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A SPATIOTEMPORAL FRAMEWORK FOR THE ANALYTICAL STUDY OF OPTIMAL GROWTH UNDER TRANSBOUNDARY POLLUTION

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ABSTRACT. We construct a spatiotemporal frame for the study of optimal growth under transboundary pollution. Space is continuous and polluting emissions originate in the intensity of use of the production input. Pollution flows across locations following a diffusion process. The objective functional of the economy is to set the optimal production policy over time and space to maximize welfare from consumption, taking into account a negative local pollution externality and the diffusive nature of pollution. Our framework allows for space and time dependent preferences and productivity, and does not restrict diffusion speed to be space-independent. This provides a comprehensive setting to analyze pollution diffusion with a close account of geographic heterogeneity. The involved optimization problem is infinite-dimensional. We propose an alternative method for an analytical characterization of the optimal paths and the asymptotic spatial distributions. The method builds on a deep economic concept of pollution spatiotemporal welfare effect, which makes it definitely useful for economic analysis.

Key words: Optimal growth, spatiotemporal modelling, transboundary pollution, infinite dimensional optimal control

JEL classification: Q53, R11, C61, R12, O41

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1. Introduction

Because air pollution is diffusive and transboundary, the problem of pollution control is deeply intricate. It raises several keys issues. One has to do with the strategic ingredients of the problem either at the international level (refer to the successive failures or at least questioning of internationals agreements to control global warming, from the Kyoto protocol in 1991 to the Paris agreement in 2016) or at the regional scale. We abstract away from these considerations here. Several papers have been written in the last decade on this topic from 2-country setting (Boucekkine et al., 2011) to continuous space modelling (see in particular, de Frutos and Martin-Herran, 2018 and 2019) through multi-country frameworks (for example, Dockner and Long, 1993).

We rather concentrate on a second set of issues, those related to the fact that the impact of air pollution is first of all local, and its magnitude and persistence depend pretty much on the local conditions. If a central planner at a country or international level has to set a pollution control policy for the benefit of all the individuals concerned, then she should take as much as possible into account the heterogeneity across locations, in addition to the fact that air pollution, being transboundary, requires the internalization of spatial externalities.

There are several spatial heterogeneity features to account for. Obvious ones are technological heterogeneity and heterogeneity in preferences (which covers cultural discrepancies with respect to the environment among others). But there are also more geographic and ecological differences, which matter a lot both in the diffusion of pollution across locations and in its local impact. The self-cleaning capacity of Nature may vary from a region to a close one, the local topography, land use and infrastructures may speed up pollution diffusion or slow it down...etc. These issues are quite known across disciplines (see Tiwari and Closs, 2010, for an excellent book on air pollution), including the economic literature. For example, Camacho and Perez-Barahona (2015) studied the problem of atmospheric transboundary pollution in the context of an optimal land use problem. Other economists have also provided highly significant contributions to the analysis of the spatiotemporal deep nature of the transboundary pollution control problem (see a recent survey in Augeraud-Véron et al., 2019). None of these papers however poses the latter problem in a full-fledged analytical spatiotemporal frame incorporating the above mentioned technological, preference, geographic and ecological spatial discrepancies.

Clearly, taking up the challenge would involve plenty of technical problems, with a very likely lack of tractability and the forced use of numerical solutions. Contrary to other disciplines (like quantitative geography, climate science or ecology) in which the use of black-box disaggregated models is routine, the economists, being more interested in identifying mechanisms, are more keen to develop parsimonious models. In this paper, we build up a spatiotemporal optimal control framework allowing to encompass the heterogeneity traits outlined above in the presence of transboundary pollution, and still producing a comprehensive analytical characterization. To this end, we build on Boucekkine et al. (2019a) who worked out a production-induced pollution social planner problem with transboundary diffusive pollution, negative environmental externalities and space-dependent environmental awareness across population. We depart from this contribution essentially in two respects.[1] First of all, we considerably enrich the geography of the model. On the technological ground, we allow productivity to be not only space-dependent but also time-dependent: any pattern of technology diffusion across time and space can be accommodated. As to preferences, we make them time and space-dependent as well. Finally, to account for the local geographic and ecological conditions discrepancy, we not only have a space-dependent self-cleaning capacity but also a space-dependent pollution diffusion speed.

Another drastic departure is the fact that we move away from the linear-quadratic setting used in Boucekkine et al. (2019a). Rather, we use the standard CRRA specification for instantaneous utility from consumption with time and space varying coefficients. This has a methodological implication though: to produce analytical results, we do not implement the specific dynamic programming method used by the authors (see also Boucekkine et al., 2019b). Nor do we invoke the maximum principle (see Brito, 2004, Brock et al, 2014, and more recently Ballestra, 2016, for the use of this technique to solve infinite-dimensional optimization problems in different contexts). Instead, we apply a functional transformation technique observing that the objective functional can be rewritten in a way that allows for a direct maximization method, ultimately finding the explicit form of the optimal control. The argument we use is related to the one employed in Barucci and Gozzi (1998, 2001) in a different economic and mathematical context. An extremely appealing feature of this method is that it builds one a pivotal spatial function (denoted $\alpha(.)$ in Section 3), which admits a neat economic interpretation: it corresponds at any location x to the the discounted sum of future disutility stream of a unit of pollutant initially located at x. Indeed, in our spatiotemporal framework with transboundary pollution, a unit of pollutant has a different effect on social welfare depending on where it is initially located and how it is going to spread over space in the future. Our alternative method has the virtue to put forward this deep economic concept, which makes it definitely transparent and useful for economic analysis.

The paper is organized as follows. Section 2 presents the basic economic problem and defends its economic relevance. Section 3 develops the method used to solve the generic spatiotemporal optimal growth model under transboundary pollution. In particular, closed-form solutions of the optimal strategies are identified. We show also further analytical results on transition dynamics and asymptotics (that's the

[1] As it will be clear in the modelling section, other differences can be put forward, in particular the geographic space chosen.

computation of asymptotic spatial distributions). Of course, our aim in this ultimate section is not to exploit entirely the richness of our setting but only to indicate how far the analytical study can go. Appendix A contains the formal proofs.

2. The basic economic problem

The basic problem builds on Boucekkine et al. (2019a) and Boucekkine et al. (2011) who study optimal growth in the presence of pollution externalities. With respect to the latter work, which uses the usual two-country framework, we introduce continuous space and pollution diffusion over space, leading to an infinite dimensional optimization problem. With respect to the former, denoted BFFG hereafter, two sets of major departures are noticeable, as mentioned in the introduction section.

(1) Contrary to BFFG, the induced optimization problem is no longer linear-quadratic. As explained in the introduction, we address the new problem using a direct method, and not the dynamic programming approach followed in BFFG. The advantage is to build the optimal paths on clear economic foundations, with a an essential role devoted to the social spatiotemporal cost of pollution, which is finely characterized in Section 3.
(2) Moreover, we also considerably generalize the spatial setting by considering space and time-dependence of both preferences and production technology and space-dependent diffusion.

