amse
école d'économie d'aix-marseille
aix-marseille school of economics

Working Papers / Documents de travail

Medium term endogenous fluctuations in three-sector optimal growth models

Kazuo Nishimura
Florian Pelgrin
Alain Venditti

WP 2022- Nr 35

Medium term endogenous fluctuations in three-sector optimal growth models*

Kazuo NISHIMURA

RIEB, Kobe University

Florian PELGRIN

EDHEC Business School

and

Alain VENDITTI

Aix-Marseille Univ., CNRS, AMSE

First version: *November 2019*; Revised: *July 2022*

Abstract: Following the recent contribution of Beaudry *et al.* [8], we exploit a three-sector optimal growth model without frictions to provide new insights regarding the emergence of endogenous medium-term fluctuations. Notably, our 3-sector model shows that matching the empirical evidence critically depends on agents' preferences, particularly the consumption of a bundle of (at least) two final goods. Endogenous fluctuations are therefore likely to occur through both relative inter-sector differences in capital intensity and intertemporal consumption allocations based on substitution effects between the two final consumed goods. We thoroughly characterize the economy's dynamics and establish the existence of clear conditions related to (Hopf) bifurcation values, as well as closely examining the theoretical periodicity of the corresponding limit cycles. Using a calibration of the US economy, our model is able to reproduce the observed peak range of spectral density at around 8 to 10 years of the cyclical component of gross domestic product, gross private investment, personal consumption expenditures, and of the corresponding price deflator series. Furthermore, such limit cycles are generated under very plausible technological parameters and estimates of the elasticities of intertemporal substitution.

Keywords: *Three-sector optimal growth models, mid-term fluctuations, Hopf bifurcation, endogenous cycle, periodicity*

Journal of Economic Literature Classification Numbers: C62, E32, O41.

*The project leading to this publication has received funding from the French National Research Agency (NRA) grant ANR-15-CE33-0001, from the French government under the "France 2030" investment plan managed by the NRA (reference: ANR-17-EURE-0020), and from the Excellence Initiative of Aix-Marseille University - A*MIDEX. We would like to thank N. Abad, R. Becker, J. Benhabib, F. Dufourt, N.-S. Pham, F. Ricci, X. Raurich, T. Seegmuller and Y. Vailakis for useful comments and suggestions. This paper also benefited from presentations at the "Long-run Dynamics in Economics" Workshop, March 30-April 1, 2022, University of Rouen and LERN, at the "International Conference on Public Economic Theory - PET 22", June 8-10, 2022, Marseille, and the "SAET 2022 Conference", July 16-22, 2022, Canberra.

1 Introduction

There is a long-standing tradition in macroeconomics of contrasting exogenous (business) cycles with endogenous fluctuations. On the one hand, cycles are believed to stem from exogenous forces (e.g., structural shocks). On the other hand, fluctuations can also be seen as the result of factors like intrinsic market behavior, certain coordination failures, and various information asymmetries, and endogenous cycles can clearly arise from the fundamental nonlinear structure of the economy. Notably, Benhabib and Nishimura [11] have shown in a very influential contribution that, even in standard models featuring forward-looking agents and a competitive equilibrium structure, the steady state or balanced growth path is inherently unstable, so that deterministic (endogenous) fluctuations are easily obtained once the nonlinear relationships between the aggregate variables are taken into account.[1] More recently, Beaudry *et al.* [6, 7, 8] proposed the existence of endogenous stochastic limit cycles able to generate alternating periods of booms and busts in the economy.[2]

The empirical relevance of endogenous fluctuation models has been questioned, and for good reasons. In this respect, Beaudry *et al.* [6, 7, 8] challenge the seminal contributions of Granger [32] and Sargent [61], claiming that macroeconomic variables do not display (very) pronounced peaks at business- to medium-term cycles frequencies and thus data are not supportive of strong internal boom-bust cycles.[3] Indeed, Beaudry *et al.* [6, 8] show the existence of a recurrent peak in several spectral densities of US trendless macroeconomic data, suggesting the presence of periodicities at medium term irrespective of the exogenous cyclical forces. At the very least, their results run counter to the idea that endogenous fluctuations are empirically irrelevant.[4] This suggests that critically evaluating (some) predictions of the endogenous cycle model would be worthwhile, obviously combined with the model specification and its calibration.[5]

[1] See also Becker and Foias [9] and Becker and Tsyganov [10] for the existence of endogenous cycles in one and two-sector optimal growth models with heterogeneous agents. Boldrin and Montrucchio [19] show more generally that any type of dynamic behavior, even complex chaotic fluctuations, can characterize the optimal solution of a standard optimal growth model.

[2] Other strands of the literature that discuss the emergence of limit cycles include contributions on innovation cycles and growth (Matsuyama [49], Growiec *et al.* [33]), on endogenous credit cycles in OLG models (Azariadis and Smith [3], Myerson [53], and Gu *et al.* [35]), on endogenous learning- and bounded rationality-based business fluctuations (Hommes, [37]).

[3] Comin and Gertler [26] first provide evidence of medium-term cycles (with a periodicity of between 8 and 50 years). See also Correa-López *et al.* [27] for an application to medium-term technology cycles.

[4] In the same vein, Growiec *et al.* [33] conclude that labor's share of GDP exhibits medium-run swings (see also Charpe *et al.* [23]). See also Dufourt *et al.* [28], where the Hopf bifurcation is shown to be relevant from an empirical perspective in two-sector models with productive externalities and sunspot fluctuations.

[5] This paper builds on a calibration. A more structural approach would involve simulating and estimating the multi-sector model in the presence of stochastic limit cycles (i.e., a deterministic limit cycle where the stochastic component is essentially an i.i.d. process). This could be done by determining the topological

The prime aim of this paper is therefore to closely examine whether, and under what minimal set of assumptions, a deterministic multi-sector optimal growth model without imperfections can generate endogenous fluctuations *à la* Hopf that are able to reproduce the spectral density peak range of the medium-term frequency component of some macro variables and are underpinned by well-grounded estimates of the preference and technological parameters.

Accordingly, our starting point is a canonical *three-sector* optimal growth model with only one consumption good, which assumes a constant relative risk-aversion-based utility function, Cobb-Douglas technologies, sectoral capital reallocations, and an exogenous labor supply. In particular, as outlined in the literature, the consideration of at least three sectors and of dimension-four dynamical systems is a *necessary* condition for the occurrence of deterministic endogenous fluctuations based on complex characteristic roots. Said differently, the 3-sector specification should be viewed as the simplest optimal intertemporal macroeconomic dynamic model (without any imperfection) capable of generating periodic cycles. In this respect, our first contribution is to specify preferences within our benchmark model: the (representative) household can consume two final goods (and, more generally, a bundle of goods), namely a pure consumption good and a mixed investment-consumption good. Note that such a specification is novel in the sense that traditional multi-sector (growth) models generally consider an economic environment where either there is only one pure non-durable consumption good and n sectors producing durable goods that are used as investment intermediate goods in the production of all sectors (Baxter [4]; Benhabib and Nishimura [11]; Huffman and Wynne [39]) or there are $n+1$ sectors producing durable goods that are both consumed and used as investment intermediate goods in all sectors (Acemoglu *et al.* [1], Long and Plosser, [44]). As explained below, such a departure from the canonical model is *sufficient* to provide richer endogenous cyclical dynamics and empirically relevant features.

Capitalizing on the specification of preferences and the presence of (at least) two consumption goods, our second contribution is to propose a new mechanism that generates endogenous fluctuations in a three-sector model. The main driver for the emergence of endogenous cycles has long been well-identified in the canonical model, and is based on relative capital intensity differences between sectors (e.g., the pure consumption good sector is more intensive in at least one capital good than the capital good sector itself). Because of the Rybczinski effect, these technological conditions engender oscillations of stocks and outputs,

normal form for the flip (respectively, Hopf) bifurcation using Taylor expansions (see Kuznetsov [42]) or perturbation methods (e.g., Galizia ([30])). We leave this issue for further research.

whereas the Stolper-Samuelson effect generates the corresponding fluctuations of relative capital stock prices, provided the discount rate is large enough to prevent intertemporal arbitrage opportunities. On the other hand, in the presence of a second consumption good, the (intertemporal) substitution effect between the two goods leads to an *intertemporal consumption allocation effect*, and thus a second core mechanism. Consequently, endogenous fluctuations stem from technology and preference parameters. Interestingly, departing from the exogenous labor supply assumption, the introduction of an (additively separable) endogenous labor supply strengthens these two key mechanisms and does not change the main conclusions.

Our third contribution is methodological and regards the full characterization of the dimension-4 dynamical system around the (unique) steady state in the canonical model and its extension. Indeed, as documented, although sufficient conditions for the existence of real roots are known,[6] simple clear-cut sufficient conditions for the existence of complex characteristic roots in dynamical systems larger than dimension-2 have not yet been derived in the literature.[7] Moreover, most of the available results are based on the extreme assumption of a linear utility function.[8] Our paper fills this gap by showing that conditions can be derived for the existence of complex characteristic roots and Hopf bifurcations provided the utility function is homogenous of either degree one (i.e., non strictly concave) or of a degree slightly lower than one (strictly concave). In the latter case, our results show that, in addition to the conditions required for the technological parameters, the occurrence of endogenous fluctuations in the three-sector model with one consumption good requires an excessive discount factor and extensive (possibly infinite) elasticity of intertemporal substitution in consumption in order to substitute enough consumption across periods and thereby smooth utility over time. In contrast, this condition no longer stands when there are two consumption goods, since the elasticities of intertemporal substitution remain finite. Finally, as a fundamental by-product of the characterization of the complex roots, we provide a closed-form solution for the periodicity of the endogenous cycles generated by the Hopf bifurcation in the case of a non-strictly concave utility function (e.g., a linear or Cobb-Douglas specification). In the case of a strictly concave utility function, while there is no closed-form solution

[6]Magill and Scheinkman [48] show that symmetric problems are necessarily characterized by real roots.

[7]Magill [45, 46, 47] shows that certain asymmetric stock-flow interaction terms need to arise in the local equations of motion about the equilibrium point in order to obtain complex roots.

[8]This assumption is usually justified by the conclusions of Rockafellar [59], who shows that the saddle-point property of the steady state is ensured as soon as the degree of concavity of the utility function is large enough. However, there are results with a non-linear utility function in Cartigny and Venditti [22] and Venditti [64], although they consider general multi-sector models, which makes it impossible to obtain precise conditions.

for the eigenvalues, the cycle length can still be evaluated numerically. In both cases, the periodicity depends on the imaginary part of the bifurcating eigenvalues, and is driven by the technological parameters and the two consumption goods' shares of the utility function.

Our last contribution is the empirical evaluation of the two-consumption-good three-sector model and the canonical three-sector model. We first rely on a quantitative assessment of the medium-term component of the main modeled variables, that is, gross domestic product, gross private domestic investment, and personal consumption expenditures (broken down into durables, non-durables, and services) as well as the corresponding price deflator series. Building on the recent low-frequency approach initiated by Müller and Watson ([51, 52]), we extract the business to medium-term cyclical component for each variable and make use of the spectral density to identify a (statistically) significant peak range around 8-10 years.[9] A second key ingredient is a numerical assessment based on a calibration of the US economy. Importantly, following Baxter [4] and Valentinyi and Herrendorf [63], the technological parameters and thus the sectoral capital shares are estimated using the input-output tables provided by the Bureau of Economic Analysis. This leads to four main results. First, the assumption of one consumption good yields implausible estimates of the elasticities of intertemporal substitution, a far too high discount rate, and far lower periodicity estimates compared with the 8-10-year peak range observed in the data. Second, if we assume two final consumed goods, endogenous cycles can explain medium-term fluctuations in the main detrended macroeconomic variables and reproduce the observed cycle periodicity. Moreover, the estimates of the elasticities of intertemporal substitution and the economy-wide capital share closely match the empirical evidence. Third, the saddle-point property of the steady state is restored with a very weak degree of concavity for the utility function. This result is consistent with the argument of Rockafellar [59], but it still means that a Cobb-Douglas utility function together with Cobb-Douglas technologies are capable of generating relevant endogenous fluctuations. Finally, results are remarkably consistent under an endogenous labor supply. More specifically, the bifurcation value of the elasticity of labor supply is in line with the recent macroeconomic estimates of Prescott and Wallenius [58] and Rogerson and Wallenius [60]. Overall, this provides strong support for the assumption of a mixed investment-consumption sector and thus the need for a finer decomposition of the standard consumption-investment two-sector model.

The rest of the paper is organized as follows: Section 2 provides empirical evidence for

[9] The seminal contribution of Beaudry *et al.* [8] reports similar empirical evidence. Our method closely follows theirs, the principal exception being that, since we need to deal with nonstationary variables, we do so using the approach of Müller and Watson ([51, 52]).

the existence of mid-term fluctuations in macroeconomic data. In Section 3, we present the model and the intertemporal equilibrium, we prove the existence and uniqueness of the steady state, and we provide the expression of the characteristic polynomial. In Section 4, we consider the standard formulation with one good consumed by the household, while Section 5 provides new results for the formulation with two consumption goods. In Section 6, we conduct a numerical evaluation based on a calibration of the US economy and show that our results are in line with the empirical evidence for both the parameter values and the medium-term periodicities obtained in Section 2. Section 7 concludes. All the proofs are contained in a final Appendix.

2 Medium term fluctuations in US data

This section explores some empirical properties of the main variables of our 3-sector model. Our treatment has three noteworthy features. First, we make use of the recent low-frequency approach initiated by Müller and Watson ([51, 52]) to identify and estimate the long-run component of each variable, and thus to build the corresponding cyclical component. Second, following the recent contribution of Beaudry *et al.* [8], we test the presence of a significant peak (range) on the spectral density, which provides some support for recurrent cyclical fluctuations at medium term and gives the periodicity of such cycles. At the same time, in the spirit of Beaudry *et al.* [8], it is worth emphasizing that we are only dealing with *indirect* empirical evidence on the variables of interest. Thus, the presence of a peak range does not necessarily imply strong endogenous cyclical forces, but simply suggests that the data do not rule out the existence of endogenous (stochastic) limit cycles. Third, armed with these empirical estimates of the peak range of medium-term fluctuations, we reconcile them with an optimal 3-sector model, that is, we are able to compare the periodicity identified from our data with that generated by our model.

2.1 Data

Using US quarterly data over the period 1960Q1-2020Q4, our first main objective is to study the spectral density of the cyclical component of the following macro variables: gross domestic product, gross private domestic investment, and personal consumption expenditures (broken down into durables, non-durables, and services) as well as the corresponding price deflator series. Unsurprisingly, a glance at Figure 1 clearly suggests that all of these variables are nonstationary, as the existence of a deterministic and/or stochastic trend cannot be ruled out for all variables. This is further confirmed by conventional unit root tests. Such nonstationarity is challenging, since spectral analysis requires (second-order) stationarity for the series, suggesting the need for a detrending procedure, which avoids spurious cycles, to isolate the long-run component.[10]

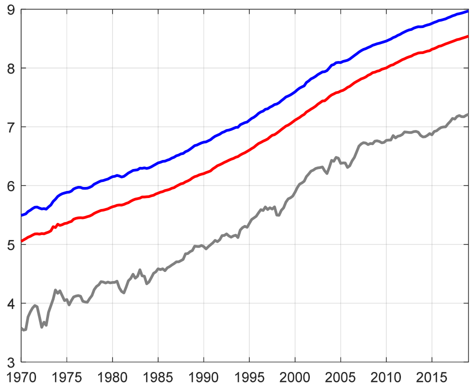

Figure 1: Gross domestic product, gross private domestic investment, and personal consumption expenditures

Note: The grey, blue and red lines depict the (log of) gross private domestic investment, gross domestic product, and personal consumption expenditures series, respectively.

