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Power-Law Distribution in the External Debt-to-Fiscal Revenue Ratios: Empirical Evidence and a Theoretical Model. Appendices

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Appendix A. Government's optimal choice

This appendix and the next ones present how the government's and household's optimal choices are calculated. The methodology is based on techniques used in continuous time optimization models (see for instance Chang (2004), Turnovsky (1999, 2000).

The government's choice is as follows:

$$\max_{\bar{C^G}(t)} = \mathbb{E}\int_0^{+\infty} \frac{1}{\gamma}\bar{C^G}(t)^{\gamma} e^{-\rho t} dt \tag{A.1}$$

subject to

$$dT(t) = [\frac{\bar{g}(t)}{T(t)} + \frac{\bar{C^G}(t)}{T(t)}]T(t)dt + [\frac{\tilde{g}dW^g(t)}{T(t)} + \frac{\tilde{C^G}dW^{C^G}(t)}{T(t)}]T(t). \tag{A.2}$$

This constraint can be expressed as:

$$\frac{dT(t)}{T(t)} = \psi^T dt + \sigma_{w^T} dw^T, \tag{A.3}$$

with

$$\begin{aligned} \psi^T &= \tfrac{\bar{g}(t)}{T(t)} + \tfrac{\bar{C^G}(t)}{T(t)}, \\ \sigma_{w^T} dw^T &= \tfrac{\tilde{g}dW^g(t)}{T(t)} + \tfrac{\tilde{C^G}dW^{C^G}(t)}{T(t)}. \end{aligned} \tag{A.4}$$

Let us define $n_g = \frac{\bar{g}(t)}{T(t)}$, $n_C = \frac{\bar{C^G}(t)}{T(t)}$, $\tilde{n_g} = \frac{\tilde{g}}{T(t)}$ and $\tilde{n_C} = \frac{\tilde{C^G}}{T(t)}$. We then have:

$$\begin{aligned} \psi^T &= n_g + n_C, \\ \sigma_{w^T} dw^T &= \tilde{n_g} dW^g(t) + \tilde{n_C} dW^{C^G}(t), \end{aligned} \tag{A.5}$$

such that

$$\sigma_{w^T}^2 = \tilde{n_g}^2 \sigma_{W^g}^2 + \tilde{n_C}^2 \sigma_{W^{C^G}}^2. \tag{A.6}$$

The program becomes

$$\max_{n^G, n^C} = \mathbb{E} \int_0^{+\infty} \frac{1}{\gamma} (T n_C)^\gamma e^{-\rho t} dt \tag{A.7}$$

subject to

$$\begin{array}{rcl} \frac{dT(t)}{T(t)} & = & \psi^T dt + \sigma_{w^T} dw^T, \\ 1 & = & n_g + n_C. \end{array} \tag{A.8}$$

The differential generator of the value function V(T,t) is defined by:

$$L\big[V(T,t)\big] \equiv \frac{\partial V}{\partial t} + \psi^T T \frac{\partial V}{\partial T} + \frac{1}{2} \sigma_{w^T}^2 T^2 \frac{\partial^2 V}{\partial T^2}. \tag{A.9}$$

We assume V to be of the following time separable form:

$$V(T,t) = e^{-\rho t} X(T). \tag{A.10}$$

And government choses n_C and n_g maximising the following Lagrangian:

$$\text{Lagrangian} \;=\; e^{-\rho t} \tfrac{1}{\gamma} (T n_C)^\gamma + L\big[e^{-\rho t} X(T)\big] + e^{-\rho t} \lambda (1 - n_g - n_C). \tag{A.11}$$

The partial derivative with respect to n_C is:

$$T^\gamma n_C^{\gamma - 1} + T X_T - T^2 X_{TT} \sigma_{W^{C^G}}^2 n_C = \lambda. \tag{A.12}$$

The partial derivative with respect to n_g is:

$$T X_T - T^2 X_{TT} \sigma_{W^g}^2 n_g = \lambda. \tag{A.13}$$

Putting these equations together with $1 = n_g + n_C$ leads to:

$$T^\gamma n_C^{\gamma - 1} = T^2 X_{TT} \big[\sigma_{W^{C^G}}^2 n_C - \sigma_{W^g}^2 n_g\big], \tag{A.14}$$

and

$$T^\gamma n_C^{\gamma - 1} = T^2 X_{TT} \big[(\sigma_{W^{C^G}}^2 + \sigma_{W^g}^2) n_C - \sigma_{W^g}^2\big]. \tag{A.15}$$

