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This paper is dedicated to Pierre Cartigny (1946-2019)

and Carine Nourry (1972-2019)

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†Pierre Cartigny was my PhD advisor. I met him first when I was a student and we became friends during my PhD. He has been a wonderful teacher and co-author. I will greatly miss him. Carine Nourry was a very close friend. I met her in 1994 as a student during her Master and we started to work jointly during her PhD. I remember as if it was yesterday how we discussed complex dynamics in optimal growth models with heterogeneous agents in our shared office in Marseille. She was a distinguished scholar and a wonderful co-author. I will also greatly miss my dear friend.

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Abstract: *This paper provides a long-run cycle perspective to explain the behavior of the annual flow of inheritance as identified by Piketty [51] for France and Atkinson [3] for the UK. Using a two-sector Barro-type [9] OLG model with non-separable preferences and bequests, we show that endogenous fluctuations are likely to occur through period-2 cycles or Hopf bifurcations. Two key mechanisms, which can generate independently or together quasi-periodic cycles, can be identified as long as agents are sufficiently impatient. The first mechanism relies on the elasticity of intertemporal substitution or equivalently the sign of the cross-derivative of the utility function whereas the second rests on sectoral technologies through the sign of the capital intensity difference across two sectors. Furthermore, building on the quasi-palindromic nature of the degree-4 characteristic equation, we derive some meaningful sufficient conditions associated to the occurrence of complex roots in a two-sector OLG model. Finally, we show that our theoretical results are consistent with some empirical evidence for medium- and long-run swings in the inheritance flows as a fraction of national income in France over the period 1896-2008.*

Keywords: *Two-sector overlapping generations model, optimal growth, endogenous fluctuations, quasi-palindromic polynomial, periodic and quasi-periodic cycles, altruism, bequest*

Journal of Economic Literature Classification Numbers: C62, E32, O41.

1 Introduction

In a very influential contribution, Piketty [51] shows that inherited wealth has again a prominent role for life-cycle income, especially with respect to human capital and labor income. Using a simple generic overlapping generations model of wealth accumulation, growth, and inheritance, Piketty [51] argues that the gap between the (steady-state) growth rate g and the rate of return on private wealth r is the core argument of this resurgence. On the one hand, when g is large and $g > r$, the new wealth accumulated out of current income, and thus human capital, contributes more to life-cycle income than past inherited wealth, especially when it is grounded on low (past) income relative to today's income.[1] In this respect, inheritance flows remains a small fraction of national income. On the other hand, when growth is low such that $r > g$, inherited wealth is capitalized at a faster (growth) rate than national income and becomes dominant with respect to current income. Consequently, inheritance flows become a larger proportion of national income. Accordingly, it turns out that the annual bequest flow relative to national income in France would be now close to its steady-state 15%-20%, and will slowly recover its level in the nineteenth century. Furthermore, it provides theoretical foundations on the very pronounced U-shaped pattern of the French inheritance-related variables.[2]

In fact, the dynamics of the inheritance flow can also be consistent with the predictions of multiple equilibria models and long-run (stochastic) limit cycles.[3] Notably the economy can converge to a stable (long-run) cyclical path where some macroeconomic variables oscillate indefinitely around the steady-state, and thus bequest flows can go back and forth to a low, steady or high level. Moreover, when examined over long periods, historical series of inheritance flows (Piketty, [51]) look part of a long cycle through its U-shapped

[1] For instance, such a pattern is observed in France during the period 1950-1970.

[2] A similar pattern is found by Atkinson [3] for the UK.

[3] Deterministic limit cycles have a long tradition in economics. Especially the seminal contributions of Benhabib and Nishimura [15, 16] have shown that even in standard models featuring forward-looking agents and a competitive equilibrium structure, the steady state or balanced growth path was inherently unstable and thus deterministic (endogenous) fluctuations were easily obtained as soon as the fundamental nonlinear structure of the model was taken into account. More recently, Beaudry *et al.* [11, 12, 13] have put forward the existence of endogenous stochastic limit cycles, i.e. a deterministic cycle where the stochastic component is essentially an i.i.d. process, that can generate alternate periods of booms and busts (see also Benhabib and Nishimura [17]). Recent strands of the literature that discuss the emergence of limit cycles include contributions on innovation-cycles and growth (Matsuyama [42], Growiec *et al.* [33]), on endogenous credit cycles in OLG models (Azariadis and Smith [4], Myerson [49], and Gu *et al.* [34]), on endogenous learning- and bounded rationality-based business fluctuations (Hommes, [35]).

pattern, especially when extracting its medium to long-run component.[4] In a broader perspective, this descriptive fact is to be reconciled with recent papers (e.g., Beaudry *et al.* [11, 12, 13]; Growiec *et al.* [32, 33]) that challenge the seminal contributions of Granger [31] and Sargent [56]: macroeconomic variables do not display (very) pronounced peaks at business cycles frequencies and thus data are not supportive of strong internal boom-bust cycles.[5] For instance, Beaudry *et al.* [11, 13] show the existence of a recurrent peak in several spectral densities of US trendless macroeconomic data suggesting the presence of periodicities around 9 to 10 years irrespective of the exogenous cyclical forces. At least their results run counter the empirical irrelevance of endogenous fluctuations.[6] We build on these results to reconcile the predictions of our model with empirical evidence.

In this paper, we view the existence of such limit cycles for the inheritance flow (in level or a as share of national income) as a complementary interpretation of Piketty [51] and thus model and rationalize it by formally characterizing the corresponding complex dynamics, and providing some empirical evidence.

Capitalizing on Michel and Venditti [43], we do so through the lens of a two-sector overlapping generation (henceforth, OLG) model with a pure consumption good and one capital good, and a constant population of finitely-lived agents. We proceed in three steps, starting successively with the central planner problem and the optimal growth solution in the absence of legs. In the third and last step, we consider the decentralized problem where altruistic agents must determine their life-cycle consumptions, savings and inheritance to their children.[7]

The central planner problem and the corresponding growth solution lead to a dimension-four dynamical system and allow us to first characterize the degree-4 characteristic polynomial associated to the linearized dynamical system around the steady state. Building on the quasi-palindromic nature of the characteristic equation, we solve it and provide a complete assessment of its characteristic roots.[8] In particular, we show that at least two

[4]For further evidence, see Section 6.

[5]Comin and Gertler [22] first provide evidence of medium-term cycles (with a periodicity between 8 and 50 years). See also Correa-López *et al.* [23] for an application to medium-term technology cycles.

[6]In the same vein, Growiec *et al.* [33] conclude that the labor's share of GDP exhibits medium-run swings. See also Charpe *et al.* [19].

[7]As shown by Weil [59], when bequests are strictly positive across generations, the solution of the Barro model is equivalent to the solution of a Ramsey-type optimal growth model where a central planner maximizes the total intertemporal welfare.

[8]If $P(x) = \sum_{i=0}^{4} a_i x^i$ is a polynomial of degree 4 and $a_i = a_{n-i}$ for $i = 0, \cdots, 2$, then P is palindromic (or reciprocal). If $P(x) = a_0x^4 + a_1x^3 + a_2x^2 + a_3mx + a_4m^2$ for some constant $m \neq 0$, then $P(mx) =$

roots or a pair of complex conjugate roots have a modulus larger than one. To the best of our knowledge, it is the first available proposition that exploits the quasi-palindromic property of the characteristic equation in the literature of macroeconomic dynamic models. This result is the cornerstone of our paper. First, it implies that the steady state can be either saddle-point stable or (totally) unstable. In the latter case, endogenous cycles can occur. Second, it is well-known that the existence of a Hopf bifurcation in models featuring forward-looking agents and a competitive equilibrium structure requires the consideration of at least three sectors and thus of dimension-four dynamical systems. And as shown in several contributions, e.g. Magill [38, 39, 40] and Magill and Scheinkman [41], the curse of dimensionality prevents the derivation of meaningful sufficient conditions for the existence of complex characteristic roots. Our paper shows that only two sectors are sufficient in an OLG economy and it makes one step further to a better understanding of the occurrence of complex roots. We provide indeed clear-cut sufficient conditions for the existence of complex roots leading to a Hopf bifurcation, and as far as we know, this is the first time such conditions are exhibited in the literature. Third, we can identify two key mechanisms that lead to quasi-periodic cycles through a Hopf bifurcation as long as agents are sufficiently impatient. The first one is based on the properties of preferences, and especially the sign of the cross-derivative of the utility function or equivalently the elasticity of intertemporal substitution. The second rests on sectoral technologies through the sign of the capital intensity difference across sectors.[9]

Furthermore, these preference and technology-based mechanisms can either generate endogenous fluctuations independently or self-sustain themselves and thus amplify or mitigate (long-run) limit cycles. In the case of non-strictly concave preferences, mild perturbations of either the capital intensity difference across sectors or of the elasticity of intertemporal substitution lead to a flip bifurcation and thus to persistent period-2 cycles.[10] The global dynamics can then be described as the product of two cycles implying complex properties of the optimal path.[11] On the one hand, the elasticity of

$\frac{x^4}{m^2}P\left(\frac{m}{x}\right)$ and P is quasi-palindromic. Importantly, using the change of variables $z = x + \frac{m}{x}$ in $\frac{P(x)}{x^2}$ produces a quadratic equation.

[9]The capital intensity difference across sectors has long been identified as a key driver of the dynamic properties of two-sector optimal growth models.

[10]The consumption good is assumed to be more capital intensive than the investment good (see Benhabib and Nishimura [15]).

[11]The quasi-palindromic polynomial can be factored as the product of two order-2 polynomials where one quadratic polynomial captures only the preference-based mechanism and the other only that based on technology.

intertemporal substitution must be large enough to allow a substitution effect between the first and second period consumptions while satisfying the standard transversality conditions and the convergence towards the period-two cycle, and, on the other hand, the consumption good sector needs to be capital intensive to generate fluctuations of the capital stock.

In contrast, these two mechanisms can no longer be separated in the case of strictly concave preferences, and a Hopf bifurcation can occur with quasi-periodic cycles. On top of the empirical implications discussed later on, this result is again drastically different from those in standard optimal growth models in which the existence of complex roots does require at least three sectors (e.g., Benhabib and Nishimura [15]). This can be explained by the fact that standard optimal growth models can only rely on the technology-based mechanism and higher-order dynamical systems (i.e., more sectors in the economy) are needed for quasi-periodic fluctuations. Here a two-sector optimal OLG growth problem with non-separable and strictly concave preferences is able to generate a Hopf bifurcation and quasi-periodic cycles taking some intermediate and plausible values for the elasticity of current consumption, the elasticity of intertemporal substitution and a upper threshold condition for the sectoral elasticities of capital-labor substitution.[12]

Turning to the decentralized problem in the presence of altruistic parents, we show that the optimal conditions and thus the dynamical system characterizing the decentralized equilibrium are equivalent to those associated with the central planner problem described above as long as bequests are strictly positive. After providing conditions for the existence of strictly stationary bequests, we argue that the (sufficient) conditions for optimal periodic and quasi-periodic cycles derived for the central planner solution also hold in the presence of positive bequests. This sharply contrasts with the predictions of the standard Barro model in which the optimal path monotonically converges toward the steady state if the life-cycle utility function of a representative generation living over two periods is additively separable. Also our result goes further than that of Michel and Venditti [43] in a one-sector model—endogenous period two-cycles can occur if the life-cycle utility function is non-additively separable with a positive cross derivative across periods. Finally, we show that both bequests and bequests as a share of GDP can be

[12]Kalra [36] and Reichlin [54] provide conditions for the existence of Hopf cycles in two-sector OLG models but do not take into account bequests. See also Ghiglino and Tvede [30] for the analysis of endogenous cycles in general OLG models, and Ghiglino [29] for the analysis of the link between wealth inequality and endogenous fluctuations in a two-sector model.

characterized by optimal periodic and quasi-periodic cycles. This result turns out to be insightful for our empirical results.

Looking at long-date inheritance flows data for France (Piketty, [51]), we finally provide a quantitative assessment of the long-run cyclical behavior of bequests as a share of national income. In so doing, we proceed in two steps. First, using the low-frequency methodology of Müller and Watson ([47], [48]) and a band pass filter at low frequencies, we extract the long-run component of the inheritance variable of interest and then we discuss some statistical features of the remaining cyclical component. Notably, we provide confidence intervals for the long-run standard deviation (variability) and contrast the filtered series with both a local-to-unit autoregressive model and a fractional model to further assess the persistence and (weakly) stationary properties. Second, following Beaudry *et al.* [11, 13], we make use of the spectral density to identify either a peak at a given frequency or a peak range over a low frequency interval, and thus to provide some support of recurrent (medium and long-run) cyclical fluctuations at that frequency (interval). Moreover, we test the presence of a shape restriction on the spectral density, i.e. the statistical significance of the "peak range" of the spectral density at low frequency interval with respect to a flat prior. Our results strongly support the view of medium-term business fluctuations with periodicity between 24 and 40 years and the fact that the corresponding peak range is statistically significant. While the presence of a peak range does not necessarily imply strong endogenous cyclical forces and the empirical relevance of a Hopf bifurcation, it says again that data can not, at least, contradict the existence of endogenous (stochastic) limit cycles.[13]

The paper is organized as follows. Section 2 presents the two-sector model with non-additively separable preferences, defines the optimal growth problem of the central planner, proves the existence of a steady state, and derives the characteristic polynomial from which the stability analysis is conducted. The existence of period-two cycles under the assumption of a non-strictly concave utility function is discussed in Section 3 together with the presentation of a simple example to illustrate the main conditions. Section 4 contains the extension to the case of a strictly concave utility function. We provide general sufficient conditions that rule out the existence of complex characteristic roots and we consider a specific class of utility functions to prove the possible existence of a Hopf

[13]See Dufourt *et al.* [26] where the Hopf bifurcation is also shown to be relevant from an empirical perspective in two-sector infinite-horizon models with productive externalities and sunspot fluctuations.

bifurcation and thus of quasi-periodic cycles. In Section 5 we show that all our previous conditions are compatible with the decentralized equilibrium characterized by strictly positive bequests. Section 6 discusses our empirical results regarding the inheritance flows as a share of national using long-dated French annual data. Concluding comments are provided in Section 7 and all the proofs are contained into a final Appendix.

2 The model

2.1 Production

We consider a two-sector economy with one pure consumption good y_0 and one capital good y. Each good is produced with a standard constant returns to scale technology:

$$y_0 = f^0(k_0, l_0), \qquad y = f^1(k_1, l_1)$$

with $k_0 + k_1 \leq k$, k being the total stock of capital, and $l_0 + l_1 \leq 1$, the total amount of labor being normalized to 1.

Assumption 1. *Each production function $f^i : \mathbb{R}^2_+ \to \mathbb{R}_+$, $i = 0, 1$, is C^2, increasing in each argument, concave, homogeneous of degree one and such that for any $x > 0$, $f^i_{k_i}(0, x) = f^i_{l_i}(x, 0) = +\infty$, $f^i_{k_i}(+\infty, x) = f^i_{l_i}(x, +\infty) = 0$.*

For any given (k, y), we define a temporary equilibrium by solving the following problem of optimal allocation of factors between the two sectors:

$$\begin{aligned} T(k,y) = \max_{k_0,k_1,l_0,l_1} \quad & f^0(k_0, l_0) \\ s.t. \quad & y \leq f^1(k_1, l_1) \\ & k_0 + k_1 \leq k \\ & l_0 + l_1 \leq 1 \\ & k_0, k_1, l_0, l_1 \geq 0. \end{aligned} \tag{1}$$

The value function $T(k, y)$ is called the social production function and describes the frontier of the production possibility set. Constant returns to scale of technologies imply that $T(k, y)$ is concave non strictly. We will assume in the following that $T(k, y)$ is at least C^2.[14]

[14] A proof of the differentiability of $T(k, y)$ under Assumption 1 and non-joint production is provided in Benhabib and Nishimura [15].

Let p denote the price of the investment good, r the rental rate of capital and w the wage rate, all in terms of the price of the consumption good, it is straightforward to show that:

$$T_k(k,y,) = r(k,y),\ T_y(k,y) = -p(k,y) \text{ and } w(k,y) = T(k,y) - r(k,y)k + p(k,y)y. \quad (2)$$

We can also characterize the second derivatives of $T(k,y)$. Using the concavity property we have:

$$T_{kk}(k,y) = \frac{\partial r}{\partial k} \leq 0,\ T_{yy}(k,y) = -\frac{\partial p}{\partial y} \leq 0.$$

As shown by Benhabib and Nishimura [16], the sign of the cross derivative $T_{ky}(k,y)$ is given by the sign of the relative capital intensity difference between the two sectors. Denoting $a_{00} = l_0/y_0$, $a_{10} = k_0/y_0$, $a_{01} = l_1/y$ and $a_{11} = k_1/y$ the capital and labor coefficients in each sector, it is easy to derive from the constant returns to scale property that:

$$\frac{dp}{dr} = a_{01}\left(\frac{a_{11}}{a_{01}} - \frac{a_{10}}{a_{00}}\right) \equiv b \quad (3)$$

with b the relative capital intensity difference, and thus

$$T_{ky} \quad = \quad T_{yk} = -\frac{\partial p}{\partial r}\frac{\partial r}{\partial k} = -T_{kk}b.$$

The sign of both b and T_{ky} is positive if and only if the investment good is capital intensive. Note also that $T_{yy}(k,y)$ can be written as:

$$T_{yy} \quad = \quad -\frac{\partial p}{\partial r}\frac{\partial r}{\partial y} = T_{kk}b^2.$$

Remark 1 : The derivative $dr/dp = b^{-1}$ is well-known in trade theory as the Stolper-Samuelson effect. Similarly, at constant prices, we can derive the associated Rybczinsky effect $dy/dk = b^{-1}$. We therefore find the well-known duality between the Rybczinsky and Stolper-Samuelson effects.

2.2 Preferences

The economy is populated by a constant population of finitely-lived agents.[15] In each period t, $N_t = N$ persons are born, and they live for two periods: they work during the first (with one unit of labor supplied) and they have preferences for consumption (c_t, when they are young, and d_{t+1}, when they are old) which are summarized by the utility

[15] An increasing population could be considered without altering all our results.

function $u(c_t, Bd_{t+1})$, with $B > 0$ a normalization constant, such that Assumption 2 is satisfied.

Assumption 2. *$u(c, Bd)$ is increasing in both arguments ($u_c(c, Bd) > 0$ and $u_d(c, Bd) > 0$), concave and C^2 over the interior of $\mathbb{R}^2_+$. Moreover, $\lim_{X\to 0} Xu_X(c, X)/u_c(c, X) = 0$ and $\lim_{X\to+\infty} Xu_X(c, X)/u_c(c, X) = +\infty$, or $\lim_{X\to 0} Xu_X(c, X)/u_c(c, X) = +\infty$ and $\lim_{X\to+\infty} Xu_X(c, X)/u_c(C, X) = 0$.*

We also introduce a standard normality assumption between the two consumption levels.

Assumption 3. *Consumptions c and d are normal goods.*

We finally consider the following useful elasticities of consumptions:

$$\epsilon_{cc} = -u_c/u_{cc}c > 0, \qquad \epsilon_{cd} = -u_c/u_{cd}Bd, \tag{4}$$

$$\epsilon_{dc} = -u_d/u_{cd}c, \quad \epsilon_{dd} = -u_d/u_{dd}Bd > 0 \tag{5}$$

It is worth noting that the normality Assumption 3 implies $1/\epsilon_{cc} - 1/\epsilon_{dc} \geq 0$ and $1/\epsilon_{dd} - 1/\epsilon_{cd} \geq 0$ and concavity in Assumption 2 implies $1/(\epsilon_{cc}\epsilon_{dd}) - 1/(\epsilon_{dc}\epsilon_{cd}) \geq 0$. Taking these elasticities, the elasticity of intertemporal substitution between c_t and d_{t+1} writes:

$$\varsigma(c_t, d_{t+1}) = \frac{\frac{u_d(c_t,d_{t+1})/u_c(c_t,d_{t+1})}{c_t/d_{t+1}}}{\frac{\partial(u_d(c_t,d_{t+1})/u_c(c_t,d_{t+1}))}{\partial(c_t/d_{t+1})}} = \frac{1}{\frac{1}{\epsilon_{cc}} - \frac{1}{\epsilon_{dc}}} \geq 0. \tag{6}$$

This elasticity will be used as a parameter driving the existence of endogenous fluctuations.