Assuming that initial series can be thought as the standard sum of a (weakly) stationary component and a nonstationary (say, of order 1) component, the cyclical component (respectively, the long-run component) is captured by the stationary component (respectively, the nonstationary component).[11] Accordingly, the use of spectral density to identify a medium-term peak range entails first extracting the long-run component and then computing the cyclical component. Before discussing the estimation of the two components, it is worth emphasizing that (stationary) time series are generally split into three ranges in the frequency domain. Indeed, the general consensus is that the periodicity of high or business cycle frequencies is below 6-8 years, whereas that of medium-term frequencies is between 6-8 and 50 years, and that of low or long-run frequencies is above 50 years.[12]

2.2 The long-run and cyclical component

We now proceed to estimate the long-run component and cyclical component. As proposed by Müller and Watson ([51, 52]), we extract the long-run sample information after isolating

[10] A short overview of spectral analysis is provided in Appendix 1.

[11] Note that we are interested in the cyclical component of the level variables and thus do not proceed with a first-difference (log-) transformation.

[12] Comin and Gertler [26] define medium-term business cycles as all cyclical fluctuations between 2 and 50 years, and then break these cycles down into a high-frequency component (below 8 years) and a low-frequency component (between 8 and 50 years).

a small number of low-frequency trigonometric weighted averages.[13]

Before presenting their regression-based filter, note that one key feature of the low-frequency approach of Müller and Watson ([51, 52]) relative to bandpass and moving average filters is its applicability beyond the (weak) stationarity I(0) assumption.[14]

Let $\{x_t, t = 1, \cdots, T\}$ denote a (scalar) time series, $\Psi(s) = [\Psi_1(s), \cdots, \Psi_q(s)]'$ denote a $\mathbb{R}^q$-valued function with $\Psi_j(s) = \sqrt{2}cos(js\pi)$, and

$$\Psi_T = \left[\Psi\left(\tfrac{1-0.5}{T}\right), \Psi\left(\tfrac{2-0.5}{T}\right), \cdots, \Psi\left(\tfrac{T-0.5}{T}\right)\right]'$$

denote the $T \times q$ matrix after evaluating $\Psi(.)$ at $s = \frac{t-0.5}{T}$, for $t = 1, \cdots, T$. The low-frequency projection is the fitted series from the OLS regression of $[x_1, \cdots, x_T]$ onto a constant and Ψ_T. In so doing, we project the series into a constant and eight (q) cosine functions with periods $\frac{2T}{j}$ for $j = 1, \cdots, 8$ in order to capture the variability for periods longer than 40 years $(2T/q)$.[15] This provides the long-run component, denoted $x_t^{\mathbf{lr}}$, and thus the cyclical component is $x_t - x_t^{\mathbf{lr}}$, for all $t = 1, \cdots, T$.

The left panel of Figure 2 displays the cyclical component of each variable (gross domestic product, domestic private investment, and personal consumption expenditures), while the right panel shows those of the corresponding implicit price deflators. As expected, all the filtered series resemble mean-reverting processes, which is further confirmed by the two (local) point-optimal unit-root tests of Müller and Watson [51], as well as the DF-GLS unit-root test of Elliott, Rothenberg, and Stock [29], and the stationarity test of Nyblom [54].[16]

[13]We also apply the Christiano and Fitzgerald ([25, 24]) approximation of the ideal band-pass filter, since it rests on the assumption that data are generated by a random-walk process. We also compute the long-run component using the HP filter. However, as shown by Phillips and Jin [56], the properties of the HP-filtered series heavily depend on the choice of smoothing parameter (say λ) that trades off the cyclical and (stochastic) trend component. More specifically, when $\lambda = O(T^4)$, where T is the sample size, the HP filter does remove the stochastic trend in the limit (i.e. as $T \to \infty$), hence explaining some "spurious cycle" effects of the HP filter. On the other hand, when $\lambda = o(T)$, the HP filter eliminates the stochastic trend. Monte-Carlo simulations show that the $\lambda = O(T^4)$ limit theory is more likely to hold in applied macroeconomics, and thus the HP long-run component can be very noisy, especially in the presence of a unit root. Finally, we also transform our data using the first difference operator and apply the Baxter and King [5] filter. This requires in turn combining the filtered series. However, the initial first-difference transformation generally emphasizes movements at higher frequencies (respectively, de-emphasizes those at lower frequencies). Overall, the approach of Müller and Watson [51] is preferable as it remains valid in both the stationary and the nonstationary case (using the appropriate limit theory).

[14]For an extensive discussion of the relationship between this approach and spectral analysis, the scarcity of low-frequency information, and the relevance of the approximation using a small q, see Müller and Watson [51].

[15]As a sensitivity analysis, we also try different upper bounds for q. All in all, our results remain robust. Detailed results are available upon request.

[16]We implement the so-called LFST and LFUR point-optimal tests, which respectively test the I(0) and I(1) null hypotheses. All detailed results and codes are available upon request.

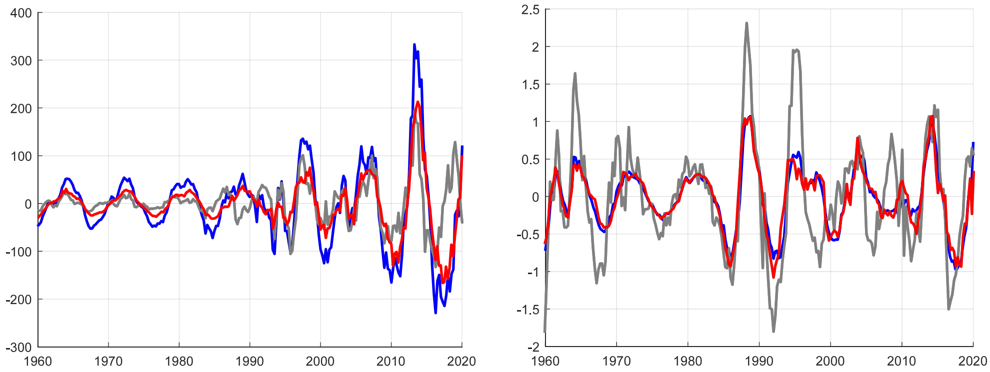

Figure 2: Cyclical component

Note: The grey, blue and red lines on the left panel (resp., right panel) depict the cyclical component of gross domestic product (resp., GDP deflator), gross private domestic investment (resp., implicit investment price deflator), and personal consumption expenditures (resp., the corresponding price deflator), respectively. The cyclical component is the difference between the raw series and the long-run component. The long-run component is estimated using the cosine-based approach of Müller and Watson ([52]).

2.3 The medium-term cyclical component

We now study the spectral density of the cyclical component of each variable. More specifically, we highlight in light (resp., dark) grey the band of frequencies corresponding to periodicities from 8 to 32 quarters (resp., 32 to 50 quarters) in Figure 3.

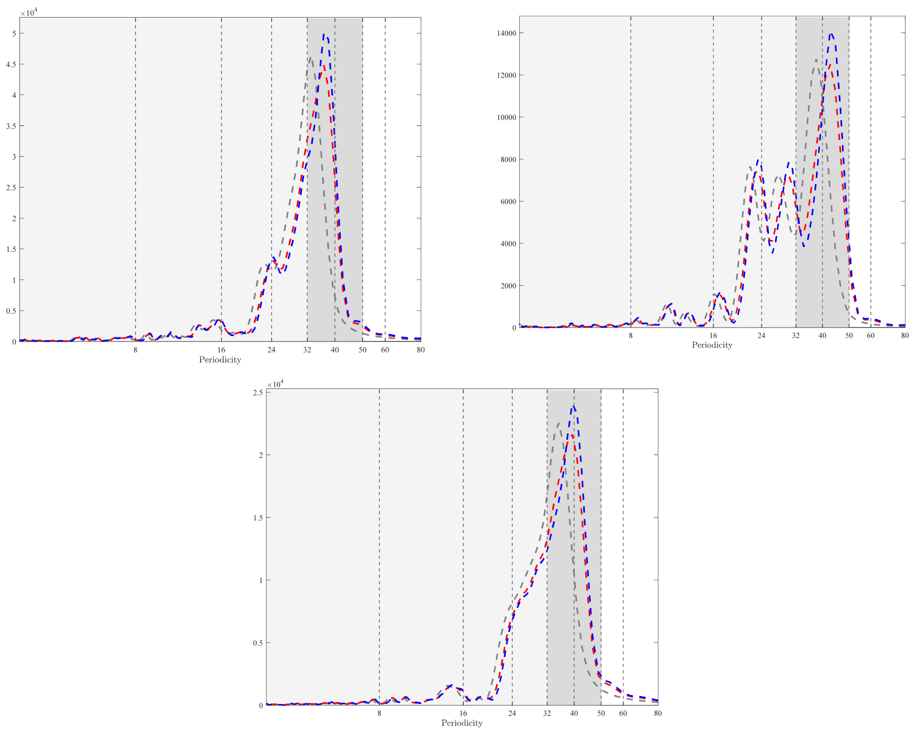

Figure 3: Periodogram

Note: The grey, red, and blue dotted lines represent the spectral density using, respectively the Hanning, Parsen, and Blackman-Tukey smoothing method on gross domestic product (top left panel), domestic private investment (top right panel), and personal consumption expenditures (bottom panel).

Irrespective of smoothing method (Hanning, Parsen or Blackman-Tukey), one dominant feature is the distinct hump in the spectral density surrounding the local peak at around 32 to 40 quarters.[17] This suggests that the three variables exhibit important recurrent cyclical phenomena at approximately 8- to 10-year intervals. Unsurprisingly, Figure 3 also shows that business fluctuations (or higher frequency movements) occur, since all spectral densities display a *local* peak within the 20-32 quarter range or are characterized by a significant contribution of this frequency range to the unconditional variance. These results are fully consistent with those outlined by Beaudry *et al.* [8]. To go one step further, we also consider the spectral densities of the implicit price deflator for each (cyclical) level variable (Figure 4).

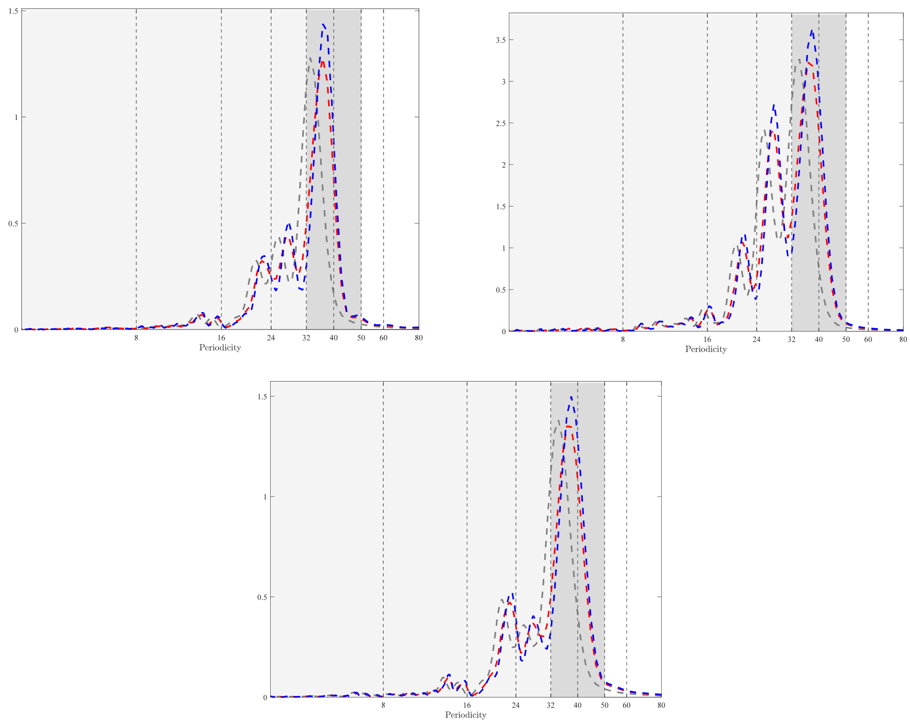

Figure 4: Periodogram

Note: The grey, red, and blue dotted lines represent spectral density using, respectively the Hanning, Parsen, and Blackman-Tukey smoothing method for the price deflator of gross domestic product (top left panel), domestic private investment (top right panel), and personal consumption expenditures (bottom panel).

Our main conclusions remain robust, showing the existence of major recurrent cyclical phenomena at approximately 8 to 10-year intervals. Interestingly, our results show that the medium-term characterization of most stationary level (macro) variables proposed by Beaudry *et al.* [8] is also strongly supported when nonstationary macro variables (e.g., gross domestic product and its price deflator) are appropriately detrended.

[17]More technical details are provided in Appendix 1.

In the same spirit as Beaudry *et al.* [8], we formally test for the presence of a shape restriction on the spectral density: we consider a "peak range" for 32-50 quarters and test the null hypothesis of a flat spectral density against a "peak range".[18] We strongly reject at 5 percent level the notion that the spectrum is flat in the "peak range". This result is robust over a narrow "peak range". Consequently, after filtering for the long-run component, the adjusted variables are predominantly dominated by medium-term fluctuations—local peaks being characteristic of traditional (short-run) business cycles. Taking into account the sampling uncertainty of the periodogram estimate, we conclude that the peaks for all the variables correspond to a periodicity of 8 to 10 years. This range is taken as a benchmark to assess the predictions of our model in Section 6.

3 The model

This section first describes the production structure and the preferences of our 3-sector model. Then we discuss the intertemporal equilibrium and the steady state. Finally we derive the characteristic polynomial associated with linearization around the steady state.

3.1 The production environment

We consider an economy producing a pure consumption good y_0, a mixed good y_1 that is both consumed and used as an investment good, and a pure investment good y_2 (Assumption A.1). While the modeling of pure and mixed consumption goods in infinite horizon models has been studied in papers looking at the existence of unique/multiple steady states,[19] this assumption is novel in the analysis of endogenous cycles. Indeed, we will show that considering the existence of a mixed investment-consumption good sector gives rise to a new generating mechanism for endogenous fluctuations. This latter rests on an intertemporal consumption arbitrage because the two consumption goods are included in the specification of the utility function. Traditional multi-sector (growth) models have generally specified an economic environment with (i) only one pure non-durable consumption good and n sectors producing durable goods that are used as investment intermediate goods in the production of all sectors (Baxter [4]; Benhabib and Nishimura [11]; Huffman and Wynne [39]) or (ii) $n + 1$ sectors

[18] The null hypothesis can be viewed as an uninformative prior for the spectral density at that periodicity range. Following Beaudry *et al.* [8] and the traditional view of a bell-shaped spectral density for most macroeconomic variables at low frequencies (Granger, [32]), we also test the null hypothesis that the spectral density at a given frequency interval is inherited from the determination of the cyclical component of a persistent autoregressive process of order 1. At standard significance levels, we reject this null hypothesis. Detailed results and codes are available upon request.

[19] See for instance Brock [21], Benhabib and Nishimura [15].

producing durable goods that are both consumed and used as investment intermediate goods in all sectors (Acemoglu *et al.* [1], Long and Plosser, [44]). In contrast, we combine these two formulations with a pure consumption good, a mixed consumption/investment good and a pure investment good. Meanwhile, we do not consider intratemporal adjustment costs, while allowing for sectoral capital reallocations.[20]

Each good is assumed to be produced by using capital k_1^j, k_2^j and labor l^j, $j = 0, 1, 2$ in different proportions *via* Cobb-Douglas production functions (Assumption A.2):

$$\begin{aligned} y_0 &= A_0(k_1^0)^{\alpha_1}(k_2^0)^{\alpha_2}(l^0)^{1-\alpha_1-\alpha_2}, \\ y_1 &= A_1(k_1^1)^{\beta_1}(k_2^1)^{\beta_2}(l^1)^{1-\beta_1-\beta_2}, \\ y_2 &= A_2(k_1^2)^{\gamma_1}(k_2^2)^{\gamma_2}(l^2)^{1-\gamma_1-\gamma_2}, \end{aligned} \tag{1}$$

where A_j denotes the total factor productivity of sector $j = 0, 1, 2$. Total labor is given by $1 = l^0 + l^1 + l^2$, and total stocks of capital are given by $k_1 = k_1^0 + k_1^1 + k_1^2$ and $k_2 = k_2^0 + k_2^1 + k_2^2$. We further assume that labor is exogenous (Assumption A.3). While endogenous labor supply obviously matters for the specification of preferences and further enriches the three-sector model dynamics, it does not alter the conclusions regarding the occurrence of bifurcations.[21] Meanwhile, note that we relax this assumption in Section 5 (see Remark 1) and provide some numerical results in Section 6 (see Remark 2).