Let us now describe briefly the basic economic problem. Consider a continuum of locations, say along the circle in $\mathbb{R}^2$. The choice of the circle is made for simplicity.[2] Call it S^1:

$$S^1 := \{x \in \mathbb{R}^2 : \ |x|_{\mathbb{R}^2} = 1\}. \tag{1}$$

Each location uses a linear (Leontief) production function: at any location x in time $t \geq 0$, production is

$$y(t,x) = a(t,x)\, i(t,x), \tag{2}$$

where $y(t,x)$ is the output, $i(t,x)$ is the capital input, and $a(t,x)$ is productivity at location x in time t. A few comments are in order already at this stage. First, and contrary to BFFG, we allow productivity per location to be not only space-dependent but also generically time-dependent, so that our setting can include for example the typical exponential exogenous technological progress in neoclassical growth theory. Actually, our modelling allows for much more: as argued in the introduction section, $a(t,x)$ can be specified to model possible technological spillovers

[2] Our approach allows generalizations to compact finite dimensional manifolds without boundary (see, e.g., Fabbri, 2016).

across locations, uneven technological development over space (that's barriers to technological diffusion) and the like. Admittedly, the latter are key features in regional development. Second, it's worth noticing that our production technology mimics the AK technology, which is a basic ingredient in endogenous growth theory (see Barro and Sala-i-Martin, 2004, Chapter 4), but with full depreciation of capital. This is essential to get our closed-form solutions, and actually this is a well known trick in optimal growth theory either to generate analytical solutions and/or to simplify the analysis (reduction of the dimension of the dynamic systems involved) and focus on other state variables (see again Barro and Sala-i-Martin, 2004, Chapter 6). In our case, both arguments hold: given the extreme complexity of our problem (infinite dimensional optimization) and the economic target (economic performance in the heterogenous space with transboundary pollution), we find it convenient to shut down capital accumulation to be able to develop a comprehensive enough analytical spatiotemporal framework to approach the latter economic objectives.

At any location, output is produced, consumed and locally invested (no trade across locations), implying:

$$c(x,t) + i(t,x) = y(t,x), \tag{3}$$

where $c(t,x)$ is consumption at location x at time t. The unique link among locations is transboundary pollution. Just like BFFG, we target air pollution, but we consider a broader specification incorporating ecological efficiency at any location x and time t. Precisely, the accumulation of pollution spatial profile is assumed to evolve following the following parabolic partial differential equation (PDE):

$$\begin{cases} \dfrac{\partial p}{\partial t}(t,x) = \dfrac{\partial}{\partial x}\left(\sigma(x)\dfrac{\partial p}{\partial x}(t,x)\right) - \delta(x)p(t,x) + \psi(t,x)i(t,x), \quad (t,x) \in \mathbb{R}^+ \times S^1, \\ \\ p(0,x) = p_0(x), \quad x \in S^1, \end{cases} \tag{4}$$

where $p(t,x)$ is the pollution stock at location x in time t. First, notice the general shape of the pollutants' emissions term $\psi(t,x)i(t,x)$ in the pollution spatiotemporal dynamics depicted above. Indeed, the unusual function $\psi(t,x)$ is meant to reflect that the pollution impact of emissions arising from the use of one unit of input may not be the same over time and space. It can be readily observed that: (i) taking $\psi(t,x) = 1$ brings to consider, as in BFFG, the case where polluting emissions at location x are exactly equal to input use intensity; (ii) if we specify $\psi(t,x) = a(t,x)\phi(t,a)$, the term $\psi(t,x)i(t,x)$ reads as $a(t,x)\phi(t,x)i(t,x) = \phi(t,x)y(t,x)$, and we are able to give a specification of the model where emissions depend on output rather than on input used; (iii) the temporal dependence allows in general to incorporate exogenous ecological efficiency technological progress (such that those conveyed by abatement activities), and finally that: (iv) independent spatial heterogeneities can also be taken into account with this more general specification.

Second, another major difference with respect to BFFG is the transboundary pollution diffusion term, that is $\frac{\partial}{\partial x}(\sigma(x)\frac{\partial p}{\partial x}(t,x))$, where $\sigma(x)$ is the pollution diffusion speed at location x. In BFFG, the latter parameter is constant (equal to a positive constant σ), yielding a transboundary pollution diffusion term $\sigma\frac{\partial^2 p}{\partial x^2}(t,x)$.[3] There is a large bunch of works documenting the role of local conditions in air pollution diffusion (see Tiwary and Colls, 2010, Chapter 1). Therefore, this generalization significantly increases the relevance of our analytical frame. Also notice that pollution diffusion also depends on nature local regeneration capacity (the term $\delta(x)p(t,x)$), and on current emissions (as captured by the term $\psi(t,x)i(t,x)$).

Finally, since our setting is spatiotemporal, an initial spatial distribution of pollution is needed, it is given by function $p_0(x)$ defined on S^1. The whole state variable dynamics follows the parabolic partial differential equation (PDE) (4). The infinite dimensional nature of the involved optimization problem derives from the latter characteristic of the state dynamics.

A last major departure from BFFG concerns the objective functional of the problem. As argued in the introduction and at the beginning of this section, BFFG consider an LQ utility function. To be precise, they consider a per location instantaneous utility function which is separable in consumption and the pollution externality, quadratic in consumption and linear in local pollution. Here we choose the instantaneous per location utility to be:

$$U\left(c(t,x),p(t,x)\right) = \frac{c(t,x)^{1-\gamma(t,x)}}{1-\gamma(t,x)} - w(x)p(t,x),$$

where $\gamma(t,x) \in (0,1) \cup (1,+\infty)$ measures the inverse of the elasticity of intertemporal substitution in consumption at location x and time t, and $w(x)$ measures for instance local environmental awareness at location x. Observe that our negative pollution externality is local. Our framework is not designed primarily to study global warming and therefore global pollution, but the diffusion of air pollutants with local health impact like particles for example (again a comprehensive account of air pollutants can be found in Tiwari and Colls, 2010, Chapter 1). The local impact of pollution is also captured via function $w(x)$, which can be indeed interpreted as local awareness and sensitivity to environmental problems. It can also reflect specific priorities of the planner to cope with particular local conditions. Notice also that our frame allows for time and space varying preferences through the parameter $\gamma(t,x)$. The utility from consumption follows a CRRA formulation, it is strictly

[3] To clarify, in BFFG the geography is modeled in $\mathbb{R}^n$ and the diffusion term is simply $\frac{\sigma^2}{2}\Delta p(t,x)$, where Δ is the Laplace operator.

concave, and no longer quadratic. This motivates the change in methodology to solve the optimization problem as explained in the introduction; full details on the alternative methodology implemented and obtained results will come in the next section. Before, let us just display the objective functional. We consider a planner problem who has to maximize the following social welfare under the above specified technological constraints of the economy and the transboundary pollution faced:

$$J\big(p_0; i\big) := \int_0^\infty e^{-\rho t}\left(\int_{S^1}\left(\frac{c(t,x)^{1-\gamma(t,x)}}{1-\gamma(t,x)} - w(x)p(t,x)\right)\mathrm{d}x\right)\mathrm{d}t,$$

where ρ is the parameter at which the planner discounts time. Further, by using equations (1) and (2), we can rewrite the functional in terms of the control $i(t,x)$:

$$J\big(p_0; i\big) = \int_0^\infty e^{-\rho t}\left(\int_{S^1}\left(\frac{\big((a(t,x)-1)i(t,x)\big)^{1-\gamma(t,x)}}{1-\gamma(t,x)} - w(x)p(t,x)\right)\mathrm{d}x\right)\mathrm{d}t. \tag{5}$$

Notice we do not incorporate population density in our analysis, and notably in the functional, nor do we introduce mortality (possibly related to pollution). We fundamentally focus on handling spatial heterogeneity abstracting away from demography, which is an already daunting task. At the minute, one could simply interpret the social welfare function above as a Benthamite welfare function summing individual welfare over locations and time with one infinite-lived individual at each location.

3. Theoretical analysis

In this section we give a precise description of our results, specifying hypotheses and formalizing the statements. To increase the readability of the text we postpone the proof in Appendix A and we divide the section in four parts.

To apply the functional transformation technique that we exploit to solve the optimal control problem (Theorem 3.6), we start by rewriting the problem in a suitable Hilbert space formalism (Subsection 3.1), then we identify a spatial function α that will be essential in the transformed expression of the functional (Subsection 3.2), and finally we will characterize the optimal control and the corresponding social welfare (Subsection 3.3). Subsection 3.4 contains transitional and long-run analysis of the dynamics via series expansions.

