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Abstract
We analyze risk-taking regulation when financial institutions are linked through shareholdings. We model regulation as an upper bound on institutions' default probability, and pin down the corresponding limits on risk-taking as a function of the shareholding network. We show that these limits depend on an original centrality measure that relies on the cross-shareholding network twice: (i) through a risk-sharing effect coming from complementarities in risk-taking and (ii) through a resource effect that creates heterogeneity among institutions. When risk is large, we find that the risk-sharing effect relies on a simple centrality measure: the ratio between Bonacich and self-loop centralities. More generally, we show that an increase in cross-shareholding increases optimal risk-taking through the risk-sharing effect, but that resource effect can be detrimental to some banks. We show how optimal risk-taking levels can be implemented through cash or capital requirements, and analyze complementary interventions through key-player analyses. We finally illustrate our model using real-world financial data and discuss extensions toward including debt-network, correlated investment portfolios and endogenous networks. (JEL: C72, D85)

Keywords: Financial Network, Risk-Taking, Prudential Regulation.

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1. Introduction

Financial institutions are highly regulated, because of the liquidity risk[1] inherent to their business, and because a default would cause important externalities, for example in terms of confidence of clients toward the whole financial system. One drawback of ex-post interventions, like bail-outs, is that they can be costly and subject to moral hazard.[2] This confers a key role on ex ante regulation. Such prudential regulation constrains the amount of risk each institution is allowed to take. During the 2007 financial crisis, these rules however proved insufficient, notably because of the importance of financial linkages – see for example the cases of Lehman Brothers and AIG discussed in Glasserman and Young (2016). Indeed, whereas financial contracts between institutions allow to share liquidity risk and diversify investment portfolios, they can also trigger contagion, i.e. the spread of negative shocks between institutions. This systemic risk, that arises through financial networks, is currently only partially taken into account in financial regulation. Regulators mostly relying on an *indicator-based* methodology, that sets up higher requirements for systemic institutions (BCBS, 2013). In this paper, we highlight the importance of the network structure of financial linkages, and discuss a *network-based* methodology for prudential regulation.

More precisely, we analyze how financial linkages, in the form of cross-shareholding, structure the risk exposure of each institution, and how prudential regulation should account for it. Focusing on equity-type contracts, that have been shown to be increasingly important among financial institutions,[3] allows us to focus on the first failure (rather than contagion), which is crucial for prudential regulation. We build a simple model in which each financial institution, financed through equity, held in part by the financial sector, and external debt, has to allocate its fund between a risky and a risk-free asset. Assuming that an extreme adverse event – in the form of a loss in its risky asset – can hurt at most one institution,[4] we pin down the optimal level of risk-taking for each institution that allows the risk of default to remain below a given threshold, set by the regulator.

Our results are threefold. First, we show that the limits of risk-taking depend on an original centrality measure that relies on the cross-shareholding network twice: (i) through a risk-sharing effect generated by complementarities

1. Liquidity risk refers to a temporary mismatch between asset and liability.

2. Moral hazard is at the root of the issue of "Too Big to Fail" financial institutions.

3. "Between 2000 and 2015 the number of banks with ownership in other banks doubled in the United States. [...] Between 2011 and 2015 the total value of ownership of banks by banks increased by 211 percent." (Pollak and Guan, 2017)

4. We consider extreme risks of high magnitude and rare frequency, in the heart of current financial regulation. Solvency II (the directive that harmonises EU insurance regulation) for example calibrates prudential regulation on the notion of bicentenary events.

in risky investments[5] and (ii) through a resource effect, inherent to available liabilities, and which is heterogeneous across institutions. Under large shock, the risk-sharing effect is aligned with a simple centrality measure: the ratio of Bonacich centrality over self-loop centrality. The resource effect makes the analysis more complex, but we are able to show for specific structures, like the star network,[6] that more central banks are allowed to take more risk.

Second, turning to comparative statics, we show that an increase in cross-shareholding increases optimal risk-taking through the risk-sharing effect channel, but the resource effect can be detrimental to some banks. In our model, cross-shareholding can increase either through integration (i.e. an increase in the amount invested by each bank in the existing network) or through diversification (i.e. new connections for a given amount of investment in the network). Simulations on random graphs suggest that those two mechanisms have very different effects. Whereas integration increases the average level of risk-taking, diversification has no significant impact.

Third, we highlight how prudential regulation, through cash or capital requirements, can implement the optimal levels of risk-taking, and discuss complementary policy interventions through key player analyses. We identify the institution whose equity injection leads to the highest global increase in risky investment; and the one on which relaxing regulation has the highest impact.

We finally bring this analysis to data using the indicators defining systemic banks, in particular the "total holding of equity issued by other financial institutions". Our computations show that the network structure (which governs how this total is shared) has significant effects both on the optimal levels of risk-taking and on the label of the bank that complementary policies should target.

Our analysis contributes to the fast-growing literature on financial network, by endogeneizing risk-taking decisions and accounting for prudential regulation. We analyze the impact of the financial network on the first failure, whereas the literature mostly focuses on contagion.

The propagation of shocks – and the resulting default contagion – is indeed at the heart of the literature on financial networks. One branch of the literature uses epidemiologic approaches to study numerically how liquidity shocks transmit in large networks (see Gai and Kapadia, 2019, for a recent survey). It provides, through simulations, quantitative assessments on the effect of network topology (in particular the average degree). Another branch of

5. The strategic complementarities between risk-taking levels stems from the risk structure we model. When an institution is hit by a negative shock, the others necessarily don't, meaning that higher investment increases their value in that state of nature, enhancing the value of the bank receiving the shock through cross-shareholding.

6. In a star network, all links involve a same agent; this agent is often called central and other agents peripheral.

the literature, more closely related to our work, seeks to analyze the effect of network structure on transmission mechanisms analytically. Bringing the seminal paper of Diamond and Dybvig (1983) to networks, Allen and Gale (2000), Acemoglu et al. (2015) and Glasserman and Young (2016) analyze the effect of financial networks on contagion, when banks are linked through debt contacts. Liquidity shocks spread through unmet obligations (i.e. unpaid debt) potentially causing default cascades. The effect of network density then depends on the size of the initial liquidity shock Acemoglu et al. (2015) and common exposures amplify contagion Glasserman and Young (2016). Focusing on this last aspect, Cabrales et al. (2017) model financial linkages as investments by banks in each other's projects and analyze the optimal network structure depending on projects' riskiness.[7] Elliott et al. (2014) consider additional frictions through default costs in a model of linear cross-holdings and discuss the effect of integration (stronger links) and diversification (more links).[8] In all the above papers, the initial risk faced by each bank is exogenous. We endogenize it through investment choices by banks, what allows discussing how the initial risk and the risk of first default depend on the network structure; and to tackle prudential regulation.[9]

Prudential regulation, through cash or capital requirements, has been shown a useful and powerful tool to deal with excessive risk-taking by banks and reduce default risk (Hellmann et al., 2000; Decamps et al., 2004).[10] It is implemented by financial regulators since the early 1990s (through the 1988 Basel Accord or Basel 1) and has been made more complex thereafter to take account for the specific natures of risk (market risk, liquidity risk and operational risk for example). It allows dampening solvency risk, without implying the social cost of bail-outs, and their induced effects – through moral hazard – when anticipated (Freixas and Rochet, 2013). We provide the first analysis of prudential financial regulation in networks, thus contributing to the nascent literature on public intervention on financial networks. Elliott et al. (2014) study the effect of reallocations of cross-holdings that leave the market value of banks unchanged and find that it doesn't allow avoiding the first failure. When banks are connected through debt-contract and subject to liquidity shocks, Leduc and Thurner (2017) study the effect of transaction-specific taxes

7. Our structure of risk with one large negative shock hurting one bank at time echoes the one modeled by Cabrales et al. (2017).

8. Although links are modeled as shareholding, Elliott et al. (2014) view them as "debt contracts around and below organizations' failure thresholds" and assume that default costs spread in the network.

9. Galeotti et al. (2016) model portfolio choice in a model of equity-holding in networks but assume away default. Jackson and Pernoud (2019) discuss a situation with both risk-taking and default but limit to simple examples with binary (and independent or fully correlated) returns on the risky assets.

10. Cash requirements correspond to constraints on the asset side whereas capital requirements lie on the liability side.

and show that it can reduce contagion. Finally, Demange (2018) and Jackson and Pernoud (2019) discuss the optimal ex-post intervention, through bailouts or cash injection. We contribute to this network literature by analysing ex-ante intervention aiming at limiting the risk-taking level of financial institutions.

The remainder of the paper is organized as follows. We introduce the model of prudential regulation in financial networks in Section 2. We characterize the optimal levels of risk-taking in Section 3 and discuss the impact of network topology in Section 4. We analyze policy interventions and real-world network in Section 5. We discuss various extensions in Section 6, and conclude in Section 7.

2. The model

2.1. *The financial network*

We consider a network of financial institutions (called banks in the following for simplicity)[11] potentially linked through cross-shareholding. In the core of the paper, the cross-shareholding network is assumed to be exogenous and debt-holding between institutions are regarded away. We discuss these issues in Section 6.

We consider a two-period model in which every institution is liquidated after risk realisation. At $t = 0$, each bank $i \in \mathcal{I} = \{1, 2, \cdots, n\}$ is financed by debt (or deposit) d_i, equity held by outside investors e_i, and equity held by other banks in the network: $\{p_{ji}\}_{j \in \mathcal{I}\setminus\{i\}}$; where p_{ji} represents the amount invested by bank j in bank i. Each bank shares this resource between investment in a risk-free asset (with normalized return equal to 1): $x_i \geq 0$, investment in a bank-specific risky asset: $z_i \in [0, d_i + e_i + \sum_{j \neq i} p_{ij}]$, and investment in the equity of other banks in the network: $\{p_{ij}\}_{j \in \mathcal{I}}$. The balance sheet of bank i at $t = 0$ (i.e. before realization of risk) can then be represented as in Figure 1:

A	L
x_i	d_i
z_i	e_i
$\sum_j p_{ij}$	$\sum_j p_{ji}$

FIGURE 1. Balance sheet of bank i at $t = 0$.

This leads to the following accounting equation at $t = 0$:

$$x_i + z_i + \sum_{j \in \mathcal{I}} p_{ij} = d_i + e_i + \sum_{j \in \mathcal{I}} p_{ji} \tag{1}$$

11. The model can also fit with the settings of insurance companies or pension funds, for example.

Defining vector $\mathbf{u} = \mathbf{e} + \mathbf{d} + (\mathbf{P}^T - \mathbf{P})\mathbf{1}$ for convenience, we have $\mathbf{z} \in [\mathbf{0}, \mathbf{u}]$ from the balance sheet equation (1) provided that $\mathbf{u} > \mathbf{0}$, which will be ensured under Assumption 1 thereafter.

At $t = 1$, risks are realized and banks are liquidated. Their values (if any) are then distributed among their shareholders. We denote by $a_{ij} = p_{ij}/(\sum_k p_{kj} + e_j)$ the share of the value of bank j held by bank i. Letting $\rho \geq 1$ represent the deterministic return on debt[12] (or deposit) and $\tilde{\mu}_i$ the stochastic return on the risky asset of bank i, Figure 2 presents the balance sheet of bank i for a given realization μ_i of the risky asset at $t = 1$, where the equity value v_i of bank i accounts for equity held both by the financial system and by external investors:

A	L
x_i	ρd_i
$\mu_i z_i$	
$\sum_j a_{ji} v_j$	v_i

FIGURE 2. Balance sheet of bank i at $t = 1$.

The equity value v_i of bank i then writes:

$$v_i = x_i + \mu_i z_i - \rho d_i + \sum_{j \neq i} a_{ij} v_j^+ \tag{2}$$

where $v_j^+ = v_j$ if $v_j > 0$ and 0 otherwise.[13] In case $v_i < 0$, all assets go to debt repayment. Using the accounting equation (1) at $t = 0$, equation (2) becomes:

$$v_i = (\mu_i - 1)z_i + \eta_i + \sum_{j \neq i} a_{ij} v_j^+ \tag{3}$$

where $\eta_i = e_i - (\rho - 1)d_i + \sum_{j \in \mathcal{I}} p_{ji} - \sum_{j \in \mathcal{I}} p_{ij}$.

ASSUMPTION 1. We assume $\eta_i > 0$ for all i, meaning that each bank remains solvent when it doesn't invest in risky asset ($v_i > 0$ when $z_i = 0$).

Assumption 1 guarantees that vector $\mathbf{u}$ is positive (this assumption also guarantees that each bank invests a positive amount in risky asset at equilibrium, when returns on risky assets are larger than unity). When the network of cross-shareholding is balanced ($\sum p_{ji} = \sum p_{ij}$), Assumption 1 boils

12. Returns on debt are assumed to be homogeneous and independent from default risk. As regulated banks will end up with the same default probability, this last assumption is reasonable.

13. Note that the market value of bank i, i.e. the share of the value held by external equity holders, is given by $v_i \cdot e_i / \left(e_i + \sum_{k \in \mathcal{I}} p_{ki}\right)$.

down to the condition $e_i > (\rho - 1)d_i$ for all i, meaning that one bank's equity is enough to finance the interest paid on debt. More generally, this assumption also depends on $\sum p_{ji} - \sum p_{ij}$, that we will denote "resource effect" thereafter, and has to be of sufficiently low magnitude.