Besides, the value function must satisfy the Bellman equation

$$\max_{n_C, n_g} \left\{ \frac{1}{\gamma} e^{-\rho t} (T n_C)^\gamma + L\big[e^{-\rho t} X(T)\big] \right\} = 0. \tag{A.16}$$

To solve it, we substitute the optimized value of n_C and n_g:

$$\tfrac{1}{\gamma}T^{\gamma}\hat{n_C}^{\gamma} - \rho X(T) + TX_T + \tfrac{1}{2}\sigma^2_{w^T}T^2X_{TT} = 0. \tag{A.17}$$

To solve the resulting equation in X(T), we postulate X(T) of the form:

$$X(T) = \delta T^{\gamma}, \tag{A.18}$$

with δ to be determined. This yields to

$$\begin{aligned} TX_T &= \gamma X(T), \\ T^2X_{TT} &= \gamma(\gamma-1)X(T). \end{aligned} \tag{A.19}$$

Using this, the Bellman equation becomes:

$$\frac{1}{\gamma}T^{\gamma}\hat{n_C}^{\gamma} - \rho X(T) + \gamma X(T) + \frac{1}{2}\sigma^2_{w^T}\gamma(\gamma-1)X(T) = 0. \tag{A.20}$$

According to (A.15), $(T\hat{n_C})^{\gamma-1}$ is given by:

$$\begin{aligned} T^{\gamma}n_C^{\gamma-1} &= T^2X_{TT}\big[(\sigma^2_{W^{C^G}} + \sigma^2_{W^g})n_C - \sigma^2_{W^g}\big] \\ &= \gamma(\gamma-1)\big[(\sigma^2_{W^{C^G}} + \sigma^2_{W^g})n_C - \sigma^2_{W^g}\big]X(T). \end{aligned} \tag{A.21}$$

We can substitute it in (A.20) and divide by $X(T)$:

$$(\sigma^2_{W^{C^G}} + \sigma^2_{W^g})\hat{n_C}^2 - \sigma^2_{W^g}\hat{n_C} + \frac{\rho-\gamma}{1-\gamma} + \frac{1}{2}\sigma^2_{w^T}\gamma = 0, \tag{A.22}$$

which leads to the second-order differential equation with:

$$\triangle = \sigma^4_{W^g} - 4(\sigma^2_{W^{C^G}} + \sigma^2_{W^g})(\frac{\rho-\gamma}{1-\gamma} + \frac{1}{2}\sigma^2_{w^T}\gamma). \tag{A.23}$$

Solutions (if $\triangle$ is positive) are of the form

$$\hat{n_C} = \frac{\sigma^2_{W^g} \pm \sqrt{\triangle}}{2(\sigma^2_{W^{C^G}} + \sigma^2_{W^g})}. \tag{A.24}$$

With $\hat{n_C}$ positive we have

$$\hat{n_C} = \frac{\sigma^2_{W^g} + \sqrt{\sigma^4_{W^g} - 4(\sigma^2_{W^{C^G}} + \sigma^2_{W^g})(\frac{\rho-\gamma}{1-\gamma} + \frac{1}{2}\sigma^2_{w^T}\gamma)}}{2(\sigma^2_{W^{C^G}} + \sigma^2_{W^g})}. \tag{A.25}$$

Appendix B. The consumer's optimal choice

Government and households are assumed to have the same time preference coefficient ρ). The objective function is

$$\max_{C^P, C^M, e, n_K, n_f} = \mathbb{E}\int_0^{+\infty} \frac{1}{\mu}\left[C^P(t)^\eta C^M(t)^{1-\eta}\right]^\mu e^{-\rho t} dt. \tag{B.1}$$

The household maximizes the intertemporal utility function subject to the constraints given by Equation (**??**), and with $W(0) = w_0$.

