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Costly agreement-based transfers and targeting on networks with synergies*

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Abstract

We consider agents organized in an undirected network of local complementarities. A principal with a limited budget offers costly bilateral contracts in order to increase the sum of agents' effort. We study excess-effort linear payment schemes, i.e. contracts rewarding effort in excess to the effort made in absence of principal. The analysis provides the following main insights. First, for all contracting costs, the optimal unit returns offered to every targeted agent are positive and generically heterogeneous. This heterogeneity is due to the presence of outsiders, who create asymmetric interaction between contracting agents. Second, when contracting costs are low, it is optimal to contract with everyone and optimal unit returns are identical for all agents. Third, when contracting costs are sufficiently high, it becomes optimal to target a subset of agents, and optimal targeting can lead to NP-hard problems. In particular, when the intensity of complementarities is sufficiently low, a correspondence is established between optimal targeting and the densest k subgraph problem. Overall, the optimal targeting problem involves a trade-off between centrality and budget spending — central agents are influential, but are also more budget-consuming. These considerations can lead the principal to not target central agents.

Keywords: Networked Synergies, Aggregate Effort, Optimal Group Targeting, Linear contract.
JEL: C72, D85

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1 Introduction

In many economic situations, organizations trade agents' effort against transfers in a context of networked synergies and positive externalities. One important dimension of such problems is that agents exert effort even in the absence of intervention by the institution. This implies that synergies not only exist between contracting agents, but also between contracting and non-contracting agents.[1] To cite a few examples: in conditional cash transfer (called CCT thereafter) grant programs, students who do not receive the grant still interact with those who obtain the grant; in R&D networks where a public fund provider allocates subsidies, a non-subsidized firm still spends on R&D and interacts with partner firms; in performance appraisal systems, firms offer workers a raise in salary (or a bonus), a worker not receiving any raise in salary still interacts with other workers.A unifying feature of such contracts is that they can involve large administration, monitoring and enforcement costs. For instance, a common criticism of CCT programs (and other social safety net programs) is that a large proportion of their budgets never reaches the intended beneficiaries but is absorbed by administration costs (see Grosh M. (1994), Caldés & Maluccio (2005)). Similarly, R&D subsidies involve large administration costs (see GAO study of 1989 by U.S. General Accounting Office, Stoffregen (1995), and Hall & Van Reenen (2000)). And finally, firms incur large administration, monitoring and enforcement costs in organizing a performance appraissal system (see Murphy & Cleveland (1995), and Aggarwal & Thakur (2013)). Given the prevalence of such instances, in this paper we investigate: how could the institution exploit synergies and positive externalities between agents? How do contracting costs influence their decisions? Would institutions contract with everyone even in the presence of large contracting costs, and if not, which set of agents would they target?

To examine the relationship between the network of synergies among agents, targeting, and contracting costs, we build a principal / multi-agent model where the principal has a limited budget and offers costly linear bilateral payment schemes in order to increase the sum of agents' effort. We consider an environment where bilateral synergies are perfectly reciprocal, and where non-contracting agents (that we may call outsiders for convenience) exert effort and interact with contracting agents, so that reservation utilities are endogenous to offered contracts. The utility functions are linear quadratic in effort with local synergies and positive externalities, and contracts are 'excess-effort' linear payment schemes — rewarding effort in excess

[1] This sort of problems is distinct from issues in which agents' exert no effort without the intervention of the institution. For latter context, see for instance pricing with interdependent consumers.

to the effort made in absence of principal. This utility specification generates a linear system of best-response effort, which leads to a unique Nash effort profile for a given set of contracts. The timing of the game is as follows. First, principal offers contracts to a set of agents, which could involve everyone or principal can also target a subgroup of agents. These contracts take the form of a transfer per unit of additional effort (which we call the unit return hereafter), and they are offered simultaneously and are made public.[2]). Second, agents decide whether to accept or reject their respective offers. And finally, all agents, including outsiders, simultaneously exert effort and transfers are realized.

Before we proceed further, it is important to highlight our interest in 'excess-effort' linear payment schemes. Due to the presence of complementarities, in general the principal has to worry about coordination among agents in the participation game. However, we show that, by offering excess-effort linear payment schemes, the principal avoids all potential coordination issues for any level of contracting costs. This allows us to study optimal excess-effort linear payment schemes, without having to worry about such coordination issues, and examine the relationship between contracting costs and optimal contracts.

The analysis brings three main messages. First, optimal unit return offered to every targeted agent is positive. Even though the networks are undirected, surprisingly these returns are are generically heterogeneous. This is due to the presence of outsiders, who create asymmetric interaction between contracting agents (whereas there are no outsiders when the principal contracts with the whole society).[3]

Second, under low contracting cost, the principal finds it optimal to contract with every agent; i.e. there is no targeting. Furthermore, all agents get the same unit return, irrespective of their position on the network. In general, contracting with more central agents involves a trade-off between centrality and budget spending – central agents generate more synergies, but are also more budget-consuming because they receive large amount of externalities and as a result, they are themselves highly responsive to transfers. When contracting costs are low, the incentives to allocate resources to more central agents in order to benefit from larger synergies is exactly counterbalanced by

[2]The principal can offer negative returns. However, as we will show, optimal unit returns are always rewards.

[3]That the presence of outsiders implies heterogeneous unit returns may in principle lead to negative returns, involving a possible participation concern leading the principal to target a strict subgroup of the initially selected group; yet, we show in the paper that the induced asymmetry of interactions between contracting agents is not strong enough to entail such consequences. Proving positiveness of returns is crucial to guarantee the absence of coordination concern; this is one of the major technical insights of this paper.

their propensity to overstretch the budget.[4]

And finally, when contracting costs are sufficiently high, it becomes optimal to target a subset of agents. We find that optimal targeting can lead to NP-hard problems under supermodular contracting cost function.[5] In particular, when intensity of complementarities is sufficiently low and the group size is fixed, optimal targeting is isomorphic to the densest k-subgraph problem, which is known to be NP-hard. Hence, the presence of contracting costs qualitatively changes the nature of the principal's problem. Furthermore, the trade-off between centrality and budget spending may lead the principal to not target the most central agents. Lastly, when the cost function only depends on the number of contracts, the optimal group size may not be a continuous function of cost value in that certain group sizes are never optimal for any cost value.

This paper contributes to three literatures, including optimal intervention in presence of synergies, optimal linear pricing with interdependent consumers, and optimality of linear contracts in moral hazard environments. We will now discuss each of these in turn. First, this paper contributes to two strands of the growing literature on optimal intervention in presence of synergies between agents. The first strand of the literature considers optimal targeting. Demange (2017) studies the optimal targeting strategies of a planner aiming to increase the aggregate action of agents embedded in a social network, allowing for non-linear interaction. Galeotti, Golub & Goyal (2020) study optimal targeting in networks, where a principal aims at maximizing utilitarian welfare or minimizing the volatility of aggregate activity. Belhaj & Deroïan (2018) take into account participation constraints, not addressed in the above papers. In this paper, we address both participation constraints and contracting costs. The second strand of the literature studies principal/multi-agent contracting in presence of synergies, taking into account participation constraints. Bernstein & Winter (2012) and Sakovics & Steiner (2012) consider coordination issues with binary actions. In particular, Bernstein and Winter (2012) study a costly participation game where participants receive positive and heterogeneous externalities from other participants, and they characterize the contracts inducing full participation while minimizing total subsidies. In Sakovics & Steiner (2012), a principal subsidizes agents facing a coordination problem akin to the adoption of a network technology. Optimal subsidies target agents who impose high externalities on others and on whom others impose low externalities. Belhaj & Deroïan

[4] This result is tightly linked to the linearity of the system of best-responses and it holds under reciprocal interactions only.

[5] That is, when the cost function only depends on the number of contracts, this means a linear or convex function.

(2018) introduce continuous effort, and they cover situations where contracting with a subset of the population can be optimal. In their paper, targeting becomes optimal because participation constraints become binding, whereas in the present paper targeting is optimal solely due to large contracting costs.

Next, our paper complements the recent literature on optimal linear pricing with interdependent consumers (Candogan, Bimpikis & Ozdaglar [2012], Bloch & Quérou [2013], and Fainmesser & Galeotti [2016]). Our result about homogeneous unit return under low contracting cost echoes a well-known result of that literature that stipulates that, on undirected networks, the price charged to each consumer does not depend on the position on the social network (Candogan et al [2012], Bloch and Quérou [2013]). However, our framework differs from theirs in that, in our model, agents exert effort in absence of principal whereas consumers do not consume without the firm. As a result, in our setting when the principal targets a subgroup, outsiders still interact with contracting agents, which then leads to heterogeneous unit returns despite the fact that interactions are undirected.

Finally, this paper is related to an interesting strand of the moral hazard literature, which helps explain the widespread use of linear contracts in the real world.[6] For instance, Holmstrom & Milgrom (1987) consider an environment where the principal and the agent have constant absolute risk-aversion (CARA) utility, and the agent controls the drift of a (possibly multidimensional) Brownian motion in continuous time. Although the principal can make payments depend on the entire path of motion, the optimal contract is simply a linear function of the endpoint. Diamond (1998) considers a model in which the agent can either choose no effort, producing a low expected output, or high effort, producing a higher expected output. For a given level of effort, the agent can choose among all distributions over output that have the same mean, and all such distributions are equally costly. With such freedom to choose the distribution, only a linear relation can tie the principal's expected profit to the agent's expected compensation. Several other papers consider models where the contractible outcome variable combines effort with mean-zero additive noise, leading to linear contract (Laffont & Tirole (1986), McAfee & McMillan (1987) and Edmans & Gabaix (2011)). Recently, Caroll (2015) showed the optimality of linear contracts in a relatively general class of moral hazard environments. His model assumes risk-neutrality and limited liability, but no other functional form assumptions. In summary, these papers have argued for the optimality of linear contracts in moral hazard environments using max-min type criteria; whereas we provide a new

[6] See Holmstrom & Milgrom (1987), Hurwitz & Shapiro (1978), Diamond (1998), Chassang (2013), and Caroll (2015). Linear contracts are also natural in environments with synergies, including in Franchising arrangements (Lafontaine [1992], and renting space in malls (Gould, Pashigian & Prendergast [2005].

explanation for the common use of linear contracts, they can help resolve coordination issues in environments with networked synergies.

Moreover, this paper also makes a technical contribution in games with interdependent agents. The existing literature has shown the emergence of NP hard problems in such games – in a monopoly setting with two prices by Candogan et al (2012), and in a sequential play by Zhou & Chen (2015). Both of these papers involve Max-Cut problem, whereas in our paper we show the emergence of a new NP-hard problem, the densest k-subgraph problem.

The paper is organized as follows. Section 2 presents the model. The case of low contracting cost is studied in Section 3, whereas large contracting cost is studied in Section 4. Section 5 concludes. All proofs are deferred to Appendix A, and Tables presenting the performance of greedy algorithms are presented in Appendix B. The matlab programs used to generate the greedy algorithms are presented in Appendix C.

2 Model

We consider a three-stage game between one principal and a finite set of agents. In the first stage, the principal offers bilateral contracts. Each contract is a linear payment scheme. In the second stage, agents simultaneously decide whether to accept or reject their respective offers. In the third stage, agents exert effort and transfers are realized. Both effort, contract and network are assumed to be publicly observable. We study Subgame Perfect Nash Equilibriums.

There is a finite set $\mathcal{N} = \{1, 2, \cdots, n\}$ of agents. We let $x_i \in \mathbb{R}_+$ represent agent i's effort, and $\mathbf{x} = (x_i)_{i \in \mathcal{N}}$ an effort profile. We let the $n \times n$ matrix $\mathbf{G} = [g_{ij}]$, with $g_{ij} \in \mathbb{R}_+$, represent the network of interaction between agents, where $g_{ij} > 0$ when agent i is influenced by agent j and $g_{ij} = 0$ otherwise. By convention, $g_{ii} = 0$ for all $i \in \mathcal{N}$. Links can be either binary or weighted. Throughout the paper, we will speak about network $\mathbf{G}$ (implicitly assuming a number of agents equal to n). The network is undirected, i.e. $\mathbf{G}^T = \mathbf{G}$ where superscript ^{T} quotes for the transpose operator (we discuss directed networks in Remark 1 in Section 3). Symbol $\mathbf{1}$ represents the n-dimensional vector of ones, $\mathbf{I}$ the n-dimensional identity matrix, d_i the degree of consumer i, $\mathbf{d} = \mathbf{G}\mathbf{1}$ the profile of consumers' degrees. We let $\mu(\mathbf{G})$ represent the largest (real) eigenvalue of network $\mathbf{G}$.