2.2. Equilibrium equity values

We introduce the following notations. Matrices are written in block and bold letters, vectors in lower case and bold letter; $\mathbf{0}, \mathbf{1}$ represent the vectors of zeros and ones respectively; the upper script ^{T} stands for the transpose operator. Numbers and entries of matrices are written in lower case. We let $\mathbf{I}$ be the identity matrix of order-n, and $\mathbf{J}$ be the n-square matrix of ones. Then, $\mathbf{d} = (d_i)_{i\in\mathcal{I}}$ is the vector of external debts, $\mathbf{z} = (z_i)_{i\in\mathcal{I}}$ the profile of investments in risky assets, $\mathbf{e} = (e_i)_{i\in\mathcal{I}}$ the vector of equity held by outside investors, $\mathbf{P} = (p_{ij})_{i,j\in\mathcal{I}^2}$ the matrix of investment in equity among bank, $\mathbf{A} = (a_{ij})_{(i,j)\in\mathcal{I}^2}$ the corresponding matrix of shares and $\mathbf{v} = (v_i)_{i\in\mathcal{I}}$ the vector of bank's equity value. We define $h_i = (\mu_i - 1)z_i + \eta_i$ and $\mathbf{h} = (h_i)_{i\in\mathcal{I}}$. Equation (3) then simply writes:

$$v_i = h_i + \sum_{j\neq i} a_{ij} v_j^+ \tag{4}$$

that is, in the absence of default (if $v_i \geq 0$ for all i):

$$\mathbf{v} = \mathbf{M}\mathbf{h} \tag{5}$$

where $\mathbf{M} = (\mathbf{I} - \mathbf{A})^{-1}$.[14] The next Lemma – reminiscent of Eisenberg and Noe (2001) – establishes uniqueness of values satisfying the system of equations (4) $\forall i$:

LEMMA 1. *For any financial network* $(\mathbf{d}, \mathbf{e}, \mathbf{P})$*, any investment profile* $\mathbf{z} \in [\mathbf{0}, \mathbf{e} + \mathbf{u}]$*, and any realization of risks* $(\mu_i)_{i\in\mathcal{I}}$*, there is a single set of values* $\mathbf{v}$ *solving system (4) for all i (with possible defaults).*

Proof. See Appendix G.1 □

The proof of Lemma 1 rests on the complementarities between banks' values, that would imply – in case of multiplicity – a minimum and a maximum configurations solving the system. Now, the total equity invested in the financial system is identical in both configurations, while the debt repayment would be larger in the maximum configuration, due to a larger number of survivors. This means there would be less wealth to distribute in the maximum configuration

14. The largest eigenvalue of any sharing matrix $\mathbf{A}$ is lower than unity (as the sum of every column is lower than 1). Therefore, $(\mathbf{I} - \mathbf{A})^{-1} = \sum_{q=0}^{\infty} \mathbf{A}^q$.

than in the minimum configuration, which contradicts that values in the maximum configuration are larger than that of the minimum configuration. Hence, both configurations coincide, implying uniqueness.[15]

2.3. The structure of risk

As explained in the Introduction, we focus here on extreme and rare events, likely to put one institution into financial distress. We therefore assume that only one institution can be hurt by this large negative shock. We however allow this shock to have a negative impact on other institutions' risky investment (on top of the effect going through cross-shareholding) for example through a fire-sale mechanism.

With probability $1-q$, the system is not stressed and the return on every banks risky asset equals $r>1$. However, with probability q, the financial system is stressed: the return on the risky investment of all banks falls to $\underline{r}<r$, and a large negative shock hits a single bank at random (with uniform probability); the bank hit by the shock suffers a stochastic loss $\tilde{s}$, defined on the non-negative support $[s_0,+\infty)$, $s_0>\underline{r}-1$ (leading to $\mu_i<1$), with cumulative function F and an average value $\overline{s}$.[16] Formally, we assume that, for every bank i:

$$\tilde{\mu}_i=\begin{cases} r & \text{with probability } 1-q \\ \underline{r} & \text{with probability } \frac{n-1}{n}q \\ \underline{r}-\tilde{s} & \text{with probability } \frac{q}{n} \end{cases} \tag{6}$$

Assumption 2. Bank-specific risky assets have positive expected values: $\mathbb{E}(\tilde{\mu}_i)>1\ \forall i$, and only the bank hit by the negative shock potentially defaults: $\underline{r}>1$.

Assumption 2 implies that investment in the risky assets is still worth it.

Assumption 3. Only the bank hit by the negative shock potentially defaults: $\underline{r}>1$.

Together with Assumption 1, Assumption 3 guarantees that v_i is positive when i is not hit by the large negative shock, even in a stressed

15. Complementarity in values also allows to build a simple algorithm that pins down the equilibrium set of surviving banks. Start with an initial set containing all banks with positive constant h_i, and compute their values in this initial setting. Then extend the set by systematically testing neighbors as newcomers, and check whether each newcomer has a positive value. If so, integrate it in the set of survivors. Such an algorithm is rather efficient as once a newcomer is surviving given the current set of survivors, it never leaves the building set of survivors. The set of survivors can then only be enlarged during the process.

16. This structure of risks echoes that of Cabrales et al. (2017) who model rare and large shocks on gross return through: a deterministic return with fixed probability, and two alternatives with either a small or a large shock.

environment. Note that Assumption 3 renders asset diversification through cross-shareholding rational.

2.4. Banks' behavior and prudential regulation

Regarding away agency issues inside the bank, we assume that managers and equity-holders objectives are aligned. Each bank then acts as risk-neutral and maximizes its expected equity value $\mathbb{E}(v_i)$.[17] The network of shareholding being assumed exogenous in the main part of the paper, the bank's behavior consists in allocating its resources u_i between the risk free asset and its specific risky asset. Using equation (3), this comes to:

$$\max_{z_i \in [0,u_i]} (\mathbb{E}(\tilde{\mu}_i) - 1)z_i + \eta_i + \sum_{j \neq i} a_{ij} v_j^+ \tag{7}$$

and by Assumptions 2 and 3, an unregulated bank optimally chooses to allocate all its resources toward risky asset: $z_i^{*u} = u_i \ \forall i$ (as $\mathbb{E}(\tilde{\mu}_i) > 1$ banks' objective is increasing in z_i). Note that $h_i < 0$ as soon as $\mu_i < \rho d / u_i$ (in this case, the survival of bank i depends upon the shareholding network).

Assume now that the regulator wishes to dampen this level of risk-taking to alleviate the social cost of default (e.g. in terms of trust in the banking system) and the cost of ex-post intervention (notably related to moral hazard considerations). Consistently with the actual regulation, we assume that the regulator sets a maximal acceptable probability of default common to all banks.[18] We discuss in section 5.1 how this objective can be achieved using capital or cash (reserve) requirement, and in section 5.2 to what extend a differentiated regulation by bank can help. Denoting by β the maximal acceptable probability of default set by the regulator, the problem of regulated banks is: $\forall i$

$$\begin{aligned} \max_{z_i \in [0,u_i]} \quad & \mathbb{E}(\tilde{v}_i) \\ \text{s.t.} \quad & \mathbb{P}(\tilde{v}_i < 0) \leq \beta \end{aligned} \tag{8}$$

where $\tilde{v}_i = (\tilde{\mu}_i - 1)z_i + \eta_i + \sum_{j \neq i} a_{ij} v_j^+$.

As clear from the above program, the optimal investment chosen by each bank then depends on the entire shareholding network $\mathbf{A}$. We characterize in

17. Our analysis remains valid with risk averse banks, as soon as the default probability resulting from their unregulated choice of risky investment is higher than the one generating the maximum acceptable default probability for the regulator.

18. This assumption echoes the use of the notion of "Value-at-risk" by financial regulators (e.g. in Basel 3 and Solvency 2). The Value-at-risk is defined by the Basel Committee on Banking Supervision as "A measure of the worst expected loss on a portfolio of instruments resulting from market movements over a given time horizon and a pre-defined confidence level" BCBS (2019). The maximal acceptable probability of defaults can here be understood as the complement to the confidence level defined by the regulator.

the next section these optimal levels of risk-taking and discuss in section 4 how the network structure impacts them.

3. Optimal risk-taking

In this section, we solve the system of optimal risk-taking for regulated banks and highlight that the shareholding network impacts it twice: (i) through a risk-sharing effect stemming from complementarities in risk-taking decisions and (ii) through a resource effect that creates heterogeneity among institutions.

3.1. Characterization

First, as noted above, the objective of bank i is increasing in its risk-taking decision z_i. For any interior solutions $z_i^* \in (0, u_i) \forall i$, the optimal levels risk-taking are therefore obtained through the system of binding constraints:

$$\mathbb{P}(\tilde{v}_i < 0) = \beta \ \forall i. \tag{9}$$

Given the structure of risk and by Assumption 2, bank i's value can only be negative when it suffers the large negative shock on its asset, in which case the values of the other banks are necessarily positive (and $\mu_j = \underline{r}$). Equation (9) then becomes:

$$\frac{q}{n}.\mathbb{P}\left((\underline{r} - \tilde{s} - 1)z_i^* + \eta_i + \sum_{j \neq i} c_{ij}\left[(\underline{r} - 1)z_j^* + \eta_j\right] < 0\right) = \beta \ \forall i \tag{10}$$

with $c_{ij} = m_{ij}/m_{ii}$ (recall here that $\mathbf{M} = (\mathbf{I} - \mathbf{A})^{-1}$).

Now define $t_{1-\frac{n\beta}{q}}$ as the $(1 - \frac{n\beta}{q})$th quantile of the distribution of $\tilde{s}$ and $\ell = \underline{r} - t_{1-\frac{n\beta}{q}}$; ℓ can then be understood as the Value-at-Risk at level $(1 - \beta)$ of each bank (see footnote 18). Equation (10) then writes:

$$(\ell - 1)z_i^* + \eta_i + \sum_{j \neq i} c_{ij}\left[(\underline{r} - 1)z_j^* + \eta_j\right] = 0 \ \forall i \tag{11}$$

Assumption 4. Regulation is constraining, which corresponds to ℓ to lower than 1. This amounts to a low enough value of β, the maximal acceptable probability of default set by the regulator.[19]

From equation (11), the optimal level of risk-taking for on isolated bank is $z_i^* = (e_i - (\rho - 1)d_i)/(1 - \ell)$, and this risk-taking level is positive under

19. In the current regulation, β is set to 1% in the banking sector (Basel 2) and 0.5% in the insurance industry (Solvency 2).

Assumption 1 and Assumption 4. Moreover, under Assumption 4, equation (11) reflects that optimal risk-taking decisions by bank are strategic complements:

$$z_i^* - \sum_{j \neq i} \varepsilon c_{ij} z_j^* = \frac{\eta_i + \sum_{j \neq i} c_{ij} \eta_j}{1 - \ell} \tag{12}$$

with $\varepsilon = (\underline{r} - 1)/(1 - \ell) > 0$ under Assumption 4 and $c_{ij} = m_{ij}/m_{ii} > 0$.

This pattern of strategic complementarities stems from the structure of risk.[20] As only one bank – say bank i – suffers a negative shock, the other banks in the network always provide support to bank i through cross-shareholding links. The value received by bank i through its shares of other banks being increasing in their own investment in risky asset (as $\underline{r} > 1$ by Assumption 2), the higher this investment the higher bank i's investment in its risky asset, for a given default probability.

The complementarities, together with the upper bounds on z_is, guarantee the existence of solution $\mathbf{z}^*$ to the system of programs (8) $\forall i$. This solution is characterized in the following theorem:

THEOREM 1. *The unique interior solution for the optimal levels of risk-taking (that satisfy the system of programs (8) $\forall i$) writes:*

$$\mathbf{z}^* = (\mathbf{I} - \varepsilon \mathbf{C})^{-1} (\mathbf{I} + \mathbf{C}) \mathbf{n} \tag{13}$$

with $\mathbf{n} = \left(\frac{\mathbf{e} - (\rho - 1)\mathbf{d} + (\mathbf{P}^T - \mathbf{P})\mathbf{1}}{1 - \ell} \right) > 0$*. Under Assumption 1, the general solution writes* $\mathbf{z}^* = \min \Big(\mathbf{u}, (\mathbf{I} - \varepsilon \mathbf{C})^{-1} (\mathbf{I} + \mathbf{C}) \mathbf{n} \Big)$.

Proof. Expression (13) is just the matrix form of the system of equations (12). Uniqueness is guaranteed by $\mathbf{n} > 0$ and $\varepsilon > 0$ (see Belhaj et al., 2014), a direct implication from Assumption 1 and Assumption 4. □

REMARK 1. *Multiple shocks.* Allowing for more than one shock makes the network less useful to banks that suffer the shocks and thereby leads to more restrictions on risk-taking. More precisely, there is strategic substitutability between risk-taking levels of the two shocked banks, what can lead to multiple equilibria. Appendix D characterizes optimal risk-taking when two banks can be hit by a negative shock (of the same size) and gives an example leading to multiple equilibria.

From now on we assume interior solutions. The interior solution $\mathbf{z}^*$ defines an original centrality measure that expresses banks' optimal risk-taking level as a function of their position in the (weighted) network of cross-shareholding

20. In Appendix E, we show that risk-taking decisions can become strategic substitute when banks can invest in each others' risky assets.

$\mathbf{A}$ (recall here that $c_{ij} = m_{ij}/m_{ii}$ and that $\mathbf{M} = (\mathbf{I} - \mathbf{A})^{-1}$). In the following, we focus on this solution, which is obtained when ℓ is high enough, that is when the maximal acceptable probability of default β is low. We now analyze the characteristics of the centrality measure $\mathbf{z}^*$ as a function of the network topology.

3.2. Network effects

As said earlier, equity-holdings impact optimal risk-taking twice: (i) through the share-holding matrix via $\mathbf{C}$, but also (ii) directly through the accounting balance via $(\mathbf{P}^T - \mathbf{P})\mathbf{1}$. This second effect comes from differences in the resources that can be allocated toward the risky asset, when investments in equities by banks don't balance: $u_i = e_i + d_i + \sum_{j\in\mathcal{I}} p_{ji} - \sum_{j\in\mathcal{I}} p_{ij}$. The quantity $(\mathbf{P}^T - \mathbf{P})\mathbf{1}$ reflects the difference between the investment of other banks in bank i's equity and the investment of bank i in other banks' equities.

Definition 1. The optimal risk taking levels $\mathbf{z}^*$ can be decomposed into:

- a pure risk-sharing effect

$$\mathbf{z}^*_{\mathbf{RS}} = (\mathbf{I} - \varepsilon\mathbf{C})^{-1}(\mathbf{I} + \mathbf{C})(\mathbf{e} - (\rho - 1)\mathbf{d})/(1 - \ell) \tag{14}$$

- a resource effect

$$\mathbf{z}^*_{\mathbf{RE}} = (\mathbf{I} - \varepsilon\mathbf{C})^{-1}(\mathbf{I} + \mathbf{C})(\mathbf{P}^T - \mathbf{P})\mathbf{1}/(1 - \ell) \tag{15}$$

with $\mathbf{z}^* = \mathbf{z}^*_{\mathbf{RS}} + \mathbf{z}^*_{\mathbf{RE}}$; the latter effect being null for bank i when $\sum_j p_{ij} = \sum_j p_{ji}$.