We define the aggregate consumption $C = C^P(t)^\eta C^M(t)^{1-\eta}$.
The consumer price index can be defined as $CPI(t) = P^P(t)^\eta P(t)^{1-\eta}$ which yields to $CPI(t) = P(t)^{1-\eta}$, as the domestic good is the numeraire
We thus have:

$$\frac{dW(t)}{W(t)} = \psi^W dt + \sigma_{z^W} dz^W, \tag{B.2}$$

with

$$\begin{aligned} \psi^W &= \left[1 - \tau + \bar{x}\tau e(t)\right] n_k^\beta n_g^{1-\beta} (\tfrac{T}{W})^{1-\beta} - \tfrac{CPI(t)C(t)}{W} + n_k r_k - (i^* + \pi) n_f, \\ \sigma_{z^W} dz^W &= \sigma\tau e(t) n_k^\beta n_g^{1-\beta} (\tfrac{T}{W})^{1-\beta} dW^W(t) + n_k du_k - n_f du_f, \end{aligned} \tag{B.3}$$

where $n_k = \frac{k}{W}$ and $n_f = \frac{PD}{W}$ $(0 < n_k < 1$ and $n_f > 0)$.
And we get

$$\sigma_{z^W}^2 = \sigma^2\tau^2 e(t)^2 n_k^{2\beta} n_g^{2(1-\beta)} (\frac{T}{W})^{2(1-\beta)} \sigma_{W^W}^2 + n_k^2 \sigma_{W^k}^2 + n_f^2 \sigma_{W^f}^2. \tag{B.4}$$

The households' decision is as follows:

$$\max_{C, e, n_K, n_f} \mathbb{E}\int_0^{+\infty} \frac{1}{\mu} C^\mu e^{-\rho t} dt, \text{ with } -\infty < \mu < 1, \rho > 0 \tag{B.5}$$

subject to

$$\frac{dW(t)}{W(t)} = \psi^W dt + \sigma_{z^W} dz^W, \tag{B.6}$$

$$n_K - n_f = 1, \tag{B.7}$$

$$W(0) = w_0. \tag{B.8}$$

Define V as the value function

$$V(W) = \max_{C,e,n_K,n_f} \mathbb{E} \int_0^{+\infty} \frac{1}{\mu} C^{\mu} e^{-\rho t} dt. \tag{B.9}$$

Then, the optimal program satisfies the Hamilton Jacobi Bellman equation

$$\rho V(W) = \max_{C,e,n_K,n_f} \tilde{F}(C,e,n_k,n_f) = \max_{C,e,n_K} F(C,e,n_k), \tag{B.10}$$

where

$$F(C,e,n_k) = \frac{1}{\mu} C^{\mu} + V'(W) W \psi^W + \frac{1}{2} V''(W) W^2 \sigma^2_{z^W}. \tag{B.11}$$

Using (B.7), which implies $n_f = n_K - 1$, we get the following necessary conditions:

$$\frac{\partial F(.)}{\partial C} = C^{\mu-1} - CPI\, V'(W) = 0, \tag{B.12}$$

$$\begin{aligned} \frac{\partial F(.)}{\partial e} &= \bar{x}\tau n_k^{\beta} n_g^{1-\beta} (\tfrac{T}{W})^{1-\beta} W V'(W) \\ &\quad + \sigma^2 \tau^2 e(t) n_k^{2\beta} n_g^{2(1-\beta)} (\tfrac{T}{W})^{2(1-\beta)} \sigma^2_{W^W} W^2 V''(W) = 0, \end{aligned} \tag{B.13}$$

$$\begin{aligned} \frac{\partial F(.)}{\partial n_k} &= \left[\beta \big[1 - \tau + \bar{x}\tau e(t)\big] n_k^{\beta-1} n_g^{1-\beta} (\tfrac{T}{W})^{1-\beta} + r_k - (i^* + \pi) \right] W V'(W) \\ &\quad + \left[\beta \sigma^2 \tau^2 e(t)^2 n_k^{2\beta-1} n_g^{2(1-\beta)} (\tfrac{T}{W})^{2(1-\beta)} \sigma^2_{W^W} + n_k \sigma^2_{W^k} + (n_k - 1) \sigma^2_{W^f} \right] W^2 V''(W) = 0. \end{aligned} \tag{B.14}$$