In the absence of principal, agents play an effort game on a social network, exhibiting both positive local externalities and local synergies. We focus on linear quadratic utilities.[7] The utility that agent i derives from exerting effort

[7] The model can be generalized to any utility function generating the same first-order conditions in

x_i on network G is given by:

$$u(x_i, \mathbf{x}_{-i}) = a_i x_i - \frac{1}{2}x_i^2 + \delta \sum_{j \in \mathcal{N}} g_{ij} x_i x_j \tag{1}$$

where $a_i > 0$ represents the private preference of agent i, and $\mathbf{a} = (a_i)_{i \in \mathcal{N}}$ the profile of private preferences. The last term represents the utility that consumer i derives from neighbors' efforts. Parameter $\delta > 0$ implies positive externalities and local complementarities: incentives increase with neighbors' efforts.[8]

In this game, Nash equilibrium efforts are shaped by Bonacich centralities, which we formally define (see Bonacich [1987]). We let the n-dimensional square matrix $\mathbf{M} = (\mathbf{I} - \delta \mathbf{G})^{-1}$. The condition $\delta \mu(\mathbf{G}) < 1$, that we assume throughout the paper, guarantees $\mathbf{M} \geq 0$. We let the n-dimensional vector $\mathbf{b} = \mathbf{M1}$, with entry i called b_i, denote the vector of *Bonacich centralities* of the network weighted by parameter δ (we avoid references to network $\mathbf{G}$ and parameter δ for convenience). Then b_i is the number of paths from agent i to others, where the weight of a path of length k from agent i to agent j is δ^k. Similarly, we define the vector $\mathbf{b_a} = (b_{\mathbf{a},i})_{i \in \mathcal{N}}$, such that $\mathbf{b_a} = \mathbf{Ma}$ represents the weighted (by $\mathbf{a}$) Bonacich centrality. We define $b = \mathbf{1}^T \mathbf{b}$ (resp. $b_{\mathbf{a}} = \mathbf{1}^T \mathbf{b_a}$) as the sum of centralities (resp. centralities weighted by $\mathbf{a}$). The condition $\delta \mu(\mathbf{G}) < 1$ guarantees the existence of a unique Nash equilibrium effort profile in the absence of principal. In this equilibrium, any agent $i \in \mathcal{N}$ exerts an effort equal to her Bonacich centrality $b_{\mathbf{a},i}$ and obtains a utility level equal to $\frac{1}{2} b_{\mathbf{a},i}^2$.

A principal offers payment schemes in the aim of increasing aggregate effort. For instance, this objective is natural when effort is about education, protection, work, R&D, etc. The principal has a limited budget t to enhance efforts. Offers are excess-effort linear contract, rewarding increased effort with respect to effort level in the absence of principal's intervention.[9] Formally, agent i is offered a transfer function $t_i(x_i) = w_i(x_i - b_{\mathbf{a},i})$, with $w_i \in \mathbb{R}$. The variable w_i represents the return per unit of excess-effort; for convenience, we shall speak of w_i as the *unit return* throughout the paper. We suppose a costly enforcement environment. The cost of contracting with group $\mathcal{S}$ is given by function $C(\mathcal{S})$, which is assumed to be supermodular throughout the paper. This means that the contribution of an agent to the

effort, and generating positive externalities with respect to neighbors' effort. Indeed, the cardinal amount of externalities between utilities plays no role in the analysis.

[8] This linear-quadratic utility specification applied to network games in economics was introduced by Ballester, Calvò and Zenou (2006).

[9] From the principal's view, this affine contract is clearly better than a linear contract $w_i x_i$; indeed, with the above affine contract, the principal does not subsidize efforts lower than $b_{\mathbf{a},i}$.

cost is non decreasing with group enlargement. Formally, for all groups $\mathcal{S}, \mathcal{T}$ with $\mathcal{S} \subset \mathcal{T}$, for all $i \notin \mathcal{T}$, $C(\mathcal{T} \cup \{i\}) - C(\mathcal{T}) \geq C(\mathcal{S} \cup \{i\}) - C(\mathcal{S})$; for instance, when contracting cost depends on the sole number of contracts called $s = |\mathcal{S}|$, then $C(\mathcal{S}) = C(s)$, and supermodularity assumption boils down to linearity or convexity of the function $C(s)$. By convention, the principal suffers no cost by offering the neutral offer $w_i = 0$. The principal's offers are gathered in a n-dimensional vector $\mathbf{w} = (w_i)_{i \in \mathcal{N}}$, where a null entry means no offer. Hence, when the principal targets group $\mathcal{S}$, $w_i > 0$ if and only if $i \in \mathcal{S}$. For any group $\mathcal{T} \subset \mathcal{S}$, we let $\mathbf{w}_{-\mathcal{T}}$ represent the set of contracts in $\mathcal{S}$ but setting $w_i = 0$ for all $i \in \mathcal{T}$. For convenience, we may abuse the notation and write $\mathbf{w} = (w_i)_{i \in \mathcal{S}}$ (excluding neutral transfers in this latter notation) when there is no confusion.

In the third stage of the game, we call by $\mathbf{x}^*(\mathbf{w})$ the equilibrium effort associated with the set of accepted contracts $\mathbf{w}$. Profile $\mathbf{x}^*(\mathbf{w})$ takes into account both the variation in effort of the contracting agent and the induced variation in other efforts on the network. The equilibrium effort profile satisfies $\mathbf{x}^*(\mathbf{w}) = \mathbf{M}\mathbf{a}'$, with $a'_i = a_i + w_i$ for all $i \in N$. The condition $\delta\mu(\mathbf{G}) < 1$ still guarantees a unique Nash equilibrium effort profile for any accepted contract. Hence, by linear contracting in a world of linear interaction, there is no coordination concern in the third stage of the game. The Nash effort profile under any set $\mathcal{S}$ of accepted contracts, $\mathbf{w} = (w_j)_{j \in \mathcal{S}}$, is given by

$$\forall i \in \mathcal{N}, x_i^*(\mathbf{w}) = b_{\mathbf{a},i} + \sum_{j \in \mathcal{S}} m_{ij} w_j \tag{2}$$

That is, the excess effort of every agent, whether this agent contracts with the principal or not, is given by the sum of returns offered to agents under contract weighted by the number of (weighted) paths to them.

We turn to the principal's program. We let $\mathbf{w}^*$ denote the optimal excess-effort linear contract, and $\mathbf{x}^* = \mathbf{x}^*(\mathbf{w}^*)$ for convenience when there is no confusion. The contract $w_i = 0$ is said to be the default offer. Throughout the paper, we shall abuse language and speak about a contract when this is not the default offer. For convenience, let $\mathcal{S}(\mathbf{w}) = \{i \in \mathcal{N}, w_i \neq 0\}$ represent the set of contracts offered by the principal through payment scheme profile $\mathbf{w}$. Taking care that the budget constraint is binding at the optimum (otherwise the principal may use the saved budget to trade increased effort even more), the optimal contract $\mathbf{w}^*$ solves:

$$\max_{\mathbf{w} \in \mathbb{R}^n} \sum_{i=1}^{n} x_i^*(\mathbf{w}) - C(\mathcal{S}(\mathbf{w}))$$

$$\text{s.t.} \begin{cases} \sum\limits_{i\in\mathcal{S}} w_i\big(x_i^*(\mathbf{w}) - b_{\mathbf{a},i}\big) = t \\ \forall i \in \mathcal{S},\ u(x_i^*(\mathbf{w}), \mathbf{x}_{-i}^*(\mathbf{w})) + w_i > u(x_i^*(\mathbf{w}_{-i}), \mathbf{x}_{-i}^*(\mathbf{w}_{-i})), \\ \forall i \in \mathcal{S}, \forall \mathcal{T} \subset \mathcal{S},\ u(x_i^*(\mathbf{w}), \mathbf{x}_{-i}^*(\mathbf{w})) + w_i > u(x_i^*(\mathbf{w}_{-\mathcal{T}}), \mathbf{x}_{-i}^*(\mathbf{w}_{-\mathcal{T}})) \end{cases}$$

The first constraint is the standard budget constraint by which the sum of rewards is equal to the available resource. The set of individual participation constraints expresses that any agent receiving an offer accepts the contract when other agents accept; note that other agents' effort varies when the agent rejects the offer. The last set of constraints deters multiple equilibriums, by guaranteeing agent's participation for all subgroups of accepted contracts in the targeted set.

Actually, the principal needs not take an explicit account of both participation constraints and multiple equilibrium issue. Indeed, it is worth emphasizing that participation is not an issue when the principal offers non-negative returns. This is because all agents increase their effort level (i.e., contracting agents and outsiders) by complementarities and positive externalities. This entails an increase of the utilities of contracting agents, and adding a positive transfer increases even more the utility from acceptance. This latter property implies:

Observation 1. *For any set of contracts $(w_i)_{i\in\mathcal{S}}$ such that $w_i \geq 0$ for all $i \in \mathcal{S}$, all agents accept their offer irrespective of the decision of the other agents.*

As will be seen below, the optimal returns are positive for any targeted group. This implies that (i) individual participation constraints are satisfied at the optimum, and (ii) there is no strict subset of the targeted group that constitutes another equilibrium of the participation game. Therefore, principal's program boils down to the program of the maximization of the principal's objective under the sole budget constraint. Hence, throughout the paper, we will focus on the simpler program:

$$\max_{\substack{\mathbf{w}\in\mathbb{R}^n \\ \text{s.t. } \sum\limits_{i\in\mathcal{S}} w_i\left(x_i^*(\mathbf{w}) - b_{\mathbf{a},i}\right) = t}} \quad \sum_{i=1}^{n} x_i^*(\mathbf{w}) - C(\mathcal{S}(\mathbf{w}))$$

3 Low contracting costs

In this section, we assume that there is no contracting cost; i.e. $C(\mathcal{S}) = 0$ for all sets $\mathcal{S} \in \mathcal{N}$. The objective function in the principal's program is thus equal to the aggregate effort.

There are at least two questions. First, are optimal returns higher for central agents? On the one hand, the principal may put more resource on central agents to exploit their larger influential power. On the other hand, central agents are themselves more responsive to returns, which may lead to overstretch the budget. Second, is there targeting? I.e., is it optimal for the principal to contract with a strict subset of the society, allocating full budget to those agents being more productive with respect to the principal's objective?

The next proposition gives a precise answer to both questions:

Proposition 1. *Under null contracting cost, there is a unique equilibrium of the two-stage game, that satisfies:*

- The principal contracts with all agents,

- The optimal return $w_i = \sqrt{\frac{t}{b}}$ *for all* $i \in \mathcal{N}$*,*

- The aggregate effort change is equal to $\sqrt{tb}$*.*

Proposition 1 calls for several remarks. First, all agents receive an offer, i.e. there is no targeting at play.

Second, since optimal returns are positive, by Observation 1 there is no coordination concern in the participation game. Each agent accepts own offer even if others reject theirs (indeed, with rewards, all efforts increase, thus all utilities increase). That is, there is no strict subgroup of the set of contracting agents that constitutes another equilibrium.

Third, agents' positions do not affect the optimal return per unit of excess-effort: network influence is exactly counter-balanced by budget effect.[10] This result echoes Candogan et al (2012) and Bloch & Quérou (2013). However this model differs in that agents exert effort in absence of principal whereas consumers do not consume without the firm. In particular, heterogeneity does not affect the per-unit rate of effort here whereas it shapes prices under monopoly pricing. And thus, both transfers and aggregate effort variation are not related to preferences.

Fourth, the variation of aggregate effort being equal to $x^* - b_{\mathbf{a}} = \sqrt{tb}$, the networks maximizing the impact of the principal's intervention are also those maximizing the sum of centralities (whatever the magnitude of the budget). One consequence is that, considering two networks such that all the links of the first are included in the second, the impact of the principal's intervention is greater on the denser network.

[10] Still, agent i's payment can be computed as $t_i^* = \frac{b_i}{b} \cdot t$; i.e., the received transfer is proportional to relative centrality.

Remark 1. *The assumption that* $\mathbf{G}^T = \mathbf{G}$ *is crucial to establish Proposition 1. Otherwise unit returns become heterogeneous and targeting is even possible. For example, it is optimal to target only the peripheral agents in a* 3*-agent directed star with two links, both originated from peripherals (meaning that the central receives all synergies, and is not providing any synergy).*[11]

4 Costly contracting

In this section, we study large contracting costs. E.g., costs could simply be contract costing, but they can also be generated by enforcement consideration (like monitoring workers' effort in firms), there may be a large number of agents or limited budget. The objective function of the principal's program is then given by the aggregate effort minus the aggregate contracting cost.

Under sufficiently large contracting cost, by limited budget the principal cannot contract with the whole society (feasibility constraint). That is, the principal raises contracts with a subgroup $S = \{1, 2, \cdots, s\}$ of size $s < n$ (agents' labels are chosen arbitrarily without loss of generality). Generally speaking, the principal finds the best target for each given number of contracts, and then selects the best number of contracts.

In what follows, we first study the general case, establishing notably that unit returns are always positive, and characterizing the performance of any group. Then, we explore NP-hardness, and finally we illustrate optimal targeting on specific network structures.

4.1 General case

The principal may be inclined to contract with central agents to exploit their influential power, but the budget constraint may qualify this tendency. Under budget constraint, the principal essentially targets a group of agents in order to maximize the performance of each contractor per dollar invested, that is the ratio of the variation of aggregate effort induced by contracting with the agent over the amount of resource transferred to this agent (what we might call agent i's productivity). The centrality of an agent has ambiguous effect on its productivity: high centrality induces a great influence on others, but it also induces a large transfer to that agent because that agent is highly incited to enhance effort through contract. In total, the productivity of an agent can be non monotonic with centralities.