We start by analyzing the pure risk-sharing effect, before accounting for the effect going through resources for specific network topologies. It is first interesting to note that

$$\lim_{\varepsilon\to 0} \mathbf{z}^*_{\mathbf{RS}} = (\mathbf{I} + \mathbf{C})(\mathbf{e} - (\rho - 1)\mathbf{d})/(1 - \ell) \tag{16}$$

In our setting $\varepsilon \to 0$ when $|\ell|$ is large, that is when β is low, which corresponds to situations a tight regulation. Recalling that $c_{ij} = m_{ij}/m_{ii}$ and denoting by $b_i^O = \sum_j m_{ij}$ the out-ward Bonacich centrality of bank i in the cross-shareholding network $\mathbf{A}$, we the obtain the next proposition:

Proposition 1. *When banks are homogeneous in terms of external financing ($e_i = e$ and $d_i = d$ $\forall i$) and when regulation is tight ($\varepsilon \to 0$), the pure risk-sharing effect of optimal risk-taking ($\mathbf{z}^*_{\mathbf{RS}}$) is proportional to*

$$(\mathbf{I} + \mathbf{C})\mathbf{1} = \left(\frac{b_i^O}{m_{ii}}\right)_{i\in\mathcal{I}} \tag{17}$$

that is, to the ratio of out-ward Bonacich centrality over self-loop centrality in cross-shareholding network $\mathbf{A}$.

Bonacich centrality aggregates the share of other banks' values got by a bank ($b_i^O = \sum_j m_{ij}$ and $\mathbf{M} = \sum_{q=0}^{\infty} \mathbf{A}^q$). Banks with higher Bonacich centrality receive more from others through the shareholding network and can therefore take more risk (for a given probability of default). Now, the network may also make a bank more exposed to its own value, and therefore to its own level of risk-taking, through self-loop (m_{ii}). Banks with higher self-loop centrality then suffer more from a shock on their risky asset and can therefore take less risk (for a given probability of default). Proposition 1 states that, under tight regulation, the pure risk-sharing effect results from a trade-off between these two effects.

More generally, the above discussion questions the effect of the size of shareholdings on optimal risk-taking. The next Lemma helps understanding the part that goes through the pure risk-sharing effect (that is when we abstract from the effect of the shareholding matrix on banks' resources).

LEMMA 2. *If* $\mathbf{A}' \leq \mathbf{A}$, *then* $\mathbf{C}' \leq \mathbf{C}$.

Proof. See Appendix G.2 □

The proof of Lemma 2 relies on the path-product property of inverse M-matrices.[21] Focusing on interior solutions, the matrix $(\mathbf{I} - \varepsilon\mathbf{C})^{-1}$ is not explosive and an increase in $\mathbf{C}$ implies an increase in $\mathbf{z}$. Increasing shareholding has an ambiguous effect a priori as (i) it propagates the negative shock on one bank's asset to the whole network, but (ii) it propagates the (necessarily positive) value of other banks to the bank hit by the negative shock. The next proposition shows that the second effect dominates when we regard away resource effects:

PROPOSITION 2. *The pure risk-sharing effect of optimal risk-taking* $\mathbf{z}^*_{\mathbf{RS}}$ *increases when cross-shareholding increases.*

Note that any change of the cross-shareholding matrix $\mathbf{A}$ potentially generates a resource effect. To address this effect, we study two types of modification of matrix $\mathbf{A}$: integration and diversification. By integration we mean that matrix $\mathbf{A}$ increases through an increase in the investment of one bank in the equity of some among its existing "neighbors"; by diversification we mean new connections, that is investments in the equity of a new bank. To

21. An M-matrix is a n-by-n matrix with nonpositive off-diagonal entries and has an entry-wise nonnegative inverse, as $(\mathbf{I} - \mathbf{A})$ in our case. $\mathbf{M} = (\mathbf{I} - \mathbf{A})^{-1}$ is then an inverse M-matrix.

analyze these various channels, we will model a binary graph $\mathbf{G}$ that supports the network of cross-investment in equity $\mathbf{P}$: $p_{ij} > 0 \Leftrightarrow g_{ij} = 0$. This allows us to analyze optimal risk-taking for particular network topologies and to analyze integration and diversification.

4. Network topology and optimal risk-taking

As explained above, to isolate the effect of network topology, we now assume that banks' balance sheets are identical except for cross-shareholding ($e_i = e$ and $d_i = d\ \forall i$) and introduce a binary network $\mathbf{G}$ supporting cross-investment in equities. We moreover denote by $\delta_i^O = (\mathbf{G1})_i$ and $\delta_i^I = (\mathbf{G}^T\mathbf{1})_i$ the out- and in-degree of bank i in this network. To relate $\mathbf{G}$ with investment in equities between banks, we analyze two specific structures.

- Either $p_{ij} = p \cdot g_{ij}$: the investment of each bank in another bank is fixed, with a ticket of size p. We then refer to **fixed participation** and

$$a_{ij} = \frac{1}{\delta_j^I + \frac{e}{p}} \cdot g_{ij} \tag{18}$$

  In this case, the resource effect is null ($\mathbf{P}^T = \mathbf{P}$ and $\mathbf{z}^* = \mathbf{z}^*_{\mathbf{RS}}$) as soon as $\delta_i^O = \delta_i^I\ \forall i$, that is when the number of banks in which one bank invests is equal to the number of banks that invest in it. This case not only covers all undirected graph (for which $\mathbf{G}^T = \mathbf{G}$), but also a class of directed graphs, as those presented in Figure 3.

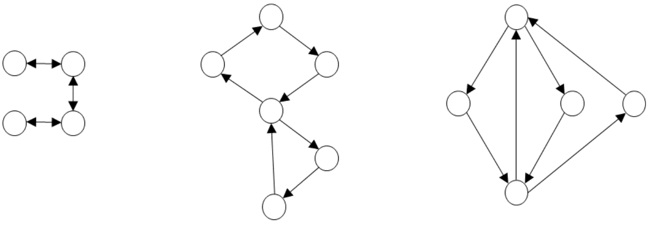

FIGURE 3. Example of graphs in which, for all nodes, in-degree equals out-degree

- Or $p_{ij} = (p \cdot g_{ij})/\delta_i^O$: the total investment of each bank in the network is fixed to p and equally shared among the banks it invests in. We then refer to **fixed investment** and

$$a_{ij} = \frac{1/\delta_i^O}{\varphi_j^{IO} + \frac{e}{p}} \cdot g_{ij} \tag{19}$$

  where $\varphi_j^{IO} = \sum_{k\in\mathcal{I}} 1/\delta_k^O \cdot g_{kj}$ represents the sum of inverse out-degrees of in-neighbors ("IO" stands for In-neighbor Out-degree). In this case, the resource effect is null ($\mathbf{P}^T = \mathbf{P}$ and $\mathbf{z}^* = \mathbf{z}^*_{\mathbf{RS}}$) when $\mathbf{G}^T\mathbf{1} = \mathbf{G}\mathbf{1} = \delta\mathbf{1}$; that is when all banks invest in the same number of banks ($\delta_i^O = \delta \ \forall i$), which is also the number of banks that invest in each ($\delta_i^I = \delta \ \forall i$).

It is worth noticing here that optimal risk-taking levels can be differentiated even on regular undirected structures and without resource effect (that are graphs on which links are reciprocated $\mathbf{G}^T = \mathbf{G}$ and where agents have the same number of links $\mathbf{G1} = \delta$). For example, on the structure represented in the left panel of Figure 4, the optimal risk-taking of agent 1 (z_1^*) is higher than the one of agent 2 (z_2^*) in any configuration with $e_i = e$ and $d_i = d \ \forall i$. This is due to the fact that agent 2 has a higher self-loop centrality than agent 1: $m_{22} > m_{11}$ (by construction all nodes have the same Bonacich centrality on regular undirected networks). To obtain homogeneous risk-taking, the key is having a matrix $\mathbf{M}$ with homogeneous diagonal entries. In Appendix A, we illustrate how this latter property entails that optimal risk-taking is independent of network position on the subclass of Directed Strongly Regular Graphs (called DSGRs thereafter) – both for fixed participation and fixed investment. In DSRGs, on top of having the same degree, all linked nodes have the same number of common friends, and all pairs of unlinked nodes also have the same number of common friends; The right panel of Figure 4 is a example of an undirected strongly regular graph. For DSRGs, we are able to show that the level of optimal risk-taking, identical for all banks, is increasing in the ratio of out-ward Bonacich centrality over self-loop centrality, this ratio being also the same for all banks.

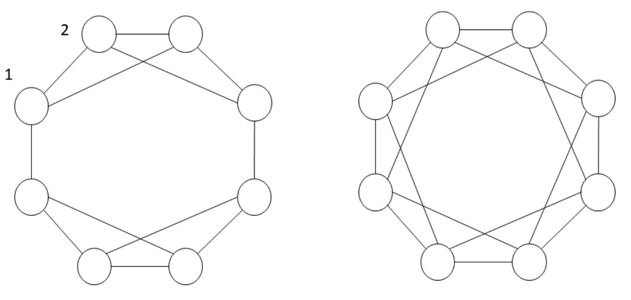

FIGURE 4. A regular (left panel) and a strongly regular (right panel) graph with eight agents

The specification of a relationship between cross-shareholding ($\mathbf{A}$) and the supporting binary graph ($\mathbf{G}$) allows us (i) to analyze what happens when investments in other banks (p) are small (in Section 4.1), (ii) to fully determine optimal risk-taking on particular structures (in Section 4.2) and (iii) to further analyze the effect of integration and diversification (in Section 4.3)

4.1. Asymptotic results

We consider low levels of cross-shareholding, by studying the case where p is close to 0. This corresponds to low investment in each bank in the fixed participation case and to low total investment in the fixed investment case. In both cases, from equations (18) and (19), we get $\mathbf{A} = p/e\mathbf{G} + o(p)$ and therefore $\mathbf{M} = \mathbf{I} + p/e\mathbf{G} + o(p)$. This allows us to state the following result that highlights again the importance of centrality and the role of resource effects.

Remark 2. When cross-shareholding is low (p close to 0), optimal risk-taking levels are approximated by

$$z_i^* = \frac{1}{1-\ell}\left(\eta + \eta(1+\varepsilon)\frac{p}{e}\delta_i^O + p(\delta_i^I - \delta_i^O)\right) + o(p) \tag{20}$$

in the fixed participation case, and by

$$z_i^* = \frac{1}{1-\ell}\left(\eta + \eta(1+\varepsilon)\frac{p}{e} + p.(\varphi_i^{IO} - 1)\right) + o(p) \tag{21}$$

in the fixed participation case where $\eta = e - (\rho - 1)d$.

It is salient from Result 2 that in the fixed participation case, risk-taking is increasing in in-degree (that is in the number of banks that invest in a given bank) but not necessarily in out-degree (that is the number of banks in which a given bank invests). This last result is fully driven by the resource effect. Indeed, in absence of resource effect, $\delta_i^I = \delta_i^O$ and z_i^* is increasing in out-degree.

Regarding the fixed investment case, we obtain from equation (21) that the banks allowed to take more risk are those that receive investment from a large number of banks that themselves invest in few others (that is banks with a large number of in-neighbors with low out-degrees). For example, it can correspond to the case of core banks in core-periphery structures,[22] that have been shown to represent pretty well real financial networks, both in terms of inter-bank lending (see e.g. Craig and von Peter, 2014) and of cross-shareholding (Rotundo and D'Arcangelis, 2014). In the following section, we analyze the case of the simplest core-periphery networks, that are star networks.

4.2. Star networks

We focus on Complete Core-Periphery networks. In such structures, banks in the core are linked to all banks, and there is no link between any pair of peripheral banks. On these networks, it is not obvious that the ratio of Bonacich

22. Core-periphery networks are networks in which highly interconnected nodes – called the core – coexist with nodes loosely connected (both to the core and between them) – called the periphery

centrality over self-loop centrality is favorable to central agents. For instance, in the undirected four-star network, this ratio can be favorable to peripheral agents under sufficiently high values of interaction.[23]

Still, the next proposition highlights that our model induces a specific interaction pattern through the definition of cross-shareholding (base of cross-investment), that leads to a higher ratio b_i/m_{ii} for the center of stars. This leads to more risk-taking for the center in the fixed participation case, in which resource effects are absent.

Proposition 3. *Consider an undirected star network with n banks ($\mathbf{G}^T = \mathbf{G}$) and the case of fixed participation ($p_{ij} = p \cdot g_{ij}$). The optimal risk-taking of the center is higher than that the one of peripheral banks.*

Proof. See Appendix G.3 □

The proof rests on the asymmetry of matrix $\mathbf{A}$ and its specification. We first prove that the ratio b_i^O/m_{ii} is larger for the center as compared to any peripheral bank. By Proposition 1, this statement proves the result for sufficiently large negative shocks. We then extend the proof to arbitrary values of ε by using the ranking of centrality in an argument by induction.

The above proposition is limited to the case of fixed participation, in which there are no resource effects when the network is undirected. This result still holds in fixed investment case, for which resource effects exist even on undirected graph.

Proposition 4. *Consider an undirected star network with n banks ($\mathbf{G}^T = \mathbf{G}$) and the case of fixed investment ($p_{ij} = p \cdot g_{ij}/\delta_i$). The optimal risk-taking of the center is higher than that the one of peripheral banks.*

Proof. See Appendix G.3 □

This result can be deduced from the preceding proposition because, in star networks with fixed investment, the resource effect is also in favor of the center. Indeed, the center receives $(n-1) \cdot p$ through the shareholding network (when it invests p) whereas a periphery bank only receives $p/(n-1)$ from the center (when it invests p).

23. Let $\Delta = 0.44$, $\mathbf{G}$ the adjacency matrix of the 4-player star network where agent 1 is the central agent, $\mathbf{M} = (\mathbf{I} - \Delta\mathbf{G})^{-1}$ and $\mathbf{b} = \mathbf{M1}$. Then $b_1/m_{11} = 2.32$ whereas $b_j/m_{jj} = 2.35$ for $j \neq 1$.

4.3. Comparative statics

More generally, using a binary graph to support the shareholding network allows us to analyse the comparative statics obtained in Section 3 into details. Proposition 2 states that an increase in the cross-shareholding matrix $\mathbf{A}$ increases the pure risk-sharing effect $(\mathbf{z}^*_{RS})$. We analyze now various mechanisms that can alter $\mathbf{A}$ and their full impact accounting for resource effects.

Through the specification of the relationship between investment in cross-shareholding and the supporting graph, we are able to fully disentangle the effect of integration (stronger links) and diversification (more links).[24] Integration corresponds to an increase in p in the fixed participation case; whereas the pure diversification effect can be obtained by adding a new link in the fixed investment case, holding constant the total investment of each bank in the network.