A firm in each industry maximizes its profit under output prices p_0, p_1 and p_2, rental rates of capital r_1 and r_2, and wage rate w. Choosing the consumption good as the *numéraire*, i.e. $p_0 = 1$, the first-order conditions subject to the technologies (1) give the following input coefficients:

$$\begin{aligned} &a_{00}(w) = \tfrac{l^0}{y_0} = \tfrac{1-\alpha_1-\alpha_2}{w}, && a_{10}(r_1) = \tfrac{k_1^0}{y_0} = \tfrac{\alpha_1}{r_1}, && a_{20}(r_2) = \tfrac{k_2^0}{y_0} = \tfrac{\alpha_2}{r_2} \\ &a_{01}(w,p_1) = \tfrac{l^1}{y_1} = \tfrac{p_1(1-\beta_1-\beta_2)}{w}, && a_{11}(r_1,p_1) = \tfrac{k_1^1}{y_1} = \tfrac{p_1\beta_1}{r_1}, && a_{21}(r_2,p_1) = \tfrac{k_2^1}{y_1} = \tfrac{p_1\beta_2}{r_2} \\ &a_{02}(w,p_2) = \tfrac{l^2}{y_2} = \tfrac{p_2(1-\gamma_1-\gamma_2)}{w}, && a_{12}(r_1,p_2) = \tfrac{k_1^2}{y_2} = \tfrac{p_2\gamma_1}{r_1}, && a_{22}(r_2,p_2) = \tfrac{k_2^2}{y_2} = \tfrac{p_2\gamma_2}{r_2}. \end{aligned} \tag{2}$$

Each coefficient a_{ij} represents the amount of "good" i, that is, labor or intermediate capital good 1 or 2, that it takes to produce one unit of good j - in other words, the consumption, mixed or investment good output. Denoting $p = (1, p_1, p_2)'$ and $\omega = (w, r_1, r_2)'$, we can then define the following matrix of input coefficients

[20] For instance, in a two-sector model, Huffman and Wynne ([39]) assume that the aggregate investment in the second sector that produces a durable investment good is a linear combination or a CES of investment in the two sectors.

[21] Bosi *et al.* [20] show that the characterization of the flip bifurcation values in a two-sector optimal growth model with endogenous labor is fundamentally the same as in an optimal growth model with exogenous labor after taking into consideration the labor elasticities. See additional comments in Sections 5 and 6.

$$\mathcal{A}(\omega,p)=\begin{pmatrix} a_{00}(w) & a_{01}(w,p_1) & a_{02}(w,p_2) \\ a_{10}(r_1) & a_{11}(r_1,p_1) & a_{12}(r_1,p_2) \\ a_{20}(r_2) & a_{21}(r_2,p_1) & a_{22}(r_2,p_2) \end{pmatrix}$$

which can basically be obtained from input-output tables available in national accounting data.

Using the results of Benhabib and Nishimura [11], and as stated in Lemma 1 and Lemma 2, the factor-price frontier and the factor market-clearing equations depend on this matrix.

Lemma 1. *$p = \mathcal{A}'(\omega,p)\omega$ and $dp = \mathcal{A}'(\omega,p)d\omega$.*

Lemma 2. *Denote $x = (1, k_1, k_2)'$ and $y = (y_0, y_1, y_2)'$. Then $\mathcal{A}(\omega,p)y = x$ and*

$$\mathcal{A}(w,p)dy+\begin{pmatrix} \left(\frac{\partial a_{00}}{\partial w}+\frac{\partial a_{01}}{\partial w}+\frac{\partial a_{02}}{\partial w}\right)dw+\frac{\partial a_{01}}{\partial p_1}dp_1+\frac{\partial a_{02}}{\partial p_2}dp_2 \\ \left(\frac{\partial a_{10}}{\partial w}+\frac{\partial a_{11}}{\partial w}+\frac{\partial a_{12}}{\partial w}\right)dw+\frac{\partial a_{11}}{\partial p_1}dp_1+\frac{\partial a_{12}}{\partial p_2}dp_2 \\ \left(\frac{\partial a_{20}}{\partial w}+\frac{\partial a_{21}}{\partial w}+\frac{\partial a_{22}}{\partial w}\right)dw+\frac{\partial a_{21}}{\partial p_1}dp_1+\frac{\partial a_{22}}{\partial p_2}dp_2 \end{pmatrix}=dx.$$

We derive that, at equilibrium, wage rate and rental rates are functions of the output prices only, that is, $w = w(p_1, p_2)$, $r_1 = r_1(p_1, p_2)$ and $r_2 = r_2(p_1, p_2)$, while outputs are functions both of the capital stocks and the output prices, $y_j = y_j(k_1, k_2, p_1, p_2)$, $j = 0, 1, 2$.

As can be expected in multi-sector optimal growth models, there is a duality between the *Rybczinski* and *Stolper-Samuelson* effects. Indeed, denoting

$$\left[\frac{\partial y}{\partial k}\right]=\begin{pmatrix} \frac{\partial y_1}{\partial k_1} & \frac{\partial y_1}{\partial k_2} \\ \frac{\partial y_2}{\partial k_1} & \frac{\partial y_2}{\partial k_2} \end{pmatrix} \text{ and } \left[\frac{\partial r}{\partial p}\right]=\begin{pmatrix} \frac{\partial r_1}{\partial p_1} & \frac{\partial r_1}{\partial p_2} \\ \frac{\partial r_2}{\partial p_1} & \frac{\partial r_2}{\partial p_2} \end{pmatrix}$$

we have

$$\left[\frac{\partial y}{\partial k}\right]=\left[\frac{\partial r}{\partial p}\right]^t. \tag{3}$$

3.2 Intertemporal equilibrium and steady state

The economy is populated by a large number of identical infinitely-lived agents. Without loss of generality, we assume that the total population is constant and normalized to one. At each period, a representative agent inelastically supplies one unit of labor. Furthermore, utility is derived from consuming the pure consumption good c_0 and the mixed good c_1 according to the following constant relative risk-aversion-based specification (Assumption A.4):

$$u(c_0,c_1)=\frac{(c_0^{\theta}c_1^{1-\theta})^{1-\sigma}-1}{1-\sigma}$$

with $\sigma \geq 0$ and $\theta \in (0,1]$. Parameter θ measures the share of good c_0 within total utility. Three points are worthy of note. First, the specification of the utility function is sufficiently flexible to embed several interesting cases that have been studied in the multi-sector (growth) model literature, e.g. linear utility function (Shea [62]), Cobb-Douglas preferences (Acemoglu *et al.* [1]), and log utility function (Shea [62]) depending on the two parameters θ and σ. Second, only two goods are consumed, a specification we consider to be a *minimal assumption.* Note that it can be shown that adding a third consumed good leads neither to a new mechanism nor to new results relative to our benchmark model. Third, the agent's preferences imply properties of interest regarding the (pure) elasticities of intertemporal substitution in consumption goods c_0 and c_1, ε_{00} and ε_{11}, and the (cross-) elasticities of intertemporal substitution between the two goods, ε_{01} and ε_{10}:

$$\begin{aligned} &\varepsilon_{00} = -\tfrac{u_1}{u_{11}c_0} = \tfrac{1}{1-\theta(1-\sigma)},\ \varepsilon_{01} = -\tfrac{u_1}{u_{12}c_1} = -\tfrac{1}{(1-\theta)(1-\sigma)}, \\ &\varepsilon_{10} = -\tfrac{u_2}{u_{21}c_0} = -\tfrac{1}{\theta(1-\sigma)},\ \varepsilon_{11} = -\tfrac{u_2}{u_{22}c_1} = \tfrac{1}{1-(1-\theta)(1-\sigma)}. \end{aligned} \tag{4}$$

Notably, ε_{00} and ε_{11} remain finite as long as $\theta < 1$ even in the case of a non-strictly concave utility with $\sigma = 0$. This property turns out to be fundamental when we look at the predictions of our 3-sector model with two consumption goods (see Section 5 and Section 6). Of course, if $\theta = 1$ we have the standard case of a unique consumption good and the elasticity of intertemporal substitution in consumption is given by $\varepsilon_{00} = 1/\sigma$, thus being infinite when the utility is linear.

The profit maximization in both sectors described in Section 3.1 yields the demands for capital and labor as functions of the capital stocks and the production levels of the investment goods, namely $l^j = l^j(k_1, k_2, y_1, y_2)$, $k_1^j = k_1^j(k_1, k_2, y_1, y_2)$ and $k_2^j = k_2^j(k_1, k_2, y_1, y_2)$, $j = 0, 1, 2$. The optimal amount of the pure consumption good is then defined by:

$$c_0 = T(k_1, k_2, y_1, y_2) = A_0(k_1^0(k_1, k_2, y_1, y_2))^{\alpha_1}(k_2^0(k_1, k_2, y_1, y_2))^{\alpha_2}(l^0(k_1, k_2, y_1, y_2))^{1-\alpha_1-\alpha_2}.$$

From the envelope theorem, we get: $r_1 = T_{k_1}(k_1, k_2, y_1, y_2)$, $r_2 = T_{k_2}(k_1, k_2, y_1, y_2)$, $p_1 = -T_{y_1}(k_1, k_2, y_1, y_2)$ and $p_2 = -T_{y_2}(k_1, k_2, y_1, y_2)$.

The intertemporal optimization problem of the representative agent is then given by:

$$\begin{aligned} \max_{\{c_0(t), c_1(t), k_1(t), k_2(t), y_1(t), y_2(t)\}} \quad & \int_0^{+\infty} \frac{\left[c_0(t)^\theta c_1(t)^{1-\theta}\right]^{1-\sigma} - 1}{1-\sigma} e^{-\delta t} dt \\ s.t. \quad & c_0(t) = T(k_1(t), k_2(t), y_1(t), y_2(t)) \\ & \dot{k}_1(t) = y_1(t) - g k_1(t) - c_1(t) \\ & \dot{k}_2(t) = y_2(t) - g k_2(t) \\ & k_1(0), k_2(0) \ given, \end{aligned} \tag{5}$$

where $\delta \geq 0$ is the discount rate and $g > 0$ is the depreciation rate of the capital stock, which is assumed to be the same for both capital goods. Substituting the expression of the pure consumption good $c_0(t)$ into the utility function, we can write the modified Hamiltonian in current value as:

$$\mathcal{H} = \frac{\left[T(k_1(t),k_2(t),y_1(t),y_2(t))^\theta c_1(t)^{1-\theta}\right]^{1-\sigma}-1}{1-\sigma} + q_1(t)\left(y_1(t) - gk_1(t) - c_1(t)\right) + q_2(t)\left(y_2(t) - gk_2(t)\right).$$

The necessary conditions, which describe the solution to problem (5), are therefore given by the following equations:

$$\begin{aligned}
q_1(t) &= p_1(t)\theta c_0(t)^{\theta(1-\sigma)-1}c_1(t)^{(1-\theta)(1-\sigma)} && (6)\\
q_2(t) &= p_2(t)\theta c_0(t)^{\theta(1-\sigma)-1}c_1(t)^{(1-\theta)(1-\sigma)} && (7)\\
q_1(t) &= (1-\theta)c_0(t)^{\theta(1-\sigma)}c_1(t)^{(1-\theta)(1-\sigma)-1} && (8)\\
\dot{k}_1(t) &= y_1(t) - gk_1(t) - c_1(t) && (9)\\
\dot{k}_2(t) &= y_2(t) - gk_2(t) && (10)\\
\dot{q}_1(t) &= (\delta+g)q_1(t) - r_1(t)\theta c_0(t)^{\theta(1-\sigma)-1}c_1(t)^{(1-\theta)(1-\sigma)} && (11)\\
\dot{q}_2(t) &= (\delta+g)q_2(t) - r_2(t)\theta c_0(t)^{\theta(1-\sigma)-1}c_1(t)^{(1-\theta)(1-\sigma)}. && (12)
\end{aligned}$$

Taking equations (6) to (12), we are now in a position to characterize an equilibrium path $\{k_1(t), k_2(t), p_1(t), p_2(t)\}_{t\geq 0}$ and to prove the existence of a unique steady state. Indeed, as shown in Section 3.1, we have $r_1 = r_1(p_1,p_2)$, $r_2 = r_2(p_1,p_2)$, $y_j = y_j(k_1,k_2,p_1,p_2)$, $j = 1,2$, and thus $c_0 = y_0 = y_0(k_1,k_2,p_1,p_2) = T(k_1,k_2,y_1(k_1,k_2,p_1,p_2),y_2(k_1,k_2,p_1,p_2)) = c_0(k_1,k_2,p_1,p_2)$. Using (6) and (8), we derive:

$$c_1(t) = c_1(k_1(t),k_2(t),p_1(t),p_2(t)) = c_0(k_1(t),k_2(t),p_1(t),p_2(t))\frac{1-\theta}{\theta p_1(t)}. \quad (13)$$

Obviously, if $\theta = 1$, we get $c_1(t) = 0$ for any $t \geq 0$. Straightforward computations then yield:

$$\frac{\partial c_1}{\partial k_1} = \frac{1-\theta}{\theta p_1}\frac{\partial c_0}{\partial k_1}, \quad \frac{\partial c_1}{\partial k_2} = \frac{1-\theta}{\theta p_1}\frac{\partial c_0}{\partial k_2}, \quad \frac{\partial c_1}{\partial p_1} = \frac{1-\theta}{\theta p_1}\frac{\partial c_0}{\partial p_1} - \frac{c_1}{p_1}, \quad \frac{\partial c_1}{\partial p_2} = \frac{1-\theta}{\theta p_1}\frac{\partial c_0}{\partial p_2}. \quad (14)$$

Considering (6)-(12) and (14), and denoting

$$\begin{aligned}
E(k_1,k_2,p_1,p_2) &\equiv 1 - \frac{[1-\theta(1-\sigma)]\left(p_1\frac{\partial c_0}{\partial p_1}+p_2\frac{\partial c_0}{\partial p_2}\right)}{c_0} + \frac{(1-\theta)(1-\sigma)\left(p_1\frac{\partial c_1}{\partial p_1}+p_2\frac{\partial c_1}{\partial p_2}\right)}{c_1}\\
&= \theta + \sigma\left[1-\theta-\frac{1}{c_0}\left(p_1\frac{\partial c_0}{\partial p_1}+p_2\frac{\partial c_0}{\partial p_2}\right)\right] \equiv E_\sigma^\theta
\end{aligned} \quad (15)$$

the motion equations write:

$$\begin{aligned}
\dot{k}_1 &= y_1(k_1,k_2,p_1,p_2) - gk_1 - c_1(k_1,k_2,p_1,p_2)\\
\dot{k}_2 &= y_2(k_1,k_2,p_1,p_2) - gk_2\\
\dot{p}_1 &= \tfrac{1}{E(k_1,k_2,p_1,p_2)}\Big[(\delta+g)p_1 - r_1(p_1,p_2) + \sigma\tfrac{\partial c_0}{\partial p_2}\tfrac{r_1(p_1,p_2)p_2 - r_2(p_1,p_2)p_1}{c_0}\\
&+ \sigma\tfrac{p_1}{c_0}\left(\tfrac{\partial c_0}{\partial k_1}\left(y_1(k_1,k_2,p_1,p_2) - gk_1 - c_1(k_1,k_2,p_1,p_2)\right) + \tfrac{\partial c_0}{\partial k_2}\left(y_2(k_1,k_2,p_1,p_2) - gk_2\right)\right)\Big] \qquad (16)\\
\dot{p}_2 &= \tfrac{1}{E(k_1,k_2,p_1,p_2)}\Big[(\delta+g)p_2 - r_2(p_1,p_2) + \tfrac{[r_2(p_1,p_2)p_1 - r_1(p_1,p_2)p_2]\left(\sigma\frac{\partial c_0}{\partial p_1} + (1-\theta)(1-\sigma)\frac{c_0}{p_1}\right)}{c_0}\\
&+ \sigma\tfrac{p_2}{c_0}\left(\tfrac{\partial c_0}{\partial k_1}\left(y_1(k_1,k_2,p_1,p_2) - gk_1 - c_1(k_1,k_2,p_1,p_2)\right) + \tfrac{\partial c_0}{\partial k_2}\left(y_2(k_1,k_2,p_1,p_2) - gk_2\right)\right)\Big].
\end{aligned}$$