F has an extremum $(\hat{C}, \hat{e}, \hat{n_k})$ defined such as to verify the last three equations. From the first two variables we obtain:

$$\hat{C} = (CPIV'(W))^{\frac{1}{\mu-1}}, \tag{B.15}$$

$$\hat{e} = \frac{\bar{x}\tau \hat{n_k}^{\beta} n_g^{1-\beta} (\frac{T}{W})^{1-\beta}}{AR(W)(\sigma\tau \hat{n_k}^{\beta} n_g^{1-\beta} (\frac{T}{W})^{1-\beta} \sigma_{W^W})^2}, \tag{B.16}$$

with $AR(W)$ being the Arrow Prat relative risk coefficient defined by $AR(W) = \frac{-WV''(W)}{V'(W)}$.
Assuming V has the form $V(W) = \delta W^{\mu}$, where δ is a constant, we get

$AR(W) = (1-\mu)$ constant.
The third equation leads to

$$\begin{aligned}&\beta\big[1-\tau+\bar{x}\tau\hat{e}\big](\tfrac{T}{W})^{1-\beta}WV'(W)n_g^{1-\beta}n_k^{\beta}\\&+\beta\sigma^2\tau^2\hat{e}^2(\tfrac{T}{W})^{2(1-\beta)}\sigma_{W^W}^2W^2V''(W)n_g^{2(1-\beta)}n_k^{2\beta}\\&+\big[n_k^2(\sigma_{W^k}^2+\sigma_{W^f}^2)-n_k\sigma_{W^f}^2\big]W^2V''(W)\\&+\big[r_k-(i^*+\pi)\big]WV'(W)n_k=0.\end{aligned}\tag{B.17}$$

Define n_y as the share of production over total wealth, that is $n_y = \frac{Y}{W}$. We then have $n_g^{1-\beta}n_k^{\beta}(\frac{T}{W})^{1-\beta} = n_y$, which leads to rewrite the condition as follows

$$\begin{aligned}&\beta\big[1-\tau+\bar{x}\tau\hat{e}\big]WV'(W)n_y\\&+\beta\sigma^2\tau^2\hat{e}^2\sigma_{W^W}^2W^2V''(W)n_y^2\\&+n_k^2(\sigma_{W^k}^2+\sigma_{W^f}^2)W^2V''(W)\\&+n_k\Big[\big[r_k-(i^*+\pi)\big]WV'(W)-\sigma_{W^f}^2W^2V''(W)\Big]=0,\end{aligned}\tag{B.18}$$

or

$$\begin{aligned}&(1-\mu)(\sigma_{W^k}^2+\sigma_{W^f}^2)n_k^2\\&+\Big[-\big[r_k-(i^*+\pi)\big]-(1-\mu)\sigma_{W^f}^2\Big]n_k\\&-\beta\big[1-\tau+\bar{x}\tau\hat{e}\big]n_y+(1-\mu)\beta\sigma^2\tau^2\hat{e}^2\sigma_{W^W}^2n_y^2=0.\end{aligned}\tag{B.19}$$

This is a second-order differential equation in n_k. The discriminant is

$$\begin{aligned}\triangle \;=\;& \Big[r_k-i^*-\pi+(1-\mu)\sigma_{W^f}^2\Big]^2\\&-4(1-\mu)(\sigma_{W^k}^2+\sigma_{W^f}^2)\big[(1-\mu)\beta\sigma^2\tau^2\hat{e}^2\sigma_{W^W}^2n_y^2-\beta\big[1-\tau+\bar{x}\tau\hat{e}\big]n_y\big].\end{aligned}\tag{B.20}$$

Considering the solutions for $\triangle > 0$, we obtain:

$$n_k^{1,2} = \frac{r_k-(i^*+\pi)+(1-\mu)\sigma_{W^f}^2 \pm \sqrt{\triangle}}{2(1-\mu)(\sigma_{W^k}^2+\sigma_{W^f}^2)}.\tag{B.21}$$

Among both solutions, the following one satisfies the condition $n_k > 0$:

$$n_k = \frac{r_k-(i^*+\pi)+(1-\mu)\sigma_{W^f}^2 + \sqrt{\triangle}}{2(1-\mu)(\sigma_{W^k}^2+\sigma_{W^f}^2)}.\tag{B.22}$$