We start by considering integration. First, as the resource effect is null for all banks in the case of full participation when the graph is undirected ($\mathbf{G}^T = \mathbf{G}$), a direct corollary of Proposition 2 is:

Corollary 1. *Integration (an increase in p in the fixed participation case) increases optimal risk-taking when the network is undirected.*

The result however doesn't extend to directed network, that is when shareholding links are not reciprocated. As shown in Appendix B, integration can then decrease the contribution of resource effects to total optimal risk-taking $(\mathbf{1}^T\mathbf{z}^*_{RE})$ in directed networks as the one presented in Figure 5.

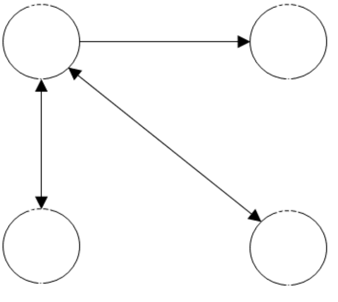

Figure 5. An directed network with 4 banks

To further explore the effect of integration, and study whether this (possibly negative) resource effect can outweigh the positive risk-sharing effect, we rely on

24. We use here the same nomenclature as Elliott et al. (2014).

simulations on random graphs. Following Elliott et al. (2014), we first generate random graphs through the Erdös-Renyi procedure: for a fixed average out-degree $\delta \in \{1, \cdots, n-1\}$, each (directed) link is created with a probability of $\delta/(n-1)$. We alternatively generate random networks with power law degree distributions, using a Barabasi-Albert like procedure (that follows a preferential attachment mechanism). More precisely, we rely on the following procedure:

1. Node 2 is attached to node 1 with probability 1. This gives $\mathbf{G}_2$ as the empty matrix plus the bilateral link 21 ($g_{12} = g_{21} = 1$).
2. For node $t = 3, 4, \cdots, n$, the probability to be linked with nodes $j = 1, 2, \cdots, t-1$ is equal to

$$\mathbb{P}_{tj}(\tau) = \frac{\tau}{n-1} \cdot \frac{\delta_j^{(t-1)}}{\sum_{k=1}^{t-1} \delta_k^{(t-1)}} \tag{22}$$

   where degrees are defined over the network $\mathbf{G}_{t-1}$ (i.e. $\delta^{(t-1)} = \mathbf{G}_{t-1}\mathbf{1}$) defined as $\mathbf{G}_t = \mathbf{G}_{t-1}$ plus the set of links created at period t. Parameter $\tau > 0$ controls for the average density of the network. To model directed networks, we randomly draw the direction on the link, once one in created.

In the following, we present the results for those two procedures. We calibrate our simulations with $n = 20$, $\rho = 1.01$, $\underline{r} = 1.01$, $\ell = 0.85$, $d = 1500$ and $e = 100$ (the two last parameters are consistent with the data we use in Section 5.3).[25] Figure 6 depicts the effect of integration on average optimal risk-taking. With the above parameters set, and the two forms of random networks, we draw 1,000 networks for each value of p in the fixed participation case ($p_{ij} = p \cdot g_{ij}$), and plot the average optimal risk-taking among banks and among runs. Figure 6 presents the results for an average degree $\delta = 5$ in the Erdös-Renyi procedure and an average density $\tau = \delta \cdot n = 100$ in the Barabasi-Albert procedure.[26]

Figure 6 highlights several features of our model. First, it illustrates that the cross-shareholding network has significant effects on average risk-taking. In our parameters set, average z_i^* can increase by more that 50% with respect to the case of isolated banks. Second, it shows that the positive risk-sharing effect of integration dominates on average the resource effect for random graphs, leading to an increase in average risk-taking. Third, it confirms that heterogeneity in network positions (which is higher in the Barabasi-Albert) tends to decrease average risk-taking through resource effects, and soften the effect of integration (this mechanism is also highlighted in Appendix B about the network presented in Figure 5).

We now turn to the study of diversification. As explained above, diversifying means distributing the same resource over a larger number of banks,

25. The maximum acceptable probability of default β is obtained for a depreciation on ones risk asset of 15%.

26. The average number of links in then the same under the two procedures.

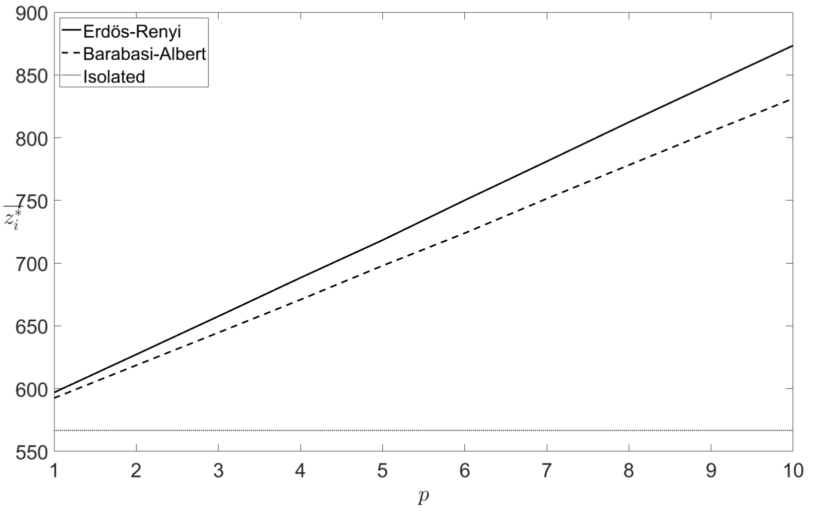

FIGURE 6. The effect of integration on average optimal risk-taking

what necessarily conveys important resource effects. Formally, there is more diversification in $\mathbf{P}$ than in $\mathbf{P}'$ whenever (i) for all (i, j): $p'_{ij} > 0$, $p_{ij} \leq p'_{ij}$, and (ii) there exists (i, j): $p_{ij} > p'_{ij} = 0$ (Elliott et al., 2014). To rigorously disentangle diversification from integration, we focus on $\mathbf{P}$ and $\mathbf{P}'$ such that $\mathbf{P1} = \mathbf{P}'\mathbf{1}$ by analyzing the effect of new links in the fixed investment case. In this case, a new directed link from i to j shifts away resources from all bank i's neighbors and increases resources of bank j. Furthermore, this lowers all cross-holding shares already invested in bank j.

Figure 7 presents the results of simulations on random graphs. As above, we use both Erdös-Renyi and Barabasi-Albert procedures (to draw $1,000$ network per parameters set), and plot the effect of diversification on the average optimal risk-taking. We keep the same set of parameters: $n = 20$, $\rho = 1.01$, $\underline{r} = 1.01$, $\ell = 0.85$, $d = 1500$, $e = 100$ and set $p = 50$ as the fixed investment of each bank in the network. We analyse values of δ from 7 to 20 (and corresponding values of $\tau = \delta \cdot n$). Below $\delta = 7$, some banks can have no out-neighbour, in which case a new link also entails an integration effect ($\sum_j p_{ij}$ increasing from 0 to p).

Figure 7 highlights that pure diversification has no significant effect on average risk-taking. The above mentioned effects therefore cancel each other. Moreover, as in the case for integration, the higher heterogeneity in degrees and thus in resources effects in the case of Barabasi-Albert graphs tends to lower average optimal risk-sharing. The absence of pure diversification effect on average in confirmed by a deeper analysis of the effect of link addition on given network structure. More precisely, we performed complementary simulations in which, starting from a random network, we add randomly one link per

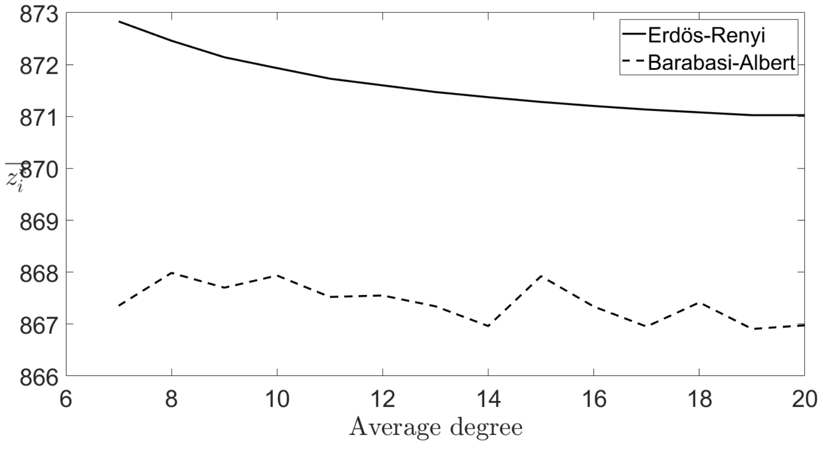

FIGURE 7. The effect of diversification on average optimal risk-taking

bank, and find no significant effect on the average risk-taking (for the above parameters and average degree from 5 to 20, the effect on the average risk-taking is always lower than 0.1 percent).

Taken together, Figure 6 and 7 suggest that integration and diversification have differentiated impacts on optimal risk-taking. This echoes the findings of Elliott et al. (2014), although we find here that pure diversification have no significant effect.[27] This absence of significant effect on average can still hide significant changes at the bank level. We analyze this on real data in the next section. This will also allow us to introduce heterogeneity in banks' balance sheets.

Before turning to this real-data analysis, we first discuss how policy interventions can implement and adjust optimal risk-taking.

5. Policy interventions

5.1. Prudential requirements

We study how a financial regulator can implement optimal risk-taking levels using prudential requirements. Equation (13), characterizing optimal risk-taking, defines a relationship between one bank's initial asset (at $t = 0$, through z_i) and its liability (through e_i and d_i):

$$\mathbf{z}^* = (\mathbf{I} - \varepsilon \mathbf{C})^{-1}(\mathbf{I} + \mathbf{C}) \left(\mathbf{e} - (\rho - 1)\mathbf{d} + (\mathbf{P}^T - \mathbf{P})\mathbf{1}\right) / (1 - \ell) \tag{23}$$

27. Elliott et al. (2014) find (different) non-linear effects of integration and diversification on contagion, but don't separate pure diversification from integration as we do here using a fixed investment framework.

It can therefore be implemented by the regulator either by constraining the asset side ex-ante, or by constraining the liability side.

Regarding the latter, equation (23) can also be written to express equity $\mathbf{e}$ as a function of $\mathbf{z}$ (and the other parameters). It then specifies the minimal level of equity e_i a bank should have to be allowed to invest z_i in its risky asset (see equation (G.29) in the Appendix). The optimal levels of risk-taking can then be achieved by setting *capital requirements* (that are lower bounds on the level of external equity) that depend on risk-exposure $\mathbf{z}$, debt $\mathbf{d}$, and the shareholding network $\mathbf{P}$.

Regarding the asset side, the accounting equation at $t = 0$: $\mathbf{x} + \mathbf{z} = \mathbf{u}$ (see equation (1)) allows to relate optimal risk-taking with cash (or risk-free asset) holdings at $t = 0$. Therefore, the optimal levels of risk-taking can be achieved by setting *cash (or reserve) requirements*, that are lower bounds on the level of investment in risk-free asset, that depend on the levels of equity $\mathbf{e}$, debt $\mathbf{d}$, and the shareholding network $\mathbf{P}$.[28]

5.2. Complementary policies

Prudential requirements or bounds on risk-taking are costly in terms of the net value created by the banking sector (assuming $\mathbb{E}(\tilde{\mu}) > 1$). One can therefore wonder whether targeted interventions on some particular banks can be useful to decrease requirements on others, keeping probabilities of default constant. We analyze two such interventions: equity injection, and tighter regulation on one bank.

We first analyze the effect of equity injection in one given bank on the sum of investments in risky assets. In other words, we study here how cash injection by the regulator in the liability side of the balance sheet can affect prudential regulation on the asset side. To simplify, suppose that the regulator can choose only one bank in which to inject equity with the objective of maximizing aggregate investment in risky asset, for a given probability of default β. The next proposition defines the bank it should target. Defining matrix $\mathbf{W}$ such that $w_{ii} = m_{ii}$ and $w_{ij} = -\varepsilon m_{ij}$ for all i, j, and vector $\mathbf{w}^S = \mathbf{W}^{-1}\mathbf{1} = (w_i^S)_{i\in\mathcal{I}}$, we obtain:

PROPOSITION 5. *The bank whose equity injection has the highest effect on total investment in risky assets ($\sum_i z_i^*$) is the one with the highest index $m_{ii}w_i^S$.*

Proof. See Appendix G.4 □

Proposition 5 allows to determine the optimal bank to inject equity based on network properties and relative magnitude of the negative shock (through

28. Note that, in a richer model, these requirements could also be used to deal with other issues such as liquidity risk or monetary policy.

parameter ε) only, even with heterogeneous financing e_i and d_i. We analyze how this targeted bank can change depending on the network in the next section.

We now analyze the pendant of this intervention on the asset side. Given that reduced risk-taking entails a lower need of external equities, and given the network of financial interdependencies, our aim is to determine the impact of a decrease in one bank's risk-taking level on the sum of external equities required to balancing regulation. The next proposition defines the bank whose decrease in risk taking entails the largest reduction in the need of total external equities. Denoting the share of bank i held by external shareholders by $\bar{e}_i = \frac{e_i}{\sum_j p_{ji}+e_i}$, we find:

Proposition 6. *The bank whose decrease in risky investment has the highest effect on total need of external assets ($\sum_i e_i$) is the one with the highest index $m_{ii}\bar{e}_i$.*

Proof. See Appendix G.5 □

Proposition 6 states that the choice of the targeted bank depends both on network effects through self-loop centrality m_{ii} (recall here that banks with higher m_{ii} are the those more exposed to their risky asset through loops in the network of shareholding) and on the structure of equity. This choice trades off bank's risk exposure through the network and the cost of its risk-taking in terms of external equity.

We now turn to real-data analysis to highlight the importance of the network structure both in the design of optimal risk-taking (and thus of optimal prudential regulation), and in the determination of the banks the two complementary interventions presented just above should target.

5.3. Real world networks

To build networks of cross-shareholding between real financial institutions, we rely on banks' annual financial reports (for the size of their liability) and on indicators they report to the financial regulator regarding their "interconnectedness" (for the size of investments in other institutions' equity).

More precisely, we focus on the set of banks defined as "global systemically important" by the Financial Stability Board (FSB) in 2019,[29] for which regulatory data are available through the Federal Financial Institutions Examination Council (FFIEC)[30] or the European Banking Authority (EBA)

29. see https://www.fsb.org/wp-content/uploads/P221119-1.pdf.