3.1. Infinite dimensional formulation and preliminary results.

On the space support S^1 introduced in (1) we consider the metrics induced by the Euclidean metrics of $\mathbb{R}^2$. In this way S^1 can be isometrically identified with $2\pi\mathbb{R}/\mathbb{Z}$ and the (class of) functions $S^1 \to \mathbb{R}$ with 2π-periodic function $\mathbb{R} \to \mathbb{R}$; differentiation of functions $S^1 \to \mathbb{R}$ is defined according to this identification. Consequently, the initial pollution distribution and the space dependent parameters δ, σ and w are measurable functions

$$p_0, \delta, \sigma, w : S^1 \to \mathbb{R}^+;$$

similarly the time and space dependent parameters γ, a, ψ are measurable functions

$$\gamma : \mathbb{R}^+ \times S^1 \to (0,1) \cup (1, +\infty), \quad a : \mathbb{R}^+ \times S^1 \to (1, +\infty), \quad \psi : \mathbb{R}^+ \times S^1 \to (0, +\infty).$$

We proceed now to our infinite dimensional reformulation of the problem. We will use the framework of Lebesgue and Sobolev spaces, for more details we refer to Brezis (2011). The infinite dimensional space H, where we will reformulate our maximization, is the Lebesgue space $L^2(S^1; \mathbb{R})$, i.e.[4]

$$H := L^2(S^1; \mathbb{R}) := \left\{ f : S^1 \to \mathbb{R} \text{ measurable} : \int_{S^1} |f(x)|^2 \mathrm{d}x < \infty \right\},$$

endowed with the usual inner product $\langle f, g \rangle = \int_{S^1} f(x) g(x) \mathrm{d}x$, which makes it a Hilbert space. We denote by $\| \cdot \|$ the associated norm, by H^+ the nonnegative cone of H, i.e.

$$H^+ := \{ f \in H : \ f \geq 0 \},$$

and by $\mathbf{1}$ the constant function equal to 1 on S^1. Moreover, we introduce the Sobolev space[5]

$$W^{2,2}(S^1; \mathbb{R}) := \left\{ f \in L^2(S^1; \mathbb{R}) : \ f \text{ is twice weakly differentiable}, f', f'' \in L^2(S^1; \mathbb{R}) \right\}.$$

Some degree of regularity of the parameters will be necessary in the analysis. We will work with the following assumptions.

Assumption 3.1.

(i) $p_0 \in L^2(S^1; \mathbb{R}^+)$, $\delta \in C(S^1; \mathbb{R}^+)$, $\sigma \in C^1(S^1; (0, +\infty))$, $w \in C(S^1; (0, +\infty))$;

(ii) the function $\gamma : \mathbb{R}^+ \times S^1 \to (0,1) \cup (1, +\infty)$ *is measurable and there exists* $\kappa \in (0,1)$ *such that, for every* $(t, x) \in \mathbb{R}^+ \times S^1$,

$$\text{either (Case (A))} \qquad \kappa \ \leq \ \gamma(t,x) \ \leq \ 1 - \kappa$$

$$\text{or (Case (B))} \qquad 1 + \kappa \ \leq \ \gamma(t,x) \ \leq \ \frac{1}{\kappa}.$$

[4]Actually, rather than a space of functions, $L^2(S^1; \mathbb{R})$ is a space of equivalence classes of functions, with the equivalence relation identifying functions which are equal *almost everywhere*, i.e. out of a null Lebesgue measure set. For details we refer again to Brezis (2011).

[5]We refer to Brezis (2011) for the notion of *weak* differentiability.

(iii) There exist $L>0$, and $g\geq 0$ such that, for every $(t,x)\in\mathbb{R}^+\times S^1$,

$$\left(\frac{a(t,x)-1}{\psi(t,x)}\right)^{\frac{1}{\gamma(t,x)}-1} \;\leq\; Le^{gt};$$

(iv) $\rho\;>\;g$.

Hereafter, our arguments will make use of the theory of unbounded linear operators and semigroups of linear operators, for which we refer to Engel and Nagel (1995). Denote by $L(H)$ the space of bounded linear operators on H. We consider the differential operator $\mathcal{L}: D(\mathcal{L})\subset H\to H$, where

$$D(\mathcal{L})=W^{2,2}(S^1;\mathbb{R}); \qquad (\mathcal{L}\varphi)(x)=(\sigma\varphi')'(x)-\delta(x)\varphi(x), \quad \varphi\in D(\mathcal{L}).$$

Proposition 3.2. *Let Assumption 3.1 hold. Then $\mathcal{L}$ generates a strongly continuous contraction semigroup $(e^{t\mathcal{L}})_{t\geq 0}\subset L(H)$. Moreover, ρ belongs to the resolvent set of $\mathcal{L}$, i.e. $\rho-\mathcal{L}:\ D(\mathcal{L})\ \to\ H$ is invertible with bounded inverse $(\rho-\mathcal{L})^{-1}: H\to D(\mathcal{L})$ and*

$$(\rho-\mathcal{L})^{-1}h=\int_0^\infty e^{-(\rho-\mathcal{L})t}h\,\mathrm{d}t \qquad \forall h\in H. \tag{6}$$

Proof. See Appendix A ☐

Given $i:\mathbb{R}^+\times S^1\to\mathbb{R}^+$, define

$$I:\mathbb{R}^+\to H^+, \qquad I(t):=i(t,\cdot).$$

Morever, define

$$\Psi:\mathbb{R}^+\to H^+, \quad \Psi(t):=\psi(t,\cdot).$$

Finally, given $h,k\in H$, define $(hk)(x):=h(x)k(x)$. Then, with the identification $P(t)=p(t,\cdot)$, we reformulate (4) in H as

$$\begin{cases} P'(t)=\mathcal{L}P(t)+\Psi(t)I(t), & t\geq 0,\\ P(0)=p_0\in H, \end{cases} \tag{7}$$

According to Definition 3.1(v), Chapter 1, Part II, of Bensoussan et al. (2007), given $I\in L^1_{loc}(\mathbb{R}^+;H^+)$, we define the *mild solution* to (7) as

$$P(t)=e^{t\mathcal{L}}p_0+\int_0^t e^{(t-s)\mathcal{L}}\Psi(s)I(s)\mathrm{d}s, \quad t\geq 0. \tag{8}$$

Setting $A(t):=a(t,\cdot)$, $\Gamma(t):=\gamma(t,\cdot)$, and

$$\left[\frac{\big((A(t)-\mathbf{1})I(t)\big)^{1-\Gamma(t)}}{1-\Gamma(t)}\right](x):=\frac{\big((a(t,x)-1)i(t,x)\big)^{1-\gamma(t,x)}}{1-\gamma(t,x)}, \quad x\in S^1,$$

the functional (5) is rewritten in this formalism as

$$J(p_0, I) = \int_0^\infty e^{-\rho t} \left[\left\langle \frac{\big((A(t) - \mathbf{1})I(t)\big)^{1-\Gamma(t)}}{1-\Gamma(t)}, \mathbf{1} \right\rangle - \langle w, P(t) \rangle \right] \mathrm{d}t. \tag{9}$$

We introduce the following set of admissible controls

$$\mathcal{A} := \left\{ I \in L^1_{loc}(\mathbb{R}^+; H^+) : \int_0^\infty e^{-\rho t} \|\Psi(t) I(t)\| \, \mathrm{d}t < \infty \right\}.$$

The following result shows that the functional is well defined on $\mathcal{A}$.

Proposition 3.3. *Let Assumption 3.1 hold. The functional $J(p_0, I)$ is well defined for all $p_0 \in H$ and $I \in \mathcal{A}$.*

Proof. See Appendix A. □

Finally, we define the value function as the optimal value of J over $\mathcal{A}$, i.e.

$$v(p_0) := \sup_{I \in \mathcal{A}} J(p_0; I).$$

Note that this function may possibly be infinite. The function

$$\alpha := (\rho - \mathcal{L})^{-1} w \in H. \tag{10}$$

will play a key role in the transformation of the functional J that we will perform: it represents the core of the solution. In the next subsection we will investigate some properties of it.