2.3 The optimal growth problem

Under complete depreciation within one period,[16] the capital accumulation equation is:

$$k_{t+1} = y_t. \tag{7}$$

Since total labor is normalized to 1, we consider from now on that $N = 1$. At each time t, total consumption is then given by the social production function, i.e. $c_t + d_t = T(k_t, y_t)$. The intertemporal objective function of the central planner combines utilities of successive generations:

[16] Considering that in an OLG model one period is approximately 30 years, complete depreciation is a realistic assumption.

$$\max_{\{c_t, d_{t+1}\}} \sum_{t=0}^{+\infty} \beta^t u(c_t, Bd_{t+1}) \tag{8}$$

where $\beta \in (0,1]$ is the discount factor.[17] Considering (7) and the fact that $c_t = T(k_t, y_t) - d_t$, the optimization program (8) can be equivalently written as follows

$$\max_{\{d_{t+1}, k_{t+1}\}} \sum_{t=0}^{+\infty} \beta^t u(T(k_t, k_{t+1}) - d_t, Bd_{t+1}) \tag{9}$$

with d_0 and k_0 given. The first order conditions are given by the following two difference equations of order two:

$$\begin{aligned} u_d(T(k_t, k_{t+1}) - d_t, Bd_{t+1})B - \beta u_c(T(k_{t+1}, k_{t+2}) - d_{t+1}, Bd_{t+2}) &= 0 \\ u_c(T(k_t, k_{t+1}) - d_t, Bd_{t+1})T_y(k_t, k_{t+1}) &+ \\ \beta u_c(T(k_{t+1}, k_{t+2}) - d_{t+1}, Bd_{t+2})T_k(k_{t+1}, k_{t+2}) &= 0. \end{aligned} \tag{10}$$

Notably, taking some (given) initial conditions (d_0, k_0), any path that satisfies equations (10) together with the following transversality conditions,

$$\lim_{t\to+\infty} \beta^t u_d(c_t, Bd_{t+1})p_{t+1}k_{t+1} = 0 \text{ and } \lim_{t\to+\infty} \beta^t u_d(c_t, Bd_{t+1})d_{t+1} = 0, \tag{11}$$

is an optimal path.

2.4 Steady state

A steady state is defined as the stationary solution, $k_t = k^*$, $d_t = d^*$, for all t of the following nonlinear system of equations:

$$\begin{aligned} \frac{u_d(T(k,k)-d, Bd)B}{u_c(T(k,k)-d, Bd)} &= \beta \\ -\frac{T_y(k,k)}{T_k(k,k)} &= \beta. \end{aligned} \tag{12}$$

Beside discussing the existence and uniqueness of the steady state, note that we further introduce the B parameter to normalize the stationary consumption d such that it remains constant when the discount factor β is modified. As in the standard two-sector model, we get the following result:

Proposition 1. *Under Assumptions 1-3, there exists a unique steady state* (k^*, d^*) *solution of equations* (12). *Moreover, there exists a unique value* B^* *such when* $B = B^*$, *the stationary consumption* d^* *can be normalized to any value* $\bar{d} \in (0, T(k^*, k^*))$.

[17] In the case $\beta = 1$, the infinite sum into the optimization program (8) may not converge. In such a case we may apply the definition of optimality as provided by Ramsey [54].

Proof. See Appendix 8.1. □

A pair (k^*, d^*) is then defined to be the Modified Golden Rule. Finally, the stationary consumption of young agents is obtained from $c^* = T(k^*, k^*) - d^*$.

2.5 Characteristic polynomial

We are now in a position to derive the characteristic polynomial from total differentiation of equations (10). Denoting $T(k^*, k^*) = T^*$, $T_k(k^*, k^*) = T_k^*$ and $T_{kk}(k^*, k^*) = T_{kk}^*$, the elasticities of the consumption good's output and the rental rate with respect to the capital stock, all evaluated at the steady state, can be written as:

$$\varepsilon_{ck} = T_k^* k^*/T^* > 0, \quad \varepsilon_{rk} = -T_{kk}^* k^*/T_k^* > 0. \tag{13}$$

Then Proposition 2 yields the degree-4 characteristic polynomial and discusses the multiplicity order of the possible (characteristic) roots.

Proposition 2. *Under Assumptions 1-3, the degree-4 characteristic polynomial is given by*

$$\mathcal{P}(\lambda) \;=\; \lambda^4 - \lambda^3 B + \lambda^2 C - \lambda \tfrac{B}{\beta} + \tfrac{1}{\beta^2} \tag{14}$$

with

$$\begin{aligned} B &= -\tfrac{\beta}{b\epsilon_{cc}}\tfrac{\varepsilon_{ck}}{\varepsilon_{rk}}\left(\tfrac{\epsilon_{cc}}{\epsilon_{dc}} - \tfrac{\epsilon_{cd}}{\epsilon_{dd}}\right) + \tfrac{\beta+b^2}{\beta b} + \tfrac{\epsilon_{dc}}{\beta\epsilon_{cc}} + \tfrac{\epsilon_{cd}}{\epsilon_{dd}} \\ C &= -\tfrac{(1+\beta)}{b\epsilon_{cc}}\tfrac{\varepsilon_{ck}}{\varepsilon_{rk}}\left(\tfrac{\epsilon_{cc}}{\epsilon_{dc}} - \tfrac{\epsilon_{cd}}{\epsilon_{dd}}\right) + \tfrac{\beta+b^2}{\beta b}\left(\tfrac{\epsilon_{dc}}{\beta\epsilon_{cc}} + \tfrac{\epsilon_{cd}}{\epsilon_{dd}}\right) + \tfrac{2}{\beta} \end{aligned} \tag{15}$$

or equivalently

$$\mathcal{P}(\lambda) = \left[\lambda^2 - \lambda\left(\tfrac{\epsilon_{dc}}{\beta\epsilon_{cc}} + \tfrac{\epsilon_{cd}}{\epsilon_{dd}}\right) + \tfrac{1}{\beta}\right]\tfrac{(\lambda b-1)(\lambda\beta-b)}{\beta b} + \lambda(\lambda-1)\left(\lambda - \tfrac{1}{\beta}\right)\tfrac{\beta}{b\epsilon_{cc}}\tfrac{\varepsilon_{ck}}{\varepsilon_{rk}}\left(\tfrac{\epsilon_{cc}}{\epsilon_{dc}} - \tfrac{\epsilon_{cd}}{\epsilon_{dd}}\right) \tag{16}$$

If λ is a characteristic root of (16), then $\bar{\lambda}$, $(\beta\lambda)^{-1}$ and $(\beta\bar{\lambda})^{-1}$ are also characteristic roots of (16). Moreover, at least two roots or a pair of complex conjugate roots have a modulus larger than one, and one of the following cases necessarily hold:

i) the four roots are real and distincts,

ii) the four roots are given by two pairs of non-real complex conjugates,

iii) there are two complex conjugates double roots or two real double roots.

Proof. See Appendix 8.2. □

Proposition 2 is of critical importance and several points are worth commenting. First, it shows that if there exists a pair of complex characteristic roots $(\lambda, \bar{\lambda})$ solutions of the quartic polynomial (16), then a second pair of complex characteristic roots, $(\beta\lambda)^{-1}$ and $(\beta\bar{\lambda})^{-1}$, are also solutions of (16). Therefore, Proposition 2 proves that the 4 characteristic roots are either all real or all complex. Second, Proposition 2 also implies that at most two characteristic roots can have a modulus lower than 1 and thus that the steady state can be either saddle-point stable or totally unstable. Of course in this last case, endogenous cycles can occur. Third, under Assumption 2, the sign of the expression $\frac{\epsilon_{cc}}{\epsilon_{dc}} - \frac{\epsilon_{cd}}{\epsilon_{dd}}$ is given by the sign of the cross derivative u_{cd}, i.e. by the opposite of the sign of $\epsilon_{cd}, \epsilon_{dc}$, which is a crucial ingredient to determine the local stability properties of the steady state. Fourth, using Eq. (16), when the utility function is non-strictly concave, i.e. if $\frac{\epsilon_{cc}}{\epsilon_{dc}} - \frac{\epsilon_{cd}}{\epsilon_{dd}} = 0$, then the degree-4 polynomial simplifies to a product of two degree-2 polynomials. In this respect, we first focus on the simpler case of a non-strictly concave utility function (Section 4), and then consider the more general case of strictly concave preferences (Section 5).

Remark 2: The degree-4 characteristic polynomial (14) is a quasi-palindromic equation that can be solved explicitly, and its roots can be determined using only quadratic equations (see Appendix 8.9 for details.). As far as we know, this is the first time this type of methodology is applied to macroeconomic dynamic models. The existence of a Hopf bifurcation in models featuring forward-looking agents and a competitive equilibrium structure has initially been emphasized in Benhabib and Nishimura [15]. However, due to the dual structure of such models, the consideration of at least three sectors and of dimension-four dynamical systems has been shown to be necessary to get such deterministic fluctuations based on the existence of complex characteristic roots. And as shown in many contributions, i.e. Magill [38, 39, 40] and Magill and Scheinkman [41], due to the high dimension of such systems, there is no obvious sufficient conditions for the existence of complex characteristic roots. In this paper, using the quasi-palindromic structure of the characteristic polynomial allows us to provide such sufficient conditions.

Remark 3: If $b = 0$, one gets the one-sector formulation with a two-dimensional dynamical system as considered in Michel and Venditti [43]. Indeed, the characteristic polynomial can be simplified as follows

$$\mathcal{P}(\lambda) \quad = \quad \lambda^2 - \lambda \frac{\frac{\epsilon_{dc}}{\beta\epsilon_{cc}} + \frac{\epsilon_{cd}}{\epsilon_{dd}} - \frac{(1+\beta)}{\epsilon_{cc}} \frac{\varepsilon_{ck}}{\varepsilon_{rk}} \left(\frac{\epsilon_{cc}}{\epsilon_{dc}} - \frac{\epsilon_{cd}}{\epsilon_{dd}} \right)}{1 - \frac{\beta}{\epsilon_{cc}} \frac{\varepsilon_{ck}}{\varepsilon_{rk}} \left(\frac{\epsilon_{cc}}{\epsilon_{dc}} - \frac{\epsilon_{cd}}{\epsilon_{dd}} \right)} + \frac{1}{\beta}$$

The same conclusions as in Michel and Venditti [43] are obviously derived.

Similarly, if the utility function is additively separable, i.e. $u_{cd} = u_{dc} = 0$, we get the two-sector optimal growth formulation with a two-dimensional dynamical system as considered in Benhabib and Nishimura [15]. The characteristic polynomial can indeed be simplified as follows

$$\mathcal{P}(\lambda) \quad = \quad \lambda^2 - \lambda(1+\beta) \frac{\frac{\beta}{\epsilon_{cc}} \frac{\varepsilon_{ck}}{\varepsilon_{rk}} + (\beta + b^2)}{\frac{\beta}{\epsilon_{cc}} \frac{\varepsilon_{ck}}{\varepsilon_{rk}} + (1+\beta)b} + \frac{1}{\beta}$$

The same conclusions as in Benhabib and Nishimura [15] are then derived.

3 Period-two cycles under non-strictly concave preferences

In this section we assume that the utility function is non-strictly concave.

Assumption 4. *The utility function $u(c, Bd)$ is concave non-strictly, i.e.* $\frac{\epsilon_{cc}}{\epsilon_{dc}} - \frac{\epsilon_{cd}}{\epsilon_{dd}} = 0$.

In so doing, one can show that the characteristic roots cannot be complex.

Lemma 1. *Under Assumptions 1-4, the characteristic roots are real.*

Proof. See Appendix 8.3. □

Following simultaneously the same methodologies as in the two-sector optimal growth model and the optimal growth solution of the aggregate OLG model, we discuss the local stability properties of equilibrium paths depending both on the sign of the capital intensity difference across sectors b and the sign of the cross derivative u_{cd}, i.e. of the two elasticities ϵ_{cd} and ϵ_{dc}.

As a first step, Proposition 3 provides some simple conditions ensuring the saddle-point property with monotone convergence.

Proposition 3. *Under Assumptions 1-4, if $b \geq 0$ and $\epsilon_{cd}, \epsilon_{dc} \geq 0$, i.e. $u_{cd} \leq 0$, then the equilibrium path is monotone and the steady-state (k^*, d^*) is a saddle-point.*

Proof. See Appendix 8.4. □

3.1 A separated mechanism for period-two cycles

We now show that convergence with oscillations and persistent competitive equilibrium cycles may occur under a quite large set of circumstances.

Proposition 4. *Under Assumptions 1-4, the following results hold:*

i) When the investment good is capital intensive, i.e. $b \geq 0$*, let* $\epsilon_{cd}, \epsilon_{dc} < 0$*, i.e.* $u_{cd} > 0$*. Then the steady state* (k^*, d^*) *is saddle-point stable with damped oscillations if and only if the elasticity of intertemporal substitution satisfies* $\varsigma \in (0, \underline{\varsigma}) \cup (\bar{\varsigma}, +\infty)$ *with*

$$\underline{\varsigma} \equiv \frac{\epsilon_{cc}\beta}{1+\beta} \quad \text{and} \quad \bar{\varsigma} \equiv \frac{\epsilon_{cc}}{2}$$

Moreover, when ς *crosses the bifurcation values* $\underline{\varsigma}$ *or* $\bar{\varsigma}$*,* (k^*, d^*) *undergoes a flip bifurcation leading to persistent period-2 cycles.*

ii) When $\epsilon_{cd}, \epsilon_{dc} \geq 0$*, i.e.* $u_{cd} \leq 0$*, let the consumption good be capital intensive, i.e.* $b < 0$*. Then the steady state* (k^*, d^*) *is saddle-point stable with damped oscillations if and only if* $b \in (-\infty, -1) \cup (-\beta, 0)$*. Moreover, if there is some* $\beta^* \in (0, 1)$ *such that* $b \in (-1, -\beta^*)$*, then there exists* $\bar{\beta} \in (0, 1)$ *such that, when* β *crosses* $\bar{\beta}$ *from above,* (k^*, d^*) *undergoes a flip bifurcation leading to persistent period-2 cycles.*

iii) When the consumption good is capital intensive, i.e. $b < 0$*, and* $\epsilon_{cd}, \epsilon_{dc} < 0$*, i.e.* $u_{cd} > 0$*, the steady state* (k^*, d^*) *is saddle-point stable with damped oscillations if and only if* $b \in (-\infty, -1) \cup (-\beta, 0)$ *and* $\varsigma \in (0, \underline{\varsigma}) \cup (\bar{\varsigma}, +\infty)$*. Moreover, if there is some* $\beta^* \in (0, 1)$ *such that* $b \in (-1, -\beta^*)$*, then there exists* $\bar{\beta} \in (0, 1)$ *such that, when* β *crosses* $\bar{\beta}$ *from above or* ς *crosses the bifurcation values* $\underline{\varsigma}$ *or* $\bar{\varsigma}$*,* (k^*, d^*) *undergoes a flip bifurcation leading to persistent period-2 cycles.*

Proof. See Appendix 8.5. □

Proposition 4 provides two independent mechanisms leading to the existence of endogenous fluctuations. The first one is based on the properties of preferences through the sign of the cross derivative u_{cd} and is the most interesting in our context since it allows to generate period-2 cycles in a two-sector model even under a capital intensive investment good sector—a condition which is known since Benhabib and Nishimura [16] to ensure monotone convergence in a standard two-sector optimal growth model.

To provide further economic insights, let us consider an instantaneous increase in the capital stock k_t. Using $c_t + d_t = T(k_t, y_t)$ and $T_k > 0$, it follows that c_t increases, and thus, taking that the marginal utility of second period consumption u_d is larger as $u_{dc} > 0$, a constant utility level $u(c_t, d_{t+1})$ can be obtained from a decrease of d_{t+1}. Using the first equation in (10) and taking d_{t+1} as given,

$$\frac{\Delta c_{t+1}}{\Delta c_t} = \frac{u_{dc}}{u_{cc}\beta} + \frac{u_{dd}}{u_{cc}\beta}\frac{\Delta d_{t+1}}{\Delta c_t} < 0.$$

Finally, since $c_{t+1}+d_{t+1} = T(k_{t+1}, y_{t+1})$, total consumption at time $t+1$ is lower, which in turn implies that a lower capital stock k_{t+1} when y_{t+1} holds constant. Endogenous fluctuations are thus generated from intertemporal consumption allocations based on some substitution effects between the first and second period consumptions. The important result is that the elasticity of intertemporal substitution needs to be large enough to allow sufficient substitution between c_t and d_{t+1} to generate aggregate oscillations, but should not be too large to be compatible with the transversality conditions (11) and a convergence process towards the period-two cycle.

The second mechanism is, as in the two-sector optimal growth model, based on the properties of sectoral technologies through the sign of the capital intensity difference across sectors. Following Benhabib and Nishimura [16], we can use the Rybczinski and Stolper-Samuelson effects to provide a simple economic intuition for this result. Assume indeed that the consumption good is capital intensive, i.e. $b < 0$, and consider an instantaneous increase in the capital stock k_t. This results in two opposing mechanisms:

- On the one hand, the trade-off in production becomes more favorable to the consumption good, and the Rybczinsky effect implies a decrease of the output of the capital good y_t. This tends to lower both the investment and the capital stock in the next period k_{t+1}.

- On the other hand, in the next period the decrease of k_{t+1} implies again through the Rybczinsky effect an increase of the output of the capital good y_{t+1}. Indeed the decrease of k_{t+1} improves the trade-off in production in favor of the investment good which is relatively less intensive in capital and this tends to increase the investment and the capital stock in period $t + 2$, k_{t+2}.

Of course, under both mechanisms, the existence of persistent fluctuations require that the oscillations in consumption and relative prices must not present intertemporal arbitrage opportunities. Consequently, a minimum level of myopia, i.e. a low enough value for the discount rate β, is thus necessary.

Note finally that in case iii) of Proposition 4, both mechanisms hold at the same time. Interestingly, using both β and ς as two bifurcation parameters allows to consider a co-dimension 2 bifurcation which corresponds to the flip bifurcation with a 1:2 resonance where two characteristic roots are equal to -1 simultaneously. As shown in Kuznetsov [37], in such a configuration, under non-degeneracy conditions, the steady state is either saddle-point stable or elliptic. This last case may give rise to the existence of quasi-periodic cycles which are usually associated to a Hopf bifurcation.

3.2 A simple illustration with homogeneous preferences

To shed more light on Proposition 4, we provide an example for all cases. In doing so, we consider the class of homogeneous of degree $\gamma \leq 1$ utility functions with $B = 1$,[18] which obviously satisfies Assumptions 2 and 3. Then we introduce the share of first period consumption within total utility $\phi(c, d) \in (0, \gamma)$ defined by:

$$\phi(c, Bd) = \frac{u_c(c,Bd)c}{u(c,Bd)}. \tag{17}$$

Accordingly, the share of second period consumption within total utility is defined as $\gamma - \phi(c, Bd) \in (0, 1)$. Consequently, using (4)-(5), the elasticities of interest are given by

$$\epsilon_{cd} = -\frac{\epsilon_{cc}}{1-\epsilon_{cc}(1-\gamma)}, \quad \epsilon_{dc} = -\frac{(\gamma-\phi)\epsilon_{cc}}{\phi[1-\epsilon_{cc}(1-\gamma)]}, \quad \epsilon_{dd} = \frac{(\gamma-\phi)\epsilon_{cc}}{\phi-\epsilon_{cc}(1-\gamma)(2\phi-\gamma)}. \tag{18}$$

Furthermore, we impose a restriction on ϵ_{cc} to ensure concavity and the normality goods assumption.

Assumption 5. $\epsilon_{cc} \leq \frac{\gamma}{\phi(1-\gamma)} \equiv \bar{\epsilon}_{cc}$

Therefore it is straightforward to get $\epsilon_{dd} > 0$ while $\epsilon_{cd}, \epsilon_{cd} < 0$ if and only if $\epsilon_{cc} < 1/(1-\gamma) \equiv \tilde{\epsilon}_{cc}(< \gamma/\phi(1-\gamma))$. Moreover, the elasticity of substitution between the two life-cycle consumption levels is now defined by:

$$\varsigma(\phi) = \frac{\epsilon_{cc}(\gamma-\phi)}{\gamma-\phi\epsilon_{cc}(1-\gamma)} \in (0, +\infty) \tag{19}$$

Notably, if $\epsilon_{cc} < \tilde{\epsilon}_{cc}$, then $\varsigma(\phi) \in (0, \epsilon_{cc})$.

Finally, from the production side, as in Baierl *et al.* [8], we assume that the consumption and investment goods are produced with Cobb-Douglas technologies as follows

$$y_0 = k_0^{\alpha_0} l_0^{1-\alpha_0}, \; y = k_1^{\alpha_1} l_1^{1-\alpha_1} \tag{20}$$

It can be shown that

$$b \;\; = \;\; \frac{\beta(\alpha_1-\alpha_0)}{1-\alpha_0} \tag{21}$$

[18]The normalization constant B is not required for this class of utility functions.

Finally, for sake of simplicity, γ is set to one, which means that Assumptions 4 and 5 hold, and that $\varsigma = \epsilon_{cc}(1-\phi) < \epsilon_{cc}$.

We first discuss case iii) of Proposition 4 where a co-dimension 2 bifurcation can arise. In that respect, we derive the following Corollary:[19]

Corollary 1. *Let the utility function be homogeneous of degree 1 and the sectoral production functions be given by (20), and assume that $\alpha_0 > (1+\alpha_1)/2$. Then the steady state (k^*, d^*) is saddle-point stable with damped oscillations if and only if $\varsigma \in (0, \underline{\varsigma}) \cup (\bar{\varsigma}, \epsilon_{cc})$ and $\beta > \underline{\beta}$, with*

$$\underline{\varsigma} = \frac{\epsilon_{cc}\beta}{1+\beta}, \quad \bar{\varsigma} = \frac{\epsilon_{cc}}{2} \quad \text{and} \quad \underline{\beta} = \frac{1-\alpha_0}{\alpha_0-\alpha_1}$$

If $\beta = \underline{\beta}$ and $\varsigma = \bar{\varsigma}$ or $\underline{\varsigma}$, then a co-dimension 2 flip bifurcation with a 1:2 resonance generically occurs.