[11] For the general case of directed networks, we define $\mathbf{M} = (\mathbf{I} - \delta\mathbf{G})^{-1}, \mathbf{M}^T = (\mathbf{I} - \delta\mathbf{G}^T)^{-1}$, $\bar{\mathbf{M}} = \left(\mathbf{I} - \delta\left(\frac{\mathbf{G}^T+\mathbf{G}}{2}\right)\right)^{-1}$. We obtain after development $\mathbf{w}^* = \frac{2}{\lambda}\left(\mathbf{1} + \left(\frac{\mathbf{G}^T-\mathbf{G}}{2}\right)\bar{\mathbf{b}}\right)$, where $\bar{\mathbf{b}} = \bar{\mathbf{M}}\mathbf{1}$ represents the Bonacich centrality of the average interaction matrix, and $\lambda = \sqrt{\frac{||(\mathbf{I}+\mathbf{M}^{-T}\mathbf{M})^{-1}\mathbf{1}||_{\mathbf{M}}}{t}}$.

Optimal targeting takes this trade off into account. Let us consider contracting with group $\mathcal{S}$, of size s. We let the $s \times s$ matrix $\mathbf{M}_{[\mathcal{S}]}$ represent the sub-matrix of $\mathbf{M}$ restricted to agents in $\mathcal{S}$. Let $\mathbf{b}_{[\mathcal{S}]} = (b_i)_{i \in \mathcal{S}}$ with b_i is the un-weighted Bonacich centrality on network $\mathbf{G}$. The next theorem summarizes our findings:

Theorem 1. *When the principal contracts with group $\mathcal{S}$, there is no participation issue, meaning that $w_i^* > 0$ for all $i \in \mathcal{S}$ and*

$$\mathbf{w}^* = \mathbf{M}_{[\mathcal{S}]}^{-1} \mathbf{b}_{[\mathcal{S}]}$$

The optimal objective is given by

$$F(\mathcal{S}) = \sqrt{t \; \mathbf{b}_{[\mathcal{S}]}^T \mathbf{M}_{[\mathcal{S}]}^{-1} \mathbf{b}_{[\mathcal{S}]}} - C(\mathcal{S}) \tag{3}$$

The optimal target $\mathcal{S}^*$ maximizes $F(\mathcal{S})$ over all possible groups.

The first message behind Theorem 1 is that the second-stage participation game suffers no coordination concern, by Observation 1. That is, there is no strict subgroup of the targeted set that constitutes a proper equilibrium of the participation game, because any non member of that subgroup finds it better off to accept the offer.

The second message behind Theorem 1 is that optimal returns are in general heterogeneous. It is important to understand why we get heterogeneous returns in the targeted group, whereas bilateral interaction is undirected. This result stems from the fact that heterogeneous number of connections with external members induce asymmetric interactions inside the group; otherwise necessarily the optimal returns would be homogeneous. To see this, we write the system of interaction between members of the targeted group as a function of their interaction with outsiders as follows. Let $\mathbf{I}_q$ be the identity matrix of size $q = s, n-s$, $\mathbf{G}_{[\mathcal{S}]}$ the submatrix of $\mathbf{G}$ restricted to agents in $\mathcal{S}$, and $\tilde{\mathbf{G}}$ the $s \times n - s$ submatrix where g_{ij} represents the link between agent $i \in \mathcal{S}$ and agent $j \in \mathcal{N} \setminus \mathcal{S}$:

$$(\mathbf{I}_s - \delta \mathbf{G}_{[\mathcal{S}]})\mathbf{x}_{[\mathcal{S}]} = \mathbf{1} + \mathbf{w} + \delta \tilde{\mathbf{G}} (\underbrace{\mathbf{I}_{n-s} + \delta \mathbf{G}_{[\mathcal{N} \setminus \mathcal{S}]} + \delta \tilde{\mathbf{G}}^T \mathbf{x}_{[\mathcal{S}]}}_{= \mathbf{x}_{[\mathcal{N} \setminus \mathcal{S}]}})$$

That is, the system of bilateral interaction among members of the set $\mathcal{S}$ is thus given by $(\mathbf{I}_s - \delta \mathbf{H}_{[\mathcal{S}]})\mathbf{x}_{[\mathcal{S}]} = \mathbf{k}$, where ($\mathbf{k}$ is an idiosyncratic constant, and) matrix $\mathbf{H} = \mathbf{G}_{[\mathcal{S}]} + \delta \tilde{\mathbf{G}} \tilde{\mathbf{G}}^T$. We observe that $\mathbf{H}^T = \mathbf{H}$, but now the diagonal is generically not null and heterogeneous; Letting $d_i^{\mathcal{N} \setminus \mathcal{S}} = |\{j \in \mathcal{N} \setminus \mathcal{S}, g_{ij} = 1\}|$ represent the number of neighbors from the complementary set $\mathcal{N} \setminus \mathcal{S}$, we get $h_{ii} = d_i^{\mathcal{N}\setminus\mathcal{S}}$. This heterogeneity of the diagonal entries of the matrix of interaction creates asymmetric interaction (the sensitiveness of agent i to agent j's move is not equal to the reciprocal sensitiveness, even if $h_{ij} = h_{ji}$). One immediate implication is that, if all members have same number of neighbors among the outsiders, then the bilateral interactions among targeted agents are no longer asymmetric, and thus:

Corollary 1. *Suppose that the targeting group is the set $\mathcal{S}$. If $d_i^{\mathcal{N}\setminus\mathcal{S}}$ is the same for all $i \in \mathcal{S}$, then optimal returns are homogeneous across agents.*

The third message behind Theorem 1 is that the level of asymmetry generated by the presence of outsiders is not sufficient to target a strict subgroup of the set $\mathcal{S}$ (what would happen if the optimal return w_i^* was negative).

Remark 2. *By Corollary 3 in Johnson [1982, p. 202], any principal submatrix of an invert M-matrix is an invert M-matrix; hence $\mathbf{M}_{[\mathcal{S}]}$ is an invert M-matrix. Therefore, matrix $\mathbf{M}_{[\mathcal{S}]}^{-1}$ is an M-matrix, which means that it is a positive diagonal, negative off-diagonal entries and positive eigenvalues. This means that the quantity $\sqrt{\mathbf{B}_{[\mathcal{S}]}^T \mathbf{M}_{[\mathcal{S}]}^{-1} \mathbf{B}_{[\mathcal{S}]}} = \|\mathbf{B}_{[\mathcal{S}]}\|_{\mathbf{M}_{[\mathcal{S}]}^{-1}}$ (this is a norm).*

Remark 3 (Key-player)**.** *Define the intercentrality index of agent i in network $\mathbf{G}$ as $c_i = \frac{b_i^2}{m_{ii}}$. Under very large contracting cost, where the principal may optimally contract with a single agent, the optimal target maximizes the intercentrality index. The intercentrality index is familiar to key-player analysis (Ballester et al 2006). We see here that linear contracting has same qualitative effect on optimal targeting as dropping an agent. In particular, by Theorem 1, the optimal return is given by $w_i^* = \frac{b_i}{m_{ii}}$. This ratio receives a simple interpretation. It represents the aggregate effort change following the increase of a unit change of agent i's effort (see Belhaj and Deroïan 2018).*

Remark 4 (Homogeneous returns in the targeted group)**.** *In many circumstances, the principal cannot discriminate among agents. In that case, every agent in the targeted group receives the same positive return (there is no coordination issue in the participation game). The optimal target then maximizes a specific group-index. Letting $\mathbf{1}_s$ denote the vector of ones of size s, and index $I(\mathcal{S}) = \frac{\left(\sum_{k\in\mathcal{S}} b_k\right)^2}{\mathbf{1}_s^T \mathbf{M}_{[\mathcal{S}]} \mathbf{1}_s}$:*

Corollary 2. *When the principal offers the same contract to all agents in $\mathcal{S}$, the group performance is proportional to the index $I(\mathcal{S})$. Hence, for all contracting costs, the optimal group maximizes this index over all groups of same size.*

Not contracting with central agents. As said before, centrality has an ambiguous effect on optimal targeting. On the one hand, agents with large centralities have a huge influence on others, i.e. centrality plays as a social multiplier effect allowing the principal to save on budget; on the other hand, a central agent is itself resource consuming: first by responding to own contract, and second by responding more to the other contracts. This force can lead the principal to separate agents in the target. To illustrate, Figure 1 shows on a five-agent line network that this trade off between centrality and budget effect can lead to an optimal target that excludes central agents.

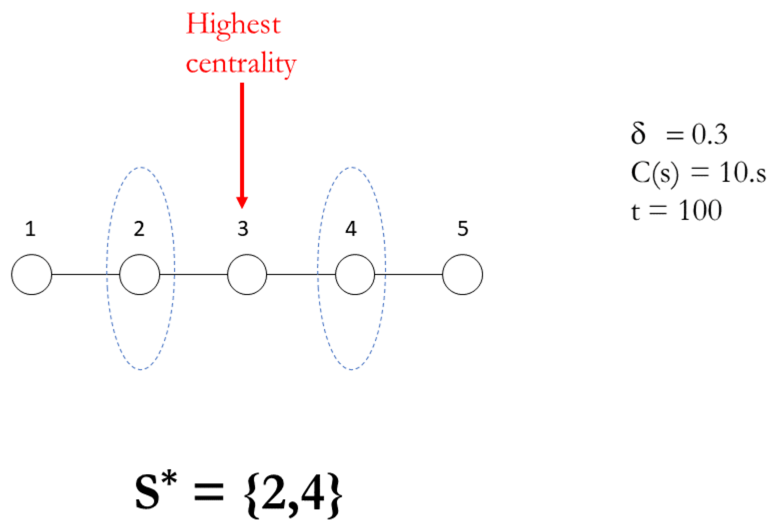

Figure 1: Optimal targeting on the five-agent line network; the agent with highest centrality is not selected.

Non continuity of the optimal group size with respect to contracting cost. In the problem, increasing the cost function induces a clear restriction in the optimal number of contracts everything equal. However, as cost increases, the reduction in size may jump from more than one unit. To illustrate, we consider the following example:

Example 1. *We assume contracting cost is a linear function of the number of contract; i.e.* $C(\mathcal{S}) = cs$ *(*$c \geq 0$*), and we consider the network depicted in Figure 2. In this network, there are three classes of agents: the central agent*

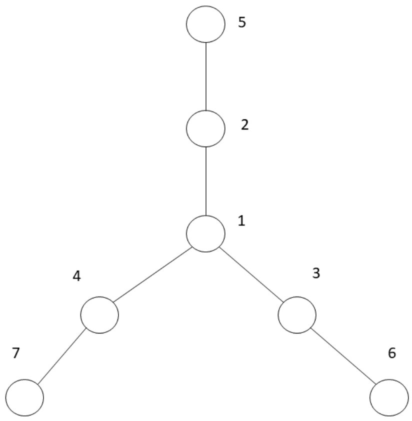

Figure 2: A 7-agent network.

(labeled as agent 1*), three intermediaries (labeled* 2, 3, 4*) and three peripheral agents (labeled* 5, 6, 7*). Table 1 presents the optimal groups as a function of the level of contracting cost (there may be multiple optimal groups due to symmetric positions; for convenience we only label one representative optimal group when multiplicity arises): In this example, there is no optimal group*

c	$[0, 1.6[$	$[1.6, 1.7[$	$[1.7, 1.8[$	$[1.8, 1.9[$	$[1.9, 3.9[$	$[3.9, 4[$	$[4, +\infty[$
$\mathcal{S}^*$	$\mathcal{N}$	$\{1, 2, 3, 4, 5, 6\}$	$\{1, 2, 3, 4, 5\}$	$\{1, 2, 3, 4\}$	$\{2, 3, 4\}$	$\{1\}$	$\emptyset$

Table 1: Optimal groups as a function of contracting cost for $\delta = 0.2$.

of size 2. The reason is inherent to the composition effect arising when the optimal group size changes, that lead to giving a premium to group of larger size.[12] *Here, at the threshold cost where the performances of the respective groups* $\{1\}$ *and* $\{1, 2\}$ *are the same, the size-*3 *group* $\{2, 3, 4\}$ *gets a higher value, and this group is therefore optimal.*[13] *This forbids any group of size* 2 *from being optimal for all costs. This jump in the group size results from the following composition effect: the best group of size* 2 *is* $\{1, 2\}$*, whereas the best group of size* 3 *is* $\{2, 3, 4\}$*; thus the central agent is excluded and replaced by an intermediary agent.*

[12]The objective function being submodular, the marginal individual contribution to group value is decreasing with group enlargement. This implies that, absent any composition effect, group enlargement is monotonic with cost. Therefore, when applying the greedy approximation, and considering the induced (nearly) optimal group size as a function of contracting cost, there is *continuity* in the sense that optimal group size is reduced one unit by one unit as the unit contracting cost is continuously increased.

