SAK model since 2013, with a special focus on the existence of **positive** optimal capital paths. We show using a combination of optimization techniques that an appropriately designed two-stage approach to the original SAK problem does allow to bring out several clarifying findings, albeit highlighting again the critical role of two parameters, the time discount rate and the spatial diffusion rate.

Beside the technical contribution, this paper, and indeed all the papers that have been published on the SAK, show all the complexity of spatiotemporal models compared to the non-spatial counterparts. Even the AK model with central planning, plus an elementary diffusion equation, is highly problematic. Accordingly, moving to decentralized equilibria in this kind of continuous space (infinite-dimensional) settings sounds as a daunting task. This does not mean that moving to discrete space and simplifying the transport equations will make things easier and existence of equilibria more straightforward, the history of this literature is full of impossibility and non-existence theorems, even in the static case (Starrett (1978)).

Appendix: Proof of Proposition 3.2

We first prepare a technical lemma which gives the "linear approximations" of the state and welfare under spike perturbations.

LEMMA .1 *Let Assumption 3.1 hold and let $y_0 \in C^{\alpha}(\mathcal{S})$ for some $\alpha \in (0,1)$. Let $\{\bar{c}, \bar{y}\}$ be a feasible pair and let $c \in \mathcal{C}$ be fixed. Then, for any $\delta \in (0,1)$ there exists a measurable* *set $E_\delta \subset \mathcal{S}_T$ and the control c_δ defined by*

$$c_\delta(t,\theta) = \begin{cases} \bar{c}(t,\theta) & \text{if } (t,\theta) \in \mathcal{S}_T \backslash E_\delta, \\ c(t,\theta) & \text{if } (t,\theta) \in E_\delta, \end{cases} \tag{51}$$

such that $|E_\delta| = \delta|\mathcal{S}_T|$ and the following hold:

$$y(\cdot, c_\delta) = \bar{y}(\cdot) + \delta z(\cdot) + o(\delta), \qquad J(c_\delta) = J(\bar{c}) + \delta l + o(\delta) \tag{52}$$

(with the first $o(\delta)$ is in space $C^{\alpha,\alpha/2}(\mathcal{S}_T)$ for some $\alpha \in (0,1)$) where z and l satisfy

$$\begin{aligned} z_t - Dz_{\theta\theta} &= a(t,\theta)z + \phi(t,\theta), \qquad \text{for } (t,\theta) \in \mathcal{S}_T, \\ z(0,\theta) &= 0 \end{aligned}$$

and

$$l = \int_0^T \int_{\mathcal{S}} \gamma(t,\theta)\, d\theta dt + \int_{\mathcal{S}} h_y(T, \bar{y}(T,\theta))\, z(T,\theta)\, d\theta,$$

respectively, with

$$\begin{aligned} \phi(t,\theta) &= b(t,\theta)[c(t,\theta) - \bar{c}(t,\theta)], \\ \gamma(t,\theta) &= g(t,\theta,c(t,\theta)) - g(t,\theta,\bar{c}(t,\theta)). \end{aligned}$$

Proof. Let

$$\mathcal{C} = \{c : \mathcal{S}_T \mapsto \mathbb{R}_+ | c \text{ is measurable}\}$$

be the metric space with the Ekeland distance $d(u,v)$ given by

$$d(u,v) = |\{(t,\theta) \in \mathcal{S}_T | u(t,\theta) \neq v(t,\theta)\}|. \tag{53}$$

(Recall that $|\Omega|$ for a Lebesgue measurable set Ω represents its measure.) It is well-known that $(\mathcal{C}, d(\cdot,\cdot))$ is a complete metric space. By the definition of c_δ in (51), $d(c_\delta, \bar{c}) \leq |E_\delta|$. Let

$$z_\delta(t,\theta) = \frac{1}{\delta}[y(t,\theta;c_\delta) - \bar{y}(t,\theta)] \qquad \text{in } \mathcal{S}_T.$$