The household's optimal choice is therefore described by the following system :

$$\begin{cases} \hat{C} = (\delta\mu)^{\frac{1}{\mu-1}} P^{\frac{1-\eta}{\mu-1}} W \\ \hat{e} = \frac{\bar{x}\tau\hat{n_k}^{\beta} n_g^{1-\beta} (\frac{T}{W})^{1-\beta}}{(1-\mu)(\sigma\tau\hat{n_k}^{\beta} n_g^{1-\beta} (\frac{T}{W})^{1-\beta}\sigma_{W^W})^2} \\ n_k = \frac{r_k-(i^*+\pi)+(1-\mu)\sigma^2_{W^f}+\sqrt{\triangle}}{2(1-\mu)(\sigma^2_{W^k}+\sigma^2_{W^f})} \\ n_f = n_k - 1 \end{cases} . \tag{B.23}$$

Appendix C. Transversality conditions

The agents' choices must also satisfy the transversality condition. We consider this condition for the consumer (the proof is similar for the government).

For the constant elasticity utility function, the transversality condition is given by:

$$\lim_{t\to\infty} \mathbb{E}\big[W(t)^{\mu} e^{-\rho t}\big] = 0. \tag{C.1}$$

The stochastic differential equation in W is

$$dW(t) = \psi^W W(t)dt + \sigma_{z^W} W(t) dz^W(t). \tag{C.2}$$

$\psi^{W(t)}$ and $\sigma_{z^{W(t)}} dz^W(t)$ (defined by Equations (**??**) and (**??**)) converge to constant terms when when $t \to \infty$, so we omit t. We first compute the solution of C.2 for $W(0) = w_0$ (initial condition of wealth), given.

We rewrite C.2 as follows

$$\frac{dW(t)}{W(t)} = \psi^W dt + \sigma_{z^W} dz^W(t). \tag{C.3}$$

Integrating this equation between 0 and t gives

$$\int_0^t \frac{dW(u)}{W(u)} = \int_0^t \psi^W \, du + \int_0^t \sigma_{z^W} \, dz^W(u) = \psi^W t + \sigma_{z^W} z^W(t). \tag{C.4}$$

We use Itô's formula:

$$df(t, W(t)) = \frac{\partial f(t, W(t))}{\partial t} dt + \frac{\partial f(t, W(t))}{\partial W} dW(t) + \frac{1}{2}\frac{\partial^2 f(t, W(t))}{\partial W^2}(dW(t))^2. \tag{C.5}$$

Taking $f(t, W(t)) = f(W) = ln(W)$, we obtain

$$d\,ln(W(t)) = \frac{dW(t)}{W(t)} - \frac{1}{2}\Big[\frac{dW(t)}{W(t)}\Big]^2. \tag{C.6}$$

Since $\big[dz^W(t)\big]^2 = dt$, $dt^2 = 0$ and $dz^W(t).dt = 0$, we have

$$\Big[\frac{dW(t)}{W(t)}\Big]^2 = \big[\psi^W dt + \sigma_{z^W} dz^W(t)\big]^2 = \sigma_{z^W}^2\, dt. \tag{C.7}$$

Thus,

$$d\,ln(W(t)) = \frac{dW(t)}{W(t)} - \frac{\sigma_{z^W}^2}{2} dt. \tag{C.8}$$

And integrating between 0 and t, we have

$$ln(W(t)) - ln(w_0) = \int_0^t \frac{dW(u)}{W(u)} - \frac{\sigma_{z^W}^2}{2} t. \tag{C.9}$$

Replacing by the expression in C.4, we get

$$ln(\frac{W(t)}{w_0}) + \frac{\sigma_{z^W}^2}{2} t = \int_0^t \frac{dW(u)}{W(u)} = \psi^W t + \sigma_{z^W} z^W(t), \tag{C.10}$$

and thus

$$W(t) = w_0\, exp\big[(\psi^W - \frac{\sigma_{z^W}^2}{2})t + \sigma_{z^W} z^W(t)\big], \tag{C.11}$$

which yields to

$$W(t)^\mu\, exp(-\rho t) = w_0^\mu\, exp\big[(\mu\psi^W - \frac{\mu\sigma_{z^W}^2}{2} - \rho)t + \mu\sigma_{z^W} z^W(t)\big]. \tag{C.12}$$

This is a geometric brownian motion. Assuming $z^W(t)$ is independent of w_0, one of the properties of such a motion is that,

$$\mathbb{E}\big[W(t)^\mu\, exp(-\rho t)\big] = \mathbb{E}\big[w_0^\mu\big]\, exp\big[(\mu\psi^W - \rho)t\big]. \tag{C.13}$$

In the end, the transversality condition can be rewritten as

$$\lim_{T\to\infty} \mathbb{E}\big[w_0^\mu\big]\, exp\big[-(\rho - \mu\psi^W)T\big] = 0. \tag{C.14}$$

This puts a lower bound on the rate of impatience ρ, since the above condition is satisfied for $\rho > \mu\psi^W$.