[13]The group $\{2, 3, 4\}$ is optimal below the threshold cost where both groups $\{1\}$ and $\{2, 3, 4\}$ get the same value, and above the threshold cost that equates its performance with that of the group $\{1, 2, 3, 4\}$.

4.2 NP-hardness

In this subsection we establish that the principal's problem is NP-hard. The difficulty of this problem can be seen when examining the case of low intensity of interaction. Then, we develop related algorithm considerations. To finish, we explore a version of the principal's problem with the additional constraint that the number of contract is bounded from above. This latter program is also NP-hard and we can make a precise correspondence with the densest k subgraph problem (which is known to be NP-hard).

We need the following notation. For any group $\mathcal{S} \subset \mathcal{N}$ and any agent $i \in \mathcal{S}$, we let $d_i^{\mathcal{S}}$ (resp. $d_i^{\mathcal{N}\setminus\mathcal{S}}$) represent the number of linked agents in the set $\mathcal{S}$ (resp. in the set $\mathcal{N} \setminus \mathcal{S}$). Then $d_i = d_i^{\mathcal{S}} + d_i^{\mathcal{S},\mathcal{N}\setminus\mathcal{S}}$. We let $L_{\mathcal{S}}$ represent the number of links among members of $\mathcal{S}$, and we let $L_{\mathcal{S},\mathcal{N}\setminus\mathcal{S}}$ be the number of links between $\mathcal{S}$ and $\mathcal{N} \setminus \mathcal{S}$ (i.e. the number of cross links between the two sets). Then $\sum_{i\in\mathcal{S}} d_i^{\mathcal{S}} = 2L_{\mathcal{S}}$ and $\sum_{i\in\mathcal{S}} d_i^{\mathcal{N}\setminus\mathcal{S}} = L_{\mathcal{S},\mathcal{N}\setminus\mathcal{S}}$. Under sufficiently low intensity of interaction, the principal's objective at optimum is given by

$$\max_{\mathcal{S}} F(\mathcal{S}) = \sqrt{t(s + 2\delta(L_{\mathcal{S}} + L_{\mathcal{S},\mathcal{N}\setminus\mathcal{S}}))} - C(\mathcal{S})$$

(see the proof of Theorem 2). When parameter δ tends to zero, it is better to have larger size for low contracting cost c whereas, under sufficiently large contracting cost, it is optimal to target a subgroup. The formulation clearly expresses that large contracting costs call for targeting. Moreover, for a given size s, the best group of given size s maximizes the sum of internal links plus twice the sum of cross-links. In a word, effort variation of agents in $N \setminus \mathcal{S}$ comes from cross links, and effort variation of agents in $\mathcal{S}$ comes from both internal links and cross links because when agent i is got a reward, increased effort takes into account his synergies with agents in $N \setminus \mathcal{S}$. For instance, on the circle, the best group of size s is $\mathcal{S} = \{1, 3, 5, \cdots, 2s+1\}$; this is because this group maximizes the number of cross links, and by grouping agents we loose cross links but we do not gain twice their number of internal links. On the star, the best group of given size s contains the central.

It is well known from mathematics on discrete optimization that maximizing a submodular function on a discrete set space leads to NP-hardness if the function is submodular.[14] The next lemma is informative in that respect:

Lemma 1. *The function $F(\mathcal{S}) = \sqrt{t(s + 2\delta(L_{\mathcal{S}} + L_{\mathcal{S},\mathcal{N}\setminus\mathcal{S}}))} - C(\mathcal{S})$ is submodular for any supermodular function $C(\mathcal{S})$.*

[14] One famous example in the context of graphs is the Maxcut problem. For general insights, see Lovász (1983), or more recently Feige, Mirrokni and Vondraák (2011).

Indeed, the marginal contribution of an agent to the sum of both internal and cross links can only decrease when the group enlarges: $F(\{i\} \cup \mathcal{S}) - F(\mathcal{S}) = d_i^{\mathcal{N} \setminus \mathcal{S}}$. That is, agent i brings internal links but all of them are lost because they are no more considered as interlinks, but i also brings new cross links to agents out of $\mathcal{S}$. In the end, the contribution of agent i is equal to the number of links between i and agents out of the set $\mathcal{S}$. Hence, as $\mathcal{S}$ is being enlarged, the set of links between i and agents out of $\mathcal{S}$ can only be decreased. This is illustrated by Figure 3. Note that if the cost function only

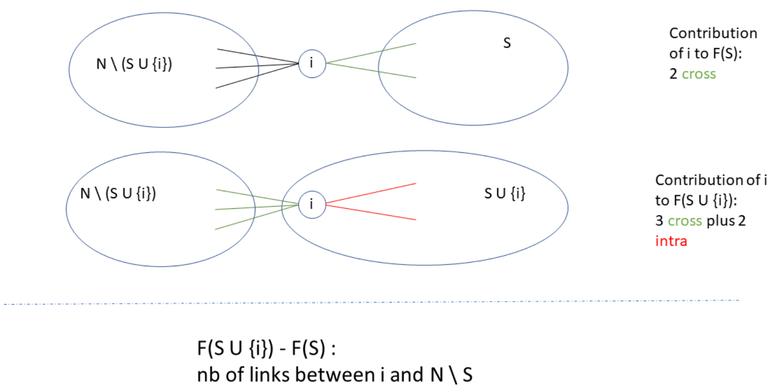

Figure 3: Submodularity of the objective function when δ is close to zero.

depends on the number of contracts, supermodularity of the cost function boils down to convexity (including linearity). By Lemma 1, NP-hardness is a potential issue. To confirm this intuition, we consider the (sub)problem of finding the optimal group among those with size equal to k.[15] The next lemma is key to establish NP-hardness formally:

Lemma 2. *The problem of finding the group maximizing the objective function $L_{\mathcal{S}} + L_{\mathcal{S},\mathcal{N}\setminus\mathcal{S}}$ among all groups of size equal to k is isomorphic with a k-densest subgraph problem of size $n - s$, which is NP-hard.*

To see why, the best target maximizes over all such groups $\mathcal{S}$ the sum of the number of links inside the group $\mathcal{S}$ and the number of links at the border,

[15] In many economic situations, there are upper bounds to the number of possible contracts, irrespective of the contracting cost. For instance, such a rationing can hold in the number of grants of a CCP program or of R&D subventions. When the principal's program contains an additional constraint that the number of contracts cannot exceed a given number $k < n$, there are complexity gains. In particular, when contracting cost is sufficiently low, the objective function is monotone (increasing) in size, and thus maximizing over groups reduces to maximizing over groups of maximal size.

which resorts to minimizing the number of links in the complementary group $\mathcal{N} \setminus \mathcal{S}$ in network $\mathbf{G}$; and equivalently, this corresponds to maximizing the number of links of group $\mathcal{N} \setminus \mathcal{S}$ in the complementary network[16], as illustrated by Figure 4. This problem is known as the densest k-subgraph problem among groups of size s, it is a NP-hard problem.[17]

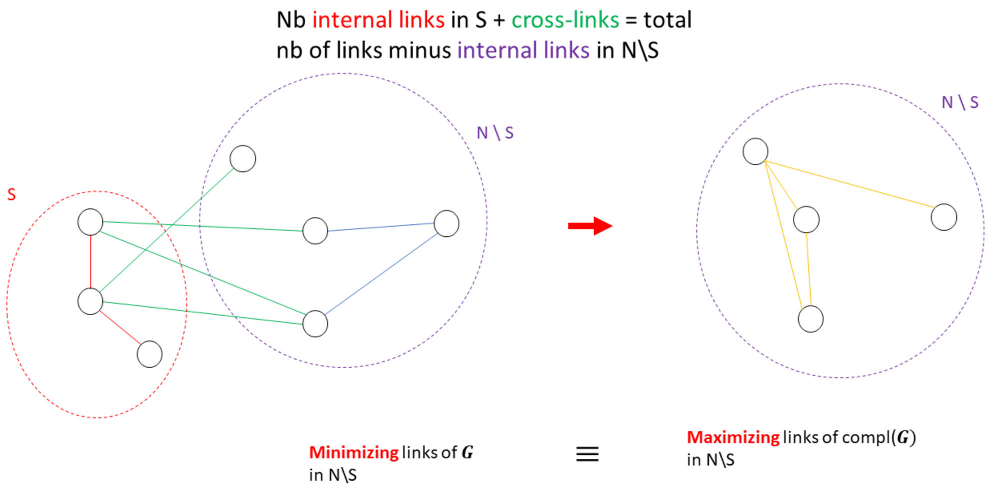

Figure 4: NP-hardness.

Going back to the principal's problem of optimal targeting under contracting cost, the next theorem confirms NP-hardness:

Theorem 2. *Under sufficiently low intensity of interaction, the principal's objective at optimum is given by*

$$\max_{\mathcal{S}} F(\mathcal{S}) = \sqrt{t(s + 2\delta(L_{\mathcal{S}} + L_{\mathcal{S},\mathcal{N}\setminus\mathcal{S}}))} - C(\mathcal{S})$$

This problem is NP-hard for all supermodular cost functions.

By the above theorem, absent any contracting cost, the principal finds it always better off to enlarge the targeted set (the objective function being monotone). Now, when the cost function is substantial, the objective function is both non-monotone and submodular, which explains that the problem becomes NP-hard.

[16] Letting $\mathbf{J}$ represent the n-square matrix of ones, the complementary network, that we call $\bar{\mathbf{G}} = \mathbf{J} - \mathbf{I} - \mathbf{G}$, is such that all active links in $\mathbf{G}$ are inactive in $\bar{\mathbf{G}}$ and vice-versa.

[17] See Feige, Kortsarz & Peleg [2001]); see also Faragó and Mojaveri (2019) for a recent survey about densest subgraph problems.

Remark 5. *If the cost function $C()$ is submodular (e.g., when the cost function depends on the number of contracts, this means that the cost function is concave), the objective function can loose submodularity. And if the objective function becomes supermodular, the problem is no longer NP-hard (by Grötschel-Lovász-Schrijver Theorem [1981]). This shows that the shape of cost function affects the level of complexity of the principal's program. At the extreme, if the cost function is reduced to a sunk cost and null marginal cost, the optimal group is trivially given by either the empty group or the whole society.*

Algorithmic considerations. Although the problem faced by the principal is NP-hard, it can be useful to know whether algorithm generate good approximations of the solution. By submodularity of the objective function (from Lemma 1) guaranties that the greedy algorithm performs rather well for problems with fixed group size (no less than $1 - 1/e \sim 63$ percent of maximum efficiency for large societies - see details in Ballester et al (2009, Appendix A, Proposition 10). However, for the unconstrained problem, best algorithms are rather combinatorial algorithms, that give a lower bound to the ratio of inefficiency of one half.

To illustrate, we performed two sets of simulations of the greedy algorithm on Erdös-Renyi random networks, one for the unconstrained problem with positive linear contracting cost on random networks, and one for the constrained problem with null contracting cost and various values of the upper bound on the number of contracts.

The greedy algorithm works as follows. In step 1, it determines the best singleton; in step 2 it determines the best pair containing the best singleton; in step 3 it determines the best triplet containing the best pair; etc, until reaching the best group. This algorithm therefore converges in no more than n steps. We implemented this algorithm with the following initial parameters: we fixed $t = 1$ in all simulations. For network size, we initiate $n = 16$ (combinatorial concerns become very important for larger values of n); for the uniform probability p of link existence, $p = 0.5$.[18] for the intensity of interaction, we chose $\delta \in \{0.01, 0.06\}$. For the unconstrained problem, we tested $c \in \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6\}$, while for the constrained problem we put $c = 0$ and we tested $k \in \{3, 5, 8, 10\}$. In each scenario, we performed a simulation generating 100 random networks. For each simulation, we found the optimal group by searching through all possible groups of size (with $n = 16$, the program searches through $2^{10} - 1 = 65535$ subsets of agents) and obtained its performance. Then, we collected the approximation using the greedy algorithm. Finally, we computed the relative error of approximation

[18] We also tested alternative values of $p \in \{0.25, 0.75\}$. They do not qualitatively affect results.

of the greedy algorithm in percentage of the optimal performance. Table 3 and Table 4 (presented in Appendix B) display the results of our numerical simulations for respectively the unconstrained and the constrained problems. The numbers in both tables are the average relative error of approximation (in percentage) over the 100 networks in each scenario. Roughly speaking, the greedy algorithm performed very well for these values in general, the average relative error of approximation being less than 1 percent in any case.[19]

4.3 Specific network structures

In this subsection, we illustrate the impact of network structure on optimal targeting on two polar network structures in terms of the distribution of centralities: the star network and regular networks. We set $C(\mathcal{S}) = cs$.