Then, z_δ satisfies

$$\begin{aligned} (z_\delta)_t - D(z_\delta)_{\theta\theta} &= b_\delta z_\delta(t,\theta) + \frac{1}{\delta}\chi_{E_\delta}(t,\theta)\,\phi(t,\theta) \qquad \text{if } t \in (0,T), \theta \in \mathcal{S}, \\ z_\delta(0,\theta) &= 0 \qquad \text{if } \theta \in \mathcal{S}, \end{aligned}$$

where

$$b_\delta = \int_0^1 a(t,\theta)\, c_\delta(t,\theta)\, ds,$$

and χ_{E_δ} is the characteristic function of E_δ. From the regularity of the parabolic equation (4) and Assumption 3.1, we see that b_δ and ϕ are uniformly bounded. Hence, by the Holder's estimate there is $\alpha \in (0,1)$ such that

$$|y(\cdot, c_\delta) - \bar{y}|_{C^{\alpha,\alpha/2}(\mathcal{S}_T)} \leq C\,|\chi_{E_\delta}|_{L^p(\mathcal{S}_T)} \to 0 \qquad \text{as } \delta \to 0,$$

where the constant C is independent of E_δ. Comparing equations for z_δ and z we derive

$$\begin{aligned} (z_\delta - z)_t - D(z_\delta - z)_{\theta\theta} &= b_\delta z_\delta - a(t,\theta)z - \left(1 - \frac{1}{\delta}\chi_{E_\delta}(t,\theta)\right)\phi(t,\theta), \\ (z_\delta - z)(0,\theta) &= 0. \end{aligned}$$

Using Lemma 3.2 in Hu and Yong (1995), we see that $|z_\delta - z|_{C^{\alpha,\alpha/2}(\mathcal{S}_T)} \to 0$ as $\delta \to 0$. This proves the first relation in (52).

To prove the second relation in (52), we let

$$\begin{aligned} l_\delta &= \frac{1}{\delta}\left[J\left(c_\delta\right)-J\left(\bar{c}\right)\right]=\frac{1}{\delta}\left\{\int_0^T\int_{\mathcal{S}}\left[g\left(t,\theta,c_\delta\left(t,\theta\right)\right)-g\left(t,\theta,\bar{c}\left(t,\theta\right)\right)\right]d\theta dt\right. \\ &\quad \left.+\left[h\left(T,y\left(T,\cdot,c_\delta\left(T,\cdot\right)\right)\right)-h\left(T,\bar{y}\left(T,\cdot\right)\right)\right]\right\}. \end{aligned}$$

By the definition of c_δ and (52), it follows that

$$l_\delta=\int_0^T\int_{\mathcal{S}}\frac{1}{\delta}\chi_{E_\delta}\left(t,\theta\right)\gamma\left(t,\theta\right)d\theta dt+\int_{\mathcal{S}}\eta_\delta\left(\theta\right)z_\delta\left(T,\theta\right)d\theta+\frac{1}{\delta}\int_{\mathcal{S}}\chi_{E_\delta}\left(t,\theta\right)\gamma\left(t,\theta\right)d\theta,$$

where

$$\eta_\delta\left(\theta\right)=\int_0^1 h_y\left(T,\bar{y}\left(T,\theta\right)+s\left(y\left(T,\theta,c_\delta\left(T,\theta\right)\right)-\bar{y}\left(T,\theta\right)\right)\right)ds.$$

It is clear that $\eta_\delta\left(\theta\right)\to h_y\left(T,\bar{y}\left(T,\theta\right)\right)$ as $\delta\to 0$.

Comparing equations for l_δ and l, we find

$$\begin{aligned} l_\delta-l &= \int_{\mathcal{S}}\left[\eta_\delta\left(\theta\right)z_\delta\left(T,\theta\right)-h_y\left(T,\bar{y}\left(T,\theta\right)\right)z\left(T,\theta\right)\right]d\theta \\ &\quad -\int_0^T\int_{\mathcal{S}}\left(1-\frac{1}{\delta}\chi_{E_\delta}\left(t,\theta\right)\right)\gamma\left(t,\theta\right)d\theta dt. \end{aligned}$$

Using again Lemma 3.2 in Hu and Yong (1995) and the convergence $z_\delta\to z$ in $C^{\alpha,\alpha/2}\left(\mathcal{S}_T\right)$, we find $l_\delta-l\to 0$ as $\delta\to 0$.

This completes the proof of the lemma.

Proof of Proposition 3.2. Note that the main difference between System (4)–(6) and that in Hu and Yong (1995) is the boundary condition and the presence of the scrap value $h\left(T,y\left(T,\cdot\right)\right)$. Thus the proof follows closely the approach in that paper. The main idea is using Ekeland's variational principle (Theorem 3.2.2 in Fattorini (1999)) together with a spike perturbation.