3.2. **The function α and its properties.** By definition α is the unique solution in $W^{2,2}(S^1; \mathbb{R})$ of the abstract ODE

$$(\rho - \mathcal{L})\, \alpha = w. \tag{11}$$

More explicitly, α, as defined in (10), is the unique solution in $W^{2,2}(S^1; \mathbb{R})$ to

$$\rho \alpha(x) - \frac{\mathrm{d}}{\mathrm{d}x} \left(\sigma(x) \frac{\mathrm{d}}{\mathrm{d}x} \alpha(x) \right) + \delta(x) \alpha(x) = w(x), \quad x \in S^1, \tag{12}$$

meaning that it verifies (12) pointwise almost everywhere in S^1. The latter ODE can be viewed as on ODE on the interval $(0, 2\pi)$ with zero-order and first-order periodic boundary conditions[6], that is

$$\begin{cases} \rho \alpha(x) - \dfrac{\mathrm{d}}{\mathrm{d}x} \left(\sigma(x) \dfrac{\mathrm{d}\alpha}{\mathrm{d}x}(x) \right) + \delta(x) \alpha(x) = w(x), \quad x \in (0, 2\pi), \\ \alpha(0) = \alpha(2\pi), \quad \alpha'(0) = \alpha'(2\pi), \end{cases}$$

[6] falling into the Sturm-Liouville theory with periodic boundary conditions (see Coddington and Levinson, 1955).

where $\alpha'(0)$ and $\alpha'(2\pi)$ are, respectively, the right derivative at 0 and the left derivative at 2π of α.

As better argued below the value of α at a certain spatial point x has the meaning of the the sum of all future (discounted) disutility of a unit of pollutant initially located at x. By Sobolev embedding $W^{2,2}(S^1;\mathbb{R}) \subset C^1(S^1;\mathbb{R})$, so $\alpha \in C^1(S^1;\mathbb{R})$. With a little more subtle analysis something more can be said about the properties of α as shown in next proposition.

Proposition 3.4. *Let Assumption 3.1 hold. Then* $\alpha \in C^2(S^1;\mathbb{R})$ *and*

$$0 < \min_{S^1} \frac{w}{\rho+\delta} \leq \alpha(x) \leq \max_{S^1} \frac{w}{\rho+\delta} \quad \forall x \in S^1.$$

Proof. See Appendix A. □

We have the following interesting result on the dependence of α on the diffusion coefficient σ when the latter is constant over space.

Proposition 3.5. *Let Assumption 3.1 hold. Denote by* α_{σ^o} *the solution to* (12) *when* $\sigma(\cdot) \equiv \sigma^o > 0$. *We have*

$$\lim_{\sigma^o \to 0^+} \alpha_{\sigma^o}(x) = \frac{w(x)}{\rho+\delta(x)}, \qquad \lim_{\sigma^o \to +\infty} \alpha_{\sigma^o}(x) = \frac{\int_{S^1} w(x)dx}{\int_{S^1}(\rho+\delta(x))dx}, \quad \forall x \in S^1. \tag{13}$$

Proof. See Appendix A. □

As will be clearer shortly, the function α has a key role both in expressing the functional in its transformed form and in describing the optimal behavior of the planner. For this reason, understanding its behavior is interesting to describe the behavior of the model.

In Proposition 3.5 two limit cases are analyzed: the case where the diffusivity vanishes and the one when it tends to infinity. The first corresponds to the case where the pollution does not move among the locations and accumulates in the production site. As recalled above, in the model the only link among the locations is the transboundary pollution. Letting the diffusivity to zero means to reset this channel of interdependence and therefore the model reduces to an independent (unidimensional) optimization problem to each point of the space whose solution can be obtained plugging the first expression of (13) in (18). Conversely the second limit of (13) correspond to the infinite diffusivity benchmark that is the case where, at each moment, the speed of the diffusion process is so fast that the pollution is instantaneously redistributed uniformly throughout the space. For this reason, whatever the specific value of δ or of w in the precise point of the emission is not relevant but only global averages of the parameters matters.

We can exploit a little more the elements we gathered so far to better describe the economic intuition about the function α. In the non-spatial version of the model or, equivalently, in a specification of the model where all the parameters are constant in space, the value of α is $\frac{w}{\rho+\delta}$. This is, not surprisingly, also the pointwise expression appearing in (13) when $\sigma^o \to 0^+$ because, as already remarked, in absence of diffusion the model reduces to a juxtaposition of independent one-dimensional problems. This expression can be rewritten as

$$\int_0^{+\infty} e^{-\rho t} e^{-\delta t} w \, dt.$$

Recalling that δ is the natural decay of the pollution, the previous expresion is exactly the sum (the integral indeed) of all future (discounted) disutility of a unit of pollutant.

When we add the space to the model a second order term, depending on σ, appears in the equation which defines α, see (12). This comes of course from the spatial diffusion process of the pollution. The solution of that equation (except when both w and δ do not depend on time) is given by a space-heterogeneous function α. This heterogeneity is due to the fact that, in the model, a unit of pollutant has different effect on the social utility depending on where it is located and how it is going to spread in the future. In terms of pollution-disutility, locations are indeed different for two reasons: the different decay of pollutions and the different unitary instantaneous disutilities $w(x)$. In the general spatial case, the function α at a point x is the (total/social) future discounted disutility of a unit of pollutant initially located at point x. This fact is particularly transparent is one look at equation (31) in the Appendix. It reads as

$$\langle \alpha, p_0 \rangle = \left\langle w, \int_0^{\infty} e^{-(\rho - \mathcal{L})t} p_0 \, \mathrm{d}t \right\rangle.$$

If one takes p_0 to be the Dirac delta in a certain spatial point x, denoted by $\Delta_{\{x\}}$, the previous expression formally gives

$$\alpha(x) = \int_0^{\infty} e^{-\rho t} \left(\int_{S^1} w(\xi) \varphi(t, \xi; x) \, \mathrm{d}\xi \right) \mathrm{d}t \tag{14}$$

where $\varphi(t, \xi; x)$ is the fundamental solution of the parabolic equation

$$\begin{cases} \frac{\partial \varphi}{\partial t}(t, \xi) = \frac{\partial}{\partial \xi} \left(\sigma(\xi) \frac{\partial \varphi}{\partial x}(t, \xi) \right) - \delta(\xi) \varphi(t, \xi), \\ \varphi(0, \xi) = \Delta_{\{x\}}(\xi) \end{cases},$$

i.e. the spatial density (with respect to the variable x) at time t of a pollutant initially concentrated at point x, once one takes into account the diffusion process and the natural decay. Thus, the term $\int_{S^1} w(\xi)\varphi(t, \xi; x) \, \mathrm{d}\xi$ measures the instantaneous disutility all over the space and the whole expression in the right side hand of (14) is the total spatial (temporally discounted) future social disutility of a unit of pollutant initially concentrated at x.

3.3. **Characterization of the optimal control.** We describe now how previous results can be used to rewrite the functional in a tranformed form and then to use it to explicitly find the optimal solution of the problem, the related trajectory and welfare. The main results are described first (Theorem 3.6) in the Hilbert space formalism introduced above and then restated (Corollary 3.7) using a more readable PDE notation.

Theorem 3.6. *Let Assumption 3.1 hold.*

(i) The functional (9) can be rewritten as

(15)
$$J\big(p_0; I\big) = -\langle \alpha, p_0\rangle + \int_0^\infty e^{-\rho t}\left[\left\langle \frac{\big((A(t)-\mathbf{1})I(t)\big)^{1-\Gamma(t)}}{1-\Gamma(t)}, \mathbf{1}\right\rangle - \langle \alpha, \Psi(t)I(t)\rangle\right] \mathrm{d}t.$$

(ii) The control I^* given by

$$I^*(t)(x) := (\psi(t,x)\alpha(x))^{-\frac{1}{\gamma(t,x)}}(a(t,x)-1)^{\frac{1}{\gamma(t,x)}-1}. \tag{16}$$

belongs to $\mathcal{A}$ and is the unique optimal control of the problem.