30. see https://www.ffiec.gov/npw/FinancialReport/FRY15Reports.

database.[31] We end up with a set of 20 banks[32] rated by the FSB in four regulatory buckets, where higher buckets correspond to higher capital requirements. For these 20 banks we retrieve data on stockholders' equity ($e_i + \sum_j p_{ij}$) and total liabilities (or total asset) from their annual report of 2018. This allows us to compute d_i as the difference between total asset liability and stockholders' equity. We use data from financial regulators (FFIEC and EBA) on "Holdings of securities issued by other financial institutions: Equity securities" (RISK-M356 for the FFIEC and GSIB-1040 for the EBA) to retrieve the quantity $\sum_j p_{ij}$. Then, we rely on assumptions on the graph structure to allocate it among other banks (assuming that all of these investments go into our set of systemic banks). This calibration aims at illustrating the impact of bank heterogeneity rather than to literally describe the financial network.

In this database, equity ($e_i + \sum_j p_{ij}$) represents about 8% of the balance sheet on average, and investment in other institutions' equity ($\sum_j p_{ij}$) less than 1% of asset.[33] In the following, we show that the form of the graph supporting cross-shareholding still significantly impacts optimal risk-taking levels, and the identity of the bank optimally targeted by our complementary interventions. To do so, we calibrate $\rho = 1.01$, $\underline{r} = 1.01$ and $\ell = 0.85$ (as in Section 4.3).

We first compare optimal risk-taking in two specific structures for graph $\mathbf{G}$: the complete graph and the star, with undirected links ($\mathbf{G}^T = \mathbf{G}$). Bilateral investments p_{ij} are therefore equal to $\sum_j p_{ij}/\delta_i$. We obtain the following findings. First, on the complete graph, optimal risk-taking equals 44.8% of bank's assets on average, with a standard deviation of 21.2% (differences between banks then mostly come from the heterogeneity in the composition of their liabilities, between debt and equity). Second, differences between optimal risk-taking on the complete graph and optimal risk-taking by each bank when it is at the center of the star appear fairly small. On average, risk-taking decreases by 1.5% when one bank is at the center of the star with respect to its level in the complete graph. This finding can be related to our result on diversification: adding links between peripheral banks in the star doesn't significantly change the optimal risk-taking level of the center. Still, when a bank is at the periphery of the star, the identity of the center is relevant. Indeed, the optimal risk-taking of a peripheral bank varies by 7% on average between its highest value and its lowest value depending on the identity of the center. This figure even

31. see https://eba.europa.eu/risk-analysis-and-data/global-systemically-important-institutions.

32. Bank of America, Bank of New York Mellon, Barclays, BNP Paribas, Citigroup, Groupe Crédit Agricole, Deutsche Bank, Goldman Sachs, Groupe BPCE, HSBC, ING Bank, JP Morgan Chase, Morgan Stanley, Santander, Société Générale, Standard Chartered, State Street, UBS, UniCredit and Wells Fargo

33. Holdings of debt securities issued by other financial institutions, that can also be retrieved on regulators' databases, represent about the same proportion as equity securities on average.

exceeds 20% for three banks: Goldman Sachs, Morgan Stanley and Société Générale.[34] This finding illustrates that the detailed structure of the graph can have substantial effects on individual optimal risk-taking. Therefore, current regulation, that relies on total investment in the equity of other banks ($\sum_j p_{ij}$) only, can be improved using detailed network data.

We finally use these data to illustrate how the network structure impacts the identity of the banks the policy interventions discussed in Propositions 5 and 6 should target. To limit comparisons, we focus on three complete-core periphery structures, varying the numbers of banks in the core based on the buckets definition of the FSB. We then alternatively model a core composed of one bank (JP Morgan Chase, the only bank in bucket 4), three banks (JP Morgan Chase plus the two banks of bucket 3: Citigroup and HSBC) and nine banks (the three previous plus the six of our database that are in bucket 2: Bank of America, Bank of China, Barclays, BNP Paribas, Deutsche Bank and Goldman Sachs). For these three networks we compute the indexes defined in Propositions 5 and 6 and find that:[35]

- when the size of the core enlarges, the optimal bank to target for equity injection (see Proposition 5) is successively JP Morgan, HSBC and BNP Paribas
- when the size of the core enlarges, the optimal bank to target for tighter regulation (see Proposition 6) is successively JP Morgan, HSBC and Goldman Sachs.

These results again highlight the importance of the shape of the financial network, on top of the individual indicators of interconnectedness used in the current regulation.

6. Extensions

We now discuss several extensions toward relaxing the main assumption of our model regarding the structure of investments.

Inter-bank debt holding. First note that our analysis extends to the modeling of inter-bank debts as long as only the bank hit by the negative shock potentially defaults. In other words, inter-bank debt positions should be such that a bank doesn't default when it doesn't invest in risky asset (see Assumption 1). In this case, we show in Appendix C that a reallocation of inter-bank debts increases average optimal risk-taking when it favors banks

34. For all three banks, the best case is when Groupe BPCE is at the center of the star. The worse case for Morgan Stanley and Société Générale is when Goldman Sachs is at the center; and the worse case for Goldman Sachs is when Barclays is at the center.

35. Remind here that m_{ij}s are defined over the matrix of shares $\mathbf{A}$ leading to exogenous values even for banks with similar position in the network $\mathbf{G}$ through differences in $\sum_j p_{ij}$.

with the highest pure risk-sharing effects (that is banks with the highest ratios of Bonacich centrality over self-loop centrality when risk is large).

Correlated investment portfolios. In the model presented above, we assume that each bank invests in a specific risky asset. Given our risk structure, this ensures that only one bank potentially defaults, and that the shareholding network always helps it. We relax this assumption in Appendix E by allowing banks to invest in each other risky assets (or projects) in the spirit of Cabrales et al. (2017). This introduces correlation in investment portfolios, making each bank less exposed to risk of its own asset, but spreading this risk among others (that then provide less support).[36] It therefore reduces the likelihood of the first failure but increases contagion. In Appendix E, we show that this extension is likely to qualitatively modify our results, as the nature of the strategic interaction between banks (in z_i^*) changes to strategic substitutability when banks invest too much of their asset into other banks' project.

Toward endogenous networks. In all the above analysis, we model only the decision of banks to allocate their resources between a risky and a risk-free asset, for a given cross-shareholding structure. Now, this structure also depends on banks' choice. We therefore analyze in Appendix F the incentives for link formation, in the simple case of bilateral links and fixed participation. When cross-shareholding is low, we find that when banks internalize the change in regulation following link formation, the set of pairwise stable networks only contains the complete network; whereas when they don't, the complete network is pairwise stable, but there can exist other stable networks defined as the union of complete components of distinct sizes. This call for more work, especially to consider higher cross-shareholding levels.

7. Conclusion

In this paper, we highlight the role of the network of cross-shareholding on optimal risk-taking by banks under prudential regulation. The two key assumptions of our model are that only one bank potentially suffers a large negative shock on its asset and that the network of cross-shareholding is exogenous. We characterize these optimal levels when the regulator fixes a maximum acceptable probability of default and show that it depends on the network twice: (i) through a pure risk-sharing effect and (ii) through a resource effect as cross-shareholding changes the size of the balance-sheet of each bank. We show that the pure risk-sharing effect is aligned with the ratio of Bonacich centrality over self-loop centrality when risk is large; and that in star networks, banks at the center are optimally allowed to take more risk. In more general

36. Note here that this corresponds to correlation as portfolio diversification and not as portfolio concentration (e.g. market portfolio) that will always lead to more regulation in our case.

networks, resource effects however complexify the analysis and we rely on simulations on random network to show that integration (an increase in the amount invested by each bank in the network) entails increased average optimal risk-taking. Turning to policy interventions, we show how the optimal levels of risk-taking can be implemented though capital or cash requirements and complemented by targeted interventions. We finally use real financial data to highlight the importance of the structure of the shareholding network, on top of the bank-specific indicators of interconnectedness used in the current regulation.

Our work can be extended in several directions. First, it would be challenging to tackle endogenous network formation for arbitrary cross-holding networks. Second, it is natural to explore possible optimal prudential regulation in conjunction with ex post regulation.

Appendix A: The case of Directed Strongly Regular Graphs

We have shown in the paper that optimal risk-taking behaviors may be differentiated in regular networks. In this sense, neutralizing network effects is more demanding. Actually, to obtain homogeneous risk-taking, the key is having a matrix $\mathbf{M}$ with homogeneous diagonal entries. In this appendix, we illustrate how this latter property entails that optimal risk-taking is independent of network position by solving for the case of Directed Strongly Regular Graphs.[37]

Definition A.1. An undirected network $\mathbf{G}$ is δ-regular whenever $\mathbf{G1} = \delta\mathbf{1}$.

Definition A.2. An undirected network $\mathbf{G}$ is strongly regular of parameters (δ, κ, θ) whenever $\mathbf{G}$ is a δ-regular graph, every pair of adjacent nodes has the same number of common neighbors κ, and every pair of non-adjacent nodes has the same number of common neighbors θ.

The class of strongly regular can moreover be extended to directed networks (see Duval 1988):

Definition A.3. An directed network $\mathbf{G}$ is strongly regular of parameters $(\delta, \kappa, \theta, t)$ if:

$$\begin{cases} \mathbf{G}^2 + (\theta - \kappa)\mathbf{G} - (t - \theta)\mathbf{I} = \theta\mathbf{J} \\ \mathbf{GJ} = \mathbf{JG} = \delta\mathbf{J} \end{cases} \tag{A.1}$$

37. A more general class of networks shares this property of homogeneous self-loop centrality: the class of distance-based regular graphs. We do not present here the results for space restriction, but the logic of our proofs extends to this latter class.

(where t represents the number of reciprocated links per node)

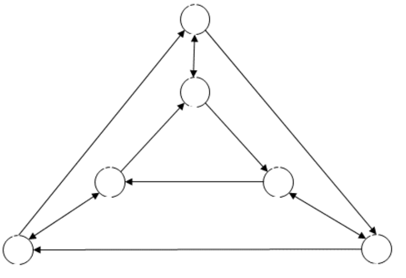

FIGURE A.1. The smallest Directed Strongly Regular Graph ($(n, \delta, \kappa, \theta, t) = (6, 2, 0, 1, 1)$)

Note first that, for both fixed investment and fixed participation, there are no resource effects ($\mathbf{P}^T = \mathbf{P}$) on directed strongly regular graphs (DSRG thereafter). We can thus focus exclusively on the pure risk-sharing effects. We know that Bonacich centralities are homogeneous in regular graphs. We show in the following that, for DSRGs, self-loop centralities ($(m_{ii})_{i\in\mathcal{I}}$) are homogeneous.

In both cases of fixed participation and fixed investment, the cross-sharing matrix $\mathbf{A}$ car be writen as $\mathbf{A} = \Delta\mathbf{G}$, with $\Delta = 1/\left(\delta + e/p\right)$ in the fixed participation case and $\Delta = 1/\left(\delta\left(1 + e/p\right)\right)$ in the fixed investment case. We note m_0 the representative diagonal entry of $\mathbf{M} = (\mathbf{I} - \delta\mathbf{G})^{-1}$ for DSRG, and b_0 the representative entry of vector $\mathbf{M1}$ (of course, m_0, b_0 depend on all parameters $(n, \delta, \kappa, \theta, t)$ defining the DSRG, we omit their enumeration for convenience). Then:

PROPOSITION A.A.1. *z_i^*s are identical in Directed Strongly regular Graphs: $z_i^* = z_0^*$ $\forall i$, and z_0^* is an increasing function of the ratio b_0/m_0.*

The intuition behind Proposition A.A.1 relies on the fact that self-loop centralities are homogeneous in DSRGs. This stems from $\mathbf{G}^2 = (\kappa - \theta)\mathbf{G} + (\delta - \theta)\mathbf{I} + \theta\mathbf{J}$, which entails that any power of $\mathbf{M}$ is a linear combination of $\mathbf{G}$, $\mathbf{I}$, $\mathbf{J}$. In the end, $\mathbf{M} = \sum_{q=0}^{+\infty} \delta^q \mathbf{G}^q$ is also a linear combination of $\mathbf{G}$, $\mathbf{I}$, $\mathbf{J}$. Now, as self-loop centralities are homogeneous across banks, matrix $\mathbf{C}$ is proportional to matrix $\mathbf{M}$. So $\sum_{q=0}^{+\infty} \varepsilon^q \mathbf{C}^q \mathbf{1}$ is a constant vector.