For any $\varepsilon>0$ we define

$$F_\varepsilon\left(c\right)=\left\{\left[J\left(c\right)-J\left(c^*\right)+\varepsilon\right]_+^2+d_0\left(y\left(\cdot;c\right)\right)^2\right\}^{1/2} \tag{54}$$

where $y\left(\cdot;c\right)$ is the solution to (4) corresponding to the control c. Then,

$$F_\varepsilon\left(c^*\right)=\varepsilon\le\inf F_\varepsilon\left(c\right)+\varepsilon.$$

This means c^* is an ε-minimum of F_ε, which is bounded below and semi-lower continuous. Hence, by Ekeland's variational principle, there exists c^ε such that

$$F_\varepsilon\left(c^\varepsilon\right)\le F_\varepsilon\left(c^*\right),\qquad d\left(c^\varepsilon,c^*\right)\le\sqrt{\varepsilon} \tag{55}$$

and

$$F_\varepsilon\left(c^\varepsilon\right)-F_\varepsilon\left(c\right)\le\sqrt{\varepsilon}d\left(c^\varepsilon,c\right)\qquad\text{for any } c\in\mathcal{C}. \tag{56}$$

Let $y^\varepsilon=y\left(\cdot;c^\varepsilon\right)$. Fix a $c\in\mathcal{C}$ and an $\varepsilon>0$. For any $\delta>0$ and a measurable set $E_\delta^\varepsilon\subset\mathcal{S}_T$ we construct the perturbation

$$c_\delta^\varepsilon\left(t,\theta\right)=\begin{cases} c^\varepsilon\left(t,\theta\right) & \text{for } \left(t,\theta\right)\in\mathcal{S}_T\backslash E_\delta^\varepsilon, \\ c\left(t,\theta\right) & \text{for } \left(t,\theta\right)\in E_\delta^\varepsilon. \end{cases} \tag{57}$$

Let $y^{\varepsilon}_{\delta} = y\left(\cdot; c^{\varepsilon}_{\delta}\right)$ denote the state corresponding to the perturbation. An application of Lemma .1 leads to

$$y^{\varepsilon}_{\delta} = y^{\varepsilon} + \delta z^{\varepsilon} + o\left(\delta\right), \qquad J\left(c^{\varepsilon}_{\delta}\right) = J\left(c^{\varepsilon}\right) + \delta l^{\varepsilon} + o\left(\delta\right), \tag{58}$$

where z^{ε} and $z^{0,\varepsilon}$ satisfy equations

$$z^{\varepsilon}_{t} - D z^{\varepsilon}_{\theta\theta} = b\left(t,\theta\right) z^{\varepsilon}\left(t,\theta\right) + \phi^{\varepsilon}\left(t,\theta\right), \tag{59}$$

and

$$l^{\varepsilon} = \int_0^T \int_{\mathcal{S}} \gamma^{\varepsilon}\left(t,\theta\right) d\theta dt + \int_{\mathcal{S}} h_y\left(T, y^{\varepsilon}\left(T,\theta\right)\right) z^{\varepsilon}\left(T,\theta\right) d\theta \tag{60}$$

respectively, with

$$\begin{aligned} \phi^{\varepsilon}\left(t,\theta\right) &= b\left(t,\theta\right)\left[c\left(t,\theta\right) - c^{\varepsilon}\left(t,\theta\right)\right], \\ \gamma^{\varepsilon}\left(t,\theta\right) &= g\left(t,\theta,c\left(t,\theta\right)\right) - g\left(t,\theta,c^{\varepsilon}\left(t,\theta\right)\right). \end{aligned}$$

We next choose a E^{ε}_{δ} so that $|E^{\varepsilon}_{\delta}| = \delta\left|\mathcal{S}_T\right|$. By (57), $d\left(c^{\varepsilon}_{\delta}, c^{\varepsilon}\right) = |E^{\varepsilon}_{\delta}|$. Hence, by (56)

$$\begin{aligned} \sqrt{\varepsilon}\left|\mathcal{S}_T\right| &\geq \frac{F_{\varepsilon}\left(c^{\varepsilon}\right) - F_{\varepsilon}\left(c^{\varepsilon}_{\delta}\right)}{\delta} \\ &= \frac{1}{\left[F_{\varepsilon}\left(c^{\varepsilon}\right) + F_{\varepsilon}\left(c^{\varepsilon}_{\delta}\right)\right]\delta}\Big\{\left[J\left(c^{\varepsilon}\right) - J\left(c^{*}\right) + \varepsilon\right]^2_{+} - \left[J\left(c^{\varepsilon}_{\delta}\right) - J\left(c^{*}\right) + \varepsilon\right]^2_{+} \\ &\quad + \left[d_0\left(y^{\varepsilon}\right) - d_0\left(y^{\varepsilon}_{\delta}\right)\right]\Big\}. \end{aligned}$$