(iii) The optimal state at time $t \geq 0$, that is $P^*(t)$, is given by

$$P^*(t) := e^{t\mathcal{L}}p_0 + \int_0^t e^{(t-s)\mathcal{L}}\Psi(s)I^*(s)ds. \tag{17}$$

(iv) The value function is finite and affine in p_0; more precisely,

$$v(p_0) = J\big(p_0; I^*\big) = \langle \alpha, p_0\rangle + q,$$

where

$$q := \int_0^\infty e^{-\rho t}\left[\left\langle \frac{\big((A(t)-\mathbf{1})I^*(t)\big)^{1-\Gamma(t)}}{1-\Gamma(t)}, \mathbf{1}\right\rangle - \langle \alpha, \Psi(t)I^*(t)\rangle\right] \mathrm{d}t.$$

Proof. See Appendix A. □

In the following corollary we summarize the results we have obtained so far rephrasing them in the PDE setting, where we use the identification of S^1 with the real interval $[0, 2\pi]$ with the identification of the extremes 0 and 2π.

Corollary 3.7. *Let Assumption 3.1 hold.*

(i) The optimal investment production input is given by

$$i^*(t,x) = (\psi(t,x)\alpha(x))^{-\frac{1}{\gamma(t,x)}}(a(t,x)-1)^{\frac{1}{\gamma(t,x)}-1}. \tag{18}$$

where α is the unique solution to the following ODE

$$\begin{cases} \rho\alpha(x) - \dfrac{\mathrm{d}}{\mathrm{d}x}\left(\sigma(x)\dfrac{\mathrm{d}\alpha}{\mathrm{d}x}(x)\right) + \delta(x)\alpha(x) = w(x), \quad x \in (0,2\pi), \\ \alpha(0) = \alpha(2\pi), \quad \alpha'(0) = \alpha'(2\pi). \end{cases}$$

(ii) The dynamics of the pollution profile p^* along the optimal path is the unique solution to the following parabolic PDE

(19)

$$\begin{cases} \dfrac{\partial p^*}{\partial t}(t,x) = \dfrac{\partial}{\partial x}\left(\sigma(x)\dfrac{\partial p^*}{\partial x}(t,x)\right) - \delta(x)p^*(t,x) + \alpha(x)^{-\frac{1}{\gamma(t,x)}}\left(\dfrac{a(t,x)-1}{\psi(t,x)}\right)^{\frac{1}{\gamma(t,x)}-1}, \quad x \in (0,2\pi), \\ p^*(t,0) = p^*(t,2\pi), \quad \dfrac{\partial p^*}{\partial x}(t,0) = \dfrac{\partial p^*}{\partial x}(t,2\pi), \quad t \geq 0, \\ p^*(0,x) = p_0(x), \quad x \in [0,2\pi]. \end{cases}$$

(iii) The social welfare is

$$v(p_0) = \int_0^{2\pi} \alpha(x)p_0(x)dx + \int_0^\infty e^{-\rho t}\Big(\int_0^{2\pi} \frac{\gamma(t,x)}{1-\gamma(t,x)}\left(\frac{a(t,x)-1}{\psi(t,x)\alpha(x)}\right)^{\frac{1}{\gamma(t,x)}-1}\mathrm{d}x\Big)\mathrm{d}t,$$

In the previous statements we have seen the explicit solution of the optimal problem of the planner. A first, eminently technical, observation concerns the transformation of the functional. In fact, the first result of Theorem 3.6 is that the functional (9) can be rewritten in the form (15). The new form is particularly useful as, in this expression, the state P no longer appears and it is only given in terms of the control I; this fact greatly simplifies the analysis. Thanks to this transformation, it is indeed much easier to find the expression of the optimal investment and the subsequent results.

Looking at the expressions that appear in Corollary 3.7, it is immediately evident that all the heterogeneities of the problem enter directly and in a non-trivial way in the solution. Partly, they appear explicitly in the expressions of the optimal spatial profile of the investment or of the parabolic equation describing the evolution of the spatial distribution of pollution and partly they contribute to these expressions through the expression of the function α. In this way the model is sensitive to heterogeneities of different nature: environmental heterogeneities (the ability to regenerate of the ecological context measured in each place by δ), productive heterogeneities (the productivity which depends exogenously on both the location and the time) and preferences heterogeneities (both through the spatial heterogeneity of the disutility of pollution w and through the spatial heterogeneity of the elasticity of intertemporal substitution).

3.4. **Transitional and long-run analysis of P^*.** In this section we analyze both the transitional dynamics of $P^*(t)$ and its limit behavior as $t \to \infty$.

3.4.1. *Transitional dynamics through series expansion.* Recall that a non identically zero function $\phi \in \mathcal{D}(\mathcal{L})$ is called *eigenfunction* of $\mathcal{L}$ if there exists a real number (called *eigenvalue*) λ such that $\mathcal{L}\phi = \lambda\phi$.

Proposition 3.8. *Let Assumption 3.1 hold. There exists a decreasing sequence $\{\lambda_n\}_{n\in\mathbb{N}} \subset (-\infty, 0]$ such that $\lambda_n \to -\infty$ and an orthonormal basis $\{\mathbf{e}_n\}_{n\in\mathbb{N}} \subset H$ such that*

$$\mathbf{e}_n \in \mathcal{D}(\mathcal{L}) \quad \textit{and} \quad \mathcal{L}\mathbf{e}_n = \lambda_n \mathbf{e}_n \quad \forall n \in \mathbb{N}.$$

Then the pollution profile along the optimal trajectory can then be expressed as a convergent series in H:

$$P^*(t) = \sum_{n\in\mathbb{N}} p_n^*(t)\mathbf{e}_n, \qquad \textit{where} \;\; p_n^*(t) := \langle P^*(t), \mathbf{e}_n \rangle \tag{20}$$

and the expressions of the coefficients $p_n^*(t)$ can be explicitly given in the following form

$$p_n^*(t) := \langle p_0, \mathbf{e}_n \rangle e^{\lambda_n t} + \int_0^t e^{\lambda_n (t-s)} \xi_n^*(s) \mathrm{d}s, \qquad \forall t \geq 0, \; \forall n \in \mathbb{N}, \tag{21}$$

where

$$\xi_n^*(t) := \langle \Psi(t) I^*(t), \mathbf{e}_n \rangle, \qquad \forall t \geq 0, \; \forall n \in \mathbb{N}. \tag{22}$$

Proof. See Appendix A. □

The previous result allows to express the solution of the equation (19) along the optimal path in terms of a series of space functions not dependent on time multiplied by time-dependent coefficients. This expression, which can be used in general to simulate the model, takes a particularly familiar form in some specific cases. For example, if the diffusivity σ and the natural decay of pollution δ are uniform in the spaces, a standard Fourier series is obtained, and the functions $\mathbf{e}_n$ are sins and cosines.

Observe also that (20) can also be used to express the total pollution $\int_{S^1} p^*(t,x)dx$ as a function of time, namely,

$$\int_{S^1} p^*(t,x)dx = \sum_{n\in\mathbb{N}} \langle \mathbf{e}_n, \mathbf{1}\rangle \, p_n^*(t), \qquad t \geq 0.$$

where $p_n^*(t)$ is given in (21). Indeed, again using this result, an even more precise description of the total pollution at time t can be given whenever the following when δ is constant in space as shown by the following proposition.