The star network. In the star network, the trade off between centrality and budget constraint effect is extreme. The following corollary shows that the centrality effect systematically dominates the budget effect:

Corollary 3. *In the star network, the optimal target always contains the central agent.*

This result is not trivial. For instance, in the case of very large cost the principal contracts with a single agent. The previous subsection has shown that the principal contracts with the agent having maximal intercentrality index $\frac{b_i^2}{m_{ii}}$. Interestingly, this ratio is favorable to peripherals on the star network (of at least four players) for large intensity of interaction. Yet, Corollary 3 shows that the balance between this effect and centrality is favorable to centrality, i.e. $\frac{b_i}{m_{ii}} \times b_i$ is always favorable to the central agent. Table 2 presents optimal group sizes on the star network for various parameter values. Like the complete network case, the optimal group size is decreasing in δ.

δ	0.09	0.12	0.15	0.18	0.21	0.24	0.27	0.3
s^*	10	10	10	10	10	5	1	1

Table 2: Optimal group size for the Star network for $n = 10, t = 1, c = 0.1$ and for various values of δ.

Regular networks. On regular networks, all agents have same degrees and thus same Bonacich centralities. Therefore, when the principal contracts with any group $\mathcal{S}$ on a regular network, the optimal objective, gross of contracting cost, is an increasing function of $F_0(\mathcal{S}) = \mathbf{1}_s^T \mathbf{M}_{[\mathcal{S}]}^{-1} \mathbf{1}_s$. In what follows, we

[19] The performance of the algorithms should be lower for larger network size.

study function $F_0()$, and then we derive properties of optimal targeting on regular networks.

For two groups $\mathcal{S}, \mathcal{S}'$ of equal size, we would like to compare $F_0(\mathcal{S})$ with $F_0(\mathcal{S}')$. Setting $\mathbf{V} = \mathbf{M}_{[\mathcal{S}]}^{-1}$ for convenience[20], we define the notion of moderate intensity of interaction:

Definition 1. *The intensity of interaction δ is moderate whenever, for every group $\mathcal{S}$, $\mathbf{V} = \mathbf{M}_{[\mathcal{S}]}^{-1}$ is diagonal dominant, i.e. $v_{ii} \geq \sum_{j \neq i} |v_{ij}|$.*

In words, under moderate intensity of interaction, the inverse of the M-matrix associated with any subgroup is diagonal dominant. By contrast, under very high intensity of interaction, the invert of an M-matrix may not be diagonal dominant in general. When the intensity of interaction is moderate, the respective performances of groups of same size can be unambiguously compared when their associated M-matrices are themselves ranked[21]:

Lemma 3. *Assume moderate intensity of interaction and consider two groups $\mathcal{S}, \mathcal{S}'$. If $\mathbf{M}_{[\mathcal{S}]} < \mathbf{M}_{[\mathcal{S}']}$, then $F_0(\mathcal{S}) > F_0(\mathcal{S}')$.*

By Lemma 3, and given that the entry m_{ij} represents a (weighted) number of paths between agents i and j, under moderate intensity of interaction the principal should, as far as possible, target groups minimizing the number of paths between the members of the group on regular networks.[22] Hence, optimal targeting among groups of size equal to two is easily identified. Indeed, on regular networks, $\mathbf{I} - \delta\mathbf{G}$ is strictly diagonal dominant. Hence, $m_{ii} > m_{ij}$ for all i, j. This implies, for $s = 2$, that $\mathbf{M}_{[\mathcal{S}]}^{-1}$ is itself diagonal dominant. This induces that, *in a regular network, a principal raising two contracts should always maximize the distance between the two contracting agents.* We performed simulations to test whether the diagonal dominance of the matrix $\mathbf{I} - \delta\mathbf{G}$ guarantees a moderate intensity of interaction: we found no counter-example. This means that in all regular networks of our tests, and whatever the group size, the principal should maximize the distance between

[20] M-matrices have positive principal minors, meaning that the diagonal entries of matrix $\mathbf{V}$ are positive.

[21] Comparing groups with minimal associated M-matrices is still an open issue.

[22] Interestingly, ultrametric matrices are inverse M-matrices whose inverse matrix is diagonal dominant (see Martinez, Michon, San Martin [1994]) - precisely, the inverse of a strictly ultrametric matrix is a strictly (row and column) diagonally dominant Stieltjes matrix, i.e., a nonsingular symmetric M-matrix. Then, ultrametricity induces a *moderate intensity of interaction.* To be more precise, a matrix $\mathbf{M} = [m_{ij}]$ in $\mathcal{R}^{n,n}$ is a strictly ultrametric matrix if:

(i) $\mathbf{M}$ is symmetric and has nonnegative entries;

(ii) $m_{ij} \geq \min(m_{ik}, m_{kj})$ for all $i, j, k \in \mathcal{N}$;

(iii) $m_{ii} \geq \max(m_{ik} : k \in \mathcal{N} \setminus \{i\})$ for all $i \in \mathcal{N}$,

where, if $n = 1$, (iii) is interpreted as $m_{11} > 0$.

By requirement (ii), ultrametric matrices imply networks of diameter 2, which is strongly restrictive for our study.

all members of the group as far as possible, given the constraints imposed by the network structure.[23]

Is there a maximal separation principle beyond the case $s = 2$? Figure 5 illustrates that this is not always the case. In this example with $s = 4$,

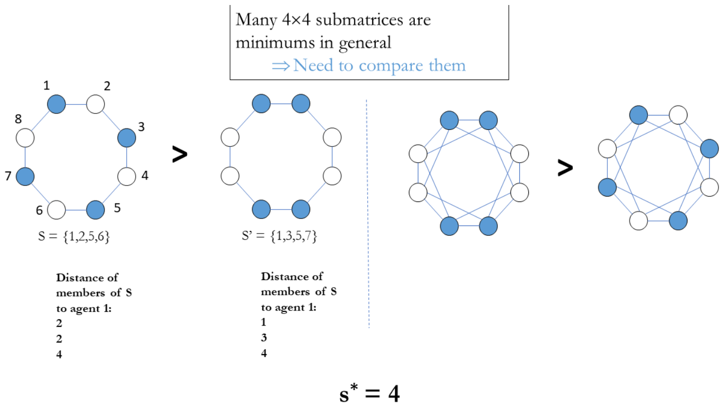

Figure 5: Optimal targeting on regular networks.

the principal has to trade between vectors of paths that cannot be ordered. For instance, in the wheel case, separating agents as in the left configuration leads to a distance from any targeted agent to the 3 others consisting in the vector $(2, 2, 4)$ (agent 1 is at distance 2 from 3, 2 from 7, 4 from 5); in the other configuration, the profile of distances to other members of the group is $(1, 3, 4)$. The comparison leads to select the left configuration that separates agents. However, in the regular network presented on the same figure, grouping agents by pairs and separating pairs is optimal.

5 Conclusion

In many economic situations, organizations trade agents' effort against transfers where synergies not only exist between contracting agents, but also between contracting and non-contracting agents. In this paper, we have considered a prevalent aspect of such situations, which is that contracting often involves large administration, monitoring and enforcement costs. In order to examine this, we built a principal-agent model where the principal has a

[23]These numerical computations raise a conjecture: is it the case that a symmetric invert M-matrix $\mathbf{M}$ such that $m_{ii} > m_{ij}$ for all i, j satisfies that $\mathbf{M}^{-1}$ is diagonal dominant? Remark that, when $\mathbf{A}$ is a non-singular diagonally dominant M-matrix, and $\mathbf{M} = \mathbf{A}^{-1}$, then $m_{ii} > m_{ij}$ for all i, j (Meltzer 1951), but the converse not true (i.e., a nonnegative matrix with $m_{ii} > m_{ij}$ for all i, j may not imply that its inverse is diagonal dominant).

limited budget and offers costly linear bilateral payment schemes in order to increase the sum of agents' effort. Our analysis provides three main insights. First, for all contracting costs, the optimal unit returns offered to every targeted agent are positive and generically heterogeneous. This heterogeneity is due to the presence of outsiders, who create asymmetric interaction between contracting agents. Second, when contracting costs are low, it is optimal to contract with everyone and optimal unit returns are identical for all agents. Third, when contracting costs are sufficiently high, it becomes optimal to target a subset of agents, and optimal targeting can lead to NP-hard problems. Overall, the optimal targeting problem involves a trade-off between centrality and budget spending — central agents are influential, but are also more budget-consuming. These considerations can lead the principal to not target central agents.

To conclude, we discuss some ideas for future research. Although our analysis highlights the complexity of solving optimal targeting problems, yet a full characterization remains open for future work. Besides contracting costs, there are also other aspects of such environment that need further investigation. Firstly, one could explore alternative objectives for the principal. In certain applications, the principal may care about other things than maximizing the sum of agents' effort. Secondly, one could consider endogenous networks. It is still unclear how network formation would react to such transfers. And finally, one could examine other policy mechanisms for the principal, besides transfers. It seems to us that exploring these aspects further are fruitful avenues for future research.

References

[1] Aggarwal, A., and G. Thakur, 2013, Techniques of Performance Appraisal – A Review, International Journal of Engineering and Advanced Technology, vol 2(3), 617-621.

[2] Ballester, C., A. Calvò-Armengol and Y. Zenou, 2006, Who's who in networks. Wanted: the key player, Econometrica, vol. 74(5), 1403-1417.

[3] Belhaj, M. and F. Deroïan, 2018, Targeting the key player: An incentive-based approach, Journal of Mathematical Economics, vol 79(C), 57-64.

[4] Belhaj, M. and F. Deroïan, 2019, Group targeting under networked synergies, Games and Economic Behavior, vol 118, 29-46.

[5] Bernstein, S. and E. Winter, 2012, Contracting with Heterogeneous Externalities, American Economic Journal: Microeconomics, 4(2), 50-76.

[6] Bloch, F. and N. Quérou, 2013, Pricing in Social Networks, Games and Economic Behavior, vol. 80, 263-281.

[7] Bonacich, P., 1987, Power and Centrality: a Family of Measures, American Journal of Sociology, 92(5), 1170-1182.

Study on How to Overcome Coordination Failure in Organizations, American Economic Review, vol. 96 (3), 669-693.

[8] Caldés, N., and J. Maluccio, 2005, The Cost of Conditional Cash Transfers, Journal of International Development, vol. 17(2), 151-168.

[9] Candogan, O., K. Bimpikis and A. Ozdaglar, 2012, Optimal pricing in networks with externalities, Operations Research, 60(4), 883-905.

[10] Demange, G., 2017, Optimal Targeting Strategies in a Network under Complementarities, Games an Economic Behavior, 107, 84-103.

[11] Edmans, A., and X. Gabaix, 2011, Tractability in Incentive Contracting, Review of Financial Studies, 24(9), 2865-2894.

[12] Faragó, A. and Z. Mojaveri, 2019, In search of the densest subgraph, Algorithms, 12, 157.

[13] Feige, U., G. Kortsarz and D. Peleg, 2001, The dense k-subgraph problem, Algorithmica, 29(3), 410-421.

[14] Feige, U., V. Mirrokni and J. Vondraák, 2011, Maximizing non-monotone submodular functions, SIAM Journal of Computing, vol. 40(4), 1133-1153.

[15] Galeotti, A., B. Golub and S. Goyal, 2020, Targeting interventions in networks, forthcoming in Econometrica.

[16] Gould, E., B. Pashigian and C. Prendergast, 2005, Contracts, Externalities, and Incentives in Shopping Malls, Review of Economics and Statistics, 87(3), 411-422.

[17] Grosh, M., 1994, Administering Targeted Social Programs in Latin America: From Platitude to Practice, The World Bank: Washington, DC.

[18] Grötschel, M., L. Lovász and A. Schrijver, 1981, The ellipsoid method and its consequences in combinatorial optimization, Combinatorica, 1(2), 169-197.

[19] Hall, B. and J. Van Reenen, 2000, How effective are fiscal incentives for R&D? A review of the evidence, Research Policy, vol. 29(4-5), 449-469.

[20] Holmstrom, B., and P. Milgrom, 1987, Aggregation and linearity in the provision of intertemporal incentives, Econometrica, vol. 55, 303-328.

[21] Johnson, C. and R. Smith, 2011, Inverse M-matrices, II, Linear algebra and its applications, 435, 953-983.

[22] Laffont, J.-J., and J. Tirole, 1986, Using Cost Observation to Regulate Firms, Journal of Political Economy, vol. 94(3), 614-641.

[23] Lafontaine, F., 1992, How and Why Do Franchisors Do What They Do: A Survey Report, in Kaufman, ed., *Franchising: Passport for Growth and World of Opportunity*, 6th Annual Proceedings of the Society of Franchising. Lincoln: International Center for Franchise Studies, College of Business Administration, University of Nebraska.

[24] Lovász, L., 1983, Submodular functions and convexity, Mathematical programming: the state of the art, Bonn, 235-257.

[25] Preston McAfee, R., and J. McMillan, 1987, Competition for Agency Contracts, RAND Journal of Economics, vol. 18(2), 296-307.

[26] Murphy, K., and J. Cleveland, 1995, Understanding Performance Appraisal: Social, Organizational, and Goal-based Perspectives, SAGE Publications.

[27] Sakovics, J. and J. Steiner, 2012, Who matters in coordination problems?, American Economic Review, 102(7), 3439-3461.