Taking $\delta \to 0$ and using (58), the right-hand side converges to

$$\frac{\left[J\left(c^{\varepsilon}\right) - J\left(c^{*}\right) + \varepsilon\right]_{+}}{F_{\varepsilon}\left(c^{\varepsilon}\right)} l^{\varepsilon} + \left\langle \frac{d_0\left(y^{\varepsilon}\right)\xi^{\varepsilon}}{F_{\varepsilon}\left(c^{\varepsilon}\right)}, z^{\varepsilon}\right\rangle \equiv \mu^{\varepsilon} l^{\varepsilon} + \left\langle \nu^{\varepsilon}, z^{\varepsilon}\right\rangle,$$

where

$$\mu^{\varepsilon} = \frac{\left[J\left(c^{\varepsilon}\right) - J\left(c^{*}\right) + \varepsilon\right]_{+}}{F_{\varepsilon}\left(c^{\varepsilon}\right)}, \qquad \nu^{\varepsilon} = \frac{d_0\left(y^{\varepsilon}\right)\xi^{\varepsilon}}{F_{\varepsilon}\left(c^{\varepsilon}\right)}$$

and

$$\xi^{\varepsilon}\left(y^{\varepsilon}\right) = \begin{cases} \nabla d_0\left(y^{\varepsilon}\right), & \text{if } y^{\varepsilon} \notin Q, \\ 0 & \text{if } y^{\varepsilon} \in Q. \end{cases}$$

Hence, by (56),

$$\sqrt{\varepsilon}\left|\mathcal{S}_T\right| \geq \mu^{\varepsilon} l^{\varepsilon} + \left\langle \nu^{\varepsilon}, z^{\varepsilon}\right\rangle. \tag{61}$$

Note that by (8),

$$\left|\xi^{\varepsilon}\left(y^{\varepsilon}\right)\right|_{C(\mathcal{S}_T)^*} = 1 \qquad \text{if } y^{\varepsilon} \notin Q.$$

It follows from (54) that $\mu^{\varepsilon} \geq 0$ and

$$\mu^{\varepsilon} + \left|\nu^{\varepsilon}\right|_{C(\mathcal{S}_T)^*} = 1 \qquad \text{for all } \varepsilon > 0.$$

Also, by (12),

$$\left\langle \nu^{\varepsilon}, z - y^{\varepsilon}\right\rangle \leq -d_0\left(y^{\varepsilon}\right) \leq 0 \qquad \text{for any } z \in Q. \tag{62}$$

Next, by (55) and Lemma .1, it follows that

$$y^{\varepsilon} = y^{*} + \delta z^{*} + o\left(\delta\right), \qquad J\left(c^{\varepsilon}\right) = J\left(c^{*}\right) + \delta l^{*} + o\left(\delta\right), \tag{63}$$

where z^{*} and l^{*} satisfy equations

$$\begin{aligned} & z^{*}_{t} - D z^{*}_{\theta\theta} = b\left(t,\theta\right) z^{*}\left(t,\theta\right) + \phi^{*}\left(t,\theta\right) \qquad \text{in } \mathcal{S}_T \\ & z^{*}\left(0,\theta\right) = 0 \end{aligned} \tag{64}$$

and

$$l^* = \int_0^T \int_{\mathcal{S}} \gamma^* (t,\theta)\, d\theta dt + \int_{\mathcal{S}} h_y (T, y^* (T,\theta))\, z^* (T,\theta)\, d\theta,$$

respectively, with

$$\begin{aligned} \phi^* (t,\theta) &= b(t,\theta) [c(t,\theta) - c^* (t,\theta)], \\ \gamma^* (t,\theta) &= g(t,\theta, c(t,\theta)) - g(t,\theta, c^* (t,\theta)). \end{aligned}$$