We now compute the optimal risk-taking z_0^* on a DSRG (with $\eta = e - (\rho - 1)d$). Given that the adjacency matrices for these networks is given by $\mathbf{A} = \Delta\mathbf{G}$, we can compute $\mathbf{M}$ analytically. Let us define α and γ such that for a DSRG of parameters $(n, \delta, \kappa, \theta, t)$, we have

$$\mathbf{G}^2 = \underbrace{(\kappa - \theta)}_{=\alpha}\mathbf{G} + \theta\mathbf{J} + \underbrace{(t - \theta)}_{=\gamma}\mathbf{I} \tag{A.2}$$

where $\gamma \geq 0$ and $\gamma > \alpha$ by construction. Then,

$$(\mathbf{I} - \Delta \mathbf{G})^{-1} = \sum_{q=0}^{\infty} \Delta^q \mathbf{G}^q = \bar{\alpha}\mathbf{G} + \bar{\theta}\mathbf{J} + \bar{\gamma}\mathbf{I} \tag{A.3}$$

with (after some computations):[38]

$$\begin{cases} \bar{\alpha} = \frac{\Delta}{1-2\Delta\alpha-4\Delta^2\gamma} \\ \bar{\gamma} = 1 + \Delta\gamma\alpha \\ \bar{\theta} = \theta\Delta^2 \left[\frac{\delta^2}{(1-\Delta\delta)(\delta^2-2\alpha\delta-4\gamma)} - \frac{4}{\delta} \cdot \frac{(\alpha^2+\gamma)(\delta+4\Delta\gamma)}{(1-2\Delta\alpha-4\Delta^2\gamma)(\delta^2-2\delta\alpha-4\gamma)} - \frac{2(\alpha+2\Delta\gamma)}{\delta(1-2\Delta\alpha-4\Delta^2\gamma)} \right] \end{cases}$$

Now, $\mathbf{C1} = \frac{b_0}{m_0}\mathbf{1}$ and $\mathbf{z}^* = \left(\sum_{q=0}^{\infty} \varepsilon^q \mathbf{C}^q\right)(\mathbf{I} + \mathbf{C})\mathbf{n}$, with $\mathbf{n} = \eta/(1-\ell)$. Therefore:

$$z_i^* = z_0^* = \eta \times \frac{b_0}{m_0} \cdot \frac{1}{1 - \varepsilon\left(\frac{b_0}{m_0} - 1\right)} \tag{A.4}$$

$\forall i$, with z_0^* increasing in b_0/m_0. Moreover, as $\mathbf{M} = \bar{\alpha}\mathbf{G} + \bar{\theta}\mathbf{J} + \bar{\gamma}\mathbf{I}$,

$$\frac{b_0}{m_0} = \frac{(\bar{\alpha}\mathbf{G} + \bar{\theta}\mathbf{J} + \bar{\gamma}\mathbf{I})\mathbf{1}}{\bar{\theta} + \bar{\gamma}} \tag{A.5}$$

That is, since $\mathbf{G1} = \delta\mathbf{1}$ and $\mathbf{J1} = n$,

$$\frac{b_0}{m_0} = \frac{n\bar{\theta} + \delta\bar{\alpha} + \bar{\gamma}}{\bar{\theta} + \bar{\gamma}}. \tag{A.6}$$

Appendix B: An example of integration with resource effect

Consider the network depicted in Figure 5 and fixed investment. Then, the resource effect is given by $p\boldsymbol{\gamma}$ where $\boldsymbol{\gamma} = (\mathbf{G}^T - \mathbf{G})\mathbf{1}$. To simplify, consider $\varepsilon = 0$, so that $\mathbf{z}^* = (\mathbf{I} + \mathbf{C})\left(\frac{e-(\rho-1)d}{1-\ell}\mathbf{1} + \frac{p}{1-\ell}\boldsymbol{\gamma}\right)$. The effect of p on the total contribution of resource effect to optimal risk-taking is then captured by: $p/(1-\ell)\mathbf{1}^T\mathbf{C}\boldsymbol{\gamma}$, and can be decreased when the level of integration of the financial network is increased. Recall here that matrix $\mathbf{C}$ depends on parameter p. Denoting $\mathbf{C}_p$ the value of this matrix under parameter p, we have for the network depicted in Figure 5: $\mathbf{1}^T\mathbf{C}_1\boldsymbol{\gamma} \sim -0.0328$ and $2\mathbf{1}^T\mathbf{C}_2\boldsymbol{\gamma} \sim -0.0789$.

38. For all $q \geq 2$: $\mathbf{G}^q = \alpha_q\mathbf{G} + \theta_q\mathbf{J} + \gamma_q\mathbf{I}$ with the convention $(\alpha_2, \theta_2, \gamma_2) = (\alpha, \theta, \gamma)$. Using the expression of $\mathbf{G}^{q-1}$, we have

$$\mathbf{G}^q = \underbrace{(\alpha\alpha_{q-1} + \gamma_{q-1})}_{=\alpha_q}\mathbf{G} + \underbrace{(\theta\alpha_{q-1} + \delta\theta_{q-1})}_{=\theta_q}\mathbf{J} + \underbrace{\gamma\alpha_{q-1}}_{=\gamma_q}\mathbf{I})$$

and $\bar{\alpha} = \Delta + \sum_{q=2}^{\infty} \Delta^q \alpha_q$; $\bar{\gamma} = 1 + \sum_{q=2}^{\infty} \Delta^q \gamma_q$; $\bar{\theta} = \sum_{q=2}^{\infty} \Delta^q \theta_q$.

Therefore, the contribution of the resource effects to total optimal risk-taking is here negative and decreasing with p. This comes from the negative correlation between the vector of out-degrees $\delta^O = (3,1,1,0)^T$ and the vector of resource effects $\boldsymbol{\gamma} = (-1,0,0,1)^T$.

Appendix C: Modeling inter-bank debt

We incorporate inter-bank debt to the model as follows. Denoting ν_{ij} the debt of bank i towards bank j and $\nu_i = \sum_j \nu_{ji} - \sum_j \nu_{ij}$, the optimal risk-taking with inter-bank debts $\mathbf{z}^*$ solves

$$\mathbf{z}^* = (\mathbf{I} - \varepsilon \mathbf{C})^{-1}(\mathbf{I} + \mathbf{C})(\mathbf{n} + (\rho - 1)\boldsymbol{\nu})/(1 - \ell) \tag{C.1}$$

and Assumption 1 becomes $\mathbf{n} + \boldsymbol{\nu} > 0$. Recall here that $\mathbf{z}^*$ guarantees that bank i is solvent when it suffers a shock of size ℓ, in which case all the other banks are also solvent.

Now consider a reallocation $\boldsymbol{\nu}'$ of inter-debt such that $\mathbf{1}^T(\boldsymbol{\nu}' - \boldsymbol{\nu}) = 0$. Then:

$$\mathbf{1}^T(\mathbf{z}' - \mathbf{z}) = (\rho - 1)((\mathbf{I} - \varepsilon \mathbf{C})^{-1}(\mathbf{I} + \mathbf{C})\mathbf{1})^T(\boldsymbol{\nu}' - \boldsymbol{\nu})/(1 - \ell) \tag{C.2}$$

Now, when $e_i = e$ and $d_i = d$, $\mathbf{z}^*_{RS} = (e - (\rho - 1)d) \cdot (\mathbf{I} - \varepsilon \mathbf{C})^{-1}(\mathbf{I} + \mathbf{C})\mathbf{1}$ (see 14) and:

$$\mathbf{1}^T(\mathbf{z}' - \mathbf{z}) = \frac{\rho - 1}{(e + (\rho - 1)d)(1 - \ell)} \cdot {\mathbf{z}^*_{RS}}^T(\boldsymbol{\nu}' - \boldsymbol{\nu}) \tag{C.3}$$

That is:

$$\sum_i (z_i^{*\prime} - z_i^*) = \frac{\rho - 1}{(e + (\rho - 1)d)(1 - \ell)} \sum_i z^*_{i,RS}(\nu_i' - \nu_i) \tag{C.4}$$

Thus, the reallocation of debts increases total optimal risk-taking if it favors banks that would have the highest pure risk-sharing effect in the absence of inter-bank debt.

Appendix D: Extension to multiple shocks

Consider the case when a shock, i.e. a return $\mu = \ell < 1$ on one's risky asset (we consider binary shocks for simplicity), can hit two banks, say i and k at the same time.[39] To avoid default, bank i must satisfy the following constraint:

$$(\ell - 1)z_i + (\ell - 1)c_{ik}z_k - (\underline{r} - 1)\sum_{l \neq i,k} c_{il}z_l \leq \eta_i + \sum_{j \neq i} c_{ij}\eta_j \tag{D.1}$$

39. The extension to more than two shocks is immediate.

That is:

$$z_i - \varepsilon \sum_{j \neq i} c_{ij} z_j \leq n_i + \sum_{j \neq i} c_{ij} n_j - \frac{r-\ell}{1-\ell} c_{ik} z_k \tag{D.2}$$

Following our definition of prudential regulation, bank i should then survive to all pairs of shocks involving its own shock, that is:

$$z_i - \varepsilon \sum_{j \neq i} c_{ij} z_j = n_i + \sum_{j \neq i} c_{ij} n_j - \frac{r-\ell}{1-\ell} \max_{k \neq i} \; c_{ik} z_k \tag{D.3}$$

Importantly, there is strategic substitutability between risk-taking levels of the two shocked banks.

We define the vector $\boldsymbol{\zeta}(\mathbf{z}) = (\max_{k \neq i} \; c_{ik} z_k)_{i \in \mathcal{I}}$. Then, a solution $\mathbf{z}^{**}$ to the problem with two shocks should satisfy equation (D.3) for all banks i. That is, it should solve the non-linear system

$$\mathbf{z}^{**} = (\mathbf{I} - \varepsilon \mathbf{C})^{-1} \left((\mathbf{I} + \mathbf{C}) \mathbf{n} - \frac{r-l}{1-l} \boldsymbol{\zeta}(\mathbf{z}^{**}) \right) \tag{D.4}$$

Remembering that the solution with a single shock is given by $\mathbf{z}^* = (\mathbf{I} - \varepsilon \mathbf{C})^{-1}(\mathbf{I} + \mathbf{C})\mathbf{n}$, we get

$$\mathbf{z}^{**} = F(\mathbf{z}^{**}) \tag{D.5}$$

where $F(\mathbf{z}) = \mathbf{z}^* - \frac{r-\ell}{1-\ell}(\mathbf{I} - \varepsilon \mathbf{C})^{-1} \boldsymbol{\zeta}(\mathbf{z})$. Then, existence follows directly from the boundedness and continuity of function F.

However, the system can now admit multiple solutions. Indeed, consider the following three-bank two-shock example. We set $e = 100, d = 100, \rho = 1.01, l = 0.8$, and let the matrix of cross-investments be

$$\mathbf{P} = \begin{pmatrix} 0 & 0.7 & 0.4 \\ 1 & 0 & 0.8 \\ 1.05 & 1 & 0 \end{pmatrix}. \tag{D.6}$$

One can then see that there are two solutions to system (D.4): $\mathbf{z}_1^{**} \sim (496.74, 493.31, 490.44)$ and $\mathbf{z}_2^{**} \sim (496.73, 494.32, 490.44)$, where no solution dominates the other one.

Appendix E: Correlated investment portfolios

We now extend our model to an environment where banks can invest not only in their own risky project but also in the risky projects of the other banks. In our baseline model we consider n banks, n risky projects, with one project receiving investment from a single bank. We instead consider here an environment with n risk projects (still attached to a bank). Each bank invests an homogeneous share λ of its risky investment in its own project, and distributes the residual share $1 - \lambda$ equally among other projects, as illustrated in Figure E.1.[40] Note that, in this extended model, the equivalent of Assumption 4 imposes $\lambda > \frac{\underline{r}-1}{\underline{r}-\ell}$.

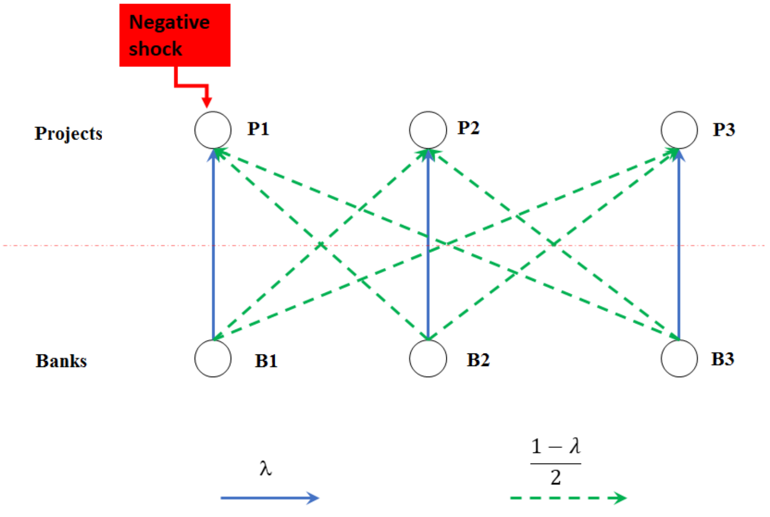

FIGURE E.1. Diversifying investment portfolios

In this case, Proposition E.E.1 shows that correlated investment portfolios can change the nature of the interaction between banks' risk-taking. Substitutability emerges when portfolios are sufficiently (positively) correlated. Under large correlation, all banks are hurt by the negative shock on bank i's project, and the cross-shareholding network does not bring value to bank i but rather transmits its low value of other banks.

PROPOSITION E.E.1. *Risk-taking decisions by banks are strategic substitutes whenever the fraction of risky investment each bank allocates to its own project is low, i.e. when*

$$\frac{1-\lambda}{n-1} > \frac{\underline{r}-1}{\underline{r}-\ell}$$

Proof. When each bank allocates a share λ of its risky investment in its own project, and distributes equally the remaining among other projects, the value

40. A more general setting will consist of n banks and an arbitrary number of projects with a bipartite network representing the amount each banks invests in each project (Elliott et al., 2014).

of firm i conditional on a negative shock on its project writes

$$\begin{aligned} v_i &= m_{ii}\left[z_i\left(\lambda\ell+(1-\lambda)\underline{r}-1\right)+\eta_i\right] \\ &+ \sum_{j\neq i} m_{ij}\left[z_j\left(\lambda\underline{r}+\frac{(n-2)(1-\lambda)}{n-1}\underline{r}+\frac{(1-\lambda)}{n-1}\ell-1\right)+\eta_j\right] \end{aligned} \tag{E.1}$$

In this case , $\mathbf{z}^*$ is defined by:

$$\mathbf{z}^* = (\mathbf{I}-\varepsilon(\lambda)\mathbf{C})^{-1}(\mathbf{I}+\mathbf{C})\mathbf{n} \tag{E.2}$$

with $n_i = \eta_i/(1-\lambda\ell-(1-\lambda)\underline{r})$ and

$$\varepsilon(\lambda) = -\frac{n-1-(1-\lambda)\ell-(\lambda+n-2)\underline{r}}{(n-1)(1-\lambda\ell-(1-\lambda)\underline{r})} \tag{E.3}$$

The numerator of $\varepsilon(\lambda)$ is negative when $\lambda < \frac{n-1-\ell-(n-2)\underline{r}}{\underline{r}-\ell}$, i.e. when $\frac{1-\lambda}{n-1} > \frac{\underline{r}-1}{\underline{r}-\ell}$, leading to substituability in z_is. □

Appendix F: Endogenous shareholding network

We examine in this Appendix the endogenous formation of cross-shareholding links. For tractability, we focus on the fixed participation case and only consider undirected networks: $\mathbf{G}^T = \mathbf{G}$ (there is therefore no resource effect in this setting). To isolate pure network effects, we assume $e_i = e$ and $d_i = d$ for all i. We simplify the analysis by examining the case of sufficiently low participation level p; and binary distribution for the return of one bank's risky asset: $\underline{r} = r > 1$ and $\tilde{\mu}_i = \ell < 1$ when bank i suffers the large shock ($\mathbb{E}(\mu_i) = \mathbb{E}(\mu) = (1-q/n)r + q/n\cdot\ell > 1$ by Assumption 2).

We consider alternatively (i) that banks do consider the change in regulation when modify their connections and (ii) that banks do not take into account modified regulation when changing links. This last setting corresponds for example to situations in which regulation are not updated often enough, with respect to banks' choices.