Proof. See Appendix 8.6. □

While providing a precise dynamic analysis of this co-dimension 2 bifurcation goes far beyond the objectives of this paper, it is worthwhile to mention that this case provides an interesting possibility of smooth endogenous fluctuations for the main aggregate variables which does not arise under a standard flip bifurcation. Indeed, while there does not *a priori* exist complex characteristic roots under a linear homogenous utility function, Kuznetsov [37] shows that under a 1:2 resonance, the steady state can be elliptic and a stable limit cycle, similar to those that arise under a Hopf bifurcation, can occur. As pointed out in the next section, a Hopf bifurcation provides another tool to describe the long-run cyclical behavior of macroeconomic variables such as bequests.

Second, we focus on case (i) of Proposition 4 in which the investment good is capital intensive. The occurrence of endogenous fluctuations is then characterized in Corollary 2. Note that the result does not depend on the sign of the capital intensity difference and also holds if the consumption good is capital intensive.

Corollary 2. *Let the utility function be homogeneous of degree 1 and the sectoral production functions be given by (20). Then, for any of $\alpha_0, \alpha_1 \in (0,1)$, the steady state (k^*, d^*) is saddle-point stable with damped oscillations if and only if $\varsigma \in (0, \underline{\varsigma}) \cup (\bar{\varsigma}, \epsilon_{cc})$. Moreover,*

[19] Similar conclusions can be derived using instead CES technologies with non unitary sectoral elasticities of capital-labor substitution. It can be shown indeed that our conclusions are robust to a wide range of values for these parameters. A proof of this claim is available upon request.

when ς crosses the bifurcation values $\underline{\varsigma}$ or $\bar{\varsigma}$, (k^*, d^*) undergoes a flip bifurcation leading to persistent period-2 cycles.

Finally, we focus on case (ii) of Proposition 4 where endogenous fluctuations arise for any given value $\varsigma \in (0, \underline{\varsigma}) \cup (\bar{\varsigma}, \epsilon_{cc})$ when the consumption good sector is capital intensive.

Corollary 3. *Let the utility function be homogeneous of degree 1 and the sectoral production functions be given by (20). Assume also that $\varsigma \in (0, \underline{\varsigma}) \cup (\bar{\varsigma}, \epsilon_{cc})$ and $\alpha_0 > (1 + \alpha_1)/2$. Then the steady state (k^*, d^*) is saddle-point stable with damped oscillations if and only if $\beta > \underline{\beta}$. Moreover, when β crosses the bifurcation value $\underline{\beta}$, (k^*, d^*) undergoes a flip bifurcation leading to persistent period-2 cycles.*

This case is presented here for illustration as it corresponds to the main conclusions of Benhabib and Nishimura [16] derived in a standard two-sector optimal growth model.

4 Quasi-periodic cycles under strictly concave preferences

As explained above, using a non-strictly concave utility function is convenient in the sense that it reduces the degree-4 characteristic polynomial to the product of two degree-2 polynomials. In such a framework, we have shown that the characteristic roots are necessarily real and that endogenous fluctuations can occur through the existence of period-two cycles. Notably the preference and technology mechanisms are separated and lead independently to the occurrence of endogenous fluctuations. Relaxing this simplifying assumption, our objective here is to prove that the preference and technology mechanisms can mix together, and then complexify and amplify the possible endogenous fluctuations in the context of strictly concave preferences.

4.1 A mixed mechanism for the existence of quasi-periodic cycles

We now need to focus on the existence of complex characteristic roots and quasi-periodic cycles occurring through a Hopf bifurcation. Therefore we start by providing general sufficient conditions allowing to rule out the existence of complex roots.

Proposition 5. *Under Assumptions 1-3, let the utility function $u(c, Bd)$ be strictly concave. Then the roots of the characteristic polynomial (16) are necessarily real in the following cases:*

i) for any sign of $\epsilon_{cd}, \epsilon_{cd}$ if the investment good sector is capital intensive, i.e. $b > 0$,

ii) if $\epsilon_{cd}, \epsilon_{cd} > 0$ and the consumption good sector is capital intensive, i.e. $b < 0$.

Proof. See Appendix 8.7. □

Necessary conditions for the existence of complex roots are therefore based on the two mechanisms that generate endogenous fluctuations in the non-strictly concave case, namely $b < 0$ **and** $\epsilon_{cd}, \epsilon_{cd} < 0$.

In order to study whether complex characteristic roots and a Hopf bifurcation with quasi-periodic cycles can occur, we consider again a utility function homogeneous of degree γ, but with $\gamma < 1$ to allow for strict concavity.

We first derive sufficient conditions to ensure saddle-point property of the steady state with real characteristic roots.

Proposition 6. *Let the utility function be homogeneous of degree $\gamma < 1$, and assume that $\epsilon_{cc} < \tilde{\epsilon}_{cc}$, $b \in (-\infty, -1) \cup (-\beta, 0)$ and*

$$-\frac{\varepsilon_{ck}}{b\varepsilon_{rk}} > 1 \tag{22}$$

Then there exist $0 < \underline{\varsigma} \leq \bar{\varsigma} < \epsilon_{cc}$ and $\hat{\epsilon}_{cc} \in (0, \tilde{\epsilon}_{cc})$ such that when $\varsigma \in (0, \underline{\varsigma}) \cup (\bar{\varsigma}, \epsilon_{cc})$ the characteristic roots are real and the steady-state is saddle-point stable. Moreover,

i) when $\varsigma \in (0, \underline{\varsigma})$, the optimal path converges towards the steady state with oscillations if $\epsilon_{cc} \in (0, \hat{\epsilon}_{cc})$ or monotonically if $\epsilon_{cc} \in (\hat{\epsilon}_{cc}, \tilde{\epsilon}_{cc})$,

ii) when $\varsigma \in (\bar{\varsigma}, \epsilon_{cc})$, the optimal path converges towards the steady state with oscillations.

Proof. See Appendix 8.8. □

Condition (22) allows to get the existence of the bound $\hat{\epsilon}_{cc}$ and thus the occurrence of oscillations when the elasticity of intertemporal substitution is low, i.e. $\varsigma \in (0, \underline{\varsigma})$. This restriction can be easily interpreted. Denoting σ_i the elasticity of capital-labor substitution in sector $i = 0, 1$ and using Drugeon [25], we can relate the ratio of elasticities $\varepsilon_{ck}/\varepsilon_{rk}$ to an aggregate elasticity of substitution between capital and labor, denoted Σ, which is obtained as a weighted sum of the sectoral elasticities σ_i:[20]

$$\frac{\varepsilon_{ck}}{\varepsilon_{rk}} = \left(\frac{T}{l_0^2}\right) \frac{s}{1-s} \frac{\Sigma}{GDP} \text{ with } \Sigma = \frac{GDP}{pykT} \left(pyk_0 l_0 \sigma_0 + Tk_1 l_1 \sigma_1\right), \tag{23}$$

$GDP = T + py$ and $s = rk/GDP$ the share of capital income in GDP. Therefore, oscillations when $\phi \in (\bar{\phi}, \gamma)$ are associated with a large aggregate elasticity of substitution between capital and labor *i.e.*, large enough sectoral elasticities of capital-labor substitution.

Proposition 6 implies that the existence of complex roots, if any, requires to consider intermediate values for the elasticity of intertemporal substitution ς between the first and second period consumptions, i.e. $\varsigma \in (\underline{\varsigma}, \bar{\varsigma})$. As mentioned previously, the degree-4 characteristic polynomial (14) is a quasi-palindromic equation that can be solved explicitly, and its roots can be determined using only quadratic equations. As shown in Appendix 8.9, we apply this methodology in order to provide sufficient conditions for the occurrence of complex roots and a Hopf bifurcation. As far as we know, this is the first time this type of sufficient conditions are exhibited in 4-dimensional optimal growth models. We therefore provide some positive issue to the previous contributions, i.e. Magill [38, 39, 40] and Magill and Scheinkman [41], where it has been shown that no obvious sufficient conditions for the existence of complex characteristic roots can be exhibited.

We can indeed derive the following result:

Proposition 7. *Let the utility function be homogeneous of degree $\gamma < 1$, and assume that $\epsilon_{cc} < \tilde{\epsilon}_{cc}$ and $b \in (-\beta, 0)$. Then there exist $\bar{b} \in (-\beta, 1)$, $\underline{\gamma} \in (0, 1)$, $\underline{\epsilon}_{cc}, \bar{\epsilon}_{cc} \in (0, \tilde{\epsilon}_{cc})$, $\bar{\varepsilon} > 0$ and four critical values $(\underline{\varsigma} \leq)\underline{\varsigma}^c < \underline{\varsigma}^H < \bar{\varsigma}^H < \bar{\varsigma}^c (\leq \bar{\varsigma})$ such that when $b \in (-\beta, \bar{b})$, $\gamma \in (\underline{\gamma}, 1)$, $\epsilon_{cc} \in (\underline{\epsilon}_{cc}, \bar{\epsilon}_{cc})$ and*

$$-\frac{\varepsilon_{ck}}{b\varepsilon_{rk}} < \bar{\varepsilon} \tag{24}$$

the following results hold:

i) the steady state (k^*, d^*) is saddle-point stable with damped oscillations if $\varsigma \in (\underline{\varsigma}^c, \underline{\varsigma}^H) \cup (\bar{\varsigma}^H, \bar{\varsigma}^c)$,

ii) when ς crosses the bifurcation values $\underline{\varsigma}^H$ or $\bar{\varsigma}^H$, (k^*, d^*) undergoes a Hopf bifurcation leading to persistent quasi-periodic cycles.

Proof. See Appendix 8.9. □

[20] The expression of Σ is derived from Proposition 2 in Drugeon [25].

From a theoretical point of view, Proposition 7 provides a strong conclusion as it shows that a Hopf bifurcation and quasi-periodic cycles can occur in a two-sector optimal growth framework as long as it is based on an OLG structure with non-separable and strictly concave preferences. More specifically, we need intermediate values for the elasticity ϵ_{cc} and the elasticity of intertemporal substitution ς between the first and second period consumptions, together with, using (23), a not too large value for the sectoral elasticities of capital-labor substitution.

Such a result is drastically different from what can be obtained in standard optimal growth models as the existence of complex roots requires to consider at least three sectors.[21] This can be explained by the fact that we can mix here two mechanisms based on both preferences and technology, while in standard optimal growth models only the technology mechanism arises and thus requires more than two sectors to generate quasi-periodic fluctuations. In our case, the preference mechanism based on a substitutability effect between first and second period consumptions, and the technology mechanism based on a capital intensive consumption good sector feed each other when the utility function is strictly concave and amplify the endogenous fluctuations of capital and consumption. More complex quasi-periodic fluctuations can thus occur for adequate values of the elasticity of intertemporal substitution in consumption.

4.2 A simple illustration

Let us now focus on a numerical illustration. Considering that the annual discount factor is often estimated to be around 0.96 and that one period in an OLG model is about 30 years, we consider here that $\beta = 0.96^{30} \approx 0.294$. Focusing on a slight deviation with respect to the linear homogeneous case with $\gamma = 0.98$, let us then assume a standard value $\epsilon_{cc} = 1$ that satisfies $\epsilon_{cc} \in (\underline{\epsilon}_{cc}, \bar{\epsilon}_{cc})$. We also consider sectoral Cobb-Douglas technologies as given by (20) with $\alpha_0 = 0.6$ and $\alpha_1 = 0.21$ so that the consumption good is capital intensive with $b \approx -0.28665$ close to $-\beta$.[22] The bounds exhibited in Proposition 7 are equal to $\underline{\varsigma}^c \approx 0.1194$ and $\bar{\varsigma}^c \approx 0.6083$. We then find that the characteristic polynomial (60) admits four characteristic roots $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ that are complex conjugate by pair

[21] See Benhabib and Nishimura [15], Cartigny and Venditti [18], Venditti [58].

[22] The existence of a Hopf bifurcation can also be obtained using instead CES technologies with non unitary sectoral elasticities of capital-labor substitution. As in the case with period-two cycles, our conclusions are robust to a wide range of values for these parameters. A proof of this claim is also available upon request.

with $\lambda_1\lambda_2 > 1$ and $\lambda_3\lambda_4 < 1$ if $\varsigma \in (\underline{\varsigma}, \underline{\varsigma}^H) \cup (\bar{\varsigma}^H, \bar{\varsigma})$ while $\lambda_3\lambda_4 > 1$ if $\varsigma \in (\underline{\varsigma}^H, \bar{\varsigma}^H)$, with $\underline{\varsigma}^H \equiv 0.3193$ and $\bar{\varsigma}^H \equiv 0.426$. Moreover $\lambda_3\lambda_4 = 1$ when $\varsigma = \underline{\varsigma}^H$ or $\bar{\varsigma}^H$. As a result $\underline{\varsigma}^H$ and $\bar{\varsigma}^H$ are Hopf bifurcation values giving rise to quasi-periodic cycles in their neighborhood.

5 The solution with altruistic agents and a bequest motive

As a final step, we need to show that the existence of optimal endogenous cycles is compatible with strictly positive bequest transmissions across generations. We thus consider a decentralized economy composed of overlapping generations of parents loving their children. As in the Barro [9] formulation, each agent is altruistic towards his descendant through a bequest motive.[23] Parents indeed care about their child's welfare by taking into account their child's utility into their own utility function. They are now price-takers, considering as given the prices p_t, w_t and r_{t+1} as defined by (2), and determine their optimal decisions with respect to their budget constraints

$$w_t + p_t x_t = c_t + \zeta_t \text{ and } R_{t+1}\zeta_t = d_{t+1} + p_{t+1}x_{t+1} \tag{25}$$

with $R_{t+1} = r_{t+1}/p_t$ the gross rate of return, ζ_t the savings of young agents born in t and x_t the amount of bequest transmitted at time t by agents born in $t-1$. Note that bequest x_t is expressed as an investment good and requires the relative price p_t to enter the budget constraints. In each period, bequests must be non-negative:

$$x_t \geq 0 \text{ for all } t \geq 0 \tag{26}$$

An altruistic agent has a utility function given by the following Bellman equation

$$\begin{aligned} V_t(x_t) &= \max_{\{c_t, d_{t+1}, s_t, x_{t+1}\}} \{u(c_t, Bd_{t+1}) + \beta V_{t+1}(x_{t+1})\} \\ &= \max_{\{c_t, d_{t+1}, s_t, x_{t+1}\}} \sum_{t=0}^{+\infty} \beta^t u(c_t, Bd_{t+1}) \end{aligned} \tag{27}$$

subject to (25) and (26). Note that β is now interpreted as the intergenerational degree of altruism.

[23] The co-existence of altruistic and non-altruistic agents as in Nourry and Venditti [50] could also be considered.

5.1 The decentralized equilibrium with positive bequests

It is well-known from the first welfare theorem that this altruistic problem is equivalent to the central planner problem (8), and the equilibrium is the unique Pareto optimum which coincides with the centralized solution. However, such an equivalence requires the non-negativity constraints of bequests (26) to hold with a strict inequality in order to preserve the link across generations. Substituting the expressions of c_t and $d_t + 1$ from the budget constraints (25) into the optimization problem (27) we get

$$\max_{\{\zeta_t, x_{t+1}\}} \sum_{t=0}^{+\infty} \beta^t u(w_t + p_t x_t - \zeta_t, B[R_{t+1}\zeta_t - p_{t+1}x_{t+1}]) \tag{28}$$

The first order conditions are given by

$$u_c(c_t, Bd_{t+1}) - u_d(c_t, Bd_{t+1})BR_{t+1} = 0 \tag{29}$$

$$\beta u_c(c_{t+1}, Bd_{t+2}) - u_d(c_t, Bd_{t+1})B \leq 0 \text{ with an equality if } x_t > 0 \tag{30}$$

Consider now the two budget constraints in (25) evaluated at the steady state. Solving with respect to ζ_t, and using the fact that $\zeta_t = p_t y_t = p_t k_{t+1}$ and $R_{t+1} = r_{t+1}/p_t$, we get

$$\begin{aligned} p^* x^* \left(1 - \tfrac{1}{R^*}\right) &= c^* + \tfrac{d^*}{R^*} - w^* = T(k^*, k^*) - w^* - d^* \left(1 - \tfrac{1}{R^*}\right) \\ &= (r^* k^* - d^*)\left(1 - \tfrac{1}{R^*}\right). \end{aligned} \tag{31}$$

If $x^* > 0$, i.e. $r^* k^* > d^*$, then we derive from the first order conditions that $R^* = r^*/p^* = \beta^{-1}$ and $u_d(c^*, Bd^*) = \beta u_c(c^*, Bd^*)$, which are exactly the same conditions as (12). Then

Proposition 8. *Under Assumptions 1-3, for any* $\beta \in (0, 1)$*, there exists a unique value* B^* *such that when* $B = B^*$*, bequests are positive in the economy with a degree of altruism equal to* β*.*

Proof. See Appendix 8.10. □

When bequests are positive at the steady state, then by continuity there are positive in a neighborhood of the steady state and we may study the local stability properties of the equilibrium path considering equation (30) with an equality. We need first to derive the precise expression of the dynamical system. Plugging equation (29) into equation (30) considered with an equality, and using the fact that $R_{t+1} = r_{t+1}/p_t$, we get the following dynamical system

$$u_c(c_t, Bd_{t+1})p_t - \beta r_{t+1} u_c(c_{t+1}, Bd_{t+2}) = 0 \quad (32)$$

$$\beta u_c(c_{t+1}, Bd_{t+2}) - u_d(c_t, Bd_{t+1})B = 0. \quad (33)$$

Taking now that $p_t = -T_y(k_t, k_{t+1})$, $r_{t+1} = T_k(k_{t+1}, k_{t+2})$ and $c_t = T(k_t, k_{t+1}) - d_t$, we immediately conclude that these equations are identical to the two difference equations of order two given by (10). Solving these two equations considering the transversality conditions (11) yields the equilibrium paths for capital $\{k_t\}_{t \geq 0}$ and second period consump-tion $\{d_t\}_{t \geq 0}$. The dynamics of bequests is then derived from the budget constraints (25) as follows

$$p_t x_t = r_t k_t - d_t \quad (34)$$

whereas the dynamics of bequests as a proportion of GDP is

$$\frac{p_t x_t}{GDP_t} = \frac{p_t x_t}{T(k_t, k_{t+1}) + p_t y_t} = s(k_t, k_{t+1}) - \frac{d_t}{T(k_t, k_{t+1}) - T_y(k_t, k_{t+1}) k_{t+1}}. \quad (35)$$

with

$$s(k_t, k_{t+1}) = \frac{T_k(k_t, k_{t+1}) k_t}{T(k_t, k_{t+1}) - T_y(k_t, k_{t+1}) k_{t+1}}$$

the share of capital income in GDP. We can then derive the following Proposition:

Proposition 9. *Under Assumptions 1-3, the local stability properties provided in Propositions 3, 4 and 7 hold for both bequests as defined by (34) and bequests as a proportion of GDP as defined by (35). In particular, bequests and bequests as a proportion of GDP can be characterized by optimal periodic and quasi-periodic cycles.*

Proof. See Appendix 8.11. □

This Proposition may explain long-run fluctuations of both bequests and bequests as a proportion of GDP, and provides a theoretical basis to explain the empirical evidence derived in Section 6 showing that the annual inheritance flow as a fraction of national income displays medium- and long-run fluctuations.

5.2 An illustration with positive bequests and endogenous fluctuations

We consider again our benchmark example in Section 3.2. with a linear homogeneous utility function ($\gamma = 1$), and the Cobb-Douglas production structure described by (20). Using the expression of the elasticity of substitution ς as given by (19) and the expression for the stationary capital stock (56) in Appendix 8.6, we derive that $r^* k^* > d^*$ and thus $x^* > 0$ if and only if

$$\frac{\alpha_0\beta(\epsilon_{cc}-\varsigma)}{\varsigma} - (1-\alpha_0-\beta\alpha_1) > 0$$

It follows immediately that if $\alpha_1 > 1-\alpha_0$ and $\beta > (1-\alpha_0)/\alpha_1$, then $1-\alpha_0-\beta\alpha_1 < 0$ and $x^* > 0$ for any $\varsigma \in (0, \epsilon_{cc})$. The existence of periodic cycles is thus compatible with positive bequests. Similarly, when $\alpha_1 < 1-\alpha_0$, straightforward computations show that $x^* > 0$ if and only if

$$\varsigma < \frac{\alpha_0\beta\epsilon_{cc}}{1-\alpha_0+\beta(\alpha_0-\alpha_1)} \equiv \tilde{\varsigma}_1$$

Therefore the conditions of Corollary 2 for the existence of period-2 cycles can be satisfied if $\tilde{\varsigma}_1 > \bar{\varsigma} = \epsilon_{cc}/2$. Sufficient conditions for this inequality to be satisfied are given by $\alpha_1 \in (1-2\alpha_0, 1-\alpha_0)$ and $\beta > (1-\alpha_0)/(\alpha_0+\alpha_1) \equiv \underline{\beta}$ with $\underline{\beta} < 1$. This example clearly shows that when the degree of altruism is large enough, endogenous optimal fluctuations are compatible with positive bequests. Moreover, this result holds for any sign of the capital intensity difference across sectors.