[28] Stoffregen, P., 1995, Giving credit where credit is due: a brief history of the administration of the R&D tax credit, Tax Notes, vol. 16, 403-416.

[29] U.S. General Accounting Office, 1989, The research tax credit has stimulated some additional research spending, Vol. GAO/GGD-89-114, U.S. General Accounting Office, Washington, DC.

[30] Zhou, J. and Y.-J. Chen, 2015, Key leaders in social networks, Journal of Economic Theory, 157, 212-235.

6 Appendix A: Proofs

Proof of Proposition 1. Agent i's best-response satisfies $x_i^{BR}(y_i) = a_i + w_i + \delta \sum_j g_{ij} x_j$, which entails the Nash profile $\mathbf{x}^*(\mathbf{w}) = \mathbf{M}(\mathbf{a} + \mathbf{w})$. This means that the sum of excess-effort at equilibrium is given by

$$\mathbf{1}^T \mathbf{x}^*(\mathbf{w}) - \mathbf{b}_{\mathbf{a}}^T = \mathbf{b}^T \mathbf{w}$$

which does not depend on the vector of preferences. As well, the budget constraint is not affected by preferences. Indeed,it is given by $t = \mathbf{w}^T(\mathbf{x}^*(\mathbf{w}) - \mathbf{b}_{\mathbf{a}})$ and, by $\mathbf{x}^*(\mathbf{w}) - \mathbf{Ma} = \mathbf{Mw}$, we get

$$\mathbf{w}^T \mathbf{M} \mathbf{w} = t$$

In the end, the principal's problem is independent of the vector of private preferences $\mathbf{a}$, and the principal wants to maximize the quantity $\mathbf{b}^T \mathbf{w}$ over $\mathbf{w} \in \mathbb{R}^n$ under the budget constraint $\mathbf{w}^T \mathbf{M} \mathbf{w} = t$.

The Lagrangian $\mathcal{L}$ is written (we ignore non-negativity constraints $w_i \geq 0$ and check that they are satisfied ex post):

$$\mathcal{L}(\mathbf{w}, \lambda) = \mathbf{b}^T \mathbf{w} + \lambda(t - \mathbf{w}^T \mathbf{M} \mathbf{w})$$

Applying the first order conditions w.r.t. w_i for all $i \in N$, we obtain

$$\frac{\mathbf{b}}{\lambda^*} = 2\mathbf{M}\mathbf{w}^*$$

that is, since $\mathbf{b} = \mathbf{M1}$,

$$\breve{\mathbf{w}} = \frac{1}{2\lambda^*} \mathbf{1}$$

To finish, given that $\mathbf{w}^{*T} \mathbf{M} \mathbf{w}^* = t$, and reminding that $\mathbf{1}^T \mathbf{M} \mathbf{1} = b$, we derive

$$\lambda^* = \frac{1}{2} \sqrt{\frac{b}{t}}$$

It follows that

$$\mathbf{w}^* = \sqrt{\frac{t}{b}} \cdot \mathbf{1} \tag{4}$$

Furthermore, the reward equation is written

$$t_i^* = w_i^* \cdot \sum_{j \in \mathcal{N}} m_{ij} w_j^* \tag{5}$$

Replacing equation (4) into equation (5), we get

$$t_i^* = \frac{b_i}{b} \cdot t$$

Last, considering aggregate excess-effort, we get

$$x^* - b_{\mathbf{a}} = \sum_{i \in \mathcal{N}} w_i^* b_i \tag{6}$$

That is, as incorporating equation (4) into equation (6), we obtain

$$x^* - b_{\mathbf{a}} = \sqrt{tb}$$

and we are done. □

Proof of Theorem 1. We define the $s \times s$ matrix $\mathbf{M}_{[\mathcal{S}]}$ representing the sub-matrix of $\mathbf{M}$ restricted to agents in $\mathcal{S}$. We also define $\mathbf{b}_{[\mathcal{S}]} = (b_1, \cdots, b_s)^T$ as the collection of centralities of agents in the set $\mathcal{S}$.

We consider by $\mathbf{w} = (w_1 \cdots, w_s)^T$ a collection of contracts offered to a given set $\mathcal{S} = \{1, 2, \cdots, s\}$. We first determine the optimal contracts by assuming that $\mathbf{w}_i > 0$ for all $i \in \mathcal{S}$. Then we show that, indeed, weights are positive, which guarantees that coordination is not a concern.

Step 1: we determine the shape of optimal contracts.

For convenience, it is useful to introduce the n-dimensional vector $\bar{\mathbf{w}} = (w_1, w_2, \cdots, w_s, 0, \cdots, 0)^T$. Let $\mathbf{x}^* = (x_i^*)_{i \in \mathcal{N}}$ be the Nash effort profile, conditional on all offers proposed to agents in the set $\mathcal{S}$ being accepted. The Nash equilibrium effort profile satisfies

$$\mathbf{x}^* - \mathbf{Ma} = \mathbf{M}\bar{\mathbf{w}} \tag{7}$$

We deduce the variation of aggregate effort induced by the introduction of the contracts (with the notation $b_{\mathbf{a}} = \mathbf{1}^T\mathbf{Ma}$, $\mathbf{b} = \mathbf{M1}$, $x^* = \mathbf{1}^T\mathbf{x}^*$, what measures the performance of group $\mathcal{S}$:

$$x^* - b_{\mathbf{a}} = \mathbf{b}^T\bar{\mathbf{w}} \tag{8}$$

Plugging agents equilibrium effort as given by equation (7) into the budget constraint, and using vector $\mathbf{w}_{[\mathcal{S}]}$, we get

$$t = \mathbf{w}^T\mathbf{M}_{[\mathcal{S}]}\mathbf{w} \tag{9}$$

The Lagrangian of the program is well defined (linear objective under convex constraint) and given by

$$\mathcal{L} = \mathbf{b}_{[\mathcal{S}]}^T\mathbf{w} + \lambda\left(t - \mathbf{w}^T\mathbf{M}_{[\mathcal{S}]}\mathbf{w}\right)$$

Exploiting first order conditions of the Lagrangian with respect to returns, the optimal returns $(w_i^*)_{i\in\mathcal{S}}$ solve the following linear system:

$$\mathbf{w}^* = \frac{1}{2\lambda}\mathbf{M}_{[\mathcal{S}]}^{-1}\mathbf{b}_{[\mathcal{S}]} \quad \text{for all } i \in \mathcal{S} \tag{10}$$

Assume that optimal returns are positive. Plugging equation (11) into the budget constraint equation (9), we get

$$2\lambda = \sqrt{\frac{\mathbf{b}_{[\mathcal{S}]}^T\mathbf{M}_{[\mathcal{S}]}^{-1}\mathbf{b}_{[\mathcal{S}]}}{t}}$$

meaning that

$$\mathbf{w}^* = \left(\sqrt{\frac{t}{\mathbf{b}_{[\mathcal{S}]}^T\mathbf{M}_{[\mathcal{S}]}^{-1}\mathbf{b}_{[\mathcal{S}]}}}\right)\mathbf{M}_{[\mathcal{S}]}^{-1}\mathbf{b}_{[\mathcal{S}]} \quad \text{for all } i \in \mathcal{S} \tag{11}$$

Plugging equation (11) into equation (8), the performance of group $\mathcal{S}$ is given by

$$F(S) = \sqrt{t\ \mathbf{b}_{[\mathcal{S}]}^T\mathbf{M}_{[\mathcal{S}]}^{-1}\mathbf{b}_{[\mathcal{S}]}}$$

Step 2: we prove that $w_i > 0$ for all $i \in \mathcal{S}$. It is the case that $\mathbf{w} > 0$ if and only if

$$\begin{pmatrix} w_1 \\ w_2 \\ \cdots \\ w_s \end{pmatrix} \sim \mathbf{M}_{[\mathcal{S}]}^{-1}\mathbf{b}_{[\mathcal{S}]} = \begin{pmatrix} m_{11} & \cdots & m_{1s} \\ m_{21} & \cdots & m_{2s} \\ & \cdots & \\ m_{s1} & \cdots & m_{ss} \end{pmatrix}^{-1} \begin{pmatrix} m_{11} + \cdots + m_{1n} \\ m_{21} + \cdots + m_{2n} \\ \cdots \\ m_{s1} + \cdots + m_{sn} \end{pmatrix} > \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \end{pmatrix}$$

Consider the return of agent $1 \in \mathcal{S}$ (this is without loss of generality). Then, few calculus shows that $w_1^* > 0$ if and only if

$$\underbrace{\begin{vmatrix} m_{11} & \cdots & m_{1s} \\ m_{21} & \cdots & m_{2s} \\ \cdots & & \\ m_{s1} & \cdots & m_{ss} \end{vmatrix}}_{>0} + (-1)^{s+1}\sum_{k>3} \underbrace{\begin{vmatrix} m_{12} & \cdots & m_{1s} & m_{1k} \\ m_{22} & \cdots & m_{2s} & m_{2k} \\ \cdots & & & \\ m_{2s} & \cdots & m_{ss} & m_{sk} \end{vmatrix}}_{\text{sign of } (-1)^{s+3}} > 0 \tag{12}$$

To see this, we decompose $w_1 = \sum_{j\in\mathcal{N}} w_{1j}$, where

$$w_{1i} = \begin{pmatrix} m_{11} & \cdots & m_{1s} \\ m_{21} & \cdots & m_{2s} \\ \cdots & & \\ m_{s1} & \cdots & m_{ss} \end{pmatrix}^{-1} \begin{pmatrix} m_{1j} \\ m_{2j} \\ \cdots \\ m_{sj} \end{pmatrix}$$

Then, by definition of inverse matrices,

$$\begin{pmatrix} m_{11} & \cdots & m_{1s} \\ m_{21} & \cdots & m_{2s} \\ \cdots & & \\ m_{s1} & \cdots & m_{ss} \end{pmatrix}^{-1} \begin{pmatrix} m_{11} \\ m_{21} \\ \cdots \\ m_{s1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \cdots \\ 0 \end{pmatrix}$$

and for all $i \in \{2, \cdots, s\}$,

$$\begin{pmatrix} m_{11} & \cdots & m_{1s} \\ m_{21} & \cdots & m_{2s} \\ \cdots & & \\ m_{s1} & \cdots & m_{ss} \end{pmatrix}^{-1} \begin{pmatrix} m_{1i} \\ m_{2i} \\ \cdots \\ m_{si} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \end{pmatrix}$$

It follows that $w_{11} = 1$ and $w_{1i} = 0$ for all $i \in \{2, \cdots, s\}$.

Then we observe that for all $k > s$,

$$w_{1k} = \begin{pmatrix} m_{11} & \cdots & m_{1s} \\ m_{21} & \cdots & m_{2s} \\ \cdots & & \\ m_{s1} & \cdots & m_{ss} \end{pmatrix}^{-1} \begin{pmatrix} m_{1k} \\ m_{2k} \\ \cdots \\ m_{sk} \end{pmatrix} = \frac{\begin{vmatrix} m_{12} & \cdots & m_{1s} & m_{1k} \\ m_{22} & \cdots & m_{2s} & m_{2k} \\ \cdots & & & \\ m_{2s} & \cdots & m_{ss} & m_{sk} \end{vmatrix}}{det(\mathbf{M}_{[\mathcal{S}]})}$$

This can be checked directly by developping the latter determinant by column k: we get $(-1)^{s+1}\Big(\sum_{i=1,\cdots,s} m_{ik}\ coF_{ik}\Big)$, and by remarking that the determinant coF_{ik} is equal to the co-factor $(1, i)$ of matrix $\mathbf{M}_{[\mathcal{S}]}$. In total, we get $w_1^* = 1 + \sum_{k>s} w_{1k}$. Multiplying all terms by $det(\mathbf{M}_{[\mathcal{S}]})$, we obtain inequality (12).

We will sign all minors given in the LHS of inequality (12); recognizing a principal minor and $n - s$ almost principal minors, we will show that the sign of almost principal minors alternate according to whether s is odd or even, but for any given size s all have the same sign; and taking into account the multiplication by $(-1)^{s+1}$, all contributions will be seen to be positive. To proceed, we define the set $\alpha = \{1, 2, \cdots, s, k\}$, for any $k \in \{s+1, \cdots, n\}$ (we write α rather than $\alpha(s, k)$ when there is no confusion). We use the informal notation $\alpha + i$ (resp. $\alpha - i$) to denote the augmentation of the set α by $i \notin \alpha$ (resp. the deletion of $i \in \alpha$ from α. The left determinant is that of the principal submatrix $\mathbf{M}_{[\mathcal{S}]} = \mathbf{M}[\alpha - k; \alpha - k]$, it is positive as being a principal minor of an inverse M-matrix. Moreover, every other determinant is that of an almost principal submatrix $\mathbf{M}[\alpha - k; \alpha - 1]$ for $k > s$. Now, it is known from inverse M-matrices theory that the sign of the determinant of this almost principal submatrix is that of $(-1)^{r+t+1}$, where r is the number of indices of α less than or equal to k, and t is the number of indices of α less than or equal to 1 (see for instance Johnson and Smith [2011, Theorem 3.1 p.961]). Thus, $r = s+1, t = 1$. In total, for any $k > s$, the sign of the almost principal minor is that of $(-1)^{2s+2}$ that is the sign of 1, i.e. it is positive. This shows that inequality (12) is positive. $\square$

Proof of Lemma 1. A function is submodular if, for all groups $\mathcal{S} \subset \mathcal{T}$, all $i \notin \mathcal{T}$, $F(\mathcal{S} \cup \{i\}) - F(\mathcal{S}) \geq F(\mathcal{T} \cup \{i\}) - F(\mathcal{T})$.