We now analyze incentive to form link. We assume that this decision is taken by existing shareholders based on the expected value of one share, which writes in our case:

$$\mathbb{E}\pi_i = \frac{\mathbb{E}(v_i)}{e+p\delta_i} = \frac{\eta}{e} + \mathbb{E}(\mu)\,\frac{\sum_{j\in\mathcal{I}} m_{ij}z_j}{e+p\delta_i} \tag{F.1}$$

with $t = e-(\rho-1)d$ and $\mathbb{E}(\mu) = q\ell+(1-q)r$.

Indeed, denoting s the state of nature $(\mu_1, ..., \mu_n)$, we have

$$v_i(s) = \sum_j m_{ij}((\mu_j-1)z_j+\eta) = \eta b_i + \sum_j m_{ij}(\mu_j-1)z_j \tag{F.2}$$

(where $b_i = \sum_j m_{ij}$ is the Bonacich centrality of i). Then,

$$\begin{aligned}\mathbb{E}(v_i) &= \eta b_i + (1-q)(r-1)\sum_j m_{ij} z_j \\ &\quad + \frac{q}{n}\sum_{k=1}^{n}\left((r-1)\sum_{j\neq k} m_{ij} z_j + (\ell-1) m_{ik} z_k\right) &\text{(F.3)} \\ &= \eta b_i + \mathbb{E}(\mu)\sum_{j\in\mathcal{I}} m_{ij} z_j &\text{(F.4)}\end{aligned}$$

Noting that, under fixed participation, $e + \sum_j p_{ji} = e + p\delta_i$ and $b_i = (e + p\delta_i)/e$ we obtain the value of $\mathbb{E}\pi_i$ in (F.1).

Note that, as $\mathbb{E}(\mu) > 0$ and $\mathbf{Mz} > \mathbf{0}$, an immediate consequence of equation (F.1) is that *the empty network is not stable.*

We consider in the following low values of p and, first, that banks take into account change in regulation when forming a link.

F.1. Link formation with regulatory change.

Considering low values of p, at order 1, we have $\mathbf{A} = \frac{p}{e}\mathbf{G} + o(p)$ leading to $\mathbf{M} = \mathbf{I} + \frac{p}{e}\mathbf{G} + o(p)$ and $\mathbf{C} = \frac{p}{e}\mathbf{G} + o(p)$ (as $m_{ii} = 1 + o(p)$). This therefore gives:

$$\mathbf{z} = \frac{\eta}{1-\ell}\left(\mathbf{I} + (1+\varepsilon)\frac{p}{e}\mathbf{G}\right)\mathbf{1} + o(p) \tag{F.5}$$

and

$$\mathbf{Mz} = \frac{\eta}{1-\ell}\left(\mathbf{I} + (2+\varepsilon)\frac{p}{e}\mathbf{G}\right)\mathbf{1} + o(p) \tag{F.6}$$

We note $\mathbf{1}_{ij}$ the square matrix with all entries equal to 0 but both entries i,j and j,i equal to 1. We then denote $\mathbf{M}'$ and $\mathbf{z}'$ the outcomes associated with network $\mathbf{G} + \mathbf{1}_{ij}$. Then, the return for bank i to forming undirected link ij writes:

$$\Delta\pi_i^{ij} = \mathbb{E}(\mu)\cdot\left[\frac{\sum_{j\in\mathcal{I}} m'_{ij} z'_j}{p\delta_i + e + p} - \frac{\sum_{j\in\mathcal{I}} m_{ij} z_j}{p\delta_i + e}\right] \tag{F.7}$$

And using F.6, bank i wants to form the link ij whenever

$$\mathbb{E}(\mu)\left[\frac{e + (2+\varepsilon)p(\delta_i+1)}{p\delta_i + e + p} - \frac{e + (2+\varepsilon)p\delta_i}{p\delta_i + e}\right] > 0 \tag{F.8}$$

that is, if

$$\mathbb{E}(\mu)\cdot\frac{p}{e}\cdot\frac{1+\varepsilon}{e + p(2\delta_i + 1)} > 0 \tag{F.9}$$

As (F.9) is always verified, *the set of pairwise stable networks is equal to the complete network* when banks take into account the change of regulation after link addition.

F.2. Link formation without regulatory change.

We now analyze the case in which banks don't account for change in regulation when choosing to form a link. We then get:

$$\Delta\pi_i^{ij} = \mathbb{E}(\mu)\left[\frac{\sum\limits_{j\in\mathcal{I}} m'_{ij}z_j}{p\delta_i + e + p} - \frac{\sum\limits_{j\in\mathcal{I}} m_{ij}z_j}{p\delta_i + e}\right] \tag{F.10}$$

Still considering low values of p, we here need to develop expressions at order 2. Then, $\mathbf{M} = \mathbf{I} + \frac{p}{e}\mathbf{G} + \frac{p^2}{e^2}\mathbf{G}^2 + o(p^2)$; $\mathbf{C} = \frac{p}{e}\mathbf{G} + \frac{p^2}{e^2}\mathbf{G}^2 + o(p^2)$;

$$\mathbf{z} = \left(\mathbf{I} + (1+\varepsilon)\frac{p}{e}\mathbf{G} + \frac{p^2}{e^2}(1+\varepsilon)^2\mathbf{G}^2\right)\mathbf{1} \tag{F.11}$$

and

$$\mathbf{Mz} = \left(\mathbf{I} + (2+\varepsilon)\frac{p}{e}\mathbf{G} + \frac{p^2}{e^2}(\varepsilon^2 + 3\varepsilon + 3)\mathbf{G}^2\right)\mathbf{1} \tag{F.12}$$

that is

$$\frac{\mathbf{Mz}_i}{e + p\delta_i} = \frac{1 + (2+\varepsilon)\frac{p}{e}\delta_i + \frac{p^2}{e^2}(\varepsilon^2 + 3\varepsilon + 3)\delta_i^2}{e + p\delta_i} \tag{F.13}$$

But since $\mathbf{M'} = \mathbf{I} + \frac{p}{e}(\mathbf{G} + \mathbf{1}_{ij}) + \frac{p^2}{e^2}(\mathbf{G} + \mathbf{1}_{ij})^2 + o(p^2)$, we get

$$\mathbf{M'z} = \mathbf{Mz} + \frac{p}{e}\mathbf{1}_{ij}\mathbf{1} + \frac{p^2}{e^2}\left(\mathbf{G}\mathbf{1}_{ij} + (2+\varepsilon)\mathbf{1}_{ij}\mathbf{G} + \mathbf{1}_{ij}^2\right)\mathbf{1} \tag{F.14}$$

Noticing that $(\mathbf{1}_{ij}\mathbf{1})_i = 1, (\mathbf{G}\mathbf{1}_{ij}\mathbf{1})_i = 0, (\mathbf{1}_{ij}\mathbf{G}\mathbf{1})_i = \delta_j$ and $(\mathbf{1}_{ij}^2)_i = 1$, we find

$$\frac{\mathbf{M'z}_i}{e + p\delta_i + p} = \frac{\mathbf{Mz}_i + \frac{p}{e} + \frac{p^2}{e^2}\left(1 + (2+\varepsilon)\delta_j\right)}{e + p\delta_i + p} \tag{F.15}$$

Using the expressions of (F.13) and (F.15) in equation (F.10), bank i wants to form the link ij whenever

$$\mathbb{E}(\mu)\left(\frac{1}{1+\varepsilon} + \left(1 + \frac{1}{1+\varepsilon}\right)\delta_j - \delta_i\right) > 0 \tag{F.16}$$

As $\mathbb{E}(\mu) > 0$ and $1/(1+\varepsilon) > 0$, all banks with same degrees prefer to form a link. This means that *the complete network is stable*. Other *stable networks are unions of complete components of distinct size*, with appropriate size distribution. Suppose that a network is the union of χ complete components of size $\sigma_1 > \sigma_2 > \cdots > \sigma_r$. From equation (F.16), it must be that (and it is sufficient that)

$$\text{For all components } i < \chi, \quad \frac{\sigma_i}{\sigma_{i+1}} > \frac{2+\varepsilon}{1+\varepsilon}$$

so that the bank with higher degree is better of refusing to form the link.

To see formally that a stable network is necessarily the union of complete components of distinct size, we observe that returns to link formation satisfy Goyal and Joshi (2006)'s proposition 4.3, which guarantees the point. Indeed, when p tends to zero and for any $e > 0$, the return of link formation satisfies

$$\Delta\pi_i^{ij} = f(\delta_i, \delta_j) \tag{F.17}$$

with f decreasing in δ_i, increasing in δ_j, and $f(\delta_i + 1, \delta_j + 1) > f(\delta_i, \delta_j)$, as

$$\Delta\pi_i^{ij} = \frac{\frac{p^2}{e}\Big(1 + (2+\varepsilon)\delta_j - (1+\varepsilon)\delta_i\Big)}{(e + p\delta_i + p)(e + p\delta_i)} + o(p^2) \tag{F.18}$$

(and $f(\delta_i + 1, \delta_j + 1) > f(\delta_i, \delta_j)$ as e/p goes to infinity when p goes to zero).

Appendix G: Proofs

G.1. Proof of Lemma 1

Consider the system:

$$v_i = h_i + \sum_{j\in\mathcal{I}} a_{ij} v_j^+ \quad \forall i \in \mathcal{I} \tag{G.1}$$

In v_is, this corresponds to a game of strategic complementarities with lower and upper bounds (for given μ_is), that is a supermodular game. Therefore, it possesses a minimum and a maximum equilibrium.

Now, consider an equilibrium with $\mathcal{S}$ non defaulting banks, i.e. with $\mathbf{v}_{\mathcal{S}} = (v_1, \cdots, v_s) > 0$ and let $\bar{a}_i = 1 - \sum_{k\in\mathcal{S}} a_{ki}$. Then,

$$\sum_{i\in\mathcal{S}} \bar{a}_i v_i = \sum_{i\in\mathcal{S}} \left(1 - \sum_{k\in\mathcal{S}} a_{ki}\right) v_i \tag{G.2}$$

and given that $\sum_{i\in\mathcal{S}} v_i = \sum_{i\in\mathcal{S}} h_i + \sum_{i\in\mathcal{S}}\sum_{k\in\mathcal{S}} a_{ik} v_k$:

$$\sum_{i\in\mathcal{S}} \bar{a}_i v_i = \sum_{i\in\mathcal{S}} h_i \tag{G.3}$$

Last, suppose that the minimum equilibrium, say $\mathcal{S}$, is distinct from the maximum equilibrium, say $\mathcal{S}'$. Then $\mathbf{v}_{\mathcal{S}} < \mathbf{v}_{\mathcal{S}'}$ (where we use here the vectorial inequality) and

$$\sum_{i\in\mathcal{S}} h_i = \sum_{i\in\mathcal{S}} \bar{a}_i v_i < \sum_{i\in\mathcal{S}} \bar{a}_i v_i' < \sum_{i\in\mathcal{S}'} \bar{a}_i v_i' = \sum_{i\in\mathcal{S}'} h_i \tag{G.4}$$

However, by construction, for all banks $i \in \mathcal{S}' \setminus \mathcal{S}$: $h_i < 0$. Indeed, by (G.1): $h_i > 0 \Rightarrow v_i > 0$ and all banks with $h_i > 0$ always belong to the surviving set. Then, $\sum_{i\in\mathcal{S}'} h_i < \sum_{i\in\mathcal{S}} h_i$, which is in contradiction with (G.4). The equilibrium values are then unique.

G.2. Proof of Lemma 2

The proof rely on the Sherman-Morrison formula, that states: Suppose $\mathbf{Q}$ is an invertible n-square matrix with real entries and $\mathbf{r}, \mathbf{s} \in \mathbb{R}^n$ are column vectors. Then $\mathbf{Q} + \mathbf{r}\mathbf{s}^T$ is invertible in and only if $1 + \mathbf{s}^T\mathbf{Q}^{-1}\mathbf{r} \neq 0$. If $\mathbf{Q} + \mathbf{r}\mathbf{s}^T$ is invertible, its inverse is given by

$$(\mathbf{Q} + \mathbf{r}\mathbf{s}^T)^{-1} = \mathbf{s}^{-1} - \frac{\mathbf{Q}^{-1}\mathbf{r}\mathbf{s}^T\mathbf{Q}^{-1}}{1 + \mathbf{s}^T\mathbf{Q}^{-1}\mathbf{r}} \tag{G.5}$$

We apply this formula with $\mathbf{Q} = \mathbf{I} - \mathbf{A}$ and $\mathbf{r}\mathbf{s}^T = -\mathbf{\Omega}$, where $\mathbf{\Omega} = [\omega_{ij}]$ is such that $\omega_{ij} = \omega$ if $(i,j) = (r,s)$, $\delta_{ij} = 0$ otherwise. Then matrix $\mathbf{\Omega}$ has a single non-zero entry, corresponding to a positive impulsion at the entry (r,s). It is easily shown that $\mathbf{\Omega} = -\mathbf{r}\mathbf{s}^T$ for $\mathbf{r} = (0, \cdots, 0, \omega, 0, \cdots, 0)^T$ with ω at entry r, and $\mathbf{s}^T = (0, \cdots, 0, 1, 0, \cdots, 0)^T$ with 1 at entry s.

Applying the formula, noting $(\mathbf{I} - \mathbf{A})^{-1} = \mathbf{M}$ and $\mathbf{s}^T\mathbf{M}\mathbf{r} = -m_{rs}\omega$, we get

$$(\mathbf{I} - \mathbf{A} - \mathbf{\Omega})^{-1} = \mathbf{M} + \frac{\mathbf{M\Omega M}}{1 - m_{rs}\omega} \tag{G.6}$$

Now the entry (i,j) of matrix $\mathbf{M\Omega M}$ is given by $[\mathbf{M\Omega M}]_{ij} = m_{ir}m_{sj}\omega$. Then,

$$[(\mathbf{I} - \mathbf{A} - \mathbf{\Omega})^{-1}]_{ij} = m_{ij} + \frac{m_{ir}m_{sj}\omega}{1 - m_{sr}\omega} \tag{G.7}$$

We want to prove that the ratio $\frac{m_{ij}}{m_{ii}}$ increases for all i,j when $\mathbf{A}$ becomes $\mathbf{A}' = \mathbf{A} + \mathbf{\Omega}$. Note that $\frac{m_{ij}}{m_{ii}} \leq \frac{m_{ij}+a}{m_{ii}+b}$ if and only if $\frac{m_{ij}}{m_{ii}} \leq \frac{a}{b}$. Then it is sufficient to prove that $\frac{m_{ij}}{m_{ii}} \leq \frac{m_{ir}m_{sj}\omega}{m_{ir}m_{si}\omega}$, i.e.

$$\frac{m_{ij}}{m_{ii}} \leq \frac{m_{sj}}{m_{si}} \tag{G.8}$$

Now the path product property of any inverse M-matrix $\mathbf{Y}$ (see for instance Johnson and Smith, 2007, p. 329) writes

$$y_{ij}y_{jk} \leq y_{ik}y_{jj} \tag{G.9}$$

Equation (G.8) can be writen:

$$m_{si}m_{ij} \leq m_{ii}m_{sj} \tag{G.10}$$

that is, permuting labels i and j:

$$m_{sj}m_{ji} \leq m_{jj}m_{si} \tag{G.11}$$

and, permuting labels i and q:

$$m_{ij}m_{js} \leq m_{jj}m_{is} \tag{G.12}$$

which corresponds to the path product property with i,j,s as shown by equation (G.9). $\mathbf{M}$ being an inverse M-matrix, we therefore have that $\mathbf{A}' > \mathbf{A}$ leads to $\mathbf{C}' > \mathbf{C}$, where $c_{ij} = m_{ij}/m_{ii}$ and $\mathbf{M} = (\mathbf{I} - \mathbf{A})^{-1}$.