It is worth noticing that if, under $\alpha_1 \in (1-2\alpha_0, 1-\alpha_0)$, we assume that $\varsigma < \tilde{\varsigma}_1$ with $\tilde{\varsigma}_1 < \underline{\varsigma}$, then bequests are positive but the conditions of Corollary 2 for the existence of period-2 cycles cannot be satisfied and the steady state is saddle-point stable. This inequality is satisfied if and only if $\alpha_1 \in (1-2\alpha_0, 1-\alpha_0)$, $\alpha_0 < 1/2$ and $\beta < (1-2\alpha_0)/\alpha_1$. Therefore, if the degree of altruism is not large enough, persistent endogenous fluctuations cannot arise.

Let us finally illustrate the possible existence of quasi-periodic cycles under positive bequests when the utility function is homogeneous of degree $\gamma < 1$ as in Section 4. Using again the expression of the elasticity of substitution ς as given by (19) and the expression for the stationary capital stock (56) in Appendix 8.6, we derive that $r^* k^* > d^*$ and thus $x^* > 0$ if and only if

$$\alpha_0\phi\beta - (\gamma-\phi)(1-\alpha_0-\beta\alpha_1) > 0$$

Consider then the particular illustration in Section 4 which is such that $1-\alpha_0-\beta\alpha_1 > 0$ and $\alpha_0 > \alpha_1$. It follows that bequests are positive if and only if

$$\varsigma < \frac{\alpha_0\beta\epsilon_{cc}}{1-\alpha_0+\beta(\alpha_0-\alpha_1)-\epsilon_{cc}(1-\gamma)(1-\alpha_0-\beta\alpha_1)} \equiv \tilde{\varsigma}_\gamma$$

This condition can be satisfied only if

$$\epsilon_{cc} < \frac{1-\alpha_0+\beta(\alpha_0-\alpha_1)}{(1-\gamma)(1-\alpha_0-\beta\alpha_1)}$$

With $\gamma = 0.98$, $\epsilon_{cc} = 1$, $\alpha_0 = 0.6$ and $\alpha_1 = 0.21$, we get $\tilde{\varsigma}_\gamma \approx 0.3566 \in (\underline{\varsigma}^H, \bar{\varsigma}^H)$. It follows that positive bequests are compatible with quasi-periodic cycles. Indeed, the steady state, which is characterized by strictly positive bequests if $\varsigma < \tilde{\varsigma}_\gamma$, is saddle-point stable with damped oscillations if and only if $\varsigma \in (\underline{\varsigma}^c, \underline{\varsigma}^H)$. Moreover, when ς crosses the bifurcation values $\underline{\varsigma}^H$ from below, the steady state undergoes a Hopf bifurcation leading to persistent quasi-periodic cycles and thus long-run fluctuations of bequests.

6 Long-run cycles and inheritance: an empirical assessment

The objective of this section is to provide a qualitative assessment of the existence of limit cycles of bequests (relative to national income) in the light of our results in Section 5. Especially, we discuss *indirect* empirical evidence that is consistent with the emergence of a period-2 cycle or of (quasi-) periodic (long-run) cycles.[24]

6.1 Empirical relevance of flip *versus* Hopf bifurcations

One critical issue is that the dynamics engendered by either a flip or a Hopf bifurcation have different implications from an empirical point of view. The former leads to a period-2 cycle whereas the latter to a (limit) periodic or quasi-periodic cycle.

On the one hand, the flip bifurcation leads to a period-2 cycle that can be seen as the stationary solution of the initial dynamic system iterated at order 2.[25] This

[24]A more structural approach will require to simulate and estimate the OLG model in the presence of stochastic limit cycles (i.e., a deterministic limit cycle where the stochastic component is essentially an i.i.d. process). This could be done by determining the topological normal form for the flip (respectively, Hopf) bifurcation using Taylor expansions (see Kuznetsov [37]) or perturbation methods (e.g., Galizia ([28])). Note that the addition of shocks to our 2-sector OLG model will eliminate the perfect predictability of the endogenous cyclical forces, but does not change the endogenous mechanisms explained before. We leave this issue for further research.

[25]For sake of illustration, and without loss of generality, consider the following (univariate) discrete dynamical system:

$$x \mapsto f(x;\alpha) \equiv f_\alpha(x) \tag{36}$$

where the map f_α is invertible for small $|\alpha|$ in the neighborhood of the origin and the system has the fixed point $x_0 = 0$ for all α. Consider now the second iterate $f_\alpha^2(x)$ of the map (36). The map f_α^2 has the trivial fixed point $x_0 = 0$ but also two nontrivial fixed points for small $\alpha > 0$

iterated system usually leads to the resolution of a degree-2 polynomial that gives a high steady-state (above the initial stationary state) and a low steady-state (below the initial stationary state), i.e. the period-2 cycle.[26] If we are in the period-2 cycle, every two periods we return to one of the two steady-states of the second iterate of the initial dynamical system. In this respect, it turns out that the flip bifurcation, and thus the period-2 cycle, rests on the long-run component of the inheritance flows as a ratio of national income (or flows). Therefore it is critical to extract this component and to characterize its dynamics.

In the case of Hopf bifurcation the dynamics are different. Notably one obtains a periodic orbit that corresponds to a "circle". If one is on this circle, the dynamics are either periodic or quasi-periodic. It depends on the curvature coefficient (or stability index) that is obtained through a tedious derivation of the (topological) normal form of the Hopf bifurcation by means of smooth invertible coordinate and parameter changes. In particular, if this coefficient is rational then the dynamics on the circle are periodic, whereas if this coefficient is irrational, the dynamics on the circle are almost periodic, i.e. dense on the circle. From an empirical point of view, in the same spirit as Beaudry et al. [13], Charpe et al. [19], and Muck et al. [33], it means that one would expect a significant contribution of the medium-term component of the inheritance variable (after extracting the long-run component).[27]

$$x_i = f_\alpha^2(x_i) \quad i = 1, 2$$

that satisfy (period-2 cycle)

$$x_2 = f_\alpha(x_1) \quad \text{and} \quad x_1 = f_\alpha(x_2)$$

with $x_1 \neq x_2$ and x_1 and x_2 are two stable or unstable equilibria depending on the sign of the coefficient associated to the degree-3 term in a normal from expression of (36) (see footnote 26). For further technical details, see Chapters 4 and 9 of Kuznetsov ([37]).

[26] Under suitable regularity conditions, starting from the map (36), one can show that there exists a topological normal form for the flip bifurcation defined by:

$$\eta \mapsto -(1+\beta)\eta + s\eta^3$$

where $\beta = g(\alpha)$ with $g(0) = 0$, and η is defined from successive changes of coordinates and parameters. On top of the trivial stationary solution $\eta = 0$, this leads to the resolution of a degree-2 polynomial. The equilibria x_1 and x_2 are stable (unstable) if $s = -1(+1)$. For further technical details, see Theorem 4.3. of Kuznetsov ([37]).

[27] While there is no consensus, it is generally considered that the high-frequency component captures the periodicity below 8 years, i.e. business fluctuations and noise, the medium-frequency component the periodicity between 8 and 30 to 50 years, and the low-frequency component the periodicity above 30 to 50 years.

6.2 The long-run component of bequests

To assess some empirical relevance of period-2 cycles, we proceed in two steps. First, we consider the (stochastic) long-run component of the inheritance flows as a fraction of national income using the low-frequency approach initiated by Müller and Watson ([47], [48]) and a bandpass filter. Second, as in Müller and Watson ([47],[45]), we make use of some parametric models of persistence to characterize the dynamics of the long-run component.

In this respect, Figure 1 depicts the inheritance flow as a fraction of national income for France using annual data from 1897 through 2008, and three filter-based long-run components. The inheritance variables displays medium- to long-run swings: it was about 20–25% of national income between 1820 and 1910, down to less than 5% in 1950, and back up to about 15% by 2010.[28] Notably, Figure 1 suggests a long-run cyclical behavior of inheritance flows.[29]. One key issue regards the stationarity of this series. Standard stationary tests report mixed evidence, especially due to our sample size and the corresponding low power of unit root tests to distinguish between non-stationary and near-stationary stochastic processes.[30] We further discuss this issue later on.

Using the structural breakpoint tests of Andrews ([1]) and Andrews and Ploberger ([2]) with a standard 15% sample trimming, both the exponential statistics and the simple average of the individual F-statistics provide no empirical support for the existence of a structural break over the period 1912-1991, when the data generating process of the observed series is assumed to be an autoregressive process of order one, AR(1).[31]. Furthermore, the (sequential) multibreak points test of Bai and Perron ([5], [6], and [7]) report no evidence against the null hypothesis of no break when using a 15% trimming, a

[28]Using Eq. (1) of Piketty [51], the inheritance flow as a ratio of national income is defined by:

$$\frac{B_t}{Y_t} = \mu_t \times m_t \times \frac{W_t}{Y_t}$$

where B_t, Y_t, W_t, μ_t and m_t denote respectively the aggregate inheritance flow, the aggregate national income, the aggregate private wealth, the mortality rate and the ratio between average wealth of the deceased and average wealth of the living. For further details, see Section 3 and the technical appendix of Piketty [51], , and Piketty and Zucman [53].

[29]A similar pattern is observed with UK data as shown by Atkinson [3]

[30]Appendix 8.12 reports the spectral density of the inheritance variable, with a typical hump shape and a peak at low frequency. As a robustness analysis, we also transform our data using the first difference operator and apply the Baxter-King filter. This requires in turn to cumulate the filtered series. On the other hand, using the appropriate asymptotic results, the low-frequency approach of Müller and Watson ([47], [48]) remains valid for I(1) models. Therefore we consider the level specification as a more plausible way to extract the long-run component, possibly at the expense of a loss of efficiency for the Baxter-King filter.

[31]Results are not reported here but available upon request

upper bound of five structural breaks, and a 5% significance level: the scaled F-statistics, which equals 5.53, is far below the critical value (11.47). In this respect, there is no compelling evidence that the inheritance ratio displays structural breaks or even some regime changes (e.g., a Markov switching model). However, the small sample size of our annual data might be an issue, as well as the absence of an incomplete long-run cycle in the observed variable.

Turning to the filtered series, the first long-run component is obtained from the approximate low bandpass filter of Baxter and King ([10]).[32] We filter the high- and medium-frequency with periodicity below 40 years.[33] On the other hand, the second trend is constructed using the methodology proposed by Müller and Watson ([47], [48]), namely by extracting long-run sample information after isolating a small number of low-frequency trigonometric weighted averages.[34] In so doing, we project the series into a constant and twelve (q) cosine functions with periods $\frac{2T}{j}$ for $j = 1, \cdots, 12$ in order to capture the variability for periods longer than 20 years ($2T/q$). Importantly, one advantage of the low-frequency approach of Müller and Watson ([47], [48]) relative to bandpass or other moving average filters, is that it is applicable beyond the I(0) assumption.[35] Finally, for sake of comparison, the third long-run component is obtained after adjusting a quadratic trend.

A quick eye inspection suggest that the bandpass and projection-based (using $q = 6$) trends contribute significantly to the total variability of the initial series, and display a nonlinear pattern. One main difference between these two measures is that the cosine transforms only use a limited number of points, which is consistent with the periodogram of the inheritance variable and the number of periodogram ordinates that fall into the low-frequency region. Intuitively, 112 years of data contain only limited information about long-run components with periodicities of more than 30 or 40 years.

To further highlight some statistical features of the inheritance variable, we make long-run inference regarding the mean, the standard deviation and conduct some persistence

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[32] We also implement the asymmetric filter proposed by Christiano and Fitzgerald ([21], [20])

[33] Qualitative conclusions remain robust when considering other periodicities.

[34] Let $\{x_t, t = 1, \cdots, T\}$ denote a (scalar) time series. Let $\Psi(s) = [\Psi_1(s), \cdots, \Psi_q(s)]'$ denote a $\mathbb{R}^q$-valued function with $\Psi_j(s) = \sqrt{2}cos(js\pi)$, and let $\Psi_T = \left[\Psi\left(\frac{1-0.5}{T}\right), \Psi\left(\frac{2-0.5}{T}\right), \cdots, \Psi\left(\frac{T-0.5}{T}\right)\right]'$ denote the $T \times q$ matrix after evaluating $\Psi(.)$ at $s = \frac{t-0.5}{T}$, for $t = 1, \cdots, T$. The low-frequency projection is the fitted series from the OLS regression of $[x_1, \cdots, x_T]$ onto a constant and Ψ_T.

[35] For an extensive discussion about the relationship of this approach with spectral analysis, the scarcity of low-frequency information, and the relevance of the approximation using a small q, see Müller and Watson ([47])

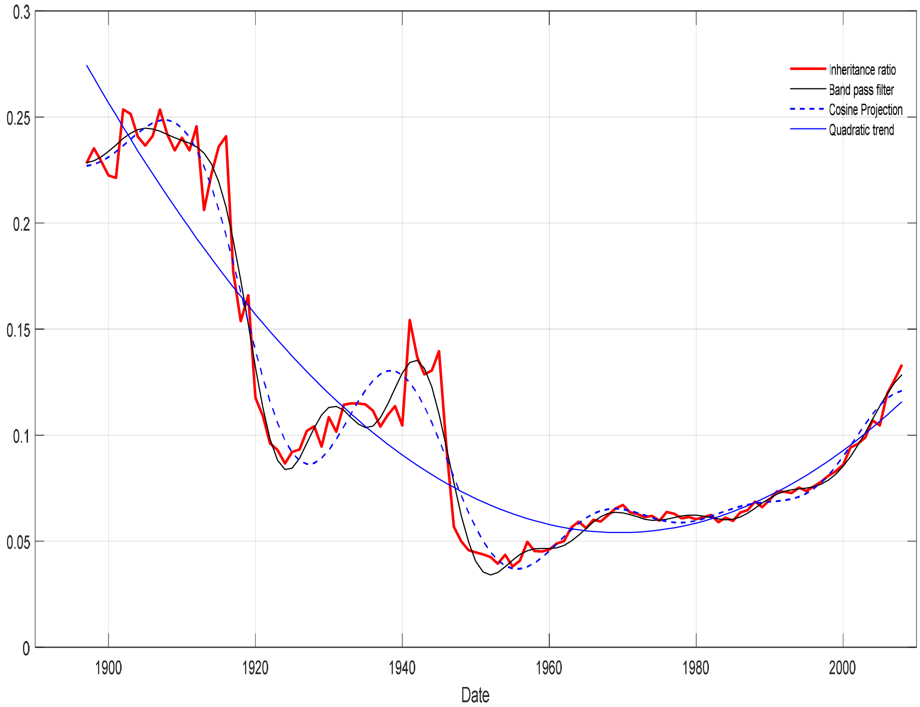


Figure 1: Inheritance and long-run components

tests (Müller and Watson, [48]). In doing so, we first suppose that the variable of interest, x_t, can be represented as $x_t = \mu + u_t$, where μ is the mean of x and u_t is a zero mean (persistent) stochastic process. We first infer about the long-run variance (ω^2) and mean (μ) of the inheritance variable. Table 1 provides the relevant descriptive statistics and the resulting confidence intervals.

Table 1: Descriptive statistics for the inheritance variable

Long-run mean μ	0.11
Long-run standard deviation ω	0.20
90% Confidence interval for μ	[0.08; 0.14]
90% Confidence interval for ω	[0.15; 0.30]

Note: (1) The 90% confidence interval for the mean is constructed as $\hat{\mu}_T \pm 1.78 \frac{\hat{\omega}_T}{\sqrt{T}}$ where 1.78 is 95th percentile of a Student distribution with twelve degrees of freedom. (2) The 90% confidence interval for ω is constructed as $[T\frac{\sum_{j=1}^{q}\hat{\beta}_{j,T}^2}{\chi_{q,0.975}^2}; T\frac{\sum_{j=1}^{q}\hat{\beta}_{j,T}^2}{\chi_{q,0.025}^2}]$ where q denotes the number of cosine functions, $\hat{\beta}_{j,T}$ is the OLS estimate of the parameter associated to the jth cosine function, and $\chi_{q,\alpha}^2$ is the αth quantile of the Chi-squared distribution with q degrees of freedom.

Several points are worth commenting. First, the long-run mean is larger than the unconditional mean of the inheritance variable (9%) as well as the values observed after the Second World War up to the last decade. It partly captures the largest values observed at the beginning of the same and before the First World War when the inheritance ratio was about 20–25% of national income. It also means that the inheritance variable is slowly reverting and, due to the scarcity of yearly data in the nineteenth century, we can only observe part of the dynamics.[36] Second, the inheritance variable displays a substantial long-run standard deviation (variability). Finally, as to be expected, the 90% confidence interval for the long-run standard deviation (and thus the mean) is rather large. Indeed, this uncertainty is explained by the fact that the long-run variance is computed using only q observations, and thus $\widehat{\omega}_T^2$ is not consistent.[37]

Moreover, these confidence intervals for ω and μ are based on the assumption that the series is weakly stationary. In this respect, we contrast our specification with a local-to-unit (LTU) AR model (e.g., Phillips [52]) in which $x_t = \mu + u_t$, and $u_t = \rho_T u_{t-1} + v_t$ for all t with an autoregressive coefficient $\rho_T = 1 - c/T$ and v_t is a weakly stationary process.[38] Notably, , we can assess the persistence of the long-run component, and especially perform the low frequency version of the Neyman-Pearson point-optimal test to assess the null of a stationary process—the so-called LFST test (see Müller and Watson, [46]).[39] Unsurprisingly, the results depend on the number of cosine functions q: there is evidence against the null of stationary when q increases. In this respect, following the approach of Müller and Watson ([45]), we determine the largest value of q such that the low-frequency transformed data are consistent with the local-level I(0) model. Results show that the null hypothesis of an I(0) model is rejected for values of $q > 5$ at 5%, i.e. a 95% confidence interval for periods for which the inheritance ratio variable behaves like a weakly stationary process includes periods greater than 45 years. To provide further evidence on the slowness of the mean reversion property, we estimate an autoregressive model of order one, after demeaning the initial series with respect to the long-run mean

[36] Interestingly, reported data with a 10-year frequency (see Figure 1 in Piketty ([51]) are likely to suggest a high and low persistent regime of the inheritance ratio with a potential structural break at the beginning of the twentieth century.

[37] Note that the asymptotic theory requires that q is held fixed as $T \to \infty$. In addition, it is well-known that consistent long-run variance estimators generally perform poorly in finite samples, and especially as the sample size is small and the series is persistent (Müller, [44]).

[38] We also consider the "local-level" model in which the initial series is the sum of I(0) and I(1) processes, as well as a fractional model in which $(1-L)^d x_t = v_t$ where L is the lag operator, $-0.5 < d < 1.5$ and v_t is a weakly stationary process.

[39] We also conduct the low frequency version of the point-optimal unit root test of Elliott et al. ([27]), the so-called LFUR test, to assess the null of non stationarity. Results are qualitatively similar. Results are available upon request.

$\tilde{x}_t = x_t - \widehat{\mu}$ for $t = 1, \cdots, T$, and determine the corresponding half-time, i.e. the time for the expected value of the inheritance ratio to reach the middle value between the current value and the long-run mean $\widehat{\mu}$ (see Table 1). This leads to a statistically significant autoregressive parameter estimate of 0.967, and thus an half-time around 20.5 years, which is consistent with our result of the recursive low-frequency stationary (or unit root) test—stationarity requires at least periods greater than 45 years. Both the half-time and the minimal periodicity are in line with a two-period model in which each period lasts approximately 35 to 40 years.

Assuming that this long-term component is captured by the balanced growth path or deviations from a steady-state, the empirical evidence is not at odds with the existence of a flip bifurcation. Nevertheless, we find no support for the presence of either structural break points nor regimes switches, which would better characterized the implications of a period-2 cycle.

6.3 The medium-term cyclical component of bequests

Capitalizing on the long-run component, we now consider the cyclical component defined as the inheritance ration variable in deviation of the long-run (stochastic) trend. Figure 2 displays the three series using the low frequency approach of Müller and Watson ([45]), the approximate Baxter-King filter and the quadratic trend.

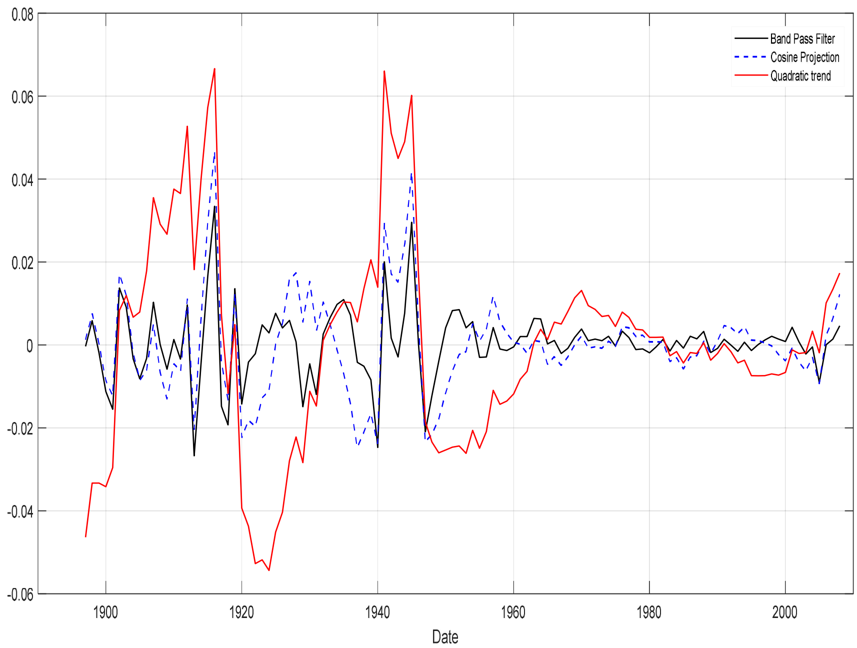


Figure 2: Inheritance and short- to medium-run components

We then focus on the spectral density, which can depicts the contribution of cycles of different frequencies in explaining the data. Especially, when the spectral density displays a substantial pick at a given frequency, this provides some support of recurrent (short- or medium-run) cyclical fluctuations at that frequency.