Any solution $\{k_1(t), k_2(t), p_1(t), p_2(t)\}_{t\geq 0}$ that also satisfies the transversality conditions:[22]

$$\lim_{t\to+\infty} e^{-\delta t} q_1(t)k_1(t) = 0 \text{ and } \lim_{t\to+\infty} e^{-\delta t} q_2(t)k_2(t) = 0$$

with $q_1(t)$ and $q_2(t)$ as given by (6) and (7), is called an equilibrium path. A steady state is defined by a vector $(c_1^*, k_1^*, k_2^*, p_1^*, p_2^*)$ solution of

$$\begin{aligned}
y_1(k_1,k_2,p_1,p_2) &= gk_1 + c_1 = gk_1 + c_0(k_1,k_2,p_1,p_2)\tfrac{1-\theta}{\theta p_1}\\
y_2(k_1,k_2,p_1,p_2) &= gk_2\\
r_1(p_1,p_2) &= (\delta+g)p_1\\
r_2(p_1,p_2) &= (\delta+g)p_2.
\end{aligned} \qquad (17)$$

We get the following result:

Proposition 1. *There exists a unique steady state $(c_1^*, k_1^*, k_2^*, p_1^*, p_2^*) > 0$ solution of the system of nonlinear equations (17) with $c_0^* = c_0(k_1^*, k_2^*, p_1^*, p_2^*)$ and $c_1^* = c_0(k_1^*, k_2^*, p_1^*, p_2^*)\frac{1-\theta}{\theta p_1^*}$.*

Proof. See Appendix 9.1

3.3 Characteristic polynomial

Linearizing the dynamical system around $(c_1^*, k_1^*, k_2^*, p_1^*, p_2^*)$ gives a 4×4 Jacobian matrix $\mathcal{J}$ which is provided in Appendix 9.2. Let us denote $\mathcal{T}$ the sum of minors of order one, $\mathcal{S}_\sigma$ the sum of minors of order two, Σ_σ the sum of minors of order three and $\mathcal{D}_\sigma$ the determinant of $\mathcal{J}$. Proposition 2 displays some properties of the eigenvalues of $\mathcal{J}$ and the expression of the characteristic polynomial.

Proposition 2. *Consider E_σ^θ as given by (15). If λ is an eigenvalue of the Jacobian matrix $\mathcal{J}$, then $\bar{\lambda}$, $\delta-\lambda$ and $\delta-\bar{\lambda}$ are also eigenvalues and thus*

$$\mathcal{T} = 2\delta \text{ and } \Sigma_\sigma^\theta = \mathcal{T}\frac{S_\sigma^\theta - \delta^2}{2} = \delta\left(S_\sigma^\theta - \delta^2\right).$$

The degree-4 characteristic polynomial is given by:

$$\mathcal{P}_\sigma^\theta(\lambda) = \lambda^4 - \lambda^3 2\delta + \lambda^2\mathcal{S}_\sigma^\theta - \lambda\delta\left(S_\sigma^\theta - \delta^2\right) + \mathcal{D}_\sigma^\theta \qquad (18)$$

or equivalently,

[22]See Michel [50] and Kamihigashi [40] for some proof of the necessity of the transversality condition.

$$
\begin{aligned}
\mathcal{P}_\sigma^\theta(\lambda) \; = \; & \left[\lambda^2 - \lambda \frac{\theta\left(\delta+g-\frac{\partial y_1}{\partial k_1}+\delta+g-\frac{\partial y_2}{\partial k_2}\right)+(1-\theta)\left(\delta+g-\frac{\partial y_1}{\partial k_1}-\frac{p_2}{p_1}\frac{\partial y_2}{\partial k_1}\right)}{\theta} + \frac{\left(\delta+g-\frac{\partial y_1}{\partial k_1}\right)\left(\delta+g-\frac{\partial y_2}{\partial k_2}\right)-\frac{\partial y_1}{\partial k_2}\frac{\partial y_2}{\partial k_1}}{\theta}\right] \\
\times \; & \left[\lambda^2 - \lambda\left(\frac{\partial y_1}{\partial k_1} - g - \frac{\partial c_1}{\partial k_1} + \frac{\partial y_2}{\partial k_2} - g\right) + \left(\frac{\partial y_1}{\partial k_1} - g - \frac{\partial c_1}{\partial k_1}\right)\left(\frac{\partial y_2}{\partial k_2} - g\right) - \left(\frac{\partial y_1}{\partial k_2} - \frac{\partial c_1}{\partial k_2}\right)\frac{\partial y_2}{\partial k_1}\right] \\
- \; & \frac{\sigma}{E_\sigma^\theta c_0}\tilde{\mathcal{P}}(\lambda) \\
\equiv \; & \mathcal{Q}_1^\theta(\lambda)\mathcal{Q}_2^\theta(\lambda) - \frac{\sigma}{E_\sigma^\theta c_0}\tilde{\mathcal{P}}(\lambda),
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{\mathcal{P}}(\lambda) &= \lambda^2\left[\frac{\Gamma_\theta\left[1-\frac{1}{c_0}\left(p_1\frac{\partial c_0}{\partial p_1}+p_2\frac{\partial c_0}{\partial p_2}\right)\right]}{\theta} + \Theta_\theta\right] - \lambda\delta\left[\frac{\Gamma_\theta\left[1-\frac{1}{c_0}\left(p_1\frac{\partial c_0}{\partial p_1}+p_2\frac{\partial c_0}{\partial p_2}\right)\right]}{\theta} + \Theta_\theta\right] \\
&+ \frac{\mathcal{D}_\sigma^\theta E_\sigma^\theta\left[(1-\theta)c_0-\left(p_1\frac{\partial c_0}{\partial p_1}+p_2\frac{\partial c_0}{\partial p_2}\right)\right]}{\theta} \\
\mathcal{D}_\sigma^\theta &= \frac{\left[\left(\frac{\partial y_1}{\partial k_1}-g-\frac{\partial c_1}{\partial k_1}\right)\left(\frac{\partial y_2}{\partial k_2}-g\right)-\left(\frac{\partial y_1}{\partial k_2}-\frac{\partial c_1}{\partial k_2}\right)\frac{\partial y_2}{\partial k_1}\right]\left[\left(\delta+g-\frac{\partial y_1}{\partial k_1}\right)\left(\delta+g-\frac{\partial y_2}{\partial k_2}\right)-\frac{\partial y_1}{\partial k_2}\frac{\partial y_2}{\partial k_1}\right]}{E_\sigma^\theta} \\
\mathcal{S}_\sigma^\theta &= \mathcal{S}_0^\theta - \sigma\frac{\Gamma_\theta\left[1-\frac{1}{c_0}\left(p_1\frac{\partial c_0}{\partial p_1}+p_2\frac{\partial c_0}{\partial p_2}\right)\right]+\theta\Theta_\theta}{\theta E_\sigma^\theta c_0}
\end{aligned}
\tag{19}
$$

and $\mathcal{S}_0^\theta$, Γ_θ, Θ_θ *are given in Appendix 9.2. Moreover, one of the following cases necessarily holds:*

i) the four roots are real and distinct,

ii) the four roots are given by two pairs of non-real complex conjugates,

iii) there are two real double roots.

Proof. See Appendix 9.2.

The results on the structure of the characteristic roots are in line with the conclusions of Kurz [41] and Levhari and Liviatan [43]. We do, however, provide more accurate results in the case of a non-linear utility function. It is also worth noting that $\tilde{\mathcal{P}}(\lambda)$ does not depend on σ.[23] This property is very useful when analyzing the case of a strictly concave utility function with $\sigma > 0$. It is also worthwhile to notice that when $\sigma = 0$, the characteristic polynomial $\mathcal{P}_\sigma^\theta(\lambda)$ simplifies to the product of two degree-2 polynomials.[24]

In what follows, using Proposition 2, we first consider a three-sector model with one consumption good (i.e., $\frac{\partial c_1}{\partial k_1} = \frac{\partial c_1}{\partial k_2} = 0$ and $\theta = 1$) in the presence of a linear utility function ($\sigma = 0$). Then we allow for a departure from the linear specification. This amounts to studying the characteristic polynomials $\mathcal{P}_0^1(\lambda)$ and $\mathcal{P}_\sigma^1(\lambda)$. Finally, we extend these results to the case of two consumption goods (i.e., $\theta \in (0,1)$ with $\frac{\partial c_1}{\partial k_1} \neq 0$ and $\frac{\partial c_1}{\partial k_2} \neq 0$) in the presence of a non-strictly (homogenous of degree one) concave utility function ($\sigma = 0$) or a strictly concave utility function ($\sigma > 0$). This means we are actually studying the characteristic polynomials $\mathcal{P}_0^\theta(\lambda)$ and $\mathcal{P}_\sigma^\theta(\lambda)$. To do so, we proceed by incremental small perturbations to our benchmark model (that is, the three-sector model with one consumption good and a linear utility function) using a continuity argument, and thus step-by-step append the

[23] Note also that the product $\mathcal{D}_\sigma^\theta E_\sigma^\theta$ does not depend on σ.

[24] This case is associated to a quasi-triangular Jacobian matrix (35) with $\mathcal{J}_3 = 0$.

set of conditions consistent with the existence of Hopf bifurcations and the occurrence of endogenous fluctuations.

4 The standard model with one consumption good

On top of Assumptions A.1-A.4, our benchmark model first assumes that the household consumes one pure consumption good ($\theta = 1$ and $c_1 = 0$) and preferences are specified as a linear utility function ($\sigma = 0$). We then proceed with a strictly concave utility function ($\sigma > 0$).

4.1 The case of a linear utility

In the case of a linear utility function such that $\sigma = 0$, $\theta = 1$ and $u(c_0, c_1) = c_0$, the characteristic polynomial simplifies as follows:

$$\begin{aligned} \mathcal{P}_0^1(\lambda) &= \left[\lambda^2 - \lambda\left[2(\delta+g) - \frac{\partial y_1}{\partial k_1} - \frac{\partial y_2}{\partial k_2}\right] + \left(\delta+g-\frac{\partial y_1}{\partial k_1}\right)\left(\delta+g-\frac{\partial y_2}{\partial k_2}\right) - \frac{\partial y_1}{\partial k_2}\frac{\partial y_2}{\partial k_1}\right] \\ &\times \left[\lambda^2 - \lambda\left(\frac{\partial y_1}{\partial k_1} + \frac{\partial y_2}{\partial k_2} - 2g\right) + \left(\frac{\partial y_1}{\partial k_1} - g\right)\left(\frac{\partial y_2}{\partial k_2} - g\right) - \frac{\partial y_1}{\partial k_2}\frac{\partial y_2}{\partial k_1}\right] \\ &\equiv \mathcal{Q}_1^1(\lambda)\mathcal{Q}_2^1(\lambda). \end{aligned} \quad (20)$$

Denoting Δ_i^1 the discriminant of polynomial $\mathcal{Q}_i^1(\lambda)$, $i = 1, 2$, one has:

$$\begin{aligned} \Delta_1^1 = \Delta_2^1 = \Delta^1 &= \left[\frac{\partial y_1}{\partial k_1} - \frac{\partial y_2}{\partial k_2}\right]^2 + 4\frac{\partial y_1}{\partial k_2}\frac{\partial y_2}{\partial k_1} \\ &= \left[2(\delta+g) - \frac{\partial y_1}{\partial k_1} - \frac{\partial y_2}{\partial k_2}\right]^2 - 4\left[\left(\delta+g-\frac{\partial y_1}{\partial k_1}\right)\left(\delta+g-\frac{\partial y_2}{\partial k_2}\right) - \frac{\partial y_1}{\partial k_2}\frac{\partial y_2}{\partial k_1}\right]. \end{aligned} \quad (21)$$

Complex characteristic roots are therefore obtained if and only if $\Delta^1 < 0$. Moreover, the four characteristic roots are given by:

$$\lambda_{1,2} = \frac{2(\delta+g) - \frac{\partial y_1}{\partial k_1} - \frac{\partial y_2}{\partial k_2} \pm i\sqrt{-\Delta^1}}{2}, \quad \lambda_{3,4} = \frac{\frac{\partial y_1}{\partial k_1} + \frac{\partial y_2}{\partial k_2} - 2g \pm i\sqrt{-\Delta^1}}{2}. \quad (22)$$

Capitalizing on the paper of Benhabib and Nishimura [11], we do not need to provide a complete and general stability analysis of the steady state. Instead, we focus on the case with complex characteristic roots, as our main objective is to propose more precise conditions for the occurrence of endogenous fluctuations and a Hopf bifurcation.

Proposition 3. *When $\sigma = 0$ and $\theta = 1$, the characteristic roots are complex if and only if*

$$\begin{aligned} &[\alpha_2(1-\gamma_1) - \gamma_2(1-\alpha_1) + \beta_1(1-\alpha_2) - \alpha_1(1-\beta_2)]^2 \\ <\ &4[\alpha_2(1-\beta_1) - \beta_2(1-\alpha_1)][\gamma_1(1-\alpha_2) - \alpha_1(1-\gamma_2)]. \end{aligned} \quad (23)$$

Moreover, under condition (23), the following results hold:

***1** - The steady state is saddle-point stable with oscillating convergence for any $\delta \geq 0$ if and only if*

$$\left(\frac{\alpha_2 - \gamma_2 + \alpha_1 - \beta_1 + \gamma_2\beta_1 - \gamma_1\beta_2}{\beta_1(\alpha_2 - \gamma_2) + \beta_2(\gamma_1 - \alpha_1) + \gamma_2\alpha_1 - \gamma_1\alpha_2}\right)^2 > 1$$

***2** - If the following condition holds:*

$$\frac{\alpha_2 - \gamma_2 + \alpha_1 - \beta_1 + \gamma_2\beta_1 - \gamma_1\beta_2}{\beta_1(\alpha_2 - \gamma_2) + \beta_2(\gamma_1 - \alpha_1) + \gamma_2\alpha_1 - \gamma_1\alpha_2} \in (-1, 1) \quad (24)$$

then the steady is saddle-point stable with oscillating convergence when $\delta \in [0, \delta^*)$ *and totally unstable when* $\delta > \delta^*$*, with*

$$\delta^* = g\frac{\beta_2(\gamma_1-\alpha_1)-(1-\beta_1)(\alpha_2-\gamma_2)+(1-\gamma_2)(\beta_1-\alpha_1)-\gamma_1(\alpha_2-\beta_2)}{\alpha_2(1-\gamma_1)-\gamma_2(1-\alpha_1)+\alpha_1(1-\beta_2)-\beta_1(1-\alpha_2)} > 0. \quad (25)$$

Moreover, δ^* *is a Hopf bifurcation value generically giving rise to non-constant saddle-point stable (or unstable) closed orbits around the steady state in a right (or left) neighborhood of* δ^**.*

Proof. See Appendix 9.3.