Consider any group $\mathcal{S} \neq \mathcal{N}$ and one agent say $i \notin \mathcal{S}$. We have $F(\mathcal{S}) = \sqrt{tf(\mathcal{S})} - C(s)$, where $f(\mathcal{S}) = s + 2\delta(L_{\mathcal{S}} + L_{\mathcal{S},\mathcal{N}\setminus\mathcal{S}})$. We have

$$f(\mathcal{S} \cup \{i\}) - f(\mathcal{S}) = 1 + 2\delta(L_{\mathcal{S}\cup i} + L_{\mathcal{N}\setminus(\mathcal{S}\cup\{i\})}) - 2\delta(L_{\mathcal{S}} + L_{\mathcal{S},\mathcal{N}\setminus\mathcal{S}})$$

i.e., observing that the difference in respective numbers of links is equal to $L_{i,\mathcal{N}\setminus\mathcal{S}}$,

$$f(\mathcal{S} \cup \{i\}) - f(\mathcal{S}) = 1 + 2\delta L_{i,\mathcal{N}\setminus\mathcal{S}}$$

Clearly $L_{i,\mathcal{N}\setminus\mathcal{S}}$ can only be reduced when the set $\mathcal{S}$ is being enlarged. By concavity of the square root and by supermodularity of the cost function, this difference is even more reduced as $\mathcal{S}$ is enlarged. In total, the submodularity property of the optimal objective holds. $\square$

Proof of Lemma 2. We show that the problem of finding the group of size $s = k$ maximizing the number of internal links plus the number of cross-links is NP-hard. Indeed, the optimum, that we call $F(\mathbf{G})$ for convenience, is given by

$$F(\mathbf{G}) = \max_{\mathcal{S},[\mathcal{S}|=s} \; L_{\mathcal{S}}(\mathbf{G}) + L_{\mathcal{S},\mathcal{N}\setminus\mathcal{S}}(\mathbf{G})$$

i.e., defining $L_{\mathcal{N}\setminus\mathcal{S}}(\mathbf{G})$ as the number of links in the set $\mathcal{N} \setminus \mathcal{S}$ in network $\mathbf{G}$, and noting that $L_{\mathcal{S}}(\mathbf{G}) + L_{\mathcal{S},\mathcal{N}\setminus\mathcal{S}}(\mathbf{G}) = L - L_{\mathcal{N}\setminus\mathcal{S}}(\mathbf{G})$,

$$F(\mathbf{G}) = \min_{\mathcal{S},|\mathcal{S}|=n-s} \; L_{\mathcal{S}}(\mathbf{G})$$

We let $\mathbf{J}$ be the n-square matrix of ones, and we let $\bar{\mathbf{G}}$ be the complementary network of $\mathbf{G}$, so that $\mathbf{G} + \bar{\mathbf{G}} = \mathbf{J} - \mathbf{I}$. Then

$$\min_{\mathcal{S},|\mathcal{S}|=n-s} \; L_{\mathcal{S}}(\mathbf{G}) = \max_{\mathcal{S},|\mathcal{S}|=n-s} \; L_{\mathcal{S}}(\bar{\mathbf{G}})$$

This last problem is the densest-k subgraph problem with sets of size $n - s$. This problem is known to be NP-hard. $\square$

Proof of Theorem 2. In the whole proof we consider that δ is close to zero. We then identify the objective function, and we deduce that the objective is submodular for supermodular cost functions by using Lemma 1. Then, using Lemma 2, we confirm NP-hardness by considering a subproblem which is itself isomorphic to a well-known NP-hard problem (the densest k subgraph problem).

We first identify the objective function. Suppose that the principal selects group $\mathcal{S}$. For any agent $i \in \mathcal{S}$, we let $d_i^{\mathcal{S}}$ (resp. $d_i^{\mathcal{N}\setminus\mathcal{S}}$) represent the number of linked agents in the set $\mathcal{S}$ (resp. in the set $\mathcal{N}\setminus\mathcal{S}$). Then $d_i = d_i^{\mathcal{S}} + d_i^{\mathcal{S},\mathcal{N}\setminus\mathcal{S}}$. We let $L_{\mathcal{S}}$ represent the number of links among members of $\mathcal{S}$, and we let $L_{\mathcal{S},\mathcal{N}\setminus\mathcal{S}}$ be the number of links between $\mathcal{S}$ and $\mathcal{N}\setminus\mathcal{S}$ (i.e. the number of cross links between the two sets). Then $\sum_{i\in\mathcal{S}} d_i^{\mathcal{S}} = 2L_{\mathcal{S}}$ and $\sum_{i\in\mathcal{S}} d_i^{\mathcal{N}\setminus\mathcal{S}} = L_{\mathcal{S},\mathcal{N}\setminus\mathcal{S}}$. Under low intensity of interaction, we approximate $b_i = 1 + \delta d_i + o(\delta)$. We also have $\mathbf{M}_{[\mathcal{S}]} = \mathbf{I} + \delta\mathbf{G}_{[\mathcal{S}]} + o(\delta)$, which means that $\mathbf{M}_{[\mathcal{S}]}^{-1} = \mathbf{I} - \delta\mathbf{G}_{[\mathcal{S}]} + o(\delta)$. We plug these Taylor approximations into $F(S)$ as defined in equation (3) and we focus on order 1. We get:

$$\mathbf{b}_{[\mathcal{S}]}^T\mathbf{M}_{[\mathcal{S}]}^{-1}\mathbf{b}_{[\mathcal{S}]} \sim \sum_{i\in\mathcal{S}} b_i^2 - \delta\sum_{i\in\mathcal{S}}\sum_{j\in\mathcal{S}} g_{ij} \sim \sum_{i\in\mathcal{S}}(1+2\delta d_i) - \delta\sum_{i\in\mathcal{S}} d_i^S$$

That is,

$$\mathbf{b}_{[\mathcal{S}]}^T\mathbf{M}_{[\mathcal{S}]}^{-1}\mathbf{b}_{[\mathcal{S}]} \sim s + 2\delta\sum_{i\in\mathcal{S}} d_i - \delta\sum_{i\in\mathcal{S}} d_i^S$$

Taking into account that $d_i = d_i^{\mathcal{S}} + d_i^{\mathcal{S},\mathcal{N}\setminus\mathcal{S}}$,

$$\mathbf{b}_{[\mathcal{S}]}^T\mathbf{M}_{[\mathcal{S}]}^{-1}\mathbf{b}_{[\mathcal{S}]} \sim s + \delta\sum_{i\in\mathcal{S}} d_i^{\mathcal{S}} + 2\sum_{i\in\mathcal{S}}\delta d_i^{\mathcal{S},\mathcal{N}\setminus\mathcal{S}} = s + 2\delta L_{\mathcal{S}} + 2\delta L_{\mathcal{S},\mathcal{N}\setminus\mathcal{S}}$$

In total,

$$F(\mathcal{S}) = \sqrt{t}\sqrt{s + 2\delta(L_{\mathcal{S}} + L_{\mathcal{S},\mathcal{N}\setminus\mathcal{S}})} - C(\mathcal{S})$$

We turn to NP-hardness. It is well-known from discrete optimization mathematics that the problems of sort $\max_{\mathcal{S}} F(S)$ can be NP-hard when function F is submodular and non-monotone (if the function is monotone, as in the case of null cost, the problem is trivial). Non-monotonicity emerges as soon as the contracting cost function is large enough. Then, by Lemma 1, we know that function F is submodular for any supermodular cost function. To show NP-hardness formally, we consider the subproblem of finding the best target among all groups of fixed size with an identical objective function (gross of contracting cost). This problem being a subproblem of the principal's problem (who has to consider all possible groups), its NP-hardness would imply that of the more general problem. Now, by Lemma 2, the problem of finding the group maximizing the number of internal links plus the number of cross links is NP-hard. Any affine transformation of that objective preserving submodularity, as well as the composition by the square root, maximizing the objective function $\sqrt{s + 2\delta(L_{\mathcal{S}} + L_{\mathcal{S},\mathcal{N}\setminus\mathcal{S}})}$ ober all groups of same size is still NP-hard. $\square$

Proof of Corollary 3. We compute the performance of a group of size s with and without the central agent, and then we show that including the central agent is always better.

In the star network, the matrix

$$\mathbf{M} = \frac{1}{1-(n-1)\delta^2}\begin{pmatrix} 1 & \delta & \delta & \cdots & \delta \\ \delta & 1-(n-2)\delta^2 & \delta^2 & \cdots & \delta^2 \\ \delta & \delta^2 & 1-(n-2)\delta^2 & \cdots & \delta^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \delta & \delta^2 & \cdots & \delta^2 & 1-(n-2)\delta^2 \end{pmatrix}$$

Consider first a group of s peripherals. The invert of matrix $\mathbf{M}_{\mathcal{S}}$ has homogeneous diagonal entries $1-(n-2)\delta^2$ and homogeneous off-diagonal entries δ^2. Hence, defining $\mathbf{V} = \mathbf{M}_{[\mathcal{S}]}^{-1}$ for convenience, matrix $\mathbf{V}$ has homogeneous diagonal entries $\frac{1-(n-s)\delta^2}{1-(n-s-1)\delta^2}$ and homogeneous off-diagonal entries $\frac{-\delta^2}{1-(n-s-1)\delta^2}$. Then the objective is given by $F_p(s) = \sqrt{t}\sqrt{f_p(s)} - cs$, with

$$f_p(s) = s\left(\frac{1+\delta}{1-(n-1)\delta^2}\right)^2$$

Second consider a group of size s containing the central agent. Then $\mathbf{M}_{\mathcal{S}}$ is a s-square matrix structurally equivalent to $\mathbf{M}$. We get the corresponding matrix

$$\mathbf{V} = \begin{pmatrix} 1-(n-s)\delta^2 & -\delta & -\delta & \cdots & -\delta \\ -\delta & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -\delta & 0 & \cdots & 0 & 1 \end{pmatrix}$$

from which we deduce the objective function $F_c(s) = \sqrt{t}\sqrt{f_c(s)} - cs$, with

$$f_c(s) = s + \frac{(n-1)\delta(2+n\delta-2(n-1)\delta^2 - n(n-1)\delta^3)}{(1-(n-1)\delta^2)^2}$$

Then, few computations indicate that $f_p(s) \le f_c(s)$ whenever

$$\delta^2 s^2 - (2+(2n-1)\delta)\delta s + (n-1)\delta(2+n\delta) \ge 0$$

which holds for all $s \le n-1$. Then for all group size, it is always better to include the central agent. $\square$

Proof of Lemma 3. We will show that: If $\mathbf{M}_{[\mathcal{S}]} < \mathbf{M}_{[\mathcal{S}']}$, then the diagonal dominance (by row and column) of $\mathbf{M}_{[\mathcal{S}]}^{-1}$ implies $\mathbf{1}_s^T\mathbf{M}_{[\mathcal{S}]}^{-1}\mathbf{1}_s > \mathbf{1}_s^T\mathbf{M}_{[\mathcal{S}']}^{-1}\mathbf{1}_s$. For that purpose, we apply the Sherman-Morrison formula to examine the impact of a small impulse in one entry of the matrix $\mathbf{M}_{[\mathcal{S}]}$ on the sum of entries of invert matrix. The Sherman-Morrison formula is given as follows. Suppose $\mathbf{Q}$ is an invertible n-square matrix with real entries and $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are column vectors. Then $\mathbf{Q} + \mathbf{u}\mathbf{v}^T$ is invertible iff $1 + \mathbf{v}^T\mathbf{Q}^{-1}\mathbf{u} \neq 0$, in which case

$$(\mathbf{Q} + \mathbf{u}\mathbf{v}^T)^{-1} = \mathbf{Q}^{-1} - \frac{\mathbf{Q}^{-1}\mathbf{u}\mathbf{v}^T\mathbf{Q}^{-1}}{1 + \mathbf{v}^T\mathbf{Q}^{-1}\mathbf{u}}$$

We apply this formula with:
. $\mathbf{Q} = \mathbf{M}$
. $\mathbf{u} = (0, \cdots, 0, \epsilon_{ij}, 0, \cdots, 0)^T$ with ϵ_{ij} at entry i
. $\mathbf{v}^T = (0, \cdots, 0, 1, 0, \cdots, 0)^T$ with 1 at entry j
Hence, $\mathbf{E} = \mathbf{u}\mathbf{v}^T$ s.t. $\mathbf{E} = [e_{pq}]$ is such that $e_{ij} = \epsilon_{ij}$, $e_{pq} = 0$ otherwise
Noting $\mathbf{V} = \mathbf{M}^{-1}$, we get

$$[(\mathbf{M} + \mathbf{E})^{-1}] = \mathbf{M}^{-1} + \zeta_{ij}\epsilon_{ij}\ \mathbf{W}$$

with $\zeta_{ij} = \frac{-1}{1+\epsilon_{ij}h_{ji}}$, and $w_{pq} = h_{pi}h_{jq}$. Hence, denoting $\mathbf{M}' = \mathbf{M} + \mathbf{E}$:

$$\mathbf{1}^T\mathbf{M}'^{-1}\mathbf{1} - \mathbf{1}^T\mathbf{M}^{-1}\mathbf{1} = \zeta_{ij}\epsilon_{ij}\Big(\sum_{k\in N} h_{jk}\Big)\Big(\sum_{k\in N} h_{ki}\Big)$$

In particular, for $j = i$, $\zeta_{ii} < 0$, and given symmetry of $\mathbf{V}$:

$$\mathbf{1}^T\mathbf{M}'^{-1}\mathbf{1} - \mathbf{1}^T\mathbf{M}^{-1}\mathbf{1} = \zeta_{ii}\epsilon_{ii}\Big(\sum_{k\in N} h_{ik}\Big)^2 \leq 0$$

And for $j \neq i$, we have $\zeta_{ij} < 0$ for ϵ_{ij} small enough, which implies that the diagonal dominance of $\mathbf{V} = \mathbf{M}^{-1}$ entails a decrease of sum of entries.