G.3. Proof of Propositions 3 and 4

Step 1. Let us first show that the ratio b_i/m_{ii} is higher for the center of the star, both in the case of fixed participation and in the case of fixed investment.

Consider an undirected star with n agents. We denote by 1 the center of the star and by 2 the representative periphery. We denote $a = a_{12}$ and $b = a_{21}$. Under fixed participation: $a = 1/(1 + e/p)$ and $b = 1/(n - 1 + e/p)$; whereas under fixed investment: $a = 1/(1 + (n - 1) \cdot e/p)$ and $b = 1/(n - 1 + e/p)$.

In both cases,

$$(\mathbf{I} - \mathbf{A}) = \begin{pmatrix} 1 & -a & -a & \cdots & -a \\ -b & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -b & 0 & \cdots & 0 & 1 \end{pmatrix} \tag{G.13}$$

and $(\mathbf{I} - \mathbf{A})^{-1} = \left(\frac{1}{1-(n-1)ab}\right) \mathbf{Q}$ with

$$\mathbf{Q} = \begin{pmatrix} 1 & a & a & \cdots & a \\ b & 1-(n-2)ab & ab & \cdots & ab \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ b & ab & ab & \cdots & 1-(n-2)ab \end{pmatrix} \tag{G.14}$$

This gives $b_1^O/m_{11} = 1 + (n - 1)a$ and $b_2^O/m_{22} = (1 + b)/(1 - (n - 2)ab)$; so that the ratio b_i^O/m_{ii} is higher for the center when

$$1 + (n - 2)a + (n - 2)(n - 1)a^2 < (n - 1)\frac{a}{b} \tag{G.15}$$

Considering the values of a and b for the fixed participation case, this corresponds to

$$(n-1)\frac{n-1+\frac{e}{p}}{1+\frac{e}{p}} > \frac{(n-1)(n-2)}{(1+\frac{e}{p})^2} + \frac{n-2}{1+\frac{e}{p}} + 1 \tag{G.16}$$

Multiplying both sides by $(1 + e/p)$ and simplifying, we get

$$\left(1+\frac{e}{p}\right)\left(n-1+\frac{e}{p}\right) > n-1 \tag{G.17}$$

which holds true as $e/p > 0$, meaning that b_i/m_{ii} is higher for the center of a star in the fixed participation case.

Now turn to the values of a and b in the fixed investment case. (G.15) then corresponds to

$$(n-1)\frac{n-1+\frac{e}{p}}{1+(n-1)\frac{e}{p}} > \frac{(n-1)(n-2)}{(1+(n-1)\frac{e}{p})^2} + \frac{n-2}{1+(n-1)\frac{e}{p}} + 1 \tag{G.18}$$

Multiplying both sides by $(1 + (n-1) \cdot e/p)$ and simplifying, we get

$$1 > \frac{1}{1 + (n-1)\frac{e}{p}} \tag{G.19}$$

which holds true as $e/p > 0$ and $n > 1$, meaning that b_i/m_{ii} is higher for the center of a star also in the fixed investment case.

Step 2. We now prove by induction that $z^*_{RS,1} > z_{RS,2}$, that is that the pure risk-sharing effect is higher for the center of the star that for each of the periphery.

For it, it is enough to show that $\forall q \; (\mathbf{C}^q\mathbf{1})_1 > (\mathbf{C}^q\mathbf{1})_2$. Now, by step 1, we now that $(\mathbf{C1})_1 > (\mathbf{C1})_2$. For convenience, let us $\psi_1 = (\mathbf{C1})_1$, $\psi_2 = (\mathbf{C1})_2$, and more generally, $\psi_1^{(q)} = (\mathbf{C}^q\mathbf{1})_1$, $\psi_2^{(q)} = (\mathbf{C}^q\mathbf{1})_2$ for all $q \geq 1$.

Let property $\mathcal{P}(q) : \varphi_c^{(q)} > \varphi_p^{(q)}$. Assume $\mathcal{P}(1), \cdots, \mathcal{P}(q-1)$. We will prove $\mathcal{P}(q)$. First note that

$$\psi_1^{(q)} = \psi_1 \psi_1^{(q-1)} \tag{G.20}$$

and

$$\psi_2^{(q)} = c_{21}\psi_1^{(q-1)} + (\psi_2 - c_{pc})\psi_2^{(q-1)} \tag{G.21}$$

The inequality $\psi_1^{(q)} > \psi_2^{(q)}$ then means

$$(\psi_1 - \psi_2)\,\psi_2^{(q-1)} > c_{21}\left(\psi_1^{(q-1)} - \psi_2^{(q-1)}\right) \tag{G.22}$$

Now, by $\mathcal{P}(q-1)$, we have

$$\psi_1\psi_2^{(q-2)} > c_{21}\psi_1^{(q-2)} + (\psi_2 - c_{21})\psi_2^{(q-2)} \tag{G.23}$$

and inequality (G.22) also writes

$$(\psi_1 - \psi_2)\psi_2^{(q-1)} > c_{21}\left((\psi_1 - \psi_2)\varphi_2^{(q-2)} - c_{21}\left(\psi_1^{(q-2)} - \psi_2^{(q-2)}\right)\right) \tag{G.24}$$

that is

$$(\psi_1 - \psi_2)\left(c_{21}\left(\psi_1^{(q-2)} - \psi_2^{(q-2)}\right) + (\psi_2 - c_{21})\psi_2^{(q-2)}\right) > -c_{21}^2\left(\psi_1^{(q-2)} - \psi_2^{(q-2)}\right)$$

what holds whenever $\psi_2 - c_{21} > 0$. Now $\psi_2 > c_{21}$ corresponds to

$$\frac{\sum_{j \neq 2} m_{2j}}{m_{22}} > \frac{m_{21}}{m_{22}} \tag{G.25}$$

which always holds as $m_{ij} \geq 0 \; \forall i, j$. Therefore $\mathcal{P}(q)$ holds, whenever $\mathcal{P}(q-1)$ holds. As $\mathcal{P}(1)$ holds by Step 1, we have that the pure risk-sharing effect is always higher for the center of the star than for the periphery.

Step 3. We now account for resource effects. In the fixed participation case, $\sum_j p_{ij} = \sum_j p_{ji}$ for all undirected graph (where $g_{ij} = g_{ji}$) so that $\mathbf{z}^*_{RE} = 0$. We thus have $\mathbf{z}^* = \mathbf{z}^*_{RS}$ and Proposition 3 holds by Step 2. In the fixed investment case, the resource effect is beneficial to the center. Indeed: $\sum_j p_{1j} = p$ and $\sum_j p_{j1} = (n-1)p$; whereas $\sum_j p_{2j} = p$ and $\sum_j p_{j1} = p/(n-1)$. Therefore $z^*_{RE,1} > z^*_{RE,2}$ what together with Step 2 gives $z^*_1 > z^*_2$.

G.4. Proof of Proposition 5

Defining $\upsilon_i = m_{ii}n_i + \sum\limits_{j\neq i} m_{ij}n_j$, the initial $\mathbf{Z}^*$ solves

$$m_{ii}z^*_i - \varepsilon \sum_{j\neq i} m_{ij}z^*_j = \upsilon_i \tag{G.26}$$

Or, in matrix notation,

$$\mathbf{W}\mathbf{z}^* = \boldsymbol{\upsilon}$$

where $\mathbf{W}$ is a n-dimensional square matrix such that $w_{ii} = m_{ii}$ and $w_{ij} = -\varepsilon m_{ij}$; and $\boldsymbol{\upsilon} = (\upsilon_i)_{i\in\mathcal{I}}$.

Suppose now that one $1-\ell$ unit of cash in the external equity of bank 1 (everything follows with bank i, we focus on bank 1 to ease exposition). Letting $\mathbf{m}_1 = (m_{11}, m_{21}, \cdots, m_{n1})^T$ be the first column of matrix $\mathbf{M}$, the optimal risk-taking $\mathbf{z}'^*$ then writes

$$\mathbf{W}\mathbf{z}'^* = \boldsymbol{\upsilon} + \mathbf{m}_1$$

and the change in total investment in risky asset is

$$\mathbf{1}^T(\mathbf{z}' - \mathbf{z}) = \mathbf{1}^T\mathbf{W}^{-1}\mathbf{m}_1$$

Noticing that

$$\mathbf{m}_1 = -\frac{1}{\varepsilon}\begin{pmatrix} m_{11} \\ -\varepsilon m_{21} \\ \cdots \\ -\varepsilon m_{n1} \end{pmatrix} + \frac{1+\varepsilon}{\varepsilon}\begin{pmatrix} m_{11} \\ 0 \\ \cdots \\ 0 \end{pmatrix}$$

we obtain

$$\mathbf{W}^{-1}\mathbf{m}_1 = -\frac{1}{\varepsilon}\begin{pmatrix} 1 \\ 0 \\ \cdots \\ 0 \end{pmatrix} + \frac{1+\varepsilon}{\varepsilon}\mathbf{W}^{-1}\begin{pmatrix} m_{11} \\ 0 \\ \cdots \\ 0 \end{pmatrix}$$

Thus, defining $\mathbf{1}^T\mathbf{W}^{-1} = (w^S_1, w^S_2, \cdots, w^S_n)$, so that w^S_i is the sum of entries of column i in matrix $\mathbf{W}^{-1}$, we obtain that

$$\mathbf{1}^T(\mathbf{z}' - \mathbf{z}) = -\frac{1}{\varepsilon} + \left(\frac{1+\varepsilon}{\varepsilon}\right)m_{11}w^S_1$$

The bank whose capital injection has the highest effect on total investments in risky assets ($\mathbf{1}^T(\mathbf{z}' - \mathbf{z})$) is the one with the highest index $m_{ii}w^S_i$.

G.5. Proof of Proposition 6

Equation (11) can be written as

$$(1-\ell)m_{ii}z_i^* - (\underline{r}-1)\sum_{j\neq i} m_{ij}z_j^* = m_{ii}\eta_i + \sum_{j\neq i} m_{ij}\eta_j \tag{G.27}$$

$\forall i$ with $\eta_i = \eta_i = e_i - (\rho-1)d_i + \sum_{j\in\mathcal{I}} p_{ji} - \sum_{j\in\mathcal{I}} p_{ij}$.

This is equivalent to:

$$\mathbf{Me} = (1-\ell)\mathbf{M}_\varepsilon \mathbf{z}^* + (\rho-1)\mathbf{Md} - \mathbf{M}(\mathbf{P^T} - \mathbf{P})\mathbf{1} \tag{G.28}$$

where matrix $\mathbf{M}_\varepsilon$ has diagonal entry (i,i) equal to m_{ii} and off-diagonal entry (i,j) equal to $-\varepsilon m_{ij}$ (with $\varepsilon = (\underline{r}-1)/(1-\ell)$). This gives:

$$\mathbf{e} = (1-\ell)(\mathbf{I}-\mathbf{A})\mathbf{M}_\varepsilon \mathbf{z}^* + (\rho-1)\mathbf{d} - (\mathbf{P^T} - \mathbf{P})\mathbf{1} \tag{G.29}$$

and

$$\mathbf{1}^T\mathbf{e} = (1-\ell)\mathbf{1}^T(\mathbf{I}-\mathbf{A})\mathbf{M}_\varepsilon \mathbf{z}^* + (\rho-1)\mathbf{1}^T\mathbf{d} \tag{G.30}$$

Now assume that z_1^* decreases to $z'^* = z_1^* - \iota$ and $z_i'^* = z_i^*$ $\forall i \neq 1$.[41] This induces a new vector of external equities $\mathbf{e}'$. The impact on total external equities is given by

$$\mathbf{1}^T(\mathbf{e}'-\mathbf{e}) = (1-\ell)\mathbf{1}^T(\mathbf{I}-\mathbf{A})\mathbf{M}_\varepsilon(\mathbf{z}'^* - \mathbf{z}^*)$$

Defining $\mathbf{1}_1 = (1,0,\cdots,0)^T$, we have by definition $\mathbf{z}'^* - \mathbf{z}^* = -\iota\mathbf{1}_1$, and

$$\mathbf{M}_\varepsilon(\mathbf{z}'-\mathbf{z}) = \iota\varepsilon\mathbf{M}_1 - \iota(1+\varepsilon)m_{11}\mathbf{1}_1 \tag{G.31}$$

Therefore, noticing that $(\mathbf{I}-\mathbf{A})\mathbf{M}_1 = \mathbf{1}_1$, we obtain

$$\mathbf{1}^T(\mathbf{e}'-\mathbf{e}) = \iota(1-\ell)\left(\varepsilon - (1+\varepsilon)m_{11}\mathbf{1}^T(\mathbf{I}-\mathbf{A})\mathbf{1}_1\right) \tag{G.32}$$

That is,

$$\mathbf{1}^T(\mathbf{e}-\mathbf{e}') = \iota(1-\ell)\left((1+\varepsilon)\Big(\frac{m_{11}e_1}{\sum_{k\neq 1} p_{k1} + e_1}\Big) - \varepsilon\right) \tag{G.33}$$

Therefore, the bank whose decrease in risky investment has the highest effect on total need of external assets ($\mathbf{1}^T(\mathbf{e}-\mathbf{e}')$) is the one with the highest index $m_{ii}\bar{e}_i$; where $\bar{e}_i = e_i/\Big(\sum_j p_{ji} + e_i\Big)$ represents the share of bank i held by external shareholders.

41. Again, we use bank 1 by convenience.

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