In addition to establishing the Hopf bifurcation, we need to justify the two conditions (23) and (24) from an economics point of view, and to assess whether there are plausible technologies that satisfy such conditions. Returning to conditions (23) and (24), it can be shown that the condition

$$\frac{\alpha_2-\gamma_2+\alpha_1-\beta_1+\gamma_2\beta_1-\gamma_1\beta_2}{\beta_1(\alpha_2-\gamma_2)+\beta_2(\gamma_1-\alpha_1)+\gamma_2\alpha_1-\gamma_1\alpha_2} \in (-1, 1)$$

is satisfied if

$$\frac{\partial y_1}{\partial k_1} + \frac{\partial y_2}{\partial k_2} > 0$$

but is not too large, while condition (23) will hold if

$$\frac{\partial y_1}{\partial k_2}\frac{\partial y_2}{\partial k_1} < 0 \text{ and } \frac{\partial y_1}{\partial k_1} - \frac{\partial y_2}{\partial k_2} \text{ is close enough to zero.}$$

Using the expressions of the input coefficient (2) together with Lemma 9.1 provided in Appendix 9.3, these conditions are satisfied under various sets of conditions on sectoral differences in relative capital intensity. The following two sets of conditions are of particular interest:

- **Set** 1: Sector 1 is more capital-intensive in capital good 2 than sector 0, which is itself more capital-intensive in capital good 2 than sector 2, i.e. $\frac{k_2^1}{l^1} > \frac{k_2^0}{l^0} > \frac{k_2^2}{l^2}$. Sector 0 is more capital-intensive in capital good 1 than sectors 1 and 2, i.e. $\frac{k_1^0}{l^0} > \frac{k_1^1}{l^1}, \frac{k_1^2}{l^2}$. Moreover, the weighted difference $\frac{l^2}{y^2}\left(\frac{k_2^0}{l^0} - \frac{k_2^2}{l^2}\right) - \frac{l^1}{y^1}\left(\frac{k_1^0}{l^0} - \frac{k_1^1}{l^1}\right)$ is small enough and the share of capital in sector 2, that is, $\gamma_1 + \gamma_2$ is large enough.

- **Set** 2: Sector 2 is more capital-intensive in capital good 2 than sector 0, which is itself more capital-intensive in capital good 2 than sector 1, i.e. $\frac{k_2^2}{l^2} > \frac{k_2^0}{l^0} > \frac{k_2^1}{l^1}$. Sectors 1 and 2 are more capital-intensive in capital good 1 than sector 0, i.e. $\frac{k_1^1}{l^1}, \frac{k_1^2}{l^2} > \frac{k_1^0}{l^0}$. Moreover, the weighted difference $\frac{l^2}{y^2}\left(\frac{k_2^2}{l^2} - \frac{k_2^0}{l^0}\right) - \frac{l^1}{y^1}\left(\frac{k_1^1}{l^1} - \frac{k_1^0}{l^0}\right)$ is small enough and the share of capital in sector 2, that is, $\gamma_1 + \gamma_2$ is large enough.[25]

In both cases, note that sector 0 of the pure consumption good is always more intensive in at least one capital good j than the sector of the capital good j itself. As shown by Benhabib and Nishimura [13], this property is at the core of the mechanism leading to endogenous fluctuations. Indeed, the use of the Rybczinski and Stolper-Samuelson effects provides a simple economic intuition for this result. For instance, in the case of Set 1 in which

[25] Similar conditions, though more restrictive, have been identified by Nishimura and Takahashi [55].

the pure consumption good sector is more capital-intensive in both capital stocks than the respective capital good sectors, suppose that there is an instantaneous increase in capital stocks $k_1(t)$ and $k_2(t)$. This results in two opposing mechanisms:

- On the one hand, the trade-off in production becomes more favorable to the consumption good, and the Rybczinsky effect implies a decrease in the outputs of capital goods $y_1(t)$ and $y_2(t)$. This tends to lower both investments and capital stocks in the subsequent period.

- On the other hand, in the subsequent period, the decrease in both capital stocks implies, again through the Rybczinsky effect, increased outputs of the capital goods. Such a decrease improves the production trade-off in favor of the investment goods, which are relatively less intensive in their own capital, and this tends to increase investments and capital stocks in the next period. We then obtain endogenous fluctuations of stocks and outputs.

Of course, under both mechanisms, the Stolper-Samuelson effect generates corresponding fluctuations in the relative prices of both capital stocks. But then, for persistent fluctuations to exist, the oscillations in consumption and relative prices must not present intertemporal arbitrage opportunities. Consequently, the discount rate δ needs to be at a minimum level. As clearly shown by the expression of the bifurcation value of the discount rate δ^*, for a given depreciation rate of capital g, choosing the coefficients of the Cobb-Douglas technologies appropriately would push δ^* as close to zero as desired from a theoretical point of view. We provide here a new proof of the main conclusion of Benhabib and Rustichini [17] showing that, for any positive discount rate, possibly arbitrarily close to zero, there exists a large family of standard Cobb-Douglas technologies with three sectors which have optimal growth paths of persistent cycles. We further discuss below the bifurcation value δ^* and the empirical relevance of this result.

Going one step further, under Proposition 3, there exists an approximation of the cyclical periodicity, denoted by T^* in Corollary 1.

Corollary 1. *When δ is close to the Hopf bifurcation value δ^*, the period of the closed orbit is approximately equal to $T^* = \frac{2\pi}{\sqrt{-\Delta^1}}\Big|_{\delta=\delta^*}$, where Δ^1 is defined in equation (21).*

Proof. See Appendix 9.4.

Looking at the expression of Δ^1, periodicity T^* depends on the size of the imaginary part of the bifurcating eigenvalues and thus on the technological parameters: the larger the value of $|\Delta^1|$, the shorter the period. Returning to the definitions of *Set 1* and *Set 2*, we can now interpret their implications in terms of cycle periodicity. In the case of *Set 1*, the more sector 0 is capital-intensive in capital good 1 compared to sector 2, i.e. $\frac{k_1^0}{l^0} >> \frac{k_1^2}{l^2}$, and the more sector 1 is capital-intensive in capital good 2 compared to sector 0, i.e. $\frac{k_2^1}{l^1} >> \frac{k_2^0}{l^0}$, the larger the value of $|\Delta^1|$ and thus the shorter the cycle. In the case of *Set 2*, the more sector 0 is capital-intensive in capital good 2 compared to sector 2, i.e. $\frac{k_2^0}{l^0} >> \frac{k_2^1}{l^1}$, and the more sector 2 is capital-intensive in capital good 1 compared to sector 0, i.e. $\frac{k_1^2}{l^2} >> \frac{k_1^0}{l^0}$, the larger the value of $|\Delta^1|$ and thus the shorter the cycle. Indeed, according to the aforementioned intuition, larger capital intensity differences between sector 0 and sector 1 and 2 amplify the dynamic adjustments of the capital stocks and decrease the periodicity of the cycle. To the best of our knowledge, with the exception of the theoretical discussion by Benhabib and Nishimura [11], no paper in the literature gives a precise approximation of cycle periodicity in terms of the model fundamentals, nor an interpretation of the relationship between relative sectoral capital intensities and cycle length.

4.2 The case of a strictly concave utility

Assuming now that $\sigma > 0$. Since $\theta = 1$ implies $\Gamma_1 = 0$ (see Appendix 9.2), our analysis is built on the property that the characteristic polynomial can be written as

$$\begin{aligned}
\mathcal{P}^1_\sigma(\lambda) &= \left[\lambda^2 - \lambda\left(\delta + g - \tfrac{\partial y_1}{\partial k_1} + \delta + g - \tfrac{\partial y_2}{\partial k_2}\right) + \left(\delta + g - \tfrac{\partial y_1}{\partial k_1}\right)\left(\delta + g - \tfrac{\partial y_2}{\partial k_2}\right) - \tfrac{\partial y_1}{\partial k_2}\tfrac{\partial y_2}{\partial k_1}\right] \\
&\times \left[\lambda^2 - \lambda\left(\tfrac{\partial y_1}{\partial k_1} - g + \tfrac{\partial y_2}{\partial k_2} - g\right) + \left(\tfrac{\partial y_1}{\partial k_1} - g\right)\left(\tfrac{\partial y_2}{\partial k_2} - g\right) - \tfrac{\partial y_1}{\partial k_2}\tfrac{\partial y_2}{\partial k_1}\right] \\
&- \tfrac{\sigma}{E^1_\sigma c_0}\tilde{\mathcal{P}}(\lambda) \\
&\equiv \mathcal{Q}^1_1(\lambda)\mathcal{Q}^1_2(\lambda) - \tfrac{\sigma}{E^1_\sigma c_0}\tilde{\mathcal{P}}(\lambda)
\end{aligned}$$

with

$$\tilde{\mathcal{P}}(\lambda) = \lambda^2\Theta_1 - \lambda\delta\Theta_1 - \mathcal{D}^1_\sigma E^1_\sigma\left(p_1\tfrac{\partial c_0}{\partial p_1} + p_2\tfrac{\partial c_0}{\partial p_2}\right)$$

and Θ_1 given in Appendix 9.2. Equivalently, any characteristic root λ must be a solution of

$$\mathcal{Q}^1_1(\lambda)\mathcal{Q}^1_2(\lambda) = \tfrac{\sigma}{E^1_\sigma c_0}\tilde{\mathcal{P}}(\lambda) \text{ or } \mathcal{P}^1_0(\lambda) = \tfrac{\sigma}{E^1_\sigma c_0}\tilde{\mathcal{P}}(\lambda).$$

Starting from the conditions established by Proposition 3, under which complex roots arise when $\sigma = 0$ as solutions of $\mathcal{P}^1_0(\lambda) = 0$, the strategy consists in analyzing the properties of $\tilde{\mathcal{P}}(\lambda)$ and characterizing how the right-hand side polynomial $\sigma\tilde{\mathcal{P}}(\lambda)/(E^1_\sigma c_0)$ evolves when σ increases from zero.

In this regard, we first state an important technical property, which establishes the monotonicity of $p_1\frac{\partial c}{\partial p_1} + p_2\frac{\partial c}{\partial p_2} \leq 0$ and $E^\theta_\sigma > 0$ with respect to σ:

Lemma 3. *Let $\theta \in (0, 1]$ and $\sigma \geq 0$. For any technological parameters $\alpha_i, \beta_i, \gamma_i$, $i = 1, 2$, $p_1\frac{\partial c}{\partial p_1} + p_2\frac{\partial c}{\partial p_2} \leq 0$ and $E^\theta_\sigma > 0$ is an increasing function of σ.*

Proof. See Appendix 9.5.

Then, using Lemma 3, the existence of complex characteristic roots is given in Proposition 4.

Proposition 4. *Under $\theta = 1$, let condition (24) hold and*

$$\begin{aligned}
&[\beta_2(\gamma_1 - \alpha_1) + (1 - \beta_1)(\gamma_2 - \alpha_2)]^2 + [\gamma_1(\beta_2 - \alpha_2) + (1 - \gamma_2)(\beta_1 - \alpha_1)]^2 \\
&< 2[\alpha_2(1 - \beta_1) - \beta_2(1 - \alpha_1)][\gamma_1(1 - \alpha_2) - \alpha_1(1 - \gamma_2)].
\end{aligned} \tag{26}$$

Then, there exists $\bar{\sigma} > 0$ such that the characteristic roots are complex if and only if $\sigma \in [0, \bar{\sigma})$. Moreover, there exists $\bar{\delta} > \delta^*$ such that when $\delta \in [0, \bar{\delta})$ and $\sigma = \bar{\sigma}$, the four characteristic roots are given by two pairs of real double roots with:

$$\lambda_1 = \tfrac{\delta + \sqrt{3\delta^2 - 2S^1_{\bar{\sigma}}}}{2} > 0 \text{ and } \lambda_2 = \tfrac{\delta - \sqrt{3\delta^2 - 2S^1_{\bar{\sigma}}}}{2} = \delta - \lambda_1 < 0,$$

where $S^1_{\bar{\sigma}}$ is defined in Proposition 19 when $\theta = 1$ and $\sigma = \bar{\sigma}$.

Proof. See Appendix 9.6.

Conditions (24) and (26) ensure that $\Theta_1 > 0$, while condition (26) also ensures that (23) is satisfied and that the characteristic roots are complex. Intuitively, condition (26) will be satisfied if the conditions defining the aforementioned *Set 1* or *Set 2* hold, together with the additional restrictions that capital intensity differences $\left(\frac{k_2^0}{l^0} - \frac{k_2^2}{l^2}\right)$ and $\left(\frac{k_1^0}{l^0} - \frac{k_1^1}{l^1}\right)$ are small enough.

Starting from the case $\sigma = 0$ in which all characteristic roots are complex, this allows us to conclude that increasing σ leads the degree-two polynomial $\sigma\tilde{\mathcal{P}}(\lambda)/(E_\sigma^1 c_0)$ to increase monotonically and to get closer and closer to $\mathcal{P}_0^1(\lambda)$ until it has two tangency points with $\mathcal{P}_0^1(\lambda)$. This occurs when $\sigma = \bar{\sigma}$, and thus the imaginary part of the four roots is equal to zero and there exist two pairs of double real roots, one positive and the other negative. Figure 5 displays how the polynomial $\sigma\tilde{\mathcal{P}}(\lambda)/(E_\sigma^1 c_0)$ (red curve) moves as σ increases, as well as $\mathcal{P}_0^1(\lambda)$ (blue curve) and the two tangency points (right panel).

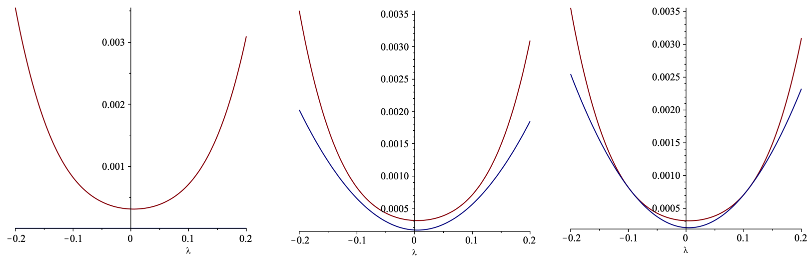

Figure 5: $\mathcal{P}_\sigma^1(\lambda)$ when $\sigma = 0$, $\sigma \in (0, \bar{\sigma})$ and $\sigma = \bar{\sigma}$

Finally, Theorem 1 establishes the existence of oscillating convergence around a stable steady state or the emergence of Hopf fluctuations.

Theorem 1. *Let $\theta = 1$ and consider the bound $\bar{\delta}$ given by Proposition 4. Under conditions (24) and (26), the following two cases hold:*

i) if $\delta \in [0, \delta^*)$, the steady state is saddle-point stable with oscillating convergence for any $\sigma \in [0, \bar{\sigma})$;

ii) if $\delta \in (\delta^*, \bar{\delta})$, there exists $\sigma^* \in (0, \bar{\sigma})$ such that the steady state is totally unstable when $\sigma \in [0, \sigma^*)$ and saddle-point stable with oscillating convergence when $\sigma \in (\sigma^*, \bar{\sigma})$. Moreover, σ^* is a Hopf bifurcation value giving rise to non-constant saddle-point stable (or unstable) closed orbits around the steady state in a left (right) neighborhood of σ^*.

Proof. See Appendix 9.7.

Theorem 1 shows that a Hopf bifurcation value for the parameter σ may exist when we start from a configuration of total instability with $\sigma = 0$. This result suggests that, for endogenous fluctuations to exist, there must be sufficiently high elasticity of intertemporal substitution in consumption to allow the representative agent to substitute consumption between periods and thus to better smooth utility over time. While this result is theoretically founded on a continuity argument, that is, the use of a small perturbation or local departure relative to the benchmark model ($\sigma = 0$), it does not say much about how close the concavity is to $\sigma = 0$. Meanwhile, as shown by Rockafellar [59], the saddle-point property is generally restored with a rather small degree of concavity for the utility function.[26] In this respect, to better evaluate δ^* and $\bar{\sigma}$, we proceed with an extensive numerical analysis as proof of concept.