Proposition 3.9. *Let Assumption 3.1 hold and assume that the function δ is constant, i.e. $\delta(\cdot) \equiv \delta_0 \geq 0$. Then*

$$\int_{S^1} p^*(t,x)\mathrm{d}x = \left(\int_{S^1} p_0(x)\mathrm{d}x\right) e^{-\delta_0 t} + \int_0^t e^{-\delta_0(t-s)} \left(\int_{S^1} \psi(s,x) i^*(s,x)\mathrm{d}x\right) \mathrm{d}s.$$

Proof. See Appendix A. □

3.4.2. *Limit behaviour in the time-homogeneous case.* We consider now the special case when the productivity coefficients and the ecological efficiency of the production process are time-independent: $a(t,x) = a(x)$, $\psi(t,x) = \psi(x)$; similarly for the inverse of the elasticity of intertemporal substitution, i.e. $\gamma(t,x) = \gamma(x)$. In this case, the expressions of the optimal control is time independent

$$I^*(t)(x) \equiv \bar{I}^*(x) = (\psi(x)\alpha(x))^{-\frac{1}{\gamma(x)}} (a(x)-1)^{\frac{1-\gamma(x)}{\gamma(x)}}, \tag{23}$$

and we have a direct characterization of the long-run profile of the pollution stock along the optimal path as described in the following proposition.

Proposition 3.10. *Let Assumption 3.1 hold. Assume that the coefficients a, γ, ψ are time-independent and that $\delta \not\equiv 0$. Then we have*

$$\lim_{t\to\infty} P^*(t) = p_\infty^* \quad \textit{in } H,$$

where p_∞^* is the unique solution to the ODE

(24)

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}x}\left(\sigma(x)\dfrac{\mathrm{d}p_\infty^*}{\mathrm{d}x}(x)\right) - \delta(x)p_\infty^*(x) + \alpha(x)^{-\frac{1}{\gamma(x)}}\left(\dfrac{a(x)-1}{\psi(x)}\right)^{\frac{1}{\gamma(x)}-1} = 0, \quad x \in (0,2\pi), \\ p_\infty^*(0) = p_\infty^*(2\pi), \quad \dfrac{\mathrm{d}p_\infty^*}{\mathrm{d}x}(0) = \dfrac{\mathrm{d}p_\infty^*}{\mathrm{d}x}(2\pi). \end{cases}$$

Proof. See Appendix A. □

4. Conclusion

In this paper, we have provided with a generic spatiotemporal non-linear-quadratic framework for transboundary pollution control. The objective functional to be maximized is a Benthamite social welfare function depending on the intertemporal stream of consumption at any location, and internalizing the spatial externalities resulting from pollution diffusion. The essential contribution of this work is to identify optimal pollution control policies with a very large account of geographic heterogeneity: (i) heterogeneity in productivity and in ecological efficiency of the production process, which also includes the broad spatio-temporal characteristics of the exogenous technological process; (ii) heterogeneity in preferences, notably in the intertemporal elasticity of substitution and in the disutility from the pollution, and finally: (iii) the heterogeneity in the environmental/ecological context, in particular in terms of speed of diffusion of pollutants and local regeneration capacity.

Despite the huge complexity of the problem, we have been able to produce a solution method which has two unexpected virtues (given the complexity of the task). First, it allows for closed-form solutions, and second, the solutions produced are based on a neatly singled out spatial function with a clear economic interpretation. We do believe that such a framework can be used in a large set of applications given the generality of most of the specifications. Clearly, one can still visualize a number of possible future extensions (for example the incorporation of demographic dynamics with space-dependent mortality depending on local pollution) but we firstly believe that the next step should be the exploitation of the variety of applications allowed by this framework.

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Appendix A. Proofs

Proof of Proposition 3.2. Due to Assumption 3.1, $\mathcal{L}$ is a closed, densely defined, unbounded linear operator on the space H (see, e.g. Lunardi, 1995, p. 71-75, Sections 3.1 and 3.1.1). A core for it is the space $C^\infty(S^1;\mathbb{R})$ (see, e.g., Engel and Nagel, 1995, pages 69-70). Let $\varphi \in C^\infty(S^1;\mathbb{R})$. Integration by parts yields

$$\begin{aligned} \langle \mathcal{L}\varphi, \varphi \rangle &= \int_{S^1} (\mathcal{L}\varphi)(x)\varphi(x)\mathrm{d}x \\ &= -\int_{S^1} \sigma(x)|\varphi'(x)|^2\mathrm{d}x - \int_{S^1} \delta(x))|\varphi(x)|^2\mathrm{d}x \leq 0 \end{aligned} \tag{25}$$

Since $C^\infty(S^1;\mathbb{R})$ is a core for $\mathcal{L}$, (26) extends to all functions $\varphi \in D(\mathcal{L})$, showing that the operator $\mathcal{L}$ is dissipative. Similarly, a double integration by parts shows that

$$\langle \mathcal{L}\varphi, \psi \rangle = \langle \varphi, \mathcal{L}\psi \rangle, \quad \forall \varphi, \psi \in C^\infty(S^1;\mathbb{R}). \tag{26}$$

Again, since $C^\infty(S^1;\mathbb{R})$ is a core for $\mathcal{L}$, (26) extends to all couples of functions in $D(\mathcal{L})$, showing that $\mathcal{L}$ is self-adjoint, i.e. $\mathcal{L} = \mathcal{L}^\star$, where $\mathcal{L}^\star$ denotes the adjoint of $\mathcal{L}$. Therefore, by Engel and Nagel (1995) (in particular, Chapter II), $\mathcal{L}$ generates a strongly continuous contraction semigroup $(e^{t\mathcal{L}})_{t\geq 0} \subset L(H)$; in particular, since $\rho > 0$, by standard theory of strongly continuous semigroup in Banach spaces (see, e.g. pages 82-83, Chapter II and Theorem 1.10, Chapter II of Engel and Nagel 1995), it follows that ρ belongs to the resolvent set of $\mathcal{L}$ and that (6) holds. □

Proof of Proposition 3.3. First of all we observe that, by Assumption 3.1(ii), the first term in the functional is always positive (Case (A)) or always negative (Case (B)), possibly infinite. Hence to prove the claim it is enough to show that, given any $p_0 \in H$, the term $\int_0^\infty e^{-\rho t}\langle w, P(t)\rangle \mathrm{d}t$ is well defined and finite for every $I \in \mathcal{A}$. We have

$$\int_0^\infty e^{-\rho t}\langle w, P(t)\rangle \mathrm{d}t = \int_0^\infty e^{-\rho t}\Big\langle w, e^{t\mathcal{L}}p_0 + \int_0^t e^{(t-s)\mathcal{L}}\Psi(s)I(s)\mathrm{d}s\Big\rangle \mathrm{d}t \tag{27}$$

Now, since w is bounded and $e^{t\mathcal{L}}$ is a contraction, the integral $\int_0^\infty e^{-\rho t}\langle w, e^{t\mathcal{L}}p_0\rangle \mathrm{d}t$ is finite. Moreover, for all $T > 0$ we get, by Fubini-Tonelli's Theorem

$$\begin{aligned}
&\int_0^T \left(\int_0^t e^{-\rho t} \left\langle w, e^{(t-s)\mathcal{L}}\Psi(s)I(s) \right\rangle \mathrm{d}s \right) \mathrm{d}t \\
&= \int_0^T \left(\int_0^t e^{-\rho s} \left\langle w, e^{-(\rho-\mathcal{L})(t-s)}\Psi(s)I(s) \right\rangle \mathrm{d}s \right) \mathrm{d}t \\
&= \int_0^T e^{-\rho s} \left\langle w, \int_s^T e^{-(\rho-\mathcal{L})(t-s)}\Psi(s)I(s)\mathrm{d}t \right\rangle \mathrm{d}s
\end{aligned}$$

Using again the fact that $e^{(t-s)\mathcal{L}}$ is a contraction and Assumption 3.1, we have, for each $s \geq 0$, $T \geq 0$

$$\left\| \int_s^T e^{-(\rho-\mathcal{L})(t-s)}\Psi(s)I(s)\mathrm{d}t \right\| \leq \int_s^\infty e^{-\rho(t-s)}\|\Psi(s)I(s)\|\mathrm{d}t \leq \frac{1}{\rho}\|\Psi(s)I(s)\|.$$

Hence, by definition of $\mathcal{A}$, the claim follows sending T to $+\infty$. □

Proof of Proposition 3.4. The fact that α solves (12) and the fact that, by Assumption 3.1, we have $\sigma(\cdot) > 0$ yield

$$\alpha''(x) = \frac{1}{\sigma(x)} \left[(\rho + \delta(x))\alpha(x) - \sigma'(x)\alpha'(x) - w(x) \right], \quad \text{for a.e. } x \in S^1.$$

Since $\alpha \in C^1(S^1; \mathbb{R})$, it follows, by Assumption 3.1, that $\alpha \in C^2(S^1; \mathbb{R})$.