From (63) we see that $y^\varepsilon \to y^*$ in $C^{\alpha,\alpha/2}(\mathcal{S}_T)$ as $\varepsilon \to 0$. Thus, from (59) and (60) we find $z^\varepsilon \to z^*$ in $C^{\alpha,\alpha/2}$ and $l^\varepsilon \to l^*$ in $\mathbb{R}$ as $\varepsilon \to 0$. Since Q is finite codimensional in $C(\mathcal{S}_T)$, it follows from Lemma 3.2 of Li and Yong (1991) that the weakly-* limit, (μ,ν), of $(\mu^\varepsilon, \nu^\varepsilon)$ as $\varepsilon \to 0$ is positive. Taking $\varepsilon \to 0$ in (62) we find $\langle \nu, z - y^* \rangle \le 0$ for any $z \in Q$. In addition, from (61) we find

$$\mu l^* + \langle v, z^* \rangle \le 0 \qquad \text{for any } c \in \mathcal{C}. \tag{65}$$

We show that the above inequality is equivalent to

$$\begin{aligned} 0 \le \int_0^T \int_{\mathcal{S}} &\{\mu [g(t,\theta,c^*(t,\theta)) - g(t,\theta,c(t,\theta))] \\ &+ \psi(t,\theta) b(t,\theta) [c^*(t,\theta) - c(t,\theta)]\} d\theta dt \end{aligned}$$

for any $c \in \mathcal{C}$. Using (10) and (64), we find

$$\begin{aligned} \int_0^T \int_{\mathcal{S}} [z^* \psi]_t \, d\theta dt &= \int_0^T \int_{\mathcal{S}} [z_t^* \psi + z^* \psi_t]\, d\theta dt \\ &= \int_0^T \int_{\mathcal{S}} \psi \phi^* d\theta dt + \langle \nu, z^* \rangle_{(0,T)\times\mathcal{S}}. \end{aligned}$$

On the other hand, since

$$z^*(0,\theta) = 0, \qquad \psi(T,\cdot) = \mu h_y (T, y^*(T,\cdot)) + \nu|_{\{T\}\times\mathcal{S}},$$

it follows that

$$\begin{aligned} \int_0^T \int_{\mathcal{S}} [z^* \psi]_t \, d\theta dt &= \int_{\mathcal{S}} z^*(T,\theta)\, \psi(T,\theta)\, d\theta \\ &= \mu \int_{\mathcal{S}} h_y (T, y^*(T,\theta))\, z^*(T,\theta)\, d\theta + \langle \nu, z^*(T,\cdot) \rangle_{\mathcal{S}}. \end{aligned}$$

As a result,

$$\begin{aligned} &\mu \textstyle\int_{\mathcal{S}} h_y (T, y^*(T,\theta))\, z^*(T,\theta)\, d\theta + \langle \nu, z^*(T,\cdot) \rangle_{\mathcal{S}} \\ &= \textstyle\int_0^T \int_{\mathcal{S}} \psi(t,\theta)\, \phi^*(t,\theta)\, d\theta dt - \langle \nu, z^* \rangle_{(0,T)\times\mathcal{S}}. \end{aligned} \tag{66}$$

By (65),

$$\mu \int_0^T \int_{\mathcal{S}} \gamma^*(t,\theta)\, d\theta dt + \mu \int_{\mathcal{S}} h_y (T, y^*(T,\theta))\, z^*(T,\theta)\, d\theta + \langle \nu, z^* \rangle_{\mathcal{S}_T} \le 0.$$

As a result, by (66),

$$\int_0^T \int_{\mathcal{S}} [\mu \gamma^*(t,\theta) + \psi(t,\theta)\, \phi^*(t,\theta)]\, dt d\theta \le 0.$$

This is equivalent to

$$0 \le \int_0^T \int_{\mathcal{S}} \{\mu [g(t,\theta,c^*(t,\theta)) - g(t,\theta,c(t,\theta))] + \psi(t,\theta) b(t,\theta) [c^*(t,\theta) - c(t,\theta)]\} d\theta dt$$

for any $c \in \mathcal{C}$. Since $c(t,\theta)$ is arbitrary, (11) follows.

The proof is complete.

References

Richard Arnott, Oded Hochman, and Gordon C Rausser. Pollution and land use: optimum and decentralization. *Journal of Urban Economics*, 64(2):390–407, 2008.

Luca Vincenzo Ballestra. The spatial ak model and the pontryagin maximum principle. *Journal of Mathematical Economics*, 67:87–94, 2016.

Alain Bensoussan, Giuseppe Da Prato, Michel C Delfour, and Sanjoy K Mitter. *Representation and control of infinite dimensional systems*. Springer, 2007.

Raouf Boucekkine, Carmen Camacho, and Benteng Zou. Bridging the gap between growth theory and the new economic geography: The spatial ramsey model. *Macroeconomic Dynamics*, 13(1):20–45, 2009.