Using a fine 3-dimensional simplex grid for each technology, $\{\vartheta_1, \vartheta_2, 1-\vartheta_1-\vartheta_2\}$ where $\vartheta = \alpha, \beta$, and γ, and a tiny grid for δ, we check all the conditions of Theorem 1 for each parameter configuration and find (if any) the solution σ^* by solving nonlinear equations that characterize the eigenvalues of the dynamical system.[27] Results show that the value of σ^*, which drives the existence of periodic cycles, is extremely close to zero.[28] This in turn confirms that the elasticity of intertemporal substitution is too great and, as such, empirically implausible.[29] Moreover, after calibrating the technological parameters in line with empirical estimates (see Appendix 3 and Section 6), the minimum admissible value of the discount factor, which corresponds to the bifurcation critical value δ^*, is much too large for the emergence of endogenous fluctuations. To summarize, a three-sector model with one consumption good can generate endogenous fluctuations but at the expense of an overly-large discount rate and an overly-high elasticity of intertemporal substitution or, equivalently, an overly-weak degree of concavity of the utility function. In other words, it involves a close-to-linear specification: the economy remains inherently saddle-point stable in the absence of further mechanisms.

As a final remark, since there is no closed-form solution of the eigenvalues when $\sigma > 0$, we cannot explicitly compute the periodicity of the Hopf cycles as in the case of a linear utility function. At the same time, cycle length can be evaluated numerically. As in Corollary 1, it depends on the imaginary part of the associated bifurcating eigenvalues, and is driven to a large extent by the technological parameters as opposed to preference parameter $\sigma > 0$. Note finally that when $\theta = 1$, it can be shown that the periodicity of the cycle is too short ($<<$ 8 years) compared to the empirical estimates provided in Section 2.

5 The model with two consumption goods

We now consider the more general model with two consumption goods, i.e. $\theta \in (0, 1)$. As shown by (4), even in the case of non-strictly concave utility with $\sigma = 0$, the elasticities of intertemporal substitution for the two goods ε_{00} and ε_{11} remain finite. We proceed as in Section 4, first considering the case $\sigma = 0$ with a non-strictly concave (homogeneous of degree one) utility function and then the case of a strictly concave utility function.

[26] Using a perturbation method, Benhabib and Rustichini [17] also prove that the existence of persistent cycles is compatible with a non-linear utility function. Our Theorem 1 provides a much more precise result, but we confirm that a very weak degree of concavity of the utility function is enough to restore the saddle-point property.

[27] Matlab codes are available upon request.

[28] See also Benhabib and Nishimura [15] for numerical illustrations leading to the same result.

[29] There is no empirical consensus regarding the estimate of the elasticity of intertemporal substitution (see Gruber [34]). An extensive literature has produced very mixed results. Time-series estimates of the elasticity of intertemporal substitution generally range between 1.5 and 5. Micro-data work has also produced a variety of estimates, ranging from 0.1 to Blundell *et al.*'s [18] estimates of 0.64 to 1.17. Taken as a whole, the empirical evidence clearly rules out an intertemporal elasticity of substitution value derived from a very small perturbation of σ around 0.

5.1 The case of a non-strictly concave utility

Suppose that $\sigma = 0$, hence the utility function $u(c_0, c_1) = c_0^\theta c_1^{1-\theta}$ is homogeneous of degree one and is non-strictly concave. The characteristic polynomial is then given by:

$$\begin{array}{rcl}\mathcal{P}_0^\theta(\lambda) & = & \left[\lambda^2 - \lambda \frac{\theta\left(\delta+g-\frac{\partial y_1}{\partial k_1}+\delta+g-\frac{\partial y_2}{\partial k_2}\right)+(1-\theta)\left(\delta+g-\frac{\partial y_1}{\partial k_1}-\frac{p_2}{p_1}\frac{\partial y_2}{\partial k_1}\right)}{\theta} + \frac{\left(\delta+g-\frac{\partial y_1}{\partial k_1}\right)\left(\delta+g-\frac{\partial y_2}{\partial k_2}\right)-\frac{\partial y_1}{\partial k_2}\frac{\partial y_2}{\partial k_1}}{\theta}\right] \\ & \times & \left[\lambda^2 - \lambda\left(\frac{\partial y_1}{\partial k_1} - g - \frac{\partial c_1}{\partial k_1} + \frac{\partial y_2}{\partial k_2} - g\right) + \left(\frac{\partial y_1}{\partial k_1} - g - \frac{\partial c_1}{\partial k_1}\right)\left(\frac{\partial y_2}{\partial k_2} - g\right) - \left(\frac{\partial y_1}{\partial k_2} - \frac{\partial c_1}{\partial k_2}\right)\frac{\partial y_2}{\partial k_1}\right] \\ & \equiv & \mathcal{Q}_1^\theta(\lambda)\mathcal{Q}_2^\theta(\lambda),\end{array}$$

where $\partial c_1/\partial k_1$ and $\partial c_1/\partial k_2$ are defined in (14). Let Δ_i^θ denote the discriminant of polynomial $i = 1, 2$, it is straightforward to get:

$$\begin{array}{rcl}\Delta_1^\theta = \Delta_2^\theta = \Delta^\theta & = & \frac{\left[\theta\left(\frac{\partial y_1}{\partial k_1}-\frac{\partial y_2}{\partial k_2}\right)-\frac{1-\theta}{p_1}\frac{\partial c_0}{\partial k_1}\right]^2+4\theta\left(\theta\frac{\partial y_1}{\partial k_2}-\frac{1-\theta}{p_1}\frac{\partial c_0}{\partial k_2}\right)\frac{\partial y_2}{\partial k_1}}{\theta^2} \\ & = & \frac{\left[\theta\left(\delta+g-\frac{\partial y_1}{\partial k_1}+\delta+g-\frac{\partial y_2}{\partial k_2}\right)+(1-\theta)\left(\delta+g-\frac{\partial y_1}{\partial k_1}-\frac{p_2}{p_1}\frac{\partial y_2}{\partial k_1}\right)\right]^2}{\theta^2} \\ & - & \frac{4\left[\left(\delta+g-\frac{\partial y_1}{\partial k_1}\right)\left(\delta+g-\frac{\partial y_2}{\partial k_2}\right)-\frac{\partial y_1}{\partial k_2}\frac{\partial y_2}{\partial k_1}\right]}{\theta}.\end{array} \tag{27}$$

Complex characteristic roots are therefore obtained if and only if $\Delta^\theta < 0$. Moreover, the four characteristic roots can be written as:

$$\begin{array}{rcl}\lambda_{1,2} & = & \frac{\theta\left(\delta+g-\frac{\partial y_1}{\partial k_1}+\delta+g-\frac{\partial y_2}{\partial k_2}\right)+(1-\theta)\left(\delta+g-\frac{\partial y_1}{\partial k_1}-\frac{p_2}{p_1}\frac{\partial y_2}{\partial k_1}\right)\pm\sqrt{\Delta^\theta}}{2\theta} \\ \lambda_{3,4} & = & \frac{\theta\left(\frac{\partial y_1}{\partial k_1}+\frac{\partial y_2}{\partial k_2}-2g\right)-\frac{1-\theta}{p_1}\frac{\partial c_0}{\partial k_1}\pm\sqrt{\Delta^\theta}}{2\theta}.\end{array} \tag{28}$$

Looking at the existence of complex roots and starting from Proposition 3, we now get the following results.

Proposition 5. *When $\sigma = 0$ and $\theta \in (0, 1]$, there exists $\underline{\theta} \in (0, 1)$ such that the characteristic roots are complex if and only if condition (23) holds and $\theta \in (\underline{\theta}, 1]$. Consider then the bound δ^* as given in Proposition 3. Under conditions (23) and (24), let $\theta \in (\underline{\theta}, 1]$ and*

$$\frac{\beta_2(1-\gamma_1)-\gamma_2(1-\beta_1)}{\beta_1(\alpha_2-\gamma_2)+\beta_2(\gamma_1-\alpha_1)+\gamma_2\alpha_1-\gamma_1\alpha_2} > 0. \tag{29}$$

Then the following results hold:

1 *- If $\delta > \delta^*$, the steady state is saddle-point stable with oscillating convergence when $\theta \in (\underline{\theta}, \theta^*)$ and totally unstable when $\theta \in (\theta^*, 1]$, with*

$$\theta^* = \frac{\beta_2(1-\gamma_1)-\gamma_2(1-\beta_1)}{(\beta_1-\alpha_1)(1-\gamma_2)+\gamma_1(\beta_2-\alpha_2)+\beta_2(1-\alpha_1)-\alpha_2(1-\beta_1)} \in (\underline{\theta}, 1) \tag{30}$$

Moreover, θ^* is a Hopf bifurcation value generically giving rise to non-constant saddle-point stable (or unstable) closed orbits around the steady state in a right (or left) neighborhood of θ^*.

2 *- If $\delta \in [0, \delta^*)$, there exists $\hat{\theta} \in (\theta^*, 1)$ such that the steady state is saddle-point stable with oscillating convergence when $\theta \in (\hat{\theta}, 1]$, totally unstable when $\theta \in (\theta^*, \hat{\theta})$ and again saddle-point stable when $\theta \in (\underline{\theta}, \theta^*)$. Moreover, θ^* and $\hat{\theta}$ are respectively Hopf bifurcation* *values generically giving rise to non-constant saddle-point stable (or unstable) closed orbits around the steady state in a right (or left) neighborhood of θ^* and in a left (or right) neighborhood of $\hat{\theta}$.*

Proof. See Appendix 9.8.

Proposition 5 is all the more important since it highlights that the structure of the consumption bundle is critical in 3-sector (growth) models, and more generally in n-sector (growth) models. Moreover, it opens up the possibility of reconciling the emergence of endogenous fluctuations with a finite elasticity of intertemporal substitution in consumption, even under the assumption of a non-strictly concave utility function.

Indeed, we prove that if the share of the mixed good within the utility function is large enough, a Hopf bifurcation giving rise to persistent fluctuations may arise. This only requires one additional condition (29) to be satisfied. Notably, the inequality (29) implies that bifurcation value θ^* exists and that $\partial c_0/\partial k_1 < 0$, or equivalently that sector 2 is more capital-intensive in capital good 2 than sector 1, that is, $\frac{k_2^2}{l^2} > \frac{k_2^1}{l^1}$. Interestingly, the second set of conditions (Set 2), which is characterized by the sectoral relative capital intensity inequalities $\frac{k_2^2}{l^2} > \frac{k_2^0}{l^0} > \frac{k_2^1}{l^1}$ and $\frac{k_1^1}{l^1}, \frac{k_1^2}{l^2} > \frac{k_1^0}{l^0}$, is sufficient to generate endogenous fluctuations, as in Proposition 5. Moreover, a second Hopf bifurcation value is obtained, depending on the value of the discount rate. Finally, it is worth emphasizing that the Hopf-based condition of the three-sector model with a single consumption good, $\delta > \delta^*$ (case 2 of Proposition 4 or case-i of Theorem 1), is extended to also cover the case $\delta < \delta^*$ (case 2 of Proposition 5). From the numerical evidence of Section 4, such a condition appears to be more easily satisfied and thus endogenous fluctuations are more likely to occur with consistent values of the discount factor (see also Section 6).

In addition to the technology-based explanation of endogenous fluctuations in the case of a single consumption good (Section 4), considering a second consumption good in the utility function provides another mechanism driving fluctuations. For instance, consider an instantaneous increase in capital stock $k_1(t)$. Since $\partial c_0/\partial k_1 < 0$, it follows that $c_0(t)$ decreases, and thus, building on the substitution between $c_0(t)$ and $c_1(t)$, a constant utility level can be obtained from an increase in $c_1(t)$. Using the accumulation equation of k_1 (9) and taking $y_1(t)$ as given, we conclude that if the share of consumption of the mixed good is large enough, the increase in $c_1(t)$ is such that the capital stock will decrease in the subsequent period. But then, building again on $\partial c_0/\partial k_1 < 0$, we find that $c_0(t)$ now increases, while $c_1(t)$ decreases due to the substitutability properties, hence generating an increase in capital stock in the next period. Endogenous fluctuations are thus generated from intertemporal consumption allocations based on substitution effects between the two consumption goods.

Note that if the discount rate δ is too small, the oscillations in consumption and relative prices will not present intertemporal arbitrage opportunities, provided the above mechanism is not too strong, that is, the share of consumption of the mixed good within total utility is not too large. This explains why only intermediary values of θ are compatible with persistent fluctuations in this case.

Finally, as in the case of a single consumption good (with a non-strictly concave utility function), we provide an approximation of cycle periodicity in Corollary 2.

Corollary 2. *Let $\Delta^{\widehat{\theta}}$ and Δ^{θ^*} be defined in equation (27). Using Proposition 5,*

- *If $\delta > \delta^*$, the period of the closed orbit is approximately equal to $T^* = \frac{2\pi}{\sqrt{-\Delta^{\theta^*}}}$ when θ is close to θ^*;*
- *If $\delta \in [0, \delta^*)$, the period of the closed orbit is approximately equal to $T^* = \frac{2\pi}{\sqrt{-\Delta^{\theta^*}}}$ or $\hat{T} = \frac{2\pi}{\sqrt{-\Delta^{\hat{\theta}}}}$ when θ is respectively close to θ^* or $\hat{\theta}$;*

Proof. See Appendix 9.9.

Accordingly, cycle length depends on the magnitude of the imaginary part of the bifurcating eigenvalues, and thus both on the technological parameters and on share θ: the larger the value of $|\Delta^\theta|$, the shorter the period. It can be shown here that, for given technological parameters, there exists $\tilde{\theta} \in (0, \hat{\theta})$ such that if $\theta \in (\tilde{\theta}, 1)$, $|\Delta^\theta|$ is a decreasing function of θ. It follows therefore that the lower θ, the shorter the cycle. Considering again an instantaneous increase in capital stock $k_1(t)$ which generates a decrease in $c_0(t)$, since a lower value of θ implies greater cross-elasticity ε_{10} in absolute value, the associated increase in $c_1(t)$ will be stronger, thereby generating a fast decrease in k_1 in the subsequent period. But then, under the same mechanism, the corresponding increase in $c_0(t)$ will generate a strong negative response of $c_1(t)$, thereby leading to a fast increase in k_1. Consequently, the periodicity of the cycle will decrease. This result can be used to compare the theoretical periodicity of the cycle with that obtained from data in Section 2.

5.2 The case of a strictly concave utility

We now study the robustness of Proposition 5. We assume that the utility is strictly concave with $\sigma > 0$ and proceed as in the case with $\theta = 1$. Starting from the case $\sigma = 0$ and the appropriate values of θ in which the characteristic roots are all complex, namely $\theta \in (\underline{\theta}, 1]$, if σ is increased, the degree-two polynomial $\sigma\tilde{\mathcal{P}}(\lambda)/(E^\theta_\sigma c_0)$ goes up monotonically and comes closer and closer to $\mathcal{P}^1_\theta(\lambda)$ until it is characterized when $\sigma = \bar{\sigma}$ by two tangency points with $\mathcal{P}^1_\theta(\lambda)$, where the imaginary part of the four roots is equal to zero and there exist two pairs of double real roots, one positive and the other negative (see Figure 1). We then get the following result:

Theorem 2. *Consider the bounds δ^*, $\bar{\delta}$, $\underline{\theta}$, θ^* and $\hat{\theta}$ as given respectively in Propositions 3, 4, and 5. Under conditions (24), (26) and (29), there exist $\tilde{\theta} \in [\underline{\theta}, \theta^*)$ and $\bar{\sigma}_\theta > 0$ such that for any given $\theta \in (\tilde{\theta}, 1]$, the characteristic roots are complex if and only if $\sigma \in [0, \bar{\sigma}_\theta)$. Moreover, the following cases hold:*

i) if $\delta \in [0, \delta^*)$, for any given $\theta \in (\tilde{\theta}, \theta^*) \cup (\hat{\theta}, 1]$, the steady state is saddle-point stable with oscillating convergence for any $\sigma \in [0, \bar{\sigma}_\theta)$. However, if $\theta \in (\theta^*, \hat{\theta})$, there exists $\sigma^*_\theta \in (0, \bar{\sigma}_\theta)$ such that the steady state is totally unstable when $\sigma \in [0, \sigma^*_\theta)$ and saddle-point stable with oscillating convergence when $\sigma \in (\sigma^*_\theta, \bar{\sigma}_\theta)$. Moreover, σ^*_θ is a Hopf bifurcation value giving *rise to non-constant saddle-point stable (or unstable) closed orbits around the steady state in a left (right) neighborhood of* σ_θ^*.

ii) if $\delta \in (\delta^*, \bar{\delta})$, *for any given* $\theta \in (\theta^*, 1]$, *the steady state is saddle-point stable with oscillating convergence for any* $\sigma \in [0, \bar{\sigma}_\theta)$. *However, if* $\theta \in (\tilde{\theta}, \theta^*)$, *there exists* $\sigma_\theta^* \in (0, \bar{\sigma}_\theta)$ *such that the steady state is totally unstable when* $\sigma \in [0, \sigma_\theta^*)$ *and saddle-point stable with oscillating convergence when* $\sigma \in (\sigma_\theta^*, \bar{\sigma}_\theta)$. *Moreover,* σ_θ^* *is a Hopf bifurcation value giving rise to non-constant saddle-point stable (or unstable) closed orbits around the steady state in a left (right) neighborhood of* σ_θ^*.