The proof is completed by starting from matrix $\mathbf{M}_{[\mathcal{S}]}$, by adding impulses iteratively in the direction of matrix $\mathbf{M}_{[\mathcal{S}']}$, the preceding argument showing a monotonic response to the sum of entries of inverse matrices along the direction. $\square$

7 Appendix B: Tables

This Appendix provides the tables that present the performance of the greedy algorithm used to approximate the optimal targeting. Table 3 shows the unconstrained program with various contracting costs c. Table 4 presents the constrained program with null contracting cost and various upper bounds k to the number of contracts. In both cases, we initiate $\mathbf{n} = 16, \mathbf{p} = 0.5$.

δ	$\mathbf{c = 0.1}$	$\mathbf{c = 0.2}$	$\mathbf{c = 0.3}$	$\mathbf{c = 0.4}$	$\mathbf{c = 0.5}$	$\mathbf{c = 0.6}$
$\delta = \mathbf{0.01}$	100	99.85	99.94	99.97	100	100
$\delta = \mathbf{0.06}$	99.97	99.68	99.78	99.65	99.84	99.82

Table 3: Targeting through greedy algorithm: Average over 100 random networks of the relative error of approximation in percentage of the optimal performance.

δ	$\mathbf{k = 3}$	$\mathbf{k = 5}$	$\mathbf{k = 8}$	$\mathbf{k = 10}$
$\delta = \mathbf{0.01}$	99.95	99.95	99.93	99.95
$\delta = \mathbf{0.06}$	99.89	99.74	99.78	99.84

Table 4: Targeting over groups of fixed size equal to k through greedy algorithm as a function of k (for $c = 0$): Average over 100 random networks of the relative error of approximation in percentage of the optimal performance.

8 Appendix C: Matlab programs

We present here the matlab program of the greedy algorithms used to fill tables in Appendix B. We first present the unconstrained problem (with cost function included in the principal's objective), and then we turn to the constrained problem of optimal targeting among groups of fixed size.

Both programs work as follows. We generate random networks. For each network, we compute the optimal group by considering the performance of all possible groups of agents; we use the function 'nchoosek'. Then we generate the greedy algorithm and we build the ratio of respective performances. We then make the average ratio over all random networks. Both programs return the average performance of the greedy algorithm. For the greedy algorithm of the constrained problem, the fixed group size is set using variable 's'.

To make further checks on programs, a matrix labeled 'Perf' gives all groups and their performance. Each row gives the label of the selected group in columns 1 to n, column $n + 1$ gives the performance of the group, and column $n+2$ gives the size of the group. A simple test is then to set $nbg = 1$ (i.e. fix the generation of a single random network), and to check directly on that matrix which is the optimal group (by sorting the matrix by column $n+1$), and also which is the greedy optimal group (which can be done by manual inspection for n low enough).

8.1 The greedy algorithm of the unconstrained problem

```
clear all
n = 16;
J = ones(n, n);
O = ones(n, 1);
id = eye(n, n);
N = O;
t = 1;
cost = 0.5;
nbg = 100;
p = 0.5;
delta = 0.01;
for i = 1 : n
N(i) = i;
end
averageperfgreedy = 0;
for nbgraphes = 1 : nbg
nbgraphes
maxPerfs = 0;
Grand = zeros(n, n);
for i = 1 : n
for j = i + 1 : n
randomlink = rand;
if randomlink < p
Grand(i, j) = 1;
Grand(j, i) = 1;
end
end
end
G = Grand;
D = G * O;
OI = zeros(2^n − 1, 1);
```

$EnsdessousensemblesdeN = zeros(2^n - 1, n);$
$cprev = 0;$
$for\ size = 1 : n$
$C = nchoosek(N, size);$
$c = factorial(n)/factorial(size)/factorial(n - size);$
$for\ i = cprev + 1 : cprev + c$
$for\ j = 1 : size$
$EnsdessousensemblesdeN(i, C(i - cprev, j)) = C(i - cprev, j);$
end
end
$cprev = cprev + c;$
end
$Perf = [EnsdessousensemblesdeN\ OI\ OI];$
$for\ z = 1 : 2^n - 1$
$size = 0;$
$for\ j = 1 : n$
$if\ EnsdessousensemblesdeN(z, j) > 0$
$size = size + 1;$
end
end
$M = inv(id - delta * G);$
$if\ min(M) < 0$
$disp('Min\ M\ negatif')$
return
end
$B = M * O;$
$MS1 = zeros(1, n);$
$for\ i = 1 : n$
$if\ EnsdessousensemblesdeN(z, i) > 0$
$MS1 = [MS1;$
$M(i, :)];$
end
end
$MS1bis = MS1(2 : size + 1, :);$
$MS1 = MS1bis;$
$MS2 = zeros(size, 1);$
$for\ j = 1 : n$
$if\ EnsdessousensemblesdeN(z, j) > 0$
$MS2 = [MS2\ MS1(:, j)];$
end
end

$MS2bis = MS2(:, 2 : size + 1);$
$MS2 = MS2bis;$
$MS = MS2;$
$BS = 0;$
$for\ i = 1 : n$
$if\ EnsdessousensemblesdeN(z, i) > 0$
$BS = [BS; B(i, 1)];$
end
end
$BS2 = BS(2 : size + 1, :);$
$BS = BS2;$
$Objectifprincipal = sqrt(t) * sqrt(transpose(BS) * inv(MS) * BS) - cost * size;$
$Perf(z, n + 1) = Objectifprincipal;$
$Perf(z, n + 2) = size;$
end
$TT = sortrows(Perf, n + 1,' descend');$
$maxPerfs = TT(1, n + 1);$
$labelPerfmax = TT(1, n + 2);$
$if\ maxPerfs < 0$
$disp('maxPerfs < 0')$
return
end
$winner = 1;$
$Perfgreedy = Perf(1, n + 1);$
$for\ io = 2 : n$
$if\ Perf(io, n + 1) > Perfgreedy + 0.000001$
$winner = io;$
$Perfgreedy = Perf(winner, n + 1);$
end
end
$winner_1 = winner;$
$winnerprec = winner;$
$for\ i = n + 1 : 2^n - 1$
$if\ Perf(i, n + 2) > Perf(i - 1, n + 2)$
$winnerprec = winner;$
end
$if\ min(Perf(i, 1 : n) - Perf(winnerprec, 1 : n)) > -0.00001$
$if\ Perf(i, n + 2) == Perf(winnerprec, n + 2) + 1$
$if\ Perf(i, n + 1) > Perf(winnerprec, n + 1) + 0.00000001$
$if\ Perf(i, n + 1) > Perf(winner, n + 1) + 0.00000001$

$Perfgreedy = Perf(i, n + 1);$
$winner = i;$
end
$if\ (i < 2^n - 1)$ && $(Perf(i+1, n+2) > Perf(i, n+2))$ && $(winner == winnerprec)$
break
end
end
end
end
end
$averageperfgreedy = averageperfgreedy + Perfgreedy/maxPerfs * 100;$
end
$averageperfgreedy = averageperfgreedy/nbg$

8.2 The greedy algorithm of the constrained problem

clear all
$n = 16;$
$J = ones(n, n);$
$O = ones(n, 1);$
$id = eye(n, n);$
$N = O;$
$t = 1;$
$nbg = 100;$
$p = 0.5;$
$delta = 0.01;$
$s = 5;$
$for\ i = 1 : n$
$N(i) = i;$
end
$averageperfgreedy = 0;$
$for\ nbgraphes = 1 : nbg$
$nbgraphes$
$maxPerfs = 0;$
$Grand = zeros(n, n);$
$for\ i = 1 : n$
$for\ j = i + 1 : n$
$randomlink = rand;$
$if\ randomlink < p$

$Grand(i,j) = 1;$
$Grand(j,i) = 1;$
end
end
end
$G = Grand;$
$D = G * O;$
$OI = zeros(2^n - 1, 1);$
$EnsdessousensemblesdeN = zeros(2^n - 1, n);$
$cprev = 0;$
$for\ size = 1 : n$
$C = nchoosek(N, size);$
$c = factorial(n)/factorial(size)/factorial(n - size);$
$for\ i = cprev + 1 : cprev + c$
$for\ j = 1 : size$
$EnsdessousensemblesdeN(i, C(i - cprev, j)) = C(i - cprev, j);$
end
end
$cprev = cprev + c;$
end
$Perf = [EnsdessousensemblesdeN\ OI\ OI];$
$for\ z = 1 : 2^n - 1$
$size = 0;$
$for\ j = 1 : n$
$if\ EnsdessousensemblesdeN(z, j) > 0$
$size = size + 1;$
end
end
$M = inv(id - delta * G);$
$if\ min(M) < 0$
$disp('Min\ M\ negatif')$
return
end
$B = M * O;$
$MS1 = zeros(1, n);$
$for\ i = 1 : n$
$if\ EnsdessousensemblesdeN(z, i) > 0$
$MS1 = [MS1; M(i, :)];$
end
end
$MS1bis = MS1(2 : size + 1, :);$

$MS1 = MS1bis;$
$MS2 = zeros(size, 1);$
$for\ j = 1 : n$
$ifEnsdessousensemblesdeN(z, j) > 0$
$MS2 = [MS2\ MS1(:, j)];$
end end
$MS2bis = MS2(:, 2 : size + 1);$
$MS2 = MS2bis;$
$MS = MS2;$
$BS = 0;$
$for\ i = 1 : n$
$ifEnsdessousensemblesdeN(z, i) > 0$
$BS = [BS; B(i, 1)];$
end
end
$BS2 = BS(2 : size + 1, :);$
$BS = BS2;$
$Objectif_principal = sqrt(t) * sqrt(transpose(BS) * inv(MS) * BS);$
$Perf(z, n + 1) = Objectif_principal;$
$Perf(z, n + 2) = size;$
end
$TT = sortrows(Perf, n + 1,' descend');$
$ia = 1;$
$whilemax(TT(ia, n + 2) - s, s - TT(ia, n + 2)) > 0$
$ia = ia + 1;$
end
$maxPerfs = TT(ia, n + 1);$
$if\ maxPerfs < 0$
$disp('maxPerfs < 0')$
return
end
$winner = 1;$
$Perfgreedy = Perf(1, n + 1);$
$for\ io = 2 : n$
$if\ Perf(io, n + 1) > Perfgreedy + 0.000001$
$winner = io;$
$Perfgreedy = Perf(winner, n + 1);$
end
end
$winner_1 = winner;$
$winnerprec = winner;$

$for\ i = n + 1 : 2^n - 2$
$if\ Perf(i, n + 2) > Perf(i - 1, n + 2)$
$winnerprec = winner;$
end
$if\ min(Perf(i, 1 : n) - Perf(winnerprec, 1 : n)) > -0.00001$
$if\ Perf(i, n + 2) == Perf(winnerprec, n + 2) + 1$
$if\ Perf(i, n + 1) > Perf(winnerprec, n + 1) + 0.00000001$
$if\ Perf(i, n + 1) > Perf(winner, n + 1) + 0.00000001$
$if\ Perf(i, n + 2) < s + 1\ Perfgreedy = Perf(i, n + 1);$
$winner = i;$
end
end
$if\ (i < 2^n - 1)\ \&\&\ (Perf(i + 1, n + 2) > Perf(i, n + 2))\ \&\&\ (winner == winnerprec)$
break
end
end
end
end
end
$averageperfgreedy = averageperfgreedy + Perfgreedy/maxPerfs * 100;$
end
$averageperfgreedy = averageperfgreedy/nbg$