For the government, the transversality condition is obtained in a similar way and implies that $\rho > \gamma\psi^T$ with ψ^T is defined by Equation(**??**).

Appendix D. Proof of Theorem 1

We first write the bureaucrats' and consumers' constraints (Equations (**??**) and(**??**) which give us the separate dynamics of $dT(t)$ and $dW(t)$):

$$\text{dW(t)} = \text{W(t)}\Big[\ 1\text{-}\tau + \bar{x}\tau\hat{e}(\lambda)\Big]\hat{n_y}(\lambda) - CPI\tfrac{\hat{C}}{W} + \hat{n_k}(\lambda)r_k - (i^* + \pi)\hat{n_f}(\lambda)\Big]dt$$
$$+ \text{W(t)}\ \Big[\ \sigma\tau\hat{e}(\lambda)\hat{n_y}(\lambda)dW^W + \hat{n_k}(\lambda)du_k - \hat{n_f}(\lambda)du_f\Big],$$
$$dT(t) = T(t)\Big[\hat{n_g}(\lambda) + \hat{n_C}(\lambda)\Big]dt + T(t)\Big[\tilde{n_g}(\lambda)dW^g(t) + \tilde{n_C}(\lambda)dW^{C^G}\Big].$$

To obtain the dynamics of the ratio $\lambda(t) = W(t)/T(t)$, we use the Itô's lemma.

Itô's lemma. Let X(t) in $\mathbb{R}^2$ be a diffusion process and F(X) a $\mathbb{C}^2$ map from $\mathbb{R}^2$ to $\mathbb{R}$, then

$$dF(X) = F_x dX + \frac{1}{2}dX' F_{xx} dX, \tag{D.1}$$

with F_x and F_{xx} representing, respectively, the matrix of partial derivatives of F and the Hessian matrix.

We define $X = (T, W)'$, $dX = (dT, dW)'$, $F(X) = \frac{W}{T}$,

$$F_x = \begin{pmatrix} \frac{\partial F}{\partial T} \\ \frac{\partial F}{\partial W} \end{pmatrix} = \begin{pmatrix} \frac{-W}{T^2} \\ \frac{1}{T} \end{pmatrix},$$

$$F_{xx} = \begin{pmatrix} \frac{\partial^2 F}{\partial T^2} & \frac{\partial^2 F}{\partial T \partial W} \\ \frac{\partial^2 F}{\partial W \partial T} & \frac{\partial^2 F}{\partial W^2} \end{pmatrix} = \begin{pmatrix} \frac{2W}{T^3} & \frac{-1}{T^2} \\ \frac{-1}{T^2} & 0 \end{pmatrix}.$$

From (D.1) we obtain

$$d(\frac{W}{T}) = \frac{-W}{T^2}dT + \frac{1}{T}dW + \frac{W}{T^3}dT^2 - \frac{1}{T^2}dTdW. \tag{D.2}$$

We get the final form of Equation (**??**) by using the Levy characterization of diffusion processes and by considering the following properties of Wiener processes. Consider two Wiener processes w_i and w_j. We have:

$$(dw_i)^2 = dt,\ < dt.dw_i >= 0\ \forall i \neq j,\ dt^2 = 0.$$

For purpose of simplicity, we assume the following correlation structure of two Wiener processes: $d < w_i, w_j >= 0$, where $< w_i, w_j >$ is the quadratic variation process for the components of the Wiener processes.

Appendix E. Asymptotic stochastic solutions of Itô diffusion processes

We first recall some mathematical properties of steady state distributions of Itô diffusion processes (see Feller (1952, 1954), Itô (1996).

Let us consider the following SDE:

$$dx = a(x)dt + b^{1/2}(x)dz,\ dz \approx N(0, dt),\ x \in \big[0, \infty\big], \tag{E.1}$$

with $a(\cdot)$ and $b(\cdot)$ being continuous and differentiable functions of x.