Now, let $x_* \in S^1$ be a minimum point of α over S^1. Then $\alpha''(x_*) \geq 0$. Plugging this into (12) we get

$$(\rho + \delta(x_*))\alpha(x_*) = \sigma(x_*)\alpha''(x_*) + w(x_*) \geq w(x_*),$$

and the estimate from below follows. The estimate from above can be obtaind symmetrically. □

Proof of Proposition 3.5. Case $\sigma \to 0^+$. First, notice that under the above assumptions (12) reads as

$$\rho\alpha_{\sigma^o}(x) - \sigma^o\alpha''_{\sigma^o}(x) + \delta(x)\alpha_{\sigma^o}(x) = \widehat{w}(x), \quad x \in S^1, \tag{28}$$

By Proposition 3.4 we have

$$\alpha_*(x) := \liminf_{\overline{\sigma^o}\to 0^+} \left\{ \alpha_{\sigma^o}(\zeta) : \ \sigma^o \leq \overline{\sigma^o}, \ \zeta \in S^1, \ |\zeta - x| \leq 1/\overline{\sigma^o} \right\} \geq \min_{S^1} \frac{w}{\rho + \delta},$$

$$\alpha^*(x) := \limsup_{\overline{\sigma^o}\to 0^+} \left\{ \alpha_{\sigma^o}(\zeta) : \ \sigma^o \leq \overline{\sigma^o}, \ \zeta \in S^1, \ |\zeta - x| \leq 1/\overline{\sigma^o} \right\} \leq \max_{S^1} \frac{w}{\rho + \delta}.$$

Clearly $\alpha^* \geq \alpha_*$. By stability of viscosity solutions (see e.g. Crandall et al., 1992), the latter functions are, respectively, (viscosity) super- and sub-solution to the limit equation

$$\rho\alpha_0(x) + \delta(x)\alpha_0(x) = w(x),$$

whose unique solution is

$$\alpha_0(x) = \frac{w(x)}{\rho + \delta(x)}.$$

By standard comparison of viscosity solutions one has $\alpha_* \geq \alpha_0 \geq \alpha^*$. It follows that

$$\exists \lim_{\sigma^o \to 0^+} \alpha_{\sigma^o}(x) = \alpha_*(x) = \alpha^*(x) = \alpha_0(x) \quad \forall x \in S^1.$$

Case $\sigma \to +\infty$. First, we rewrite (28) as

$$\alpha''_{\sigma^o}(x) = \frac{1}{\sigma^o}\left[\rho\alpha_{\sigma^o}(x) + \delta(x)\alpha_{\sigma^o}(x) - w(x)\right], \quad x \in S^1, \tag{29}$$

From this and from Proposition 3.4, we see that α_{σ^o} is equi-bounded and equi-uniformly continuous with respect to $\sigma^o \geq 1$. Hence, by Ascoli-Arzelà Theorem we have that, from each sequence $\sigma_n \to +\infty$, we can extract a subsequence σ_{n_k} such that

$$\lim_{k\to+\infty} \alpha_{\sigma_{n_k}} = \alpha_\infty \quad \text{uniformly on } x \in S^1,$$

for some $\alpha_\infty \in C(S^1;\mathbb{R})$. Again by stability viscosity solutions, α_∞ must solve, in the viscosity sense, the limit equation

$$\alpha''_\infty(x) = 0, \quad x \in S^1.$$

Hence, it must be $\alpha_\infty \equiv c_0$ for some $c_0 \geq 0$. To find the value of c_0 we may integrate (28) over S^1 getting

$$\int_{S^1} (\rho + \delta(x))\alpha_{\sigma^o}(x)\mathrm{d}x = \int_{S^1} w(x)\mathrm{d}x.$$

Letting $\sigma^o \to +\infty$ above, we get

$$c_0 = \frac{\int_{S^1} w(x)\mathrm{d}x}{\int_{S^1}(\rho + \delta(x))\mathrm{d}x}.$$

As this value does not depend on the sequence σ_n chosen, the claim follows. □

Proof of Theorem 3.6. (i) Using (8) it is possible to rewrite the second part of (9). We first set

$$e^{-(\rho-\mathcal{L})t} := e^{-\rho t}e^{t\mathcal{L}}, \quad t \geq 0,$$

and we write

$$\begin{aligned}(30)\qquad &\int_0^\infty e^{-\rho t}\langle w, P(t)\rangle \mathrm{d}t = \int_0^\infty e^{-\rho t}\Big\langle w, e^{t\mathcal{L}}p_0 + \int_0^t e^{(t-s)\mathcal{L}}\Psi(s)I(s)ds\Big\rangle \mathrm{d}t \\ &= \left\langle w, \int_0^\infty e^{-(\rho-\mathcal{L})t}p_0\,\mathrm{d}t\right\rangle + \int_0^\infty e^{-\rho t}\left\langle w, \int_0^t e^{(t-s)\mathcal{L}}\Psi(s)I(s))\mathrm{d}s\right\rangle \mathrm{d}t\end{aligned}$$

Note that the first term of the right hand side is the only one which depends on the initial datum and, by (6), it can be rewritten as

$$(31)\qquad \left\langle w, \int_0^\infty e^{-(\rho-\mathcal{L})t}p_0\,\mathrm{d}t\right\rangle = \langle w, (\rho-\mathcal{L})^{-1}p_0\rangle = \langle (\rho-\mathcal{L})^{-1}w, p_0\rangle = \langle \alpha, p_0\rangle\,,$$

where α is defined in (10).

We look now at the last term of the last line of (30). It can be rewritten by exchanging the integrals as follows:

$$\begin{aligned}&\int_0^\infty \left(\int_0^t e^{-\rho t}\left\langle w, e^{(t-s)\mathcal{L}}\Psi(s)I(s)\right\rangle \mathrm{d}s\right)\mathrm{d}t \\ &= \int_0^\infty \left(\int_0^t e^{-\rho s}\left\langle w, e^{-(\rho-\mathcal{L})(t-s)}\Psi(s)I(s)\right\rangle \mathrm{d}s\right)\mathrm{d}t \\ &= \int_0^\infty e^{-\rho s}\left\langle w, \int_s^\infty e^{-(\rho-\mathcal{L})(t-s)}\Psi(s)I(s)dt\right\rangle \mathrm{d}s \\ &= \int_0^\infty e^{-\rho s}\left\langle w, (\rho-\mathcal{L})^{-1}\Psi(s)I(s)\right\rangle \mathrm{d}s \\ &= \int_0^\infty e^{-\rho s}\left\langle (\rho-\mathcal{L})^{-1}w, \Psi(s)I(s)\right\rangle \mathrm{d}s\end{aligned}$$

Hence, we can finally rewrite (9) as (15).

(ii) After writing explicitely the inner products in (9), the integral can be optimized pointwisely. We end up, for $(t,x)\in\mathbb{R}^+\times S^1$ fixed, with the optimization

$$\sup_{\iota\geq 0}\left\{\frac{\big((a(t,x)-1)\iota\big)^{1-\gamma(t,x)}}{1-\gamma(t,x)} - \alpha(x)\psi(t,x)\iota\right\};$$

so, we easily get the claimed expression (16) of the candidate unique optimal control I^*. On the other hand, we need to verify that $I^*\in\mathcal{A}$. Indeed, by Assumption 3.1, we have

$$\begin{aligned}\Psi(t)(x)I^*(t)(x) &= \alpha(x)^{-\frac{1}{\gamma(t,x)}}\psi(t,x)^{1-\frac{1}{\gamma(t,x)}}(a(t,x)-1)^{\frac{1}{\gamma(t,x)}-1} \\ &= \alpha(x)^{-\frac{1}{\gamma(t,x)}}\left(\frac{a(t,x)-1}{\psi(t,x)}\right)^{\frac{1}{\gamma(t,x)}-1}.\end{aligned}$$

Since α is bounded from above and from below by positive constants, then so is $\alpha(x)^{-\frac{1}{\gamma(t,x)}}$ by Assumption 3.1(ii). Consequently, by Assumption 3.1(iii), we get, for some $C_0 > 0$

$$0 \leq \Psi(t)(x)I^*(t)(x) \leq C_0 e^{gt}, \quad \forall x \in S^1.$$

Since $\rho > g$ by Assumption 3.1(iv), we get $I^* \in \mathcal{A}$.