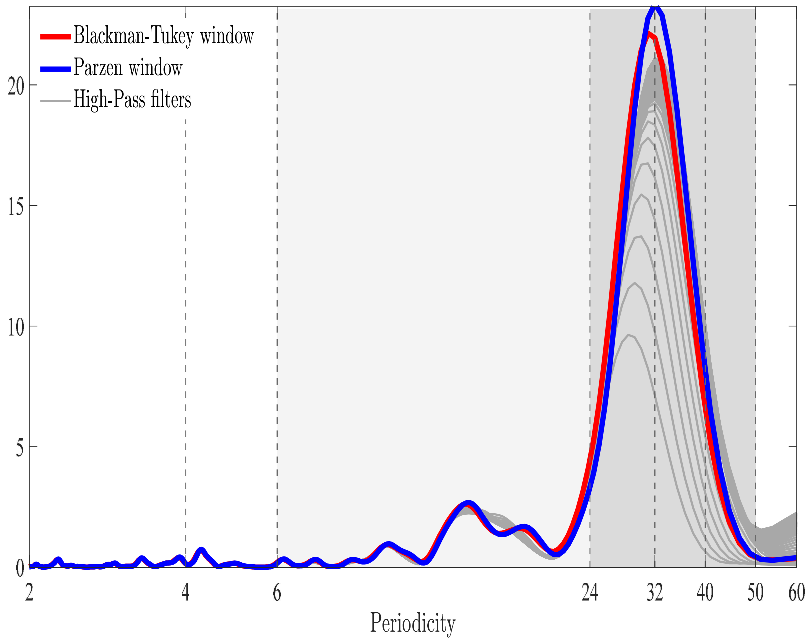


Figure 3: Spectral density of the inheritance variable

Figure 3 displays the spectral density of the projection-based cyclical component (using $q = 6$). We first report two nonparametric power spectral density estimates by applying a Blackman-Tukey window (red line) and a Parzen window (blue line) to the covariogram of the filtered series before using a fast Fourier transform algorithm. For comparison, we also report on the same axes the spectra when first passing the series through various high bandpass filters that remove some remaining low-frequency. We highlight in dark grey the band of frequencies corresponding to periodicities from 24 to 40 (or 50) years. Irrespective of the method used, one dominant feature is the distinct peak in the spectral density around 32 years and the local hump in its neighborhood. This suggests that the inheritance variable exhibits important recurrent cyclical phenomena at approximatively 30-year intervals, which is consistent with the occurrence of Hopf-based limit cycles. Following Beaudry *et al.* [13], we formally test the presence of a shape restriction on the spectral density. In so doing, we consider a "peak range" for 24-40 years and test the null hypothesis of a flat spectral density against a "peak range". We strongly reject at 5 percent level that the spectrum is flat in the "peak range". This result is robust when considering a narrow "peak range".

To summarize, while the presence of a peak range does not necessarily imply strong endogenous cyclical forces and the empirical relevance of a Hopf bifurcation, it says again that data can not, at least, contradict the existence of endogenous (stochastic) limit cycles.

7 Concluding comments

This paper explores the existence of limit cycles to explain the behavior of the annual flow of inheritance (in level or a as share of national income) as a complementary interpretation of the results of Piketty [51].

Using a two-sector Barro-type [9] OLG model with non-separable preferences and bequests, we show that two endogenous mechanisms, which can operate independently or together, can be identified as long as agents are sufficiently patient. The first mechanism relies on the elasticity of intertemporal substitution or equivalently the sign of the cross-derivative of the utility function whereas the second rests on sectoral technologies through the sign of the capital intensity difference across the two sectors. Accordingly, mild and plausible perturbations of these parameters can lead to endogenous fluctuations through period-2 cycles or Hopf bifurcations.

From a methodological point of view, we exploit the quasi-palindromic nature of the characteristic equation associated to the optimal growth solution without bequest to derive some meaningful sufficient conditions associated to the occurrence of complex roots in a two-sector OLG model. We then show that the decentralized problem in the presence of altruistic parents is equivalent to the central planner problem (without bequest). Finally, our theoretical results are consistent with some empirical evidence for medium- and long-run swings in the inheritance flows as a fraction of national income in France over the period 1896-2008. Notably, the contribution of the medium term component does not run counter the existence of Hopf bifurcations.

A first avenue of future research would be to consider a stochastic version of our model and thus to characterize the existence of stochastic limit cycles. Another research perspective would be to study more deeply the econometrics of long-run/endogenous cycles.

8 Appendix

8.1 Proof of Proposition 1

Consider in a first step the second equation in (12). Notice that the steady state value for k only depends on the characteristics of the technologies and is independent from the utility function. Moreover, this equation is equivalent to the equation which defines the stationary capital stock of a standard two-sector optimal growth model. The proof of Theorem 3.1 in Becker and Tsyganov [14] restricted to the case of one homogeneous agent applies so that there exists one unique k^* solution of this equation.

Consider now the first equation in (12) evaluated at k^*. We get:

$$\frac{u_d(T(k^*,k^*)-d,Bd)B}{u_c(T(k^*,k^*)-d,Bd)} \equiv h(d) \quad = \quad \beta \tag{37}$$

The function $h(d)$ is defined over $(0, T(k^*,k^*))$ and satisfies

$$h'(d) = \frac{\frac{Bu_{dd}}{u_d} - \frac{u_{cd}}{u_c} + \frac{u_{cc}}{u_c} - \frac{Bu_{cd}}{u_d}}{u_c u_d} = -\beta\left[\frac{1}{d}\left(\frac{1}{\epsilon_{dd}} - \frac{1}{\epsilon_{cd}}\right) + \frac{1}{c}\left(\frac{1}{\epsilon_{cc}} - \frac{1}{\epsilon_{dc}}\right)\right]$$

Assumption 3 implies that $h'(d) < 0$. This monotonicity property together with the boundary conditions in Assumption 2 finally ensure the existence and uniqueness of a solution $d^* \in (0, T(k^*,k^*))$ of equation (37).

For a given k^*, consider a particular value $d^* = \bar{d} \in (0, T(k^*,k^*))$. $\bar{d}$ is a steady state if

$$\frac{u_d(T(k^*,k^*)-\bar{d},B\bar{d})B}{u_c(T(k^*,k^*)-\bar{d},B\bar{d})} \equiv g(B) \quad = \quad \beta \tag{38}$$

We easily get

$$g'(B) \quad = \quad -\frac{u_d}{u_c}\left[\frac{1}{\epsilon_{dd}} - \frac{1}{\epsilon_{cd}} - 1\right]$$

which is generically different from zero. Therefore, under the boundary conditions in Assumption 2, there generically exists a unique value B^* such that when $B = B^*$, $d^* = \bar{d}$ is a normalized steady state.

□

8.2 Proof of Proposition 2

Using (4)-(5) and the fact that at the steady state $-T_y^* = \beta T_k^*$, total differentiation of the first order equations (10) gives after tedious but straightforward computations:

$$
\begin{aligned}
& -\Delta k_t \tfrac{\beta T_k^* \epsilon_{cc}}{\epsilon_{dc}} + \Delta k_{t+1} \beta T_k^* \left(1 + \tfrac{\beta \epsilon_{cc}}{\epsilon_{dc}}\right) + \Delta d_t \tfrac{\beta \epsilon_{cc}}{\epsilon_{dc}} - \Delta d_{t+1} \beta \left(1 + \tfrac{\beta \epsilon_{cc} \epsilon_{cd}}{\epsilon_{dc} \epsilon_{dd}}\right) \\
= \; & \Delta k_{t+2} \beta^2 T_k^* - \Delta d_{t+2} \tfrac{\beta^2 \epsilon_{cc}}{\epsilon_{dc}} \\
& \Delta k_t \left(\tfrac{\beta T_k^{*2}}{\epsilon_{cc} c^* T_{kk}^*} - b\right) - \Delta k_{t+1} \left(\tfrac{\beta(1+\beta) T_k^{*2}}{\epsilon_{cc} c^* T_{kk}^*} - \Delta - b^2\right) - \Delta d_t \tfrac{\beta T_k^*}{\epsilon_{cc} c^* T_{kk}^*} \\
+ \; & \Delta d_{t+1} \tfrac{\beta T_k^*}{\epsilon_{cc} c^* T_{kk}^*} \left(1 + \tfrac{\beta \epsilon_{cc}}{\epsilon_{dc}}\right) = -\Delta k_{t+2} \beta \left(\tfrac{\beta T_k^{*2}}{\epsilon_{cc} c^* T_{kk}^*} - b\right) + \Delta d_{t+2} \tfrac{\beta^2 T_k^*}{\epsilon_{cc} c^* T_{kk}^*}
\end{aligned}
$$

Denoting $\Delta \kappa_t = \Delta k_{t+1}$ and $\Delta \delta_t = \Delta d_{t+1}$, we get the following matrix expression of the previous linear system:

$$
\begin{aligned}
& \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \beta^2 T_k^* & -\frac{\beta^2 \epsilon_{cc}}{\epsilon_{dc}} \\ 0 & 0 & -\left(\frac{\beta T_k^{*2}}{\epsilon_{cc} c^* T_{kk}^*} - b\right) & \frac{\beta^2 T_k^*}{\epsilon_{cc} c^* T_{kk}^*} \end{pmatrix} \begin{pmatrix} \Delta k_{t+1} \\ \Delta d_{t+1} \\ \Delta \kappa_{t+1} \\ \Delta \delta_{t+1} \end{pmatrix} \\
= & \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{\beta T_k^* \epsilon_{cc}}{\epsilon_{dc}} & \frac{\beta \epsilon_{cc}}{\epsilon_{dc}} & \beta T_k^* \left(1 + \frac{\beta \epsilon_{cc}}{\epsilon_{dc}}\right) & -\beta \left(1 + \frac{\beta \epsilon_{cc} \epsilon_{cd}}{\epsilon_{dc} \epsilon_{dd}}\right) \\ \frac{\beta T_k^{*2}}{\epsilon_{cc} c^* T_{kk}^*} - b & \frac{\beta T_k^*}{\epsilon_{cc} c^* T_{kk}^*} & -\frac{\beta(1+\beta) T_k^{*2}}{\epsilon_{cc} c^* T_{kk}^*} + \beta + b^2 & \frac{\beta T_k^*}{\epsilon_{cc} c^* T_{kk}^*} \left(1 + \frac{\beta \epsilon_{cc}}{\epsilon_{dc}}\right) \end{pmatrix} \begin{pmatrix} \Delta k_t \\ \Delta d_t \\ \Delta \kappa_t \\ \Delta \delta_t \end{pmatrix} \\
\Leftrightarrow \; & A \begin{pmatrix} \Delta k_{t+1} \\ \Delta d_{t+1} \\ \Delta \kappa_{t+1} \\ \Delta \delta_{t+1} \end{pmatrix} = B \begin{pmatrix} \Delta k_t \\ \Delta d_t \\ \Delta \kappa_t \\ \Delta \delta_t \end{pmatrix}
\end{aligned}
$$

with

$$
A = \begin{pmatrix} I & 0 \\ 0 & A_{22} \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & I \\ B_{21} & B_{22} \end{pmatrix}
$$

Matrix A is invertible as $\det A = \det A_{22} = \delta^3 b \epsilon_{cc} / \epsilon_{dc}$, and we get

$$
A^{-1} = \begin{pmatrix} I & 0 \\ 0 & A_{22}^{-1} \end{pmatrix} \text{ with } A_{22}^{-1} = \begin{pmatrix} \frac{T_k^*}{\beta b \epsilon_{cc} c^* T_{kk}^*} & \frac{1}{\beta b} \\ \frac{\epsilon_{dc}}{\beta^2 \epsilon_{cc}} \left(\frac{\beta T_k^{*2}}{b \epsilon_{cc} c^* T_{kk}^*} - 1\right) & \frac{\epsilon_{dc} T_k^*}{\beta b \epsilon_{cc}} \end{pmatrix}
$$

The linearized dynamical system can then be expressed as follows

$$\begin{pmatrix} \Delta k_{t+1} \\ \Delta d_{t+1} \\ \Delta \xi_{t+1} \\ \Delta \zeta_{t+1} \end{pmatrix} = A^{-1}B \begin{pmatrix} \Delta k_{t} \\ \Delta d_{t} \\ \Delta \xi_{t} \\ \Delta \zeta_{t} \end{pmatrix} = \begin{pmatrix} 0 & I \\ A_{22}^{-1}B_{21} & A_{22}^{-1}B_{22} \end{pmatrix} \begin{pmatrix} \Delta k_{t} \\ \Delta d_{t} \\ \Delta \xi_{t} \\ \Delta \zeta_{t} \end{pmatrix} \equiv J \begin{pmatrix} \Delta k_{t} \\ \Delta d_{t} \\ \Delta \xi_{t} \\ \Delta \zeta_{t} \end{pmatrix}$$

Using (13), tedious but straightforward computations give the characteristic polynomial

$$\mathcal{P}(\lambda) \;=\; \lambda^4 - \lambda^3 B + \lambda^2 C - \lambda \tfrac{B}{\beta} + \tfrac{1}{\beta^2} \tag{39}$$

with

$$\begin{aligned} B &= -\tfrac{\beta}{b\epsilon_{cc}} \tfrac{\varepsilon_{ck}}{\varepsilon_{rk}} \left(\tfrac{\epsilon_{cc}}{\epsilon_{dc}} - \tfrac{\epsilon_{cd}}{\epsilon_{dd}} \right) + \tfrac{\beta + b^2}{\beta b} + \tfrac{\epsilon_{dc}}{\beta \epsilon_{cc}} + \tfrac{\epsilon_{cd}}{\epsilon_{dd}} \\ C &= -\tfrac{(1+\beta)}{b\epsilon_{cc}} \tfrac{\varepsilon_{ck}}{\varepsilon_{rk}} \left(\tfrac{\epsilon_{cc}}{\epsilon_{dc}} - \tfrac{\epsilon_{cd}}{\epsilon_{dd}} \right) + \tfrac{\beta + b^2}{\beta b} \left(\tfrac{\epsilon_{dc}}{\beta \epsilon_{cc}} + \tfrac{\epsilon_{cd}}{\epsilon_{dd}} \right) + \tfrac{2}{\beta} \end{aligned} \tag{40}$$

After simplifications we get the expression (16).

Consider now that λ is a root of the characteristic polynomial (16), i.e. $\mathcal{P}(\lambda) = 0$. It follows obviously that if λ is complex then its conjugate $\bar{\lambda}$ is also a characteristic root. Let us then consider $\mathcal{P}((\beta\lambda)^{-1})$, namely

$$\begin{aligned} \mathcal{P}\left(\tfrac{1}{\beta\lambda}\right) &= \left[\tfrac{1}{\beta^2\lambda^2} - \tfrac{1}{\beta\lambda} \left(\tfrac{\epsilon_{dc}}{\beta\epsilon_{cc}} + \tfrac{\epsilon_{cd}}{\epsilon_{dd}} \right) + \tfrac{1}{\beta} \right] \tfrac{\left(\frac{b}{\beta\lambda} - 1\right)\left(\frac{1}{\lambda} - b\right))}{\beta b} \\ &+ \tfrac{1}{\beta\lambda} \left(\tfrac{1}{\beta\lambda} - 1 \right) \left(\tfrac{1}{\beta\lambda} - \tfrac{1}{\beta} \right) \tfrac{\beta}{b\epsilon_{cc}} \tfrac{\varepsilon_{ck}}{\varepsilon_{rk}} \left(\tfrac{\epsilon_{cc}}{\epsilon_{dc}} - \tfrac{\epsilon_{cd}}{\epsilon_{dd}} \right) \\ &= \tfrac{1}{\beta^4\lambda^4} \Bigg\{ \left[\lambda^2 - \lambda \left(\tfrac{\epsilon_{dc}}{\beta\epsilon_{cc}} + \tfrac{\epsilon_{cd}}{\epsilon_{dd}} \right) + \tfrac{1}{\beta} \right] \tfrac{(\lambda b - 1)(\lambda\beta - b)}{\beta b} \\ &+ \lambda(\lambda - 1) \left(\lambda - \tfrac{1}{\beta} \right) \tfrac{\beta}{b\epsilon_{cc}} \tfrac{\varepsilon_{ck}}{\varepsilon_{rk}} \left(\tfrac{\epsilon_{cc}}{\epsilon_{dc}} - \tfrac{\epsilon_{cd}}{\epsilon_{dd}} \right) \Bigg\} = 0 \end{aligned}$$

If follows that $(\beta\lambda)^{-1}$ is also a characteristic root. The same argument applies for $(\beta\bar{\lambda})^{-1}$. It follows that the four characteristic roots are either all real, or given by two pairs of complex conjugates. Moreover, at least two roots or a pair of complex conjugate roots have a modulus larger than one.

The nature of the characteristic roots can be derived considering the following expressions:

$$\begin{aligned} \Delta &= \tfrac{256}{\beta^6} - \tfrac{192B^2}{\beta^5} - \tfrac{128C^2}{\beta^4} + \tfrac{288B^2C}{\beta^4} - \tfrac{60B^4}{\beta^4} - \tfrac{80B^2C^2}{\beta^3} + \tfrac{36B^4C}{\beta^3} \\ &- \tfrac{4B^6}{\beta^3} + \tfrac{16C^4}{\beta^2} - \tfrac{8B^2C^3}{\beta^2} + \tfrac{B^4C^2}{\beta^2} \\ D &= \tfrac{64}{\beta^2} - 16C^2 + 16B^2C - \tfrac{16B^2}{\beta} - 3B^4 \\ P &= 8C - 3B^2 \\ R &= B\left[B^2 + \tfrac{8}{\beta} - 4C \right] \end{aligned} \tag{41}$$

Since we already know that the characteristic roots are either all real, or all complex, we immediately derive that $\Delta \geq 0$. Tedious but straightforward computations also show that

$$\begin{array}{rcl} D & = & \frac{R}{B}\left[\frac{8}{\beta} - 3B^2 + 4C\right] \\ \Delta & = & \frac{\left(\beta^2C^2 - 4\beta B^2 + 4\beta C + 4\right)R^2}{\beta^4 B^2} \end{array} \quad (42)$$

It follows that if $R = 0$ then $D = 0$ and $\Delta = 0$. This implies the following characterization of the roots:

i) when $\Delta > 0$ the characteristic roots are real and distincts if $P < 0$ and $D < 0$, and given by two pairs of non-real complex conjugates if $P > 0$ or $D > 0$;

ii) when $\Delta = R = D = 0$, there are two complex conjugates double roots or two real double roots depending on whether $P > 0$ or $P < 0$.

□

8.3 Proof of Lemma 1

Under Assumption 4, let us denote the two degree-2 polynomials as follows

$$\mathcal{P}_1(\lambda) = \lambda^2 - \lambda\left(\frac{\epsilon_{dc}}{\beta\epsilon_{cc}} + \frac{\epsilon_{cd}}{\epsilon_{dd}}\right) + \frac{1}{\beta}, \quad \mathcal{P}_2(\lambda) = \frac{(\lambda b - 1)(\lambda\beta - b)}{\beta b} \quad (43)$$

The discriminant of $\mathcal{P}_1(\lambda)$ is equal to:

$$\Delta_1 = \left(\frac{\epsilon_{dc}}{\beta\epsilon_{cc}} + \frac{\epsilon_{cd}}{\epsilon_{dd}} + \frac{2}{\sqrt{\beta}}\right)\left(\frac{\epsilon_{dc}}{\beta\epsilon_{cc}} + \frac{\epsilon_{cd}}{\epsilon_{dd}} - \frac{2}{\sqrt{\beta}}\right)$$

Using (4)-(5) we get

$$\begin{array}{rcl} \Delta_1 & = & \left(\frac{1}{u_{cd}}\right)^2\left(u_{cc} + \frac{2u_{cd}}{\sqrt{\beta}} + \frac{u_{dd}}{\beta}\right)\left(u_{cc} - \frac{2u_{cd}}{\sqrt{\beta}} + \frac{u_{dd}}{\beta}\right) \\ & = & \left(\frac{1}{u_{cd}}\right)^2 \begin{pmatrix} 1 & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} u_{cc} & u_{cd} \\ u_{dc} & u_{dd} \end{pmatrix} \begin{pmatrix} 1 \\ \frac{1}{\sqrt{\beta}} \end{pmatrix} \\ & \times & \begin{pmatrix} 1 & -\frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} u_{cc} & u_{cd} \\ u_{dc} & u_{dd} \end{pmatrix} \begin{pmatrix} 1 \\ -\frac{1}{\sqrt{\beta}} \end{pmatrix} \end{array}$$

Under the concavity property in Assumption 2, the Hessian matrix of the utility function $u(c, d)$ is quasi-negative definite which implies $\Delta_1 \geq 0$ and the associated characteristic roots are necessarily real. From $\mathcal{P}_2(\lambda)$ we obviously conclude that for any sign of the capital intensity difference b the associated characteristic roots are also necessarily real.