Raouf Boucekkine, Carmen Camacho, and Giorgio Fabbri. Spatial dynamics and convergence: the spatial ak model. *Journal of Economic Theory*, 148:2719–2736, 2013.

Raouf Boucekkine, Giorgio Fabbri, Salvatore Federico, and Fausto Gozzi. Growth and agglomeration in the heterogeneous space: a generalized ak approach. *Journal of Economic Geography*, 19:1287–1318, 2019.

Raouf Boucekkine, Giorgio Fabbri, Salvatore Federico, and Fausto Gozzi. A dynamic theory of spatial externalities. *Games and Economic Behavior*, 132:133–165, 2022.

Paulo Brito. The dynamics of growth and distribution in a spatially heterogeneous world. 2004.

Buzz Brock and Anastasios Xepapadeas. Diffusion-induced instability and pattern formation in infinite horizon recursive optimal control. *Journal of Economic Dynamics and Control*, 32:2745–2787, 2008.

B Malcolm Brown, Michael SP Eastham, and Karl Michael Schmidt. *Periodic differential operators*, volume 228. Springer Science & Business Media, 2012.

Carmen Camacho, Benteng Zou, and Maya Briani. On the dynamics of capital accumulation across space. *European Journal of Operational Research*, 186(2):451–465, 2008.

Eduardo Casas and Karl Kunisch. Infinite horizon optimal control problems for a class of semilinear parabolic equations. *SIAM Journal on Control and Optimization*, 60(4): 2070–2094, 2022.

Eduardo Casas and Karl Kunisch. Infinite horizon optimal control for a general class of semilinear parabolic equations. *Applied Mathematics & Optimization*, 88(2):47, 2023.

Giorgio Fabbri. Geographical structure and convergence: A note on geometry in spatial growth models. *Journal of Economic Theory*, 162:114–136, 2016.

Hector O Fattorini. *Infinite dimensional optimization and control theory*, volume 54. Cambridge University Press, 1999.

Masahisa Fujita and Jacques-Francois Thisse. *Economics of Agglomeration: Cities, Industrial Location, and Regional Growth*. Cambridge University Press, 2002.

Fausto Gozzi and Marta Leocata. A stochastic model of economic growth in time-space. *SIAM Journal on Control and Optimization*, 60(2):620–651, 2022.

Bei Hu and Jiongmin Yong. Pontryagin maximum principle for semilinear and quasilinear parabolic equations with pointwise state constraints. *SIAM journal on control and optimization*, 33(6):1857–1880, 1995.

Emma Hutchinson and Peter W Kennedy. State enforcement of federal standards: Implications for interstate pollution. *Resource and Energy Economics*, 30(3):316–344, 2008.

Paul Krugman. Increasing returns and economic geography. *Journal of Political Economy*, 99:483–499, 1991.

Paul Krugman. Models of Spatial Self-Organization. In *The Self-Organizing Economy*, pages 75–99. Blackwell Publishers, Cambridge, MA, 1996.

Xunjing Li and Jiongmin Yong. Necessary conditions for optimal control of distributed parameter systems. *SIAM Journal on Control and Optimization*, 29(4):895–908, 1991.

Jacques Louis Lions. *Optimal control of systems governed by partial differential equations*, volume 170. Springer, 1971.

Pascal Mossay. A theory of rational spatial agglomerations. *Regional Science and Urban Economics*, 43(2):385–394, 2013.

Jean-Pierre Raymond and Hasnaa Zidani. Pontryagin's principle for state-constrained control problems governed by parabolic equations with unbounded controls. *SIAM Journal on Control and Optimization*, 36(6):1853–1879, 1998.

Jean Pierre Raymond and Housnaa Zidani. Hamiltonian pontryagin's principles for control problems governed by semilinear parabolic equations. *Applied Mathematics and Optimization*, 39(2):143–177, 1999.

Cristiano Ricci. A non-invariance result for the spatial ak model: C. ricci. *Decisions in Economics and Finance*, 48(1):465–484, 2025.

Joel Smoller. *Shock waves and reaction—diffusion equations*, volume 258. Springer Science & Business Media, 2012.

David Starrett. Market allocations of location choice in a model with free mobility. *Journal of economic theory*, 17(1):21–37, 1978.

Anastasios Xepapadeas and Athanasios N Yannacopoulos. Spatial growth theory: Optimality and spatial heterogeneity. *Journal of Economic Dynamics and Control*, 146:104584, 2023.