Consider $X(t)$ the solution of the SDE and define the transition probability as

$$P(x, t; x_0, t_0) = Pr\big[X(t) \leq x \,\big|X(t_0) = x_0\big].\text{(E.2)}$$

The probability density $\pi(x, t, x_0)$ satisfies the Kolmogorov-Fokker-Planck equation:

$$\frac{1}{2}\frac{\partial^2}{\partial x^2}\big[b(x)\pi(x, t, x_0)\big] - \frac{\partial}{\partial x}\big[a(x)\pi(x, t, x_0)\big] = \frac{\partial \pi(x, t, x_0)}{\partial t}. \tag{E.3}$$

The steady state density function, obtained by integrating (E.3), must satisfy

$$p(x) = c_1 m(x) + c_2 S(x),\ p(x) = \lim_{t\to\infty} \pi(x, t, x_0), \tag{E.4}$$

where

$$M(x) \equiv \int_{x_0}^{x} m(u)du,\ S(x) \equiv \int_{x_0}^{x} s(u)du, \tag{E.5}$$

with

$$m(x) \equiv \frac{exp\big[2J(x)\big]}{b(x)},\ s(x) \equiv exp\big[-2J(x)\big],\ J(x) \equiv \int_{x_0}^{x} \frac{a(u)}{b(u)}du. \tag{E.6}$$

c_1 and c_2 are constants of integration ensuring that $p(x)$ is a true probability density. $s(x)$, $S(x)$ and $m(x)$ are called, respectively, the scale density function, the scale function and the speed density function of the stochastic process X(t).

Existence of steady state distribution. A time-invariant distribution function $P(x)$ exists if and only if

$$\lim_{\substack{x \to 0 \\ x \to \infty}} S(x) = \mp\infty \text{ and M(x) is finite at the boundaries 0 and } +\infty,$$

$$\text{i.e. } \lim_{\substack{x \to 0 \\ x \to \infty}} S(x) = \mp\infty \text{ and } |M(b)| = \int_{x_0}^{b} m(u)du < \infty \text{ for } b = 0 \text{ and } +\infty. \tag{E.7}$$

The existence of steady state distribution implies that the boundaries of the process are inaccessible. A corollary is that, if the boundaries are inaccessible, then $\pi(x, t, x_0)$ converges towards a probability density function defined by (E.4) with $c_2 = 0$ (for rigorous proofs, see **?**, **?**, **?**).

A theorem of the existence of a steady state distribution for the wealth-to-tax revenues ratio $\lambda(t)$

The concept of steady state for diffusion processes is defined in a stochastic sense. Instead of a point, $\lambda(t)$ converges to a set of values in a basin of attraction. This means that, once λ has reached its long-term attractor λ^*, all the variables in the model which depend upon λ will also reach their own basin of attraction. For Itô diffusion processes, the properties of the values in the basin of attraction of the variable of interest can be studied by constructing their distribution called the steady state distribution of the diffusion process. The issue here is therefore to study the convergence in distribution of the variable $\lambda(t)$. To do this, we use the mathematical tools of the theory of Markov chains to prove the existence and derive stationary probability measures. The techniques are similar to those used in a few papers dealing with continuous time stochastic growth models (see Bourguignon (1974), Merton (1975), Chang and Maliaris (1987), Jensen and Richter (2007).

Now, we must prove that the boundaries 0 and $+\infty$ are inaccessible for the wealth-to-tax revenues ratio $\lambda(t)$. To do that, several preliminary remarks are in order.

First, $\lambda(t)$ is the ratio of two variables $W(t)$ and $T(t)$. We assume that $T \in (0, +\infty)$ and $W \in (0, +\infty)$. By assumption 0 is thus an inaccessible boundary for W and T. This means that we do not consider the extreme situation in which corruption and tax evasion are so important that this yields a depletion of tax revenues ($T \to 0$) and therefore entirely annihilates net wealth ($W \to 0$, because of a high level of foreign debt).

Second, the fact that by assumption, T and W do not reach the zero boundary, does not mean that 0 and $+\infty$ are inaccessible for λ. Indeed,

$$\text{for a finite T, } \lim_{W\to+\infty} \lambda = +\infty, \text{ and for a finite W, } \lim_{T\to\infty} \lambda = 0. \tag{E.8}$$

Therefore to prove that 0 and $+\infty$ are inaccessible boundaries for λ, a sufficient condition consists in proving that $+\infty$ is an inaccessible boundary for both W and T.