□

8.4 Proof of Proposition 3

Under Assumptions 1-4, let $b \geq 0$ and $\epsilon_{cd}, \epsilon_{dc} \geq 0$, i.e. $u_{cd} \leq 0$. Using the fact that $\frac{\epsilon_{cc}}{\epsilon_{dc}} = \frac{\epsilon_{cd}}{\epsilon_{dd}}$, we derive the following expression

$$\mathcal{P}_1(\lambda) = \left(\lambda - \frac{\epsilon_{cc}}{\epsilon_{dc}}\right)\left(\lambda - \frac{\epsilon_{dc}}{\beta\epsilon_{cc}}\right) \tag{44}$$

The associated characteristic roots λ_1 and λ_2 are therefore both positive. Moreover we get:

$$\begin{aligned}\mathcal{P}_1(0) &= \tfrac{1}{\beta} \geq 1\\ \mathcal{P}_1(1) &= -\epsilon_{cc}\epsilon_{dc}\left(\tfrac{1}{\epsilon_{cc}} - \tfrac{1}{\epsilon_{dc}}\right)\left(\tfrac{1}{\beta\epsilon_{cc}} - \tfrac{1}{\epsilon_{dc}}\right)\end{aligned}$$

The normality Assumption 3 implies $\mathcal{P}_1(1) < 0$ and we conclude that the associated characteristic roots λ_1 and λ_2 are such that $\lambda_1 < 1$ and $\lambda_2 > 1$.

From $\mathcal{P}_2(\lambda)$, the associated characteristic roots λ_1 and λ_2 are both positive. Moreover we derive:

$$\mathcal{P}_2(0) = \tfrac{1}{\beta} \geq 1, \quad \mathcal{P}_2(1) = -\tfrac{(\beta-b)(1-b)}{\beta b}$$

From constant returns to scale, we get $wa_{01} + ra_{11} = p$ with $a_{01} = l_1/y$ and $a_{11} = k_1/y$. The second equation in (12) rewrites as $p = \beta r$. We then obtain after substitution in the previous equation $r(\beta - a_{11}) = wa_{01} > 0$ and thus

$$\beta - b = \tfrac{a_{00}(\beta - a_{11}) + a_{10}a_{01}}{a_{00}} > 0$$

When $b \geq 0$ we then necessarily have $b < \beta \leq 1$. It follows that $\mathcal{P}_2(0) < 0$ and we conclude that the associated characteristic roots λ_1 and λ_2 are such that $\lambda_1 < 1$ and $\lambda_2 > 1$. The steady state is therefore a saddle-point.

□

8.5 Proof of Proposition 4

i) Under Assumptions 1-4, let $b \geq 0$ and $\epsilon_{cd}, \epsilon_{dc} < 0$, i.e. $u_{cd} > 0$. As shown previously, we derive from $\mathcal{P}_2(\lambda) = 0$ that there exist two positive characteristic roots, one being lower than 1 and the other larger. From $\mathcal{P}_1(\lambda)$ as given by (44), the associated characteristic roots λ_1 and λ_2 are both negative. Moreover, we get:

$$\mathcal{P}_1(-1) = \left(1 + \tfrac{\epsilon_{cc}}{\epsilon_{dc}}\right)\left(1 + \tfrac{\epsilon_{dc}}{\beta\epsilon_{cc}}\right) = \tfrac{(\epsilon_{cc}+\epsilon_{dc})(\beta\epsilon_{cc}+\epsilon_{dc})}{\beta\epsilon_{cc}\epsilon_{dc}}$$

We conclude easily that

$$\mathcal{P}_1(-1) < 0 \quad \Leftrightarrow \quad \epsilon_{cc} \in (0, -\epsilon_{dc}) \cup (-\epsilon_{dc}/\beta, +\infty)$$

$$\mathcal{P}_1(-1) > 0 \quad \Leftrightarrow \quad \epsilon_{cc} \in (-\epsilon_{dc}, -\epsilon_{dc}/\beta)$$

It follows that the steady state is a saddle-point with damped oscillations when $\epsilon_{cc} \in (0, -\epsilon_{dc}) \cup (-\epsilon_{dc}/\beta, +\infty)$ and there exists a flip bifurcation with persistent period-2 cycles when ϵ_{cc} crosses the bifurcation values $-\epsilon_{dc}$ or $-\epsilon_{dc}/\beta$. Considering the expression of the elasticity of intertemporal substitution in consumption (6), these conditions can be equivalently stated in terms of ε. Namely, the steady state is a saddle-point with damped oscillations when $\varsigma \in (0, \underline{\varsigma}) \cup (\bar{\varsigma}, +\infty)$ with $\underline{\varsigma} = (\epsilon_{cc}\beta)/(1+\beta)$ and $\bar{\varsigma} = \epsilon_{cc}/2$, and there exists a flip bifurcation with persistent period-2 cycles when ς crosses the bifurcation values $\underline{\varsigma}$ or $\bar{\varsigma}$.

ii) Under Assumptions 1-4, let $\epsilon_{cd}, \epsilon_{dc} \geq 0$, i.e. $u_{cd} \leq 0$, and $b < 0$. As shown previously, we derive from $\mathcal{P}_1(\lambda) = 0$ that there exist two positive characteristic roots, one being lower than 1 and the other larger. From $\mathcal{P}_2(\lambda)$, the associated characteristic roots λ_1 and λ_2 are both negative. Moreover we get:

$$\mathcal{P}_2(-1) \quad = \quad \tfrac{(1+b)(b+\beta)}{\beta b}$$

We conclude easily that

$$\mathcal{P}_1(-1) < 0 \quad \Leftrightarrow \quad b \in (-\infty, -1) \cup (-\beta, 0)$$

$$\mathcal{P}_1(-1) > 0 \quad \Leftrightarrow \quad b \in (-1, -\beta)$$

It follows that the steady state is a saddle-point with damped oscillations when $b \in (-\infty, -1) \cup (-\beta, 0)$. Moreover, if there is some $\beta^* \in (0,1)$ such that $b \in (-1, -\beta^*)$, then there exists $\bar{\beta} \in (0,1)$ such that, when β crosses $\bar{\beta}$ from above, (k^*, d^*) undergoes a flip bifurcation leading to persistent period-2 cycles.

iii) The case where the consumption good is capital intensive, i.e. $b < 0$, and $\epsilon_{cd}, \epsilon_{dc} < 0$, i.e. $u_{cd} > 0$, is obviously derived from the two previous cases.

□

8.6 Proof of Corollary 1

Under a linear homogeneous utility function, standard Euler equalities based on the homogeneity of degree 1, namely $u = u_c c + u_d Bd$, $0 = u_{cc}c + u_{cd}Bd$ and $0 = u_{dc}c + u_{dd}Bd$, lead to

$$u_{cd} = -\frac{u_{cc}}{Bd}, \quad u_{dc} = -\frac{u_{dd}Bd}{c} \quad \text{and thus } u_{dd} = u_{cc}\left(\frac{c}{Bd}\right)^2$$

Moreover, we get from the first order condition $u_d B = \beta u_c$ and (17)

$$\frac{c}{Bd} = \frac{\beta\phi}{1-\phi}$$

Substituting all this into (4)-(5) implies

$$\epsilon_{cd} = -\epsilon_{cc}, \quad \epsilon_{dc} = -\epsilon_{cc}\frac{1-\phi}{\phi}, \quad \epsilon_{dd} = \epsilon_{cc}\frac{1-\phi}{\phi}$$

We consider now Cobb-Douglas technologies as given by (20). We follow the same methodology as in Baierl *et al.* [8]. The Lagrangian associated with the optimization program (1) is:

$$\mathcal{L} = k_0^{\alpha_0} l_0^{1-\alpha_0} + w\big(1 - l_0 - l_1\big) + r\big(k - k_0 - k_1\big) + p\left[k_1^{\alpha_1} l_1^{1-\alpha_1} - y\right] \tag{45}$$

The first order conditions are:

$$r = \alpha_0 k_0^{\alpha_0 - 1} l_0^{1-\alpha_0} = p\alpha_1 k_1^{\alpha_1 - 1} l_1^{1-\alpha_1} \tag{46}$$

$$w = (1-\alpha_0) k_0^{\alpha_0} l_0^{-\alpha_0} = p(1-\alpha_1) k_1^{\alpha_1} l_1^{-\alpha_1} \tag{47}$$

Using $k_0 = k - k_1$, $l_0 = 1 - l_1$, and merging the above equations gives:

$$l_0^* = \frac{(1-\alpha_0)\alpha_1(k - k_1^*)}{(\alpha_0 - \alpha_1)k_1^* + (1-\alpha_0)\alpha_1 k} \tag{48}$$

$$l_1^* = \frac{\alpha_0(1-\alpha_1)k_1^*}{(\alpha_0 - \alpha_1)k_1^* + (1-\alpha_0)\alpha_1 k} \tag{49}$$

$$K_c^* = k - k_1^* \tag{50}$$

$$k_1^* = g(k, y) \equiv g \tag{51}$$

where

$$g(k, y) = \left\{ k_1 \in [0, k^{\alpha_1}] \ / \ y = \frac{[\alpha_0(1-\alpha_1)]^{1-\alpha_1} k_1}{[(1-\alpha_0)\alpha_1 k + (\alpha_0 - \alpha_1)k_1]^{1-\alpha_1}} \right\} \tag{52}$$

From (46), (48) and (50) we obtain:

$$T_k = r^* = \alpha_0 \left[\frac{(1-\alpha_0)\alpha_1}{(1-\alpha_0)\alpha_1 k + (\alpha_0 - \alpha_1)g}\right]^{1-\alpha_0} \tag{53}$$

and from (46), (49), (51) and (53):

$$T_y = p^* = \frac{\alpha_0[(1-\alpha_0)\alpha_1]^{1-\alpha_0}[\alpha_0(1-\alpha_1)]^{-(1-\alpha_1)}[(1-\alpha_0)\alpha_1 k + (\alpha_0 - \alpha_1)g]^{\alpha_0 - \alpha_1}}{\alpha_1} \tag{54}$$

By the derivation of g, we have, for any equilibrium path, the identity $(1-\alpha_0)\alpha_1 k + (\alpha_0 - \alpha_1)g = \alpha_0(1-\alpha_1)(g/y)^{1/(1-\alpha_1)}$. Substituting this into (53) and (54) gives after simplifications:

$$
\begin{aligned}
T_k(k,y) &= \alpha_0 \left(\frac{(1-\alpha_0)\alpha_1}{\alpha_0(1-\alpha_1)}\right)^{1-\alpha_0} \left(\frac{y}{g}\right)^{\frac{1-\alpha_0}{1-\alpha_1}} \\
T_y(k,y) &= -\frac{\alpha_1}{\beta_1}\left(\frac{(1-\alpha_0)\alpha_1}{\alpha_0(1-\alpha_1)}\right)^{1-\alpha_0} \left(\frac{y}{g}\right)^{\frac{\alpha_1-\alpha_0}{1-\alpha_1}} \\
T_{kk}(k,y) &= -T_k(k,y)\frac{g_1}{g}
\end{aligned}
$$

with $g_1 = \partial g(k,y)/\partial k$. A steady state k^* is then defined as $T_k(k^*,k^*) + \beta T_y(k^*,k^*)$. Denote $g^* = g(k^*,k^*)$ and $y^* = k^*$. Using the derivatives of T in the definition of k^* gives:

$$
g^* = \beta\alpha_1 k^* \tag{55}
$$

Substituting (55) into the definition of g, we find

$$
k^* = \frac{\alpha_0(1-\alpha_1)(\beta\alpha_1)^{\frac{1}{1-\alpha_1}}}{\alpha_1[1-\alpha_0+\beta(\alpha_0-\alpha_1)]} \tag{56}
$$

Considering (52), we easily derive

$$
g_1 = \frac{\beta\alpha_1(1-\alpha_0)(1-\alpha_1)}{1-\alpha_0+\beta(\alpha_0-\alpha_1)} \tag{57}
$$

From all these results and (3), we get

$$
\begin{aligned}
c^* = T(k^*,k^*) &= \left(\frac{\alpha_0(1-\alpha_1)}{(1-\alpha_0)\alpha_1}\right)^{\alpha_0} \frac{(1-\alpha_0)(1-\beta\alpha_1)(\beta\alpha_1)^{\frac{\alpha_0}{1-\alpha_1}}}{1-\alpha_0+\beta(\alpha_0-\alpha_1)} \\
r^* = T_k(k^*,k^*) &= \alpha_0\left(\frac{(1-\alpha_0)\alpha_1}{\alpha_0(1-\alpha_1)}\right)^{1-\alpha_0} (\beta\alpha_1)^{-\frac{1-\alpha_0}{1-\alpha_1}} \\
T_{kk}(k^*,k^*) &= -\frac{T_k(k^*,k^*)}{k^*}\frac{(1-\alpha_0)^2}{1-\alpha_0+\beta\alpha_1(\alpha_0-\alpha_1)} \\
b &= \frac{\beta(\alpha_1-\alpha_0)}{1-\alpha_0}
\end{aligned}
$$

We then easily derive

$$
\varepsilon_{ck} = \frac{\alpha_0}{1-\beta\alpha_1} \text{ and } \varepsilon_{rk} = \frac{(1-\alpha_0)^2}{1-\alpha_0+\beta\alpha_1(\alpha_0-\alpha_1)}
$$

Considering (19) with $\gamma = 1$, the characteristic polynomial (16) becomes here

$$
\mathcal{P}(\lambda) = \left(\lambda + \frac{\epsilon_{cc}-\varsigma}{\varsigma}\right)\left(\lambda + \frac{\varsigma}{\beta(\epsilon_{cc}-\varsigma)}\right)\frac{(\lambda b-1)(\lambda\beta-b)}{\beta b} \tag{58}
$$

The characteristic roots are

$$
\lambda_1 = -\frac{\epsilon_{cc}-\varsigma}{\varsigma},\ \lambda_2 = -\frac{\varsigma}{\beta(\epsilon_{cc}-\varsigma)},\ \lambda_3 = \frac{1}{b} \text{ and } \lambda_4 = \frac{b}{\beta} \tag{59}
$$

The critical values $\underline{\varsigma}$ and $\bar{\varsigma}$ are given in Proposition 4. Assume that $\alpha_0 > (1+\alpha_1)/2$. We immediately derive from the expression of b that $\lambda_3 > -1$ if and only if $\beta > (1-\alpha_0)/(\alpha_0-\alpha_1) \equiv \underline{\beta}$ while $\lambda_4 < -1$. The result follows from the fact that if $\beta = \underline{\beta}$ and $\varsigma = \bar{\varsigma}$ or $\underline{\varsigma}$ then two characteristic roots are simultaneously equal to -1.

□

8.7 Proof of Proposition 5

The characteristic polynomial (16) can be expressed as follows

$$\left[\lambda^2 - \lambda\left(\frac{\epsilon_{dc}}{\beta\epsilon_{cc}} + \frac{\epsilon_{cd}}{\epsilon_{dd}}\right) + \frac{1}{\beta}\right]\frac{(\lambda b-1)(\lambda\beta-b)}{\beta b} = -\lambda(\lambda-1)\left(\lambda - \frac{1}{\beta}\right)\frac{\beta}{b\epsilon_{cc}}\frac{\varepsilon_{ck}}{\varepsilon_{rk}}\left(\frac{\epsilon_{cc}}{\epsilon_{dc}} - \frac{\epsilon_{cd}}{\epsilon_{dd}}\right)$$

or equivalently, using the notations of Lemma 1,

$$P_1(\lambda)P_2(\lambda) = P_3(\lambda)$$

with $P_3(\lambda)$ a degree-3 polynomial while $P_1(\lambda)P_2(\lambda)$ is a degree-4 polynomial. If these two polynomials intersect four times, then the four characteristic roots are real. To determine the number of intersections of these polynomials, we can use informations derived from the location of their respective roots. The roots of $P_3(\lambda) = 0$ are quite obvious, namely $\lambda_{31} = 0$, $\lambda_{32} = 1$ and $\lambda_{33} = 1/\beta$. Moreover, depending of the sign of $\epsilon_{cd}, \epsilon_{dc}$ we get

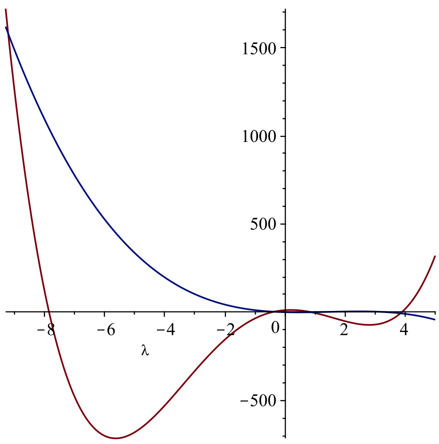
- if $\epsilon_{cd}, \epsilon_{dc} < 0$, then $\frac{\epsilon_{cc}}{\epsilon_{dc}} - \frac{\epsilon_{cd}}{\epsilon_{dd}} > 0$ and $\lim_{\lambda\to+\infty} P_3(\lambda) = -\infty$ while $\lim_{\lambda\to-\infty} P_3(\lambda) = +\infty$;

- if $\epsilon_{cd}, \epsilon_{dc} > 0$, then $\frac{\epsilon_{cc}}{\epsilon_{dc}} - \frac{\epsilon_{cd}}{\epsilon_{dd}} < 0$ and $\lim_{\lambda\to+\infty} P_3(\lambda) = +\infty$ while $\lim_{\lambda\to-\infty} P_3(\lambda) = -\infty$;

The roots of $P_1(\lambda)P_2(\lambda) = 0$ are obviously given by the respective roots of $P_1(\lambda) = 0$ and $P_2(\lambda) = 0$.

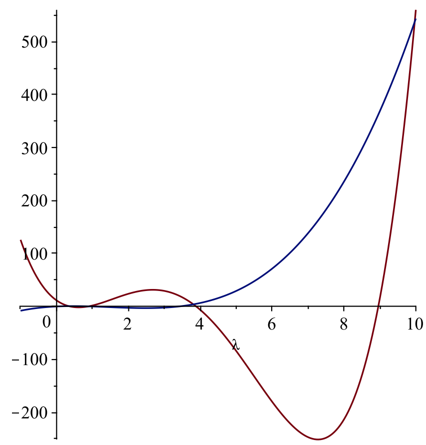
i) Assume first that $b > 0$. We have shown in the proof of Proposition 3 that $b < \beta \leq 1$. The roots of $P_2(\lambda) = 0$ are then quite obvious, namely $\lambda_{21} = 1/b > 1$ and $\lambda_{22} = b/\beta < 1$. Finally, the roots of $P_1(\lambda) = 0$ are necessarily real and negative if $\epsilon_{cd}, \epsilon_{dc} < 0$, or positive if $\epsilon_{cd}, \epsilon_{dc} > 0$. Moreover, we have $\lim_{\lambda\to\pm\infty} P_1(\lambda)P_2(\lambda) = +\infty$ and $P_1(0)P_2(0) > 0$.

If $\epsilon_{cd}, \epsilon_{dc} < 0$, we derive from the above informations that $P_1(b/\beta)P_2(b/\beta) = 0 > P_3(b/\beta)$ while $P_1(1)P_2(1 < P_3(b/\beta) = 0$ implying a first intersection between $P_1(\lambda)P_2(\lambda)$ and $P_3(\lambda)$ in the positive orthant. Moreover, since $P_1(1/\beta)P_2(1/\beta) < P_3(1/\beta) = 0$ while $P_1(1/b)P_2(1/b) = 0 > P_3(b/\beta)$, we get a second intersection $P_1(\lambda)P_2(\lambda)$ and $P_3(\lambda)$ in the positive orthant. Since $P_1(0)P_2(0) > 0$, $P_1(\lambda)P_2(\lambda) = 0$ admits two roots in the negative horthant, $P_3(0) = 0$ and $P_3(\lambda)$ is an increasing function in the negative hortant, we conclude that there necessarily exists a third intersection between $P_1(\lambda)P_2(\lambda)$ and $P_3(\lambda)$ in the positive orthant. The last intersection, which also occurs in the negative orthant, is obtained because $\lim_{\lambda\to-\infty} P_1(\lambda)P_2(\lambda) > \lim_{\lambda\to-\infty} P_3(\lambda)$. Indeed $P_3(\lambda)$ a degree-3 polynomial while $P_1(\lambda)P_2(\lambda)$ is a degree-4 polynomial. We then get the following graphical illustration



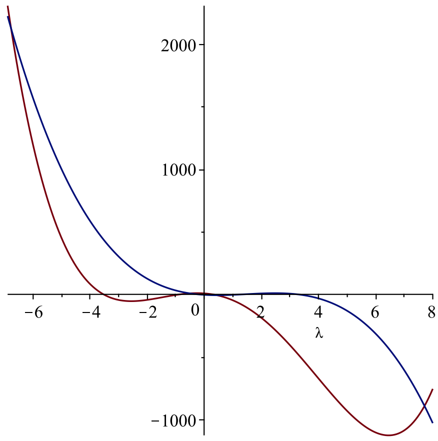
It follows that the four roots of the characteristic polynomial (16) are real.