Proof. See Appendix 9.10.

Theorem 2, which builds on Propositions 3, 4 and 5, is a generalization of previous results. As in the case with a unique consumption good, Theorem 2 shows that the conclusions of Proposition 5 still hold in a 3-sector model with two consumption goods when the preferences are specified with a strictly concave utility function. More specifically, there exists a Hopf bifurcation value for σ. In contrast to the 3-sector model with one consumption good, endogenous fluctuations can now be reconciled with a reasonable value for the discount rate ($\delta < \delta^*$) and a plausible elasticity of intertemporal substitution in consumption. Indeed, the specification of preferences allows us to disentangle the one-to-one relationship between the degree of concavity and the elasticity of intertemporal substitution when there is one consumption good and $\theta = 1$. In particular, a (nonnegative) close-to-zero value of σ can now be associated with any finite elasticity of intertemporal substitution. To some extent, the degree of concavity is less of an issue and the key parameter is now the elasticity of intertemporal substitution. Moreover, the driving mechanism based on substitution between the two consumption goods allows us to obtain endogenous fluctuations under milder restrictions concerning the intertemporal arbitrage opportunities. Finally, note that when $\sigma > 0$, we cannot explicitly compute cycle periodicity as in the case of a 3-sector model, with one consumption good and a strictly concave utility function, and thus resort to a numerical evaluation in Section 6.

Remark 1. ***The role of endogenous labor***
So far, we have assumed that labor supply is inelastic and is not included in the specification of the utility function. To assess the robustness of Assumption A.3, we now consider that labor is endogenous and additively separable:

$$u(c_0, c_1, l) = \frac{(c_0^\theta c_1^{1-\theta})^{1-\sigma} - 1}{1-\sigma} - \frac{l^{1+\chi}}{1+\chi},$$

where $\chi \geq 0$ *drives the elasticity of labor supply* $\varepsilon_l = 1/\chi$. *Proceeding as before, two main results emerge in the presence of endogenous labor supply. First, all the results of Propositions 3 and 5 remain unaltered in the case of a non-strictly concave utility function with respect to consumption* ($\sigma = 0$), *irrespective of the value of* $\theta \in (0, 1]$. *More specifically, the conditions for the existence of a Hopf bifurcation do not depend on the elasticity of labor supply.*[30] *Second, in the case of a strictly concave utility function* ($\sigma > 0$), *for any given* $\theta \in (0, 1]$,

[30]Indeed, when $\sigma = 0$, for any given $\chi \geq 0$, the Jacobian matrix (35) provided in Appendix 9.2 is quasi-triangular and the characteristic roots are obtained as solutions of the same polynomials $\mathcal{P}_0^1(\lambda)$ or $\mathcal{P}_0^\theta(\lambda)$.

the existence of a Hopf bifurcation requires $\sigma < \sigma_\theta^*$—σ_θ^* being the same as in Theorem 2—irrespective of the value of the elasticity of labor supply ε_l. Indeed, it can be shown that for any given value $\sigma \in [0, \sigma_\theta^*)$, there exists $\varepsilon_l^* > 0$ such that the steady state is totally unstable when $\varepsilon_l \in [0, \varepsilon_l^*)$ and becomes saddle-point stable with damped oscillations when $\varepsilon_l > \varepsilon_l^*$, with ε_l^* a Hopf bifurcation value.[31] *To summarize, introducing an (additively separable) endogenous labor supply strengthens the two key mechanisms underlined by our 3-sector model with a bundle of a consumption good and a mixed good: the differences in sectoral capital intensities and the intertemporal consumption allocation induced by the substitution effect. Notably, the conditions on σ remain unchanged irrespective of the elasticity of labor supply.*[32]

6 A comparative study

This section proposes a numerical evaluation of the results of the 3-sector model with two consumption goods. We first explain the calibration of the 3-sector model. We then examine the conditions and bifurcation values associated with Proposition 5 and Theorem 2. Finally, we discuss the empirical relevance of such results by computing the corresponding elasticity of intertemporal substitution for each good and the critical values of the discount factor, and by comparing the theoretical cycle periodicity (using the expression in Corollary 2 or a numerical approximation) with the empirical evidence described in Section 2, that is, periodicities of between 8 and 10 years for the variables of interest.

6.1 Calibration

The numerical experiment requires calibrating the depreciation rate, the discount factor, and the Cobb-Douglas technological parameters (Assumption A.3). The discount factor is set at $\delta = 0.01$ and the annual rate of depreciation at 10%, that is, a standard quarterly depreciation rate of 2.5%. However, the lack of clear evidence from the literature regarding sectoral decomposition into an "aggregate" consumption sector and investment sector (2-sector model) makes it more challenging to calibrate technological parameters in a 3-sector model including a mixed investment-consumption sector. Indeed, while the assumption that entire industries are exclusively consumption or investment industries is a very useful theoretical short-cut, it is not necessarily supported by input-output tables.[33] Moreover, accounting for the final use of each output (including intermediate goods) of each industry and then building a two-sector or three-sector decomposition through an aggregation at the product level requires allocations and detailed (granular) information that are simply not available. We therefore proceed with two approaches.[34] Following the methodology of Valentinyi and Herrendorf [63], our first approach considers that the economy is composed of five "global" industries, namely agriculture, manufactured consumption, services, equipment,

[31] A proof is available upon request.

[32] Similar results are obtained in Garnier *et al.* [31] for the existence of sunspot fluctuations in a two-sector continuous-time model with sector-specific externalities.

[33] For example, as pointed out by Valentinyi and Herrendorf [63], cars sold to consumers are counted as consumption whereas they are counted as investment when sold to firms.

[34] Further details are provided in Appendix 3.

and construction, and then aggregate them to a two-sector model in a first step. The consumption sector includes manufactured consumption, agriculture, and services and the investment sector includes construction and equipment. After estimating the corresponding shares of capital (Valentinyi and Herrendorf, [63]), the second step determines some linear combinations of the technological parameters such that (i) the steady-state value of the capital share in our model, which is given by:

$$S_K = \frac{r_1 k_1 + r_2 k_2}{y_0 + p_1 y_1 + p_2 y_2}$$

is consistent with the estimate of the US economy, and (ii) the three sectoral capital shares are consistent with those of the two-sector decomposition, that is, the capital share of the aggregate consumption sector is greater than that of the aggregate investment sector (Valentinyi and Herrendorf [63]). In the latter, the disaggregation implicitly assumes that some outputs from both the consumption sector and the investment sector are reallocated to a mixed consumption-investment sector. Following Baxter [4] and Huffman and Wynne [39], our second approach categorizes the different industries according to final use of each industry's output in consumption or investment goods. Notably, we make use of the two-digit Input-Output Table of the US Bureau of Economic Analysis and thus the decomposition of sectoral output by final use in 2017. The last step is the same as in the first approach.[35] Building on these two different approaches, Table 1 displays the estimates of capital income shares (in producer prices) and a two-sector decomposition.

Table 1: Capital income shares and decomposition

Sector	Categorization	"Aggregate" industries
Investment $\alpha_{I,K}$	0.32	0.29
Consumption $\alpha_{C,K}$	0.35	0.36

In this respect, based on the second step, Table 2 reports three sets of technological parameters $\Theta_j = \{\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2\}$. While the second (respectively, third) set Θ_2 (respectively, Θ_3) is derived from classifying each industry as a consumption or investment sector (respectively, the five-aggregate-sector economy), the first set Θ_1 stems from a numerical search procedure such that cycle periodicity is in the vicinity of 9 years.[36]

Table 2: Calibration

	Sector 0 Consumption		Sector 1 Mixed		Sector 2 Investment		Two-sector aggregate Invest.	Two-sector aggregate Cons.	Total capital share
	α_1	α_2	β_1	β_2	γ_1	γ_2	$\alpha_{I,K}$	$\alpha_{C,K}$	S_K
Θ_1	0.01	0.36	0.34	0.01	0.37	0.05	0.36	0.37	0.37
Θ_2	0.01	0.36	0.30	0.01	0.30	0.08	0.32	0.37	0.35
Θ_3	0.01	0.33	0.29	0.01	0.30	0.065	0.31	0.34	0.33

[35] As a robustness check, we also classify industries into three sectors (using either a two-digit or a three-digit classification) by examining the percentage of total sector output allocated to personal consumption expenditures and private investment (Baxter [4]), thresholding the corresponding percentages and then estimating the corresponding capital shares according to Valentinyi and Herrendorf [63]. See Appendix 3.

[36] The search procedure is initialized using Θ_2 and Θ_3, whereas the discount factor and the depreciation rate are left unchanged.

The first set Θ_1 preserves the empirical result that the capital share is larger in consumption than in investment, but leads to an economy-wide capital share greater than the standard estimate of 1/3. In contrast, the two sets of technological parameters Θ_2 and Θ_3 provide very plausible empirical values for the capital shares in a two-sector consumption-investment model and at the aggregate GDP level (total capital share), as shown by the results (Table 1 p. 826) of Valentinyi and Herrendorf [63] and by our estimates (Table 1). In the following, we thus opt for the last two sets of technological parameters Θ_2 and Θ_3. As a final remark, note that the allocation between the two capital goods k_1 and k_2 (e.g., α_1 and α_2) is set arbitrarily, but satisfies the conditions of *Set 2*, that is, the investment and mixed investment-consumption sectors are more capital-intensive in capital good 1 than the consumption sector.

6.2 Results

Capitalizing on the results of Section 5 and the calibration of our 3-sector model, we first discuss the emergence of endogenous fluctuations in the presence of a non-strictly concave utility function. We then turn to the case of strictly concave preferences.

The case of a non-strictly concave utility According to Proposition 5 and Corollary 2, Table 3 displays the threshold values for the discount factor (δ^*) and the preference parameter θ^* or $\hat{\theta}$, the elasticities of intertemporal substitution for the two goods ε_{00} and ε_{11}, cycle periodicity T, and aggregate capital shares.

Table 3: Numerical results with a non-strictly concave utility

	δ^*	θ^*	ε^*_{00}	ε^*_{11}	T^*	S^*_K	$\hat{\theta}$	$\hat{\varepsilon}_{00}$	$\hat{\varepsilon}_{11}$	$\hat{T}$	$\hat{S}_K$
Θ_1	0.056	0.62	2.57	1.636	8.7	0.37	0.67	3.1	1.475	9.17	0.37
Θ_2	0.336	0.61	2.565	1.638	7.94	0.35	0.6396	2.774	1.563	8.13	0.35
Θ_3	0.109	0.668	3.01	1.497	7.83	0.33	0.709	3.437	1.141	8.07	0.33

Several points are worth commenting on. First, all general technical conditions (equations (23), (24) and (29)) are satisfied and thus we can closely examine the implications of Proposition 5. Second, since $\delta < \delta^*$ irrespective of the technological parameters, this is case 2, that is, there exist two Hopf bifurcation values θ^* and $\hat{\theta}$ and a period of closed orbit approximately equal to $T^* = 2\pi/\sqrt{-\Delta^{\theta^*}}$ and $\hat{\theta} = 2\pi/\sqrt{-\Delta^{\hat{\theta}}}$, where Δ^{θ^*} and $\Delta^{\hat{\theta}}$ are given by equation (27).[37] Third, these bifurcation values lead to plausible empirical values for elasticities of intertemporal substitution irrespective of the technological set, and are consistent with those reported in the literature (e.g., Gruber [34]). Fourth, we observe that the larger the share of capital in the pure investment good sector, the greater the periodicity of the cycle. Indeed, as discussed in Section 5, since the determinant of the imaginary part of the bifurcating eigenvalues depends on both the technological parameters and preference parameter θ, Table 3 provides evidence that cycle periodicity is less sensitive to variation

[37]Note that the critical values δ^* are the same as those of the three-sector model with one consumption good, thus providing additional support for a possible family of Cobb-Douglas parameter estimates. This is compatible with the existence of Hopf bifurcations (Benhabib and Rustichini, [17]) under a close-to-zero discount rate, but at the expense of unrealistic values for the technological parameters.

in bifurcation value θ^* or $\hat{\theta}$ than to variation in the capital share of the investment sector. Interestingly, when $\theta = \hat{\theta}$, the estimate of cycle length is in line with the empirical evidence in Section 2, that is, a periodicity of 8 to 10 years for cyclical nominal and price variables. It is important to bear in mind that assuming a one-consumption good 3-sector model with $\theta = 1$ leads to implausible estimates of the elasticity of intertemporal substitution, an overly-large discount rate ($\delta > \delta^*$),[38] and far lower periodicity estimates. Fifth, consistent with Table 1, the economy-wide capital share varies between 0.33 and 0.37 and is much larger for the first set Θ_1 than for the last two technological sets. There is an apparent trade-off between cycle periodicity and aggregate capital share. However, comparative statics provides numerical evidence that decreasing the depreciation rate generally leads to a longer cycle and thus preserves the attractive results of the two sets Θ_2 and Θ_3 without requiring an overly-large capital share for the investment sector.[39]

The case of a strictly concave utility We are now in a position to study the case of a strictly concave utility function. As reported in Table 4, provided that conditions (24), (26), and (29) are satisfied, there exist $\tilde{\theta} \in [\underline{\theta}, \theta^*)$ and $\bar{\sigma}_\theta > 0$ such that the characteristic roots are complex for any given $\theta \in (\tilde{\theta}, 1]$. Moreover, since $\delta < \delta^*$, the occurrence of endogenous Hopf fluctuations is driven by case-i of Theorem 2.

Table 4: Simulation results with a strictly concave utility

	θ	$\bar{\sigma}_\theta$	σ^*_θ	ε^*_{00}	ε^*_{11}	T^*	S^*_K
Set 1	0.65	0.1193	5×10^{-6}	2.857	1.538	9	0.37
Set 2	0.63	0.123	3.27×10^{-6}	2.7	1.587	8.07	0.35
Set 3	0.7	0.1243	2.8×10^{-6}	3.33	1.428	8.02	0.33

Using the results of Table 3 and, in particular, the two bifurcation values θ^* and $\hat{\theta}$, we choose $\theta \in (\theta^*, \hat{\theta})$ such that the steady state is totally unstable when $\sigma \in [0, \sigma^*_\theta)$ and saddle-point stable with damped oscillations when $\sigma \in (\sigma^*_\theta, \bar{\sigma}_\theta)$. Accordingly, σ^*_θ is a Hopf bifurcation value. Unsurprisingly, one key feature is that the values of σ^*_θ are nearly equal to zero irrespective of the technological set. Said differently, in an optimal multi-sector growth model without imperfections, the saddle-point property of the steady state is restored with a very weak degree of concavity for the utility function (Rockafellar [59]), and a Cobb-Douglas specification of the utility function is capable of generating relevant endogenous fluctuations. Meanwhile, as expected given the magnitude of σ^*, the values for elasticities of intertemporal substitution and cycle periodicity are quite close to those in Table 3.