Theorem Appendix E.1. *Let us consider the SDE of T and W in a compact form using the Levy characterization*

$$dT(t) = a_1(T, \lambda^*)dt + b_1^{\frac{1}{2}}((T, \lambda^*))dw^T,$$

where

$$a_1(T, \lambda^*) = T(t)\Big[n_g(\lambda^*) + n_C(\lambda^*)\Big] = T(t)a(\lambda^*),$$

$$b_1(T, \lambda^*) = T^2(t)\Big[\tilde{n}_g^2(\lambda^*) + \tilde{n}_C^2(\lambda^*)\Big] = T^2(t)b(\lambda^*),$$

$$dW(t) = d_1(W, \lambda^*)dt + h_1^{\frac{1}{2}}((W, \lambda^*))dz^W,$$

where

$$d_1(W, \lambda^*) = W(t)\Big[\big[1 - \tau + \bar{x}\tau\hat{e}(\lambda^*)\big]\hat{n}_y(\lambda^*) - CPI\tfrac{\hat{C}}{W} +$$
$$\hat{n}_k(\lambda^*)r_k - (i^* + \pi)\hat{n}_f(\lambda^*)\Big] = W(t)d(\lambda^*),$$
$$h_1(W, \lambda^*) = W^2(t)\Big[\sigma^2\tau^2\hat{e}(\lambda^*)^2\hat{n}_y^2(\lambda^*) + \hat{n}_k^2(\lambda^*) + \hat{n}_f^2(\lambda^*)\Big] = W^2(t)h(\lambda^*).$$

Sufficient conditions for the existence of a steady state distribution for the ratio of wealth-to-fiscal revenue are:

$$a)\ 2a(\lambda^*) - b(\lambda^*) < 0,\ and\ b)\ 2d(\lambda^*) - h(\lambda^*) < 0 \tag{E.9}$$

Proof. *i)*

We first prove that $\lim_{T\to+\infty} S(T, \lambda^*) = +\infty$

Using the Levy representation of the SDE of T, as given in the theorem, we compute the scale density function of $T(t)$ as

$$s(T,\lambda^*) = exp\left\{-2\int_{T_0}^{T}\frac{a_1(u,\lambda^*)}{b_1(u,\lambda^*)}du\right\},\ T_0 = T(0), \tag{E.10}$$

or

$$s(T,\lambda^*) = exp\left\{-2\frac{a(\lambda^*)}{b(\lambda^*)}\int_{T_0}^{T}\frac{1}{u}du\right\} = [\frac{T}{T_0}]^{-2\frac{a(\lambda*)}{b(\lambda^*)}}, \tag{E.11}$$

Then, we calculate the scale function

$$S(T,\lambda^*) = \int_{T_0}^{T} s(u)du = \frac{b(\lambda^*)(T_0)^{\frac{2a(\lambda^*)}{b(\lambda^*)}}}{-2a(\lambda^*)+b(\lambda^*)}\left\{[T]^{\frac{-2a(\lambda^*)+b(\lambda^*)}{b(\lambda^*)}} - [T_0]^{\frac{-2a(\lambda^*)+b(\lambda^*)}{b(\lambda^*)}}\right\}, \tag{E.12}$$

We see that

$$\lim_{T\to+\infty} S(T,\lambda^*) = +\infty,\ \text{if } 2a(\lambda^*) - b(\lambda^*) < 0. \tag{E.13}$$

ii)

Using a similar approach by considering the SDE of W(t), we get

$$\lim_{W\to+\infty} S(W,\lambda^*) = +\infty,\ \text{if }\ 2d(\lambda^*) - h(\lambda^*) < 0 \tag{E.14}$$

■

Remark 1. *Condition b) in (E.9) means that to avoid an infinite increase of wealth, the risk-adjusted return of net wealth must be capped, which implies that the risk-adjusted return of tax evasion and the marginal productivity of capital should not exceed a threshold value, and that the cost of borrowing abroad cannot be too low. Condition a) implies that there is a minimal risk associated to tax evasion and corruption. Indeed, would they be no risk for corrupted bureaucrats and frauding taxpayers, no tax revenues would be left to finance public spending. It would lead to the extreme case described in section Appendix E.*