If $\epsilon_{cd}, \epsilon_{dc} > 0$, the roots of $P_3(\lambda) = 0$ and $P_2(\lambda) = 0$ are the same as before while the roots of $P_1(\lambda) = 0$ are now real and positive. Since $P_1(0)P_2(0) > 0$, $P_1(1/b)P_2(1/b) = 0$ and $P_1(1)P_2(1) > 0$, there necessarily exists a second root of $P_1(\lambda)P_2(\lambda) = 0$ between 0 and $1/b$ implying two intersections between $P_1(\lambda)P_2(\lambda)$ and $P_3(\lambda)$. The two others are obtained since $P_1(1/\beta)P_2(1/\beta) > P_3(1/\beta) = 0$, $P_1(b/\beta)P_2(b/\beta) = 0 < P_3(b/\beta)$ and $\lim_{\lambda\to+\infty} P_1(\lambda)P_2(\lambda) > \lim_{\lambda\to+\infty} P_3(\lambda)$. We then get the following graphical illustration



Here again, it follows that the four roots of the characteristic polynomial (16) are real.

ii) Assume now that $b < 0$ and $\epsilon_{cd}, \epsilon_{dc} > 0$. The roots of $P_2(\lambda) = 0$ become negative, namely $\lambda_{21} = 1/b < \lambda_{22} = b/\beta < 0$. We easily get $P_1(0)P_2(0) > 0$, $P_1(1)P_2(1) < P_3(1) = 0$,$P_1(1/\beta)P_2(1/\beta) < P_3(1/\beta) = 0$, $\lim_{\lambda\to+\infty} P_1(\lambda)P_2(\lambda) = +\infty$ and $\lim_{\lambda\to+\infty} P_3(\lambda) = -\infty$. It follows that there are three intersections between $P_1(\lambda)P_2(\lambda)$ and $P_3(\lambda)$ in the positive orthant. Moreover, we have $\lim_{\lambda\to-\infty} P_1(\lambda)P_2(\lambda) > \lim_{\lambda\to-\infty} P_3(\lambda)$ implying the existence of two additional intersections between $P_1(\lambda)P_2(\lambda)$ and $P_3(\lambda)$ in the negative orthant. We then get the following graphical illustration



and it follows that the four roots of the characteristic polynomial (16) are real.

□

8.8 Proof of Proposition 6

Using a homogeneous of degree $\gamma < 1$ utility function, the degree-4 characteristic polynomial as given by Proposition 2 becomes

$$\begin{aligned} \mathcal{P}(\lambda) &= \left[\lambda^2 + \lambda\left(\frac{(\gamma-\phi)^2+\beta\phi^2-\beta\phi\epsilon_{cc}(1-\gamma)(2\phi-\gamma)}{\beta\phi(\gamma-\phi)[1-\epsilon_{cc}(1-\gamma]}\right) + \frac{1}{\beta}\right] \frac{(\lambda b-1)(\lambda\beta-b)}{\beta b} \\ &+ \lambda(\lambda-1)\left(\lambda-\frac{1}{\beta}\right)\frac{\beta}{b}\frac{\varepsilon_{ck}}{\varepsilon_{rk}}\frac{(1-\gamma)[\gamma-\epsilon_{cc}\phi(1-\gamma)]}{(\gamma-\phi)[1-\epsilon_{cc}(1-\gamma)]} \end{aligned} \tag{60}$$

and can be expressed as $\mathcal{Q}_1(\lambda) = \mathcal{Q}_2(\lambda)$ with

$$\begin{aligned} \mathcal{Q}_1(\lambda) &\equiv \frac{1}{\gamma-\phi}\left[\lambda^2(\gamma-\phi) + \lambda\left(\frac{(\gamma-\phi)^2+\beta\phi^2-\beta\phi\epsilon_{cc}(1-\gamma)(2\phi-\gamma)}{\beta\phi[1-\epsilon_{cc}(1-\gamma]}\right) + \frac{(\gamma-\phi)}{\beta}\right]\frac{(\lambda b-1)(\lambda\beta-b)}{\beta b} \\ \mathcal{Q}_2(\lambda) &\equiv -\frac{1}{\gamma-\phi}\lambda(\lambda-1)\left(\lambda-\frac{1}{\beta}\right)\frac{\beta}{b}\frac{\varepsilon_{ck}}{\varepsilon_{rk}}\frac{(1-\gamma)[\gamma-\epsilon_{cc}\phi(1-\gamma)]}{[1-\epsilon_{cc}(1-\gamma)]} \end{aligned}$$

Considering the limit $\phi \to \gamma$ we immediately conclude that one root λ_1 is necessarily real and equal to $\pm\infty$ and we get

$$\begin{aligned} \mathcal{Q}_1(\lambda) &= \lambda\gamma\frac{(\lambda b-1)(\lambda\beta-b)}{\beta b} \\ \mathcal{Q}_2(\lambda) &= -\lambda\gamma(\lambda-1)\left(\lambda-\frac{1}{\beta}\right)\frac{\beta}{b}\frac{\varepsilon_{ck}}{\varepsilon_{rk}}(1-\gamma) \end{aligned}$$

It follows that a second root λ_2 is real and equal to 0. Computing now the derivatives $\mathcal{Q}_1'(\lambda)$ and $\mathcal{Q}_2'(\lambda)$, and evaluating them at 0 gives

$$\begin{aligned} \mathcal{Q}_1'(0) &= \frac{\gamma}{\beta} \\ \mathcal{Q}_2'(0) &= -\frac{\gamma}{b}\frac{\varepsilon_{ck}}{\varepsilon_{rk}}(1-\gamma) \end{aligned}$$

It follows that $\mathcal{Q}_1'(0) \gtreqless \mathcal{Q}_2'(0)$ if and only if $\epsilon_{cc} \lesseqgtr \hat{\epsilon}_{cc}$ with

$$\hat{\epsilon}_{cc} \equiv -\frac{b}{(1-\gamma)}\frac{\varepsilon_{rk}}{\varepsilon_{ck}} \in (0, \tilde{\epsilon}_{cc})$$

Note that $\hat{\epsilon}_{cc} \in (0, \tilde{\epsilon}_{cc})$ if and only if

$$-\frac{\varepsilon_{ck}}{b\varepsilon_{rk}} > 1 \tag{61}$$

We conclude therefore that under condition (61) there exist two additional intersections between $\mathcal{Q}_1(\lambda)$ and $\mathcal{Q}_2(\lambda)$ implying that the two last characteristic roots λ_3, λ_4 are also real. Let us then assume that $b \in (-\infty, -1) \cup (-\beta, 0)$. We derive that

i) if $\epsilon_{cc} < \hat{\epsilon}_{cc}$ then $\mathcal{Q}_1'(0) > \mathcal{Q}_2'(0)$ with $\mathcal{Q}_1(1/b) = \mathcal{Q}_1(b/\beta) = 0$ which implies that one intersection must occur between -1 and 0, say $\lambda_3 \in (-1, 0)$. Moreover we derive also that $\lambda_1 = -\infty$ and $\lambda_4 < -1$;

ii) if $\epsilon_{cc} \in (\hat{\epsilon}_{cc}, \tilde{\epsilon}_{cc})$ then $\mathcal{Q}_1'(0) < \mathcal{Q}_2'(0)$ with $\mathcal{Q}_2(1) = 0$ which implies that one intersection must occur between 0 and 1, say $\lambda_3 \in (0, 1)$. Moreover we derive $\lambda_1 = +\infty$ and $\lambda_4 > 1$.

We then conclude by continuity that there exists $0 < \bar{\phi} < \gamma$ such that when $\phi \in (\bar{\phi}, \gamma)$, the above results hold with $\lambda_1 \in (-\infty, -1)$ and $\lambda_2 \in (-1, 0)$ when $\epsilon_{cc} < \hat{\epsilon}_{cc}$ or $\lambda_1 \in (1, \infty)$ and $\lambda_2 \in (0, 1)$ when $\epsilon_{cc} \in (\hat{\epsilon}_{cc}, \tilde{\epsilon}_{cc})$. Considering the expression of ς as given by (19) which is a decreasing function of ϕ, we derive that there exists a corresponding value $\underline{\varsigma} = \varsigma(\bar{\phi})$, and it follows that the above results hold for $\varsigma \in (0, \underline{\varsigma})$.

Note now that the characteristic polynomial (60) can be also expressed as $\mathcal{Q}_1(\lambda) = \mathcal{Q}_2(\lambda)$ with

$$\begin{aligned}
\mathcal{Q}_1(\lambda) &\equiv \frac{1}{\phi}\left[\lambda^2\phi + \lambda\left(\frac{(\gamma-\phi)^2+\beta\phi^2-\beta(\gamma-\phi)\epsilon_{cc}(1-\gamma)(2\phi-\gamma)}{\beta(\gamma-\phi)[1-\epsilon_{cc}(1-\gamma]}\right) + \frac{\phi}{\beta}\right]\frac{(\lambda b-1)(\lambda\beta-b)}{\beta b} \\
\mathcal{Q}_2(\lambda) &\equiv -\frac{1}{\phi}\lambda(\lambda-1)\left(\lambda-\frac{1}{\beta}\right)\frac{\beta}{b}\frac{\varepsilon_{ck}}{\varepsilon_{rk}}\frac{\phi(1-\gamma)[\gamma-\epsilon_{cc}\phi(1-\gamma)]}{(\gamma-\phi)[1-\epsilon_{cc}(1-\gamma)]}
\end{aligned}$$

Considering the limit $\phi \to 0$ we immediately conclude that one root λ_1 is necessarily real and equal to $-\infty$ as $b < 0$, and we get

$$\begin{aligned}
\mathcal{Q}_1(\lambda) &= \frac{\lambda\gamma^2}{\beta[1-\epsilon_{cc}(1-\gamma)]}\frac{(\lambda b-1)(\lambda\beta-b)}{\beta b} \\
\mathcal{Q}_2(\lambda) &= 0
\end{aligned}$$

It follows that $\lambda_2 = 0$, $\lambda_3 = 1/b$ and $\lambda_4 = b/\beta$ with one larger than -1 and the other lower than -1 as $b \in (-\infty, -1) \cup (-\beta, 0)$. We then conclude by continuity that there exists $0 < \underline{\phi} \leq \bar{\phi}$ such that when $\phi \in (0, \underline{\phi})$, the above results hold with $\lambda_1 \in (-\infty, -1)$ and $\lambda_2 \in (-1, 0)$. Considering again the expression of ς as given by (19) which is a decreasing function of ϕ, we derive that there exists a corresponding value $\bar{\varsigma} = \varsigma(\underline{\phi}) \geq \underline{\varsigma}$, and it follows that the above results hold for $\varsigma \in (\bar{\varsigma}, \epsilon_{cc})$.

$\square$

8.9 Proof of Proposition 7

The expressions in (40) become here

$$\begin{array}{rcl} B & = & -\frac{\beta}{b}\frac{\varepsilon_{ck}}{\varepsilon_{rk}}\frac{(1-\gamma)[\gamma-\epsilon_{cc}\phi(1-\gamma)]}{(\gamma-\phi)[1-\epsilon_{cc}(1-\gamma)]} + \frac{\beta+b^2}{\beta b} - \left(\frac{(\gamma-\phi)^2+\beta\phi^2-\beta\phi\epsilon_{cc}(1-\gamma)(2\phi-\gamma)}{\beta\phi(\gamma-\phi)[1-\epsilon_{cc}(1-\gamma]}\right) \\ C & = & -\frac{(1+\beta)}{b}\frac{\varepsilon_{ck}}{\varepsilon_{rk}}\frac{(1-\gamma)[\gamma-\epsilon_{cc}\phi(1-\gamma)]}{(\gamma-\phi)[1-\epsilon_{cc}(1-\gamma)]} - \frac{\beta+b^2}{\beta b}\left(\frac{(\gamma-\phi)^2+\beta\phi^2-\beta\phi\epsilon_{cc}(1-\gamma)(2\phi-\gamma)}{\beta\phi(\gamma-\phi)[1-\epsilon_{cc}(1-\gamma]}\right) + \frac{2}{\beta} \end{array} \quad (62)$$

As $\epsilon_{cc} < \tilde{\epsilon}_{cc}$ and $b \in (-\infty, -1) \cup (-\beta, 0)$, we immediately get $C > 0$ for any $\phi \in (0, \gamma)$. Moreover, when $\epsilon_{cc} = 0$, we get

$$B \;=\; \frac{\beta+b^2}{\beta b} - \frac{\beta}{b}\frac{\varepsilon_{ck}}{\varepsilon_{rk}}\frac{\gamma(1-\gamma)}{\gamma-\phi} - \left(\frac{(\gamma-\phi)^2+\beta\phi^2}{\beta\phi(\gamma-\phi)}\right) < 0$$

for any $\phi \in (0, \gamma)$ if and only if

$$-\frac{\varepsilon_{ck}}{b\varepsilon_{rk}} < \frac{\gamma-\phi}{\beta\gamma(1-\gamma)}\left[\frac{(\gamma-\phi)^2+\beta\phi^2}{\beta\phi(\gamma-\phi)} - \frac{\beta+b^2}{\beta b}\right] \quad (63)$$

As the right-hand-side of (63) is a decreasing function of ϕ, we conclude that it is always satisfied if

$$-\frac{\varepsilon_{ck}}{b\varepsilon_{rk}} < \frac{1}{\beta(1-\gamma)} \equiv \varepsilon^1 \quad (64)$$

with $\varepsilon^1 > 1$. Therefore, under condition (64) there exists $\bar{\epsilon}^1_{cc} \in (0, \tilde{\epsilon}_{cc})$ such that $B < 0$ for any $\phi \in (0, \gamma)$ if $\epsilon_{cc} \in (0, \bar{\epsilon}^1_{cc})$.

Let us consider now the expression $P = 8C - 3B^2$. We derive from (62) that P is a hump-shaped function of ϕ over $(0, \gamma)$. When $\epsilon_{cc} = 0$, we get

$$\begin{array}{rl} C = & -\frac{1+\beta}{b}\frac{\varepsilon_{ck}}{\varepsilon_{rk}}\frac{\gamma(1-\gamma)}{\gamma-\phi} - \frac{\beta+b^2}{\beta b}\left(\frac{(\gamma-\phi)^2+\beta\phi^2}{\beta\phi(\gamma-\phi)}\right) + \frac{2}{\beta} \\ \equiv & -\frac{1+\beta}{b}x - \frac{\beta+b^2}{\beta b}z + \frac{2}{\beta} \\ B = & \frac{\beta+b^2}{\beta b} - z - \frac{\beta}{b}x \end{array} \quad (65)$$

and

$$P \;<\; -\frac{8(1+\beta}{b}x - \left(\frac{\beta+b^2}{\beta b} + z\right)^2 - 2\left[\left(\frac{\beta+b^2}{\beta b}\right)^2 - \frac{8}{\beta} + z^2\right]$$

Straightforward computations yield $z \geq 2/\sqrt{\beta}$ and thus

$$\begin{array}{rcl} \left(\frac{\beta+b^2}{\beta b}\right)^2 - \frac{8}{\beta} + z^2 > \left(\frac{\beta+b^2}{\beta b}\right)^2 - \frac{4}{\beta} & = & \left(\frac{\beta+b^2}{\beta b} - \frac{2}{\sqrt{\beta}}\right)\left(\frac{\beta+b^2}{\beta b} + \frac{2}{\sqrt{\beta}}\right) \\ & = & \frac{(b-\sqrt{\beta})^2}{\beta b}\frac{(b+\sqrt{\beta})^2}{\beta b} > 0 \end{array}$$

for any $\phi \in (0, \gamma)$. Therefore, $P < 0$ for any $\phi \in (0, \gamma)$ when $\epsilon_{cc} = 0$ if and only if

$$-\frac{\varepsilon_{ck}}{b\varepsilon_{rk}} < \frac{\gamma-\phi}{8(1+\beta)\gamma(1-\gamma)}\left\{\left(\frac{\beta+b^2}{\beta b}+z\right)^2+2\left[\left(\frac{\beta+b^2}{\beta b}\right)^2-\frac{8}{\beta}+z^2\right]\right\} \tag{66}$$

We can show that the right-hand-side of (66) is a U-shaped function of ϕ over $(0,\gamma)$ and there exists a unique minimum value $\varepsilon^2 > 1$ such that condition (66) holds if

$$-\frac{\varepsilon_{ck}}{b\varepsilon_{rk}} < \varepsilon^2 \tag{67}$$

It follows that under condition (67) there exists $\bar{\epsilon}_{cc}^2 \in (1, \bar{\epsilon}_{cc}^1)$ such that $P < 0$ for any $\phi \in (0,\gamma)$ if $\epsilon_{cc} \in (0, \bar{\epsilon}_{cc}^2)$.

Let us consider finally R and D as given by (41) and (42). Straightforward computations yield:

$$\lim_{\phi\to 0} B = -\infty \text{ and } \lim_{\phi\to 0} C = -\infty \text{ so that } \lim_{\phi\to 0} R = -\infty \text{ and } \lim_{\phi\to 0} D = -\infty$$

and there exists $\underline{\gamma}^1 \in (0,1)$ such that when $\gamma \in (\underline{\gamma}^1, 1)$

$$\lim_{\phi\to \gamma} B = -\infty \text{ and } \lim_{\phi\to \gamma} C = -\infty \text{ so that } \lim_{\phi\to \gamma} R = -\infty \text{ and } \lim_{\phi\to \gamma} D = -\infty$$

We need now to show that there exists a subset of values of ϕ for which R and D can be positive. Let us consider the particular values $\epsilon_{cc} = 0$, and $b = -\beta$. It follows from (65) that

$$B^2 + \frac{8}{\beta} - 4C = \left(z(\phi) - x - \frac{1+\beta}{\beta}\right)^2 - \frac{8(1+\beta)x}{\beta} \equiv F(\phi)$$

with

$$z(\phi) = \frac{(\gamma-\phi)^2+\beta\phi^2}{\beta\phi(\gamma-\phi)} \text{ and } x = \frac{\varepsilon_{ck}}{\varepsilon_{rk}}\frac{\gamma(1-\gamma)}{\gamma-\phi}$$

Obviously, $F(\phi) = 0$ can be solved through the degree two polynomial

$$z(\phi) - x - \frac{1+\beta}{\beta} = 2\sqrt{\frac{2(1+\beta)x}{\beta}}$$

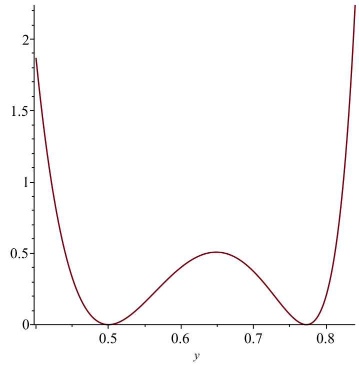
It follows therefore that there exists $\underline{\gamma}^2 \in (0,1)$ such that when $\gamma \in (\underline{\gamma}^2, 1)$ the two roots for which $F(\phi) = 0$ satisfy $\phi_1, \phi_2 \in (0,1)$. In the particular case $\gamma = 1$, these roots are indeed such that

$$\phi_1 = \frac{1}{2} \text{ and } \phi_2 = \frac{1}{1+\beta}$$

Moreover, there exists $\underline{\gamma}^3 \in (0,1)$ such that when $\gamma \in (\underline{\gamma}^3, 1)$ there is a value $\phi_3 \in (\phi_1, \phi_2)$ such that $F'(z) = 0$ when $\phi = \phi_1, \phi_2, \phi_3$. Notice indeed that in the particular case $\gamma = 1$, we have

$$\phi_3 = \frac{1}{1+\sqrt{\beta}}$$

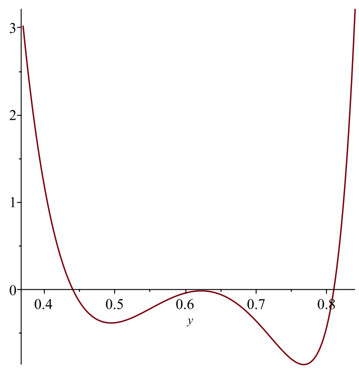
Obviously, $F(\phi) > 0$ when $\phi = \phi_3$. Note also that $\lim_{\phi\to 0} F(\phi) = \lim_{\phi\to 1} F(\phi) = +\infty$. As a result, we conclude that $F(\phi) \geq 0$ for any $\phi \in (0,1)$ with the following shape



Consider now $\gamma < 1$, $\epsilon_{cc} > 0$ and the expressions of B and C as given by (62), and let us define

$$B^2 + \frac{8}{\beta} - 4C \equiv G(\epsilon_{cc}, \gamma, \phi) \tag{68}$$

By continuity, there exists $\underline{\gamma}^4 \in (0,1)$ close to 1 and $\tilde{\phi}_3$ close to ϕ_3 such that for any given $\gamma \in (\underline{\gamma}^4, 1)$, $\partial G(\epsilon_{cc}, \gamma, \tilde{\phi}_3)/\partial\phi = 0$. Moreover, since $G(\epsilon_{cc}, \gamma, \phi)$ is a decreasing function of ϵ_{cc} with $\lim_{\epsilon_{cc}\to\bar{\epsilon}^2_{cc}} G(\epsilon_{cc}, \gamma, \tilde{\phi}_3) < 0$, we conclude that there exists $\underline{\epsilon}_{cc} \in (0, \bar{\epsilon}^2_{cc})$ such that for any given $\gamma \in (\underline{\gamma}^4, 1)$, when $\epsilon_{cc} = \underline{\epsilon}_{cc}$ and $\phi = \tilde{\phi}_3$ we have $G(\underline{\epsilon}_{cc}, \gamma, \tilde{\phi}_3) = \partial G(\underline{\epsilon}_{cc}, \gamma, \tilde{\phi}_3)/\partial\phi = 0$ such that



We conclude therefore that there exist $\bar{b} \in (-\beta, 0)$, $\underline{\phi}^c \in (0, \phi_1)$ and $\bar{\phi}^c \in (\phi_2, \gamma)$ such that if $\gamma \in (\max\{\underline{\gamma}^1, \underline{\gamma}^2, \underline{\gamma}^3, \underline{\gamma}^4\}, 1)$, $b \in (-\beta, \bar{b})$ and $\epsilon_{cc} \in (\underline{\epsilon}_{cc}, \bar{\epsilon}^2_{cc})$, then $R > 0$ when $\phi \in (\underline{\phi}^c, \bar{\phi}^c)$ and $R < 0$ when $\phi \in (0, \underline{\phi}^c) \cup (\bar{\phi}^c, \gamma)$. Considering the expression of ς as given by (19) which is a decreasing function of ϕ, we derive that there exist a corresponding values $\bar{\varsigma}^c = \varsigma(\underline{\phi}^c)$ and $\underline{\varsigma}^c = \varsigma(\bar{\phi}^c)$, and it follows that $R > 0$ when $\varsigma \in (\underline{\varsigma}^c, \bar{\varsigma}^c)$ and $R < 0$ when $\varsigma \in (0, \underline{\varsigma}^c) \cup (\bar{\varsigma}^c, +\infty)$.