(iii)-(iv) These claims immediately follow by straightforward computations. □

Proof of Proposition 3.8. Consider the operator $(1-\mathcal{L})^{-1} \in L(H)$. The range of this operator is the space $D(\mathcal{L}) = W^{2,2}(S^1;\mathbb{R})$, which by Kondrachov's Theorem is embedded in $H = L^2(S^1;\mathbb{R})$ with compact embedding. It follows that $(1-\mathcal{L})^{-1}$ is compact; being also self-adjoint, by standard spectral theory in Hilbert spaces, there exists an orthonormal basis of eigenvectors for it, hence also for $\mathcal{L}$. Hence, considering also that $\mathcal{L}$ is dissipative and unbounded, there exists a decreasing sequence $\{\lambda_n\}_{n\in\mathbb{N}} \subset (-\infty, 0]$ such that $\lambda_n \to -\infty$ and an orthonormal basis $\{\mathbf{e}_n\}_{n\in\mathbb{N}} \subset H$ such that

$$\mathbf{e}_n \in \mathcal{D}(\mathcal{L}) \quad \text{and} \quad \mathcal{L}\mathbf{e}_n = \lambda_n \mathbf{e}_n \quad \forall n \in \mathbb{N}. \tag{32}$$

Consider the Fourier series expansion

$$P^*(t) = \sum_{n\in\mathbb{N}} p_n^*(t)\mathbf{e}_n, \qquad \text{where} \;\; p_n^*(t) := \langle P^*(t), \mathbf{e}_n\rangle.$$

We can write explicitly the Fourier coefficients $p_n^*(t)$ by the following argument. By Proposition 3.2, Chapter 1, Part II of Bensoussan et al. (2007), the function P^* defined in (17) is also a *weak solution* to (7) with $I = I^*$, i.e., taking into account that $\mathcal{L}$ is self-adjoint, i.e. $\mathcal{L} = \mathcal{L}^\star$, it holds

$$\langle P^*(t), \varphi\rangle = \langle p_0, \varphi\rangle + \int_0^t \Big(\langle P^*(s), \mathcal{L}\varphi\rangle + \langle \Psi(s)I^*(s), \varphi\rangle\Big)\mathrm{d}s \qquad \forall t \geq 0, \;\; \forall \varphi \in D(\mathcal{L}).$$

In particular, taking into account (32), we have

$$\begin{aligned} p_n^*(t) = \langle P^*(t), \mathbf{e}_n\rangle &= \langle p_0, \mathbf{e}_n\rangle + \int_0^t \Big(\lambda_n \langle P^*(s), \mathbf{e}_n\rangle + \langle \Psi(s)I^*(s), \mathbf{e}_n\rangle\Big)\mathrm{d}s \\ &= \langle p_0, \mathbf{e}_n\rangle + \int_0^t \Big(\lambda_n p_n^*(s) + \xi_n^*(s)\Big)\mathrm{d}s \qquad \forall t \geq 0, \;\; \forall n \in \mathbb{N}. \end{aligned}$$

Then

$$\int_{S^1} p^*(t,x)\mathrm{d}x = \langle P^*(t), \mathbf{1}\rangle = \Big\langle \sum_{n\in\mathbb{N}} p_n^*(t)\mathbf{e}_n, \mathbf{1} \Big\rangle = \sum_{n\in\mathbb{N}} \langle \mathbf{e}_n, \mathbf{1}\rangle \, p_n^*(t), \qquad t \geq 0,$$

as claimed. □

Proof of Proposition 3.9. First we observe that, in this case, $\mathbf{e}_0(\cdot) \equiv \frac{1}{\sqrt{2\pi}}$, $\lambda_0 = -\delta_0$. Hence

$$\int_{S^1} p^*(t,x)\mathrm{d}x = \langle P^*(t), \mathbf{1}\rangle = \sqrt{2\pi}\langle P^*(t), \mathbf{e}_0\rangle.$$

From the mild form of P given in (8) we now get, for $t \geq 0$,

$$\begin{aligned}\langle P^*(t), \mathbf{e}_0\rangle &= \langle e^{t\mathcal{L}} p_0, \mathbf{e}_0\rangle + \int_0^t \langle e^{(t-s)\mathcal{L}}\Psi(s)I^*(s), \mathbf{e}_0\rangle \mathrm{d}s \\ &= \langle p_0, e^{t\mathcal{L}}\mathbf{e}_0\rangle + \int_0^t \langle \Psi(s)I^*(s), e^{(t-s)\mathcal{L}}\mathbf{e}_0\rangle \mathrm{d}s \\ &= \langle p_0, e^{-\delta_0 t}\mathbf{e}_0\rangle + \int_0^t \langle \Psi(s)I^*(s), e^{-\delta_0(t-s)}\mathbf{e}_0\rangle \mathrm{d}s,\end{aligned}$$

where we used that $e^{-\delta_0 t}$ is the eigenvalue of $e^{t\mathcal{L}}$ associated to $\mathbf{e}_0$. The claim immediately follows. □

Proof of Proposition 3.10. In this case $I^*(\cdot) \equiv \bar{I}^* \in H$ is time independent too. Since $\delta \not\equiv 0$, we have $\lambda_0 < 0$. Let us write

$$\mathcal{L} = \mathcal{L}_0 - \lambda_0, \quad \text{where} \quad \mathcal{L}_0 := \mathcal{L} + \lambda_0,$$

and note that $\mathcal{L}_0$ is dissipative by definition, hence $e^{s\mathcal{L}_0}$ is a contraction. Then, setting $\bar{\Psi} := \psi(\cdot) \in H$, we can rewrite

$$P^*(t) = e^{\lambda_0 t}e^{t\mathcal{L}_0}p_0 + \int_0^t e^{\lambda_0(t-s)}e^{(t-s)\mathcal{L}_0}\bar{\Psi}\bar{I}^*\mathrm{d}s = e^{\lambda_0 t}e^{t\mathcal{L}_0}p_0 + \int_0^t e^{\lambda_0 t}e^{t\mathcal{L}_0}\bar{\Psi}\bar{I}^*\mathrm{d}s,$$

and take the limit above when $t \to \infty$. Since $e^{s\mathcal{L}_0}$ is a contraction, the first term of the right hand side converges to 0, whereas the second one converges to

$$P^*_\infty := \int_0^\infty e^{-\delta_0 s}e^{s\mathcal{L}_0}\bar{\Psi}\bar{I}^*\mathrm{d}s \in H.$$

Then, the limit state $P^*_\infty \in H$ can be expressed using again Proposition 3.14, page 82 and Theorem 1.10, Chapter II of Engel and Nagel (1995) as

$$P^*_\infty = (\delta_0 - \mathcal{L}_0)^{-1}\bar{\Psi}\bar{I}^*,$$

i.e. P^*_∞ is the solution to

$$(\delta_0 - \mathcal{L}_0)P^*_\infty = \bar{\Psi}\bar{I}^*,$$

equivalently

$$\mathcal{L}P^*_\infty + \bar{\Psi}\bar{I}^* = 0,$$

i.e., in the PDE formalism, $p^*_\infty(\cdot) := P^*_\infty$ solves (24). □

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