Let us consider now D. We have proved that for any given $\gamma \in (\underline{\gamma}^4, 1)$, if $\epsilon_{cc} \in (0, \bar{\epsilon}^2_{cc})$ then $P < 0$ for any $\phi \in (0, \gamma)$. This implies that $-3B^2 < -8C$ and thus

$$\frac{8}{\beta} - 3B^2 + 4C \quad < \quad \frac{8}{\beta} - 4C = -4\left\{ - \frac{(1+\beta)}{b}\frac{\varepsilon_{ck}}{\varepsilon_{rk}}\frac{(1-\gamma)[\gamma-\epsilon_{cc}\phi(1-\gamma)]}{(\gamma-\phi)[1-\epsilon_{cc}(1-\gamma)]} \right.$$

$$\left. - \quad \frac{\beta+b^2}{\beta b}\left(\frac{(\gamma-\phi)^2+\beta\phi^2-\beta\phi\epsilon_{cc}(1-\gamma)(2\phi-\gamma)}{\beta\phi(\gamma-\phi)[1-\epsilon_{cc}(1-\gamma]}\right)\right\} < 0$$

It follows that if $\gamma \in (\max\{\underline{\gamma}^1, \underline{\gamma}^2, \underline{\gamma}^3, \underline{\gamma}^4\}, 1)$, $b \in (-\beta, \bar{b})$ and $\epsilon_{cc} \in (\underline{\epsilon}_{cc}, \bar{\epsilon}^2_{cc})$, then D has the same sign as R for any $\phi \in (0, \gamma)$, and the characteristic roots are complex when $\phi \in (\underline{\phi}^c, \bar{\phi}^c)$ and real when $\phi \in (0, \underline{\phi}^c) \cup (\bar{\phi}^c, \gamma)$. Moreover, when $\phi = \underline{\phi}^c$ or $\bar{\phi}^c$, $R = D = 0$. It follows therefore that the characteristic roots are complex when $\varsigma \in (\underline{\varsigma}^c, \bar{\varsigma}^c)$ and real when $\varsigma \in (0, \underline{\varsigma}^c) \cup (\bar{\varsigma}^c, \epsilon_{cc})$. Moreover, when $\varsigma = \underline{\varsigma}^c$ or $\bar{\varsigma}^c$, $R = D = 0$.

As explained in Remark 1, the polynomial (39) belongs to the class of quasi-palindromic equation and the exact solutions can be computed. Dividing $\mathcal{P}(\lambda)$ by λ^2 gives

$$\mathcal{P}(\lambda) \quad = \quad \lambda^2 + \left(\frac{1}{\lambda\beta}\right)^2 - B\left(\lambda + \frac{1}{\lambda\beta}\right) + C = 0$$

and denoting $z = \lambda + 1/(\lambda\beta)$ yields to the following degree-2 polynomial in z

$$\mathcal{P}(z) \quad = \quad z^2 - zB + C - \frac{2}{\beta}$$

The corresponding discriminant is then

$$\Delta_z = B^2 + \frac{8}{\beta} - 4C = \frac{R}{B}$$

and under the previous conditions we have $\Delta_z < 0$. The roots are then

$$z_1 = \frac{B+i\sqrt{-\frac{R}{B}}}{2} \text{ and } z_2 = \frac{B-i\sqrt{-\frac{R}{B}}}{2}$$

Plugging this into the definition of z gives the following two degree-2 polynomials in λ:

$$\lambda^2\beta - \lambda z_1\beta + 1 = 0 \text{ and } \lambda^2\beta - \lambda z_2\beta + 1 = 0$$

Denoting $\Delta_1 = (z_1\beta)^2 - 4\beta$ and $\Delta_2 = (z_2\beta)^2 - 4\beta$, straightforward computations give

$$\sqrt{\Delta_1} = \frac{\beta\left(\sqrt{\frac{B^2+\frac{R}{B}-\frac{16}{\beta}+\sqrt{\left(B^2+\frac{R}{B}-\frac{16}{\beta}\right)^2-4BR}}{2}}+i\frac{B\sqrt{\frac{-R}{B}}}{\sqrt{\frac{B^2+\frac{R}{B}-\frac{16}{\beta}+\sqrt{\left(B^2+\frac{R}{B}-\frac{16}{\beta}\right)^2-4BR}}{2}}}\right)}{2}$$

$$\sqrt{\Delta_2} = \frac{\beta\left(\sqrt{\frac{B^2+\frac{R}{B}-\frac{16}{\beta}+\sqrt{\left(B^2+\frac{R}{B}-\frac{16}{\beta}\right)^2-4BR}}{2}}-i\frac{B\sqrt{\frac{-R}{B}}}{\sqrt{\frac{B^2+\frac{R}{B}-\frac{16}{\beta}+\sqrt{\left(B^2+\frac{R}{B}-\frac{16}{\beta}\right)^2-4BR}}{2}}}\right)}{2}$$

and we finally derive the characteristic roots

$$\lambda_1 = \frac{B+\sqrt{\frac{B^2+\frac{R}{B}-\frac{16}{\beta}+\sqrt{\left(B^2+\frac{R}{B}-\frac{16}{\beta}\right)^2-4BR}}{2}}+i\sqrt{\frac{-R}{B}}\left[1+\frac{B}{\sqrt{\frac{B^2+\frac{R}{B}-\frac{16}{\beta}+\sqrt{\left(B^2+\frac{R}{B}-\frac{16}{\beta}\right)^2-4BR}}{2}}}\right]}{4}$$

$$\lambda_2 = \frac{B+\sqrt{\frac{B^2+\frac{R}{B}-\frac{16}{\beta}+\sqrt{\left(B^2+\frac{R}{B}-\frac{16}{\beta}\right)^2-4BR}}{2}}-i\sqrt{\frac{-R}{B}}\left[1+\frac{B}{\sqrt{\frac{B^2+\frac{R}{B}-\frac{16}{\beta}+\sqrt{\left(B^2+\frac{R}{B}-\frac{16}{\beta}\right)^2-4BR}}{2}}}\right]}{4}$$

$$\lambda_3 = \frac{B-\sqrt{\frac{B^2+\frac{R}{B}-\frac{16}{\beta}+\sqrt{\left(B^2+\frac{R}{B}-\frac{16}{\beta}\right)^2-4BR}}{2}}+i\sqrt{\frac{-R}{B}}\left[1-\frac{B}{\sqrt{\frac{B^2+\frac{R}{B}-\frac{16}{\beta}+\sqrt{\left(B^2+\frac{R}{B}-\frac{16}{\beta}\right)^2-4BR}}{2}}}\right]}{4}$$

$$\lambda_4 = \frac{B-\sqrt{\frac{B^2+\frac{R}{B}-\frac{16}{\beta}+\sqrt{\left(B^2+\frac{R}{B}-\frac{16}{\beta}\right)^2-4BR}}{2}}-i\sqrt{\frac{-R}{B}}\left[1-\frac{B}{\sqrt{\frac{B^2+\frac{R}{B}-\frac{16}{\beta}+\sqrt{\left(B^2+\frac{R}{B}-\frac{16}{\beta}\right)^2-4BR}}{2}}}\right]}{4}$$

with $\lambda_3 = 1/(\beta\lambda_1)$ and $\lambda_4 = 1/(\beta\lambda_2)$. The existence of a Hopf bifurcation amounts to show that the product $\lambda_1\lambda_2$ can cross the value 1 when the parameter ϕ is varied over the interval $(\underline{\phi}^c, \bar{\phi}^c)$. Obviously we get

$$\lambda_1\lambda_2 = \left(\frac{B+\sqrt{\frac{B^2+\frac{R}{B}-\frac{16}{\beta}+\sqrt{\left(B^2+\frac{R}{B}-\frac{16}{\beta}\right)^2-4BR}}{2}}}{4}\right)^2 \frac{B^2-\frac{R}{B}-\frac{16}{\beta}+\sqrt{\left(B^2+\frac{R}{B}-\frac{16}{\beta}\right)^2-4BR}}{B^2+\frac{R}{B}-\frac{16}{\beta}+\sqrt{\left(B^2+\frac{R}{B}-\frac{16}{\beta}\right)^2-4BR}}$$

By definition we know that if $\phi = \underline{\phi}^c$ or $\bar{\phi}^c$, we get $R = 0$ and thus

$$\lambda_1\lambda_2 = \left(\frac{B+\sqrt{B^2-\frac{16}{\beta}}}{4}\right)^2$$

Considering that $B < 0$, we then derive that $\lambda_1\lambda_2 < 1$ if and only if

$$B < -\frac{2(1+\beta)}{\beta} \tag{69}$$

But since $R = 0$, $B^2 = 4C - 8/\beta$ and, using (62) and assuming $b = -\beta$, inequality (72) becomes

$$\frac{\varepsilon_{ck}}{\varepsilon_{rk}}\frac{(1-\gamma)[\gamma-\epsilon_{cc}\phi(1-\gamma)]}{(\gamma-\phi)[1-\epsilon_{cc}(1-\gamma)]} + \frac{(\gamma-\phi)^2+\beta\phi^2-\beta\phi\epsilon_{cc}(1-\gamma)(2\phi-\gamma)}{\beta\phi(\gamma-\phi)[1-\epsilon_{cc}(1-\gamma]} > \frac{1+\beta}{\beta} \tag{70}$$

When $\epsilon_{cc} = 0$, this inequality becomes

$$\frac{\varepsilon_{ck}}{\varepsilon_{rk}}\frac{\gamma(1-\gamma)}{(\gamma-\phi)} + \frac{\gamma-2\phi}{\phi(1-\phi)}\frac{\gamma-\phi(1+\beta)}{\beta} > 0 \tag{71}$$

There exists $\underline{\gamma}^5 \in (0,1)$ such that when $\gamma \in (\underline{\gamma}^5, 1)$, (71) is obviously satisfied when $\phi = \underline{\phi}^c$ or $\bar{\phi}^c$. Since the left-hand-side of inequality (70) is an increasing function of ϵ_{cc}, we conclude that $\lambda_1\lambda_2 < 1$ when $\gamma \in (\max\{\underline{\gamma}^1, \underline{\gamma}^2, \underline{\gamma}^3, \underline{\gamma}^4, \underline{\gamma}^5\}, 1)$, $b \in (-\beta, \bar{b})$, $\epsilon_{cc} \in (\underline{\epsilon}_{cc}, \bar{\epsilon}_{cc}^2)$ and $\phi = \underline{\phi}^c$ or $\bar{\phi}^c$.

Tedious but straightforward computations also show that $\lambda_1\lambda_2$ is a hump-shaped function of ϕ over $(\underline{\phi}^c, \bar{\phi}^c)$. Consider the critical values $\underline{\epsilon}_c$ and $\tilde{\phi}_3$ previously defined such that when $\epsilon_{cc} = \underline{\epsilon}_{cc}$ and $\phi = \tilde{\phi}_3$ we have $G(\underline{\epsilon}_{cc}, \gamma, \tilde{\phi}_3) = \partial G(\underline{\epsilon}_{cc}, \gamma, \tilde{\phi}_3)/\partial\phi = 0$ with $G(.)$ as defined by (68). We know that $\tilde{\phi}_3$ is in a neighborhood of $\phi_3 = 1/(1+\sqrt{\beta})$. It follows that when $\epsilon_{cc} = \underline{\epsilon}_{cc}$ and $\phi = \tilde{\phi}_3$ we get again $R = 0$ and following the same argument as above we conclude that $\lambda_1\lambda_2 > 1$ if and only if

$$B > -\frac{2(1+\beta)}{\beta} \tag{72}$$

Assuming $b = -\beta$ and $\phi = \phi_3$, this inequality is approximated by

$$\frac{\varepsilon_{ck}}{\varepsilon_{rk}} \frac{(1-\gamma)[1+\sqrt{\beta}-\underline{\epsilon}_{cc}\phi(1-\gamma)]}{1-\underline{\epsilon}_{cc}(1-\gamma)} + \frac{2-\underline{\epsilon}_{cc}(1-\gamma)(1-\sqrt{\beta})}{1-\underline{\epsilon}_{cc}(1-\gamma)} < \frac{1+\beta}{\sqrt{\beta}} \tag{73}$$

When $\gamma = 1$, this inequality is obviously satisfied. Therefore, there exists $\underline{\gamma}^6 < 1$ such that $\lambda_1\lambda_2 > 1$ when $\gamma \in (\underline{\gamma}^6, 1)$, $\epsilon_{cc} = \underline{\epsilon}_{cc}$ and $\phi = \tilde{\phi}_3$. We conclude that there exists $\bar{\epsilon}_{cc}^3 \in (\underline{\epsilon}_{cc}, \bar{\epsilon}_{cc}^2]$, $\underline{\phi}^H \in (\underline{\phi}^c, \tilde{\phi}_3)$ and $\bar{\phi}^H \in (\tilde{\phi}_3, \bar{\phi}^c)$ such that when $\gamma \in (\max\{\underline{\gamma}^1, \underline{\gamma}^2, \underline{\gamma}^3, \underline{\gamma}^4, \underline{\gamma}^5, \underline{\gamma}^6\}, 1)$, $b \in (-\beta, \bar{b})$, $\epsilon_{cc} \in (\underline{\epsilon}_{cc}, \bar{\epsilon}_{cc}^3)$ then $\lambda_1\lambda_2 < 1$ when $\phi \in (\underline{\phi}^c, \underline{\phi}^H) \cup (\bar{\phi}^H, \bar{\phi}^c)$ and $\lambda_1\lambda_2 > 1$ when $\phi \in (\underline{\phi}^H, \bar{\phi}^H)$. The result follows denoting $\underline{\gamma} = \max\{\underline{\gamma}^1, \underline{\gamma}^2, \underline{\gamma}^3, \underline{\gamma}^4, \underline{\gamma}^5, \underline{\gamma}^6\}$, $\bar{\varepsilon} = \min\{\varepsilon^1, \varepsilon^2\}$ and $\bar{\epsilon}_{cc} = \min\{\bar{\epsilon}_{cc}^1, \bar{\epsilon}_{cc}^2, \bar{\epsilon}_{cc}^3\}$. Considering one more time the expression of ς as given by (19) which is a decreasing function of ϕ, we derive that there exist a corresponding values $\bar{\varsigma}^H = \varsigma(\underline{\phi}^H)$ and $\underline{\varsigma}^H = \varsigma(\bar{\phi}^H)$, and it follows that $\lambda_1\lambda_2 < 1$ when $\varsigma \in (\underline{\varsigma}^c, \underline{\varsigma}^H) \cup (\bar{\varsigma}^H, \bar{\varsigma}^c)$ and $\lambda_1\lambda_2 > 1$ when $\varsigma \in (\underline{\varsigma}^H, \bar{\varsigma}^H)$. □

8.10 Proof of Proposition 8

As shown in the proof of Proposition 1, there exists a unique steady state (k^*, d^*) solution of equations $R^* = r^*/p^* = \beta^{-1}$ and $u_d(c^*, Bd^*) = \beta u_c(c^*, Bd^*)$. Moreover, k^* does not depend on the utility function $u(c, Bd)$. Since the stationary bequest x^* is strictly positive if and only if $r^*k^* = T_k(k^*, k^*)k^* > d^*$, let us consider a particular value $d^* = \bar{d} \in (0, \min\{T_k(k^*, k^*), T_k(k^*, k^*)k^*\})$. Then, for any $\beta \in (0, 1)$, the same argument as in the proof of Proposition 1 holds: there generically exists a unique value B^* such that when $B = B^*$, $d^* = \bar{d}$ is a normalized steady state such that $x^* > 0$.

□

8.11 Proof of Proposition 9

Considering that $p_t = -T_y(k_t, k_{t+1})$, $r_{t+1} = T_k(k_{t+1}, k_{t+2})$ and $c_t = T(k_t, k_{t+1}) - d_t$, since equations (32) and (33) are identical to the two difference equations of order two given by (10), we derive that the local stability properties of the model with altruistic agents and a bequest motive are equivalent to those of the optimal growth model as described in Propositions 3, 4 and 7. From the equilibrium paths for capital $\{k_t\}_{t\geq 0}$ and second period consumption $\{d_t\}_{t\geq 0}$, the budget constraints (25) allow to derive the dynamics of bequests

$$p_t x_t = r_t k_t - d_t \tag{74}$$

and the dynamics of bequests as a proportion of GDP

$$\frac{p_t x_t}{GDP_t} = \frac{p_t x_t}{T(k_t,k_{t+1})+p_t y_t} = s(k_t, k_{t+1}) - \frac{d_t}{T(k_t,k_{t+1})-T_y(k_t,k_{t+1})k_{t+1}}. \tag{75}$$

with

$$s(k_t, k_{t+1}) = \frac{T_k(k_t,k_{t+1})k_t}{T(k_t,k_{t+1})-T_y(k_t,k_{t+1})k_{t+1}}$$

the share of capital income in GDP. If $\{k_t\}_{t\geq 0}$ and $\{d_t\}_{t\geq 0}$ are characterized by periodic or quasi-periodic dynamics, this is also true for bequests. Indeed, consider first the case of period-2 cycles which are characterized for $\{k_t, d_t\}_{t\geq 0}$ by the existence of two pairs (k_1, d_1) and (k_2, d_2) such that $(k_t, d_t) = (k_1, d_1)$ and $(k_{t+1}, d_{t+1}) = (k_2, d_2)$. It follows that a period-2 cycle also exists for bequests as $p_t x_t = T_k(k_1, k_2)k_1 - d_1$ and $p_{t+1}x_{t+1} = T_k(k_2, k_1)k_2 - d_2$. A similar argument can be applied for quasi-periodic cycles.

Similarly, since the share of capital income in GDP, $s(k_t, k_{t+1})$, and the share of consumption of old agents in GDP, d_t/GDP_t, are generically non-constant and non-equal, if $\{k_t\}_{t\geq 0}$ and $\{d_t\}_{t\geq 0}$ are characterized by periodic or quasi-periodic dynamics, this is also true for bequests as a proportion of GDP. Indeed, considering again the case of period-2 cycles for $\{k_t, d_t\}_{t\geq 0}$, it follows that a period-2 cycle also exists for bequests as a proportion of GDP as

$$\frac{p_t x_t}{GDP_t} = s(k_1, k_2) - \frac{d_1}{T(k_1,k_2)-T_y(k_1,k_2)k_2} \text{ and } \frac{p_{t+1}x_{t+1}}{GDP_{t+1}} = s(k_2, k_1) - \frac{d_2}{T(k_2,k_1)-T_y(k_2,k_1)k_1}$$

A similar argument can be applied for quasi-periodic cycles. □

8.12 Spectral density of the inheritance as a ratio of national income

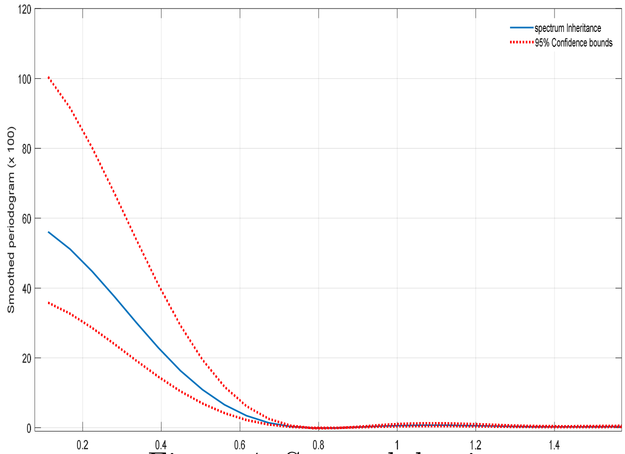


Figure 4: Spectral density

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