Remark 2. *To complement Remark 1 and as a robustness check, we also study the implications of Hopf bifurcation values in the case of an endogenous labor supply. As explained in Remark 1, when $\sigma < \sigma^*_\theta$, there exists a Hopf bifurcation value for the elasticity of labor*

[38]As shown in Table 3, compared to the empirically relevant value $\delta = 0.01$, the critical value δ^* for each set of parameter values is far too large.

[39]Note that the empirical macroeconomics literature uses a quarterly capital depreciation rate of between 1.5% and 3%. Additional numerical results are available upon request.

supply, denoted by ε_l^* and the restrictions on σ remain the same as in the case of a strictly concave utility function (without labor supply).

Table 5: Simulation results with endogenous labor

	θ	σ	ε_l^*	ε_{00}^*	ε_{11}^*	T^*	S_K^*
Set 1	0.65	3.5×10^{-6}	1.282	2.857	1.538	9	0.37
Set 2	0.63	2×10^{-6}	2	2.7	1.587	8.07	0.35
Set 3	0.7	1.5×10^{-6}	2.857	3.33	1.428	8.02	0.33

From Table 5, since inelastic labor is associated with $\varepsilon_l = 0$, it is clear that the introduction of endogenous labor does not affect cycle length, nor any other elasticities and shares. It should however be noted that, especially with Set 3, the Hopf bifurcation value of the elasticity of labor supply is in line with the recent macroeconomic estimates of Prescott and Wallenius [58] and Rogerson and Wallenius [60].

Overall, our theoretical results, together with the numerical experiment, indicate that the most important and empirically relevant conclusions are provided by Proposition 5. We show that, under non-strictly concave utility with respect to two consumption goods, endogenous cycles may arise, explaining mid-term fluctuations in the main detrended macroeconomic variables with a periodicity of between 8 and 10 years. Moreover, all the values of the structural parameters are in line with empirical estimates and, since none of the conditions depend on the elasticity of labor supply, the value of this parameter can easily be set in line with the estimates provided by Prescott and Wallenius [58] and Rogerson and Wallenius [60].

7 Concluding comments

This paper explores the existence of periodic limit cycles to explain the mid-term periodicities of some major macroeconomic (cyclical) variables. Following Beaudry *et al.* [6, 8], we identify a significant spectral density peak range at around 8 to 10 years in US quarterly detrended data on gross domestic product, gross private investment and personal consumption expenditures, as well as the corresponding price deflators. Capitalizing on this result, we consider a 3-sector optimal growth model with a non-linear utility function. We then provide clear conditions for the existence of endogenous fluctuations through a Hopf bifurcation and characterize the theoretical cycle periodicity. Notably, we show that the assumption of one consumption good is inadequate to reproduce reliable periodicity of endogenous fluctuations and generates implausible discount rates and elasticities of intertemporal substitution. In contrast, considering two consumption goods (or a bundle of consumption goods), can circumvent these two critical issues that arise with the standard one-good two-sector (optimal) growth model. At the same time, endogenous fluctuations are likely to occur due to the relative sectoral capital intensity differences **and** the intertemporal consumption allocations based on substitution effects between the two consumption goods.

Taken together, our conclusions suggest that considering a linear approximation of the dynamical system is not sufficient to ensure locally stable limit cycles. Complex higher-order Taylor approximations need to be computed for such a result to be reached.[40] Another option is to simulate the model, in particular using the examples from Section 6. However, a discrete-time version of the three-sector model should be considered, and the numerical procedure outlined by Galizia [30] should then be used to locate the saddle-point limit cycle and to study its local stability. Building on the contribution of Benhabib and Nishimura [14], our analysis could be extended to stochastic models focusing on the concept of stochastic limit cycles, followed by simulations. In this respect, Benhabib *et al.* [16] show that 3-sector models subject to stochastic technological shocks are able to provide a good fit with the data on macroeconomic fluctuations. Finally, our numerical analysis calls for clearer empirical evidence regarding the calibration of multi-sector models (e.g., sectoral capital shares). We leave all these research avenues for future work.

References

[1] Acemoglu, D., V.M. Carvalho, A. Ozdaglar and A. Tahbaz-Salehi (2012): "The Network Origins of Aggregate Fluctuations," *Econometrica*, 80(5), 1977–2016.

[2] Alessio, S.M. (2016): *Digital Signal Processing and Spectral Analysis for Scientists: Concepts and Applications*, Springer.

[3] Azariadis, C. and B. Smith (1998): "Financial Intermediation and Regime Switching in Business Cycles," *American Economic Review*, vol. 88(3), 516-536.

[4] Baxter, M. (1996): "Are Consumer Durables Important for Business Cycles," *The Review of Economics and Statistics*, 78(1), 147-155.

[5] Baxter, M., and R.G. King (1999): "Measuring Business Cycles: Approximate Band-Pass Filters for Economic Time Series," *The Review of Economics and Statistics*, vol. 81, 575-593.

[6] Beaudry, P., D. Galizia and F. Portier (2015): "Reviving the Limit Cycle View of Macroeconomic Fluctuations," *NBER Working Paper* 21241.

[7] Beaudry, P., D. Galizia and F. Portier (2017): "Is the Macroeconomy Locally Unstable and Why Should We Care?," In: *NBER Macroeconomics Annual*, Vol. 31, ed: M. Eichenbaum and J. A. Parker, 479–530. University of Chicago Press.

[8] Beaudry, P., D. Galizia and F. Portier (2020): "Putting the Cycle Back into Business Cycle Analysis," *American Economic Review*, 110(1), 1-47.

[9] Becker, R. and C. Foias (1994): "The Local Bifurcation of Ramsey Equilibrium," *Economic Theory*, 4(5), 719-744.

[10] Becker, R. and E. Tsyganov (2002): "Ramsey Equilibrium in a Two-Sector Model with Heterogeneous Households," *Journal of Economic Theory*, 105(1), 188-225.

[40] See Guckenheimer and Holmes [36] and the comments provided in Appendices 9.3 and 9.7.

[11] Benhabib, J., and K. Nishimura (1979): "The Hopf Bifurcation and the Existence and Stability of Closed Orbits in Multisector Models of Optimal Economic Growth," *Journal of Economic Theory*, 21, 421-444.

[12] Benhabib, J., and K. Nishimura (1979): "On the Uniqueness of Steady States in an Economy with Heterogeneous Capital Goods," *International Economic Review*, 20, 59-82.

[13] Benhabib, J., and K. Nishimura (1985): "Competitive Equilibrium Cycles," *Journal of Economic Theory*, 35, 284-306.

[14] Benhabib, J., and K. Nishimura (1989): "Stochastic Equilibrium Oscillations," *International Economic Review*, 30, 85-102.

[15] Benhabib, J., and K. Nishimura (1998): "Indeterminacy and Sunspots with Constant Returns," *Journal of Economic Theory*, 81, 58-96.

[16] Benhabib, J., R. Perli and P. Sakellaris (2006): "Persistence of Business Cycles in Multisector Real Business Cycle Models," *International Journal of Economic Theory*, 2, 181-197.

[17] Benhabib, J., and A. Rustichini (1990): "Equilibrium Cycling with Small Discounting," *Journal of Economic Theory*, 52, 423-432.

[18] Blundell, R., Browning, M., and C. Meghir (1994): "Consumer Demand and the Life-Cycle Allocation of Household Expenditures," *The Review of Economic Studies*, 61, 57-80.

[19] Boldrin, M., and L. Montrucchio (1986): "On the Indeterminacy of Capital Accumulation Paths," *Journal of Economic Theory*, 40, 26-39.

[20] Bosi, S., F. Magris and A. Venditti (2005): "Competitive Equilibrium Cycles with Endogenous Labor," *Journal of Mathematical Economics*, 41, 325-349.

[21] Brock, W. (1973): "Some Results on the Uniqueness of Steady States in Multisector Models of Optimum Growth When Future Utilities are Discounted," *International Economic Review*, 14, 535-559.

[22] Cartigny, P., and A. Venditti (1994): "Turnpike theory: Some New Results on the Saddle Point Property of Equilibria and on the Existence of Endogenous Cycles," *Journal of Economic Dynamics and Control*, 18, 957-974.

[23] Charpe, M., Bridge S., and P. McAdam (2020): "Labor Share and Growth in the Long-Run," *Macroeconomic Dynamics*, vol. 24, 1720 - 1757.

[24] Christiano, L.J., and T.J. Fitzgerald (1999): "The Band Pass Filter," NBER Working Paper 7257.

[25] Christiano, L.J., and T.J. Fitzgerald (2003): "The Band Pass Filter," *International Economic Review*, vol. 44(2), 435-465.

[26] Comin, D., and M. Gertler (2006): "Medium-term Business Cycles," *American Economic Review*, vol. 96(3), 523-551.

[27] Correa-López, M. and B. De Blas (2012): "International Transmission of Medium-term Technology Cycles: Evidence from Spain as a Recipient Country," *The B.E. Journal of Macroeconomics*, vol. 12(1), 1-52.

[28] Dufourt, F., K. Nishimura and A. Venditti (2015): "Indeterminacy and Sunspots in Two-Sector RBC Models with Generalized No-Income Effect Preferences," *Journal of Economic Theory*, 157, 1056-1080.

[29] Elliott, G., T. J. Rothenberg and J.H. Stock (1996): "Efficient Test for an Autoregressive Unite Root," *Econometrica*, vol. 64, 813–836.

[30] Galizia, D. (2021): "Saddle Cycles: Solving Rational Expectations Models Featuring Limit Cycles (or Chaos) using Perturbation Methods," *Quantitative Economics*, 12, 869-901.

[31] Garnier, J.-P., K. Nishimura and A. Venditti (2007): "Intertemporal Substitution in Consumption, Labor Supply Elasticity and Sunspot Fluctuations in Continuous-Time Models," *International Journal of Economic Theory*, 3, 235-259.

[32] Granger, C.W.J. (1966): "The Typical Shape of an Economic Variable," *Econometrica*, vol. 34(1), 150-161.

[33] Growiec, J., P. McAdam and J. Mućk (2018): "Endogenous Labor Share Cycles: Theory and Evidence," *Journal of Economic Dynamics and Control*, vol. 87(2), 74-93.

[34] Gruber, J. (2013): "Tax-Based Estimate of the Elasticity of Substitution," *Quarterly Journal of Finance*, 3, 1-20.

[35] Gu, C., Mattesini, F., Monnet, C. and R. Wright (2013): "Endogenous Credit Cycles," *Journal of Political Economy*, vol. 121(5), 940-965.

[36] Guckenheimer, J., and P. Holmes (1983): *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer-Verlag, New York.

[37] Hommes, C. (2013): *Behavior Rationality and Heterogenous Expectations in Complex Economic Systems*, New York: Cambridge University Press.

[38] Hopf, E. (1942): "Bifurcation of a Periodic Solution from a Stationary Solution of a System of Differential Equations," translated by L. Howard and N. Koppel, in: *The Hopf Bifurcation and its Applications*, J. Marsden and M. McCracken (Eds.), pp. 163-194, Springer-Verlag, New York, 1976.

[39] Huffman, G.W. and M.A. Wynne (1999): “The role of Intratemporal Adjustment Costs in a Multisector Economy,” *Journal of Monetary Economics*, 43(2), 317-350.

[40] Kamihigashi, T. (2001): “Necessity of Transversality Conditions for Infinite Horizon Problems,” *Econometrica*, 69, 995-1012.

[41] Kurz, M. (1968): ”The General Instability of a Class of Competitive Growth Processes,” *Review of Economic Studies*, 25, 155-174.

[42] Kuznetsov, Y. (1998): *Elements of Applied Bifurcation Theory*, 2nd Edition, Springer, New York.

[43] Levhari, L., and N. Liviatan (1972): “On Stability in the Saddle Point Sense,” *Journal of Economic Theory*, 4, 88-93.

[44] Long, J., and C. Plosser (1983): “Real Business Cycles,” *Journal of Political Economy*, 91, 39–69.

[45] Magill, M. (1977): “Some New Results on the Local Stability of the Process of Capital Accumulation,” *Journal of Economic Theory*, 15, 174-210.

[46] Magill, M. (1979): “On Cyclical Motion in Dynamic Economics,” *Journal of Economic Dynamics and Control*, 1, 199-218.

[47] Magill, M. (1979): “The Stability of Equilibrium,” *International Economic Review*, 20, 577-597.

[48] Magill, M., and J. Scheinkman (1979): “Stability of Regular Equilibria and the Correspondence Principle for Symmetric Variational Problems,” *International Economic Review*, 20, 297-315.

[49] Matsuyama, K. (1999): “Growing Through Cycles,” *Econometrica*, vol. 67(2), 335-347.

[50] Michel, P. (1982): “On the Transversality Condition in Infinite Horizon Optimal Problems,” *Econometrica*, 50, 975-985.

[51] Müller, U., and M.W. Watson (2016): “Measuring Uncertainty about Long-run Predictions,” *Review of Economic Studies*, vol. 83, 1711-1740.

[52] Müller, U., and M.W. Watson (2017): “Low Frequency Econometrics,” in Advances in Economics and Econometrics: Eleventh World Congress of the Econometric Society, vol. II, Cambridge University Press, 53-94.

[53] Myerson, R.B. (2012): “A Model of Moral-Hazard Credit Cycles,” *Journal of Political Economy*, vol. 120(5), 847-878.

[54] Nyblom, J. (1989): “Testing for the Constancy of Parameters Over Time,” *Journal of the American Statistical Association*, vol. 84, 223–230.

[55] Nishimura, K., and H. Takahashi (1992): "Factor Intensity and Hopf Bifurcations," in: G. Feichtinger (Ed.), *Dynamic Economic Models and optimal Control*, Elsevier, 135-149.

[56] Phillips, P.C.B., and S. Jin (2021): "Business Cycles, Trend Elimination, and the HP Filter," *International Economic Review*, vol. 62, 469-520.

[57] Priestley, M. (1981): *Spectral Analysis and Time Series, Volume 1, Univariate Series*, Academic Press, London.

[58] Prescott, E., and J. Wallenius (2011): "Aggregate Labor Supply," Federal Reserve Bank of Minneapolis, *Research Department Staff Report* 457.

[59] Rockafellar, R. (1976): "Saddle Points of Hamiltonian Systems in Convex Lagrangian Problems with a Nonzero Discount Rate," *Journal of Economic Theory*, 12, 71-113.

[60] Rogerson, R., and J. Wallenius (2009): "Micro and Macro Elasticities in a Life Cycle Model with Taxes," *Journal of Economic Theory*, 144, 2277-2292.

[61] Sargent, T.J. (1987): *Macroeconomic Theory*, 2nd ed. Bingley, UK: Emerald Group Publishing Limited.

[62] Shea, J. (2002): "Complementarities and Comovements," *Journal of Money, Credit, and Banking*, 34, 412–433.

[63] Valentinyi, A., and B. Herrendorf (2008): "Measuring Factor Income Shares at the Sectoral Level," *Review of Economic Dynamics*, 11, 820-835.

[64] Venditti, A. (1996): "Endogenous Cycles with Small Discounting in Multisector Optimal Growth Models: Continuous-Time Case," *Journal of Optimization Theory and Applications*, vol. 88, 453-474.

8 Appendix 1: Spectral analysis

Spectral density The second-order properties of a second-order stationary, zero mean time series $\{X_t\}_{-\infty<t<+\infty}$ may be described either by the autocovariances $\gamma_X(h) = E(X_{t+h}X_t)$ or, equivalently, by the spectral density $f(\omega)$ of the process, which is given by:

$$f_X(\omega) = \frac{1}{2\pi}\sum_{h=-\infty}^{+\infty} \gamma_X(h)e^{-i\omega h}, \quad \omega \in [-\pi;\pi].$$

provided that $\sum_{h=-\infty}^{+\infty} |\gamma_{ij}(h)| < \infty$. On the other hand, using the inverse of the Fourier transform, one can retrieves the autocovariances as:

$$\gamma_X(h) = \int_{-\pi}^{\pi} e^{i\omega h} f_X(\omega)d\omega.$$

In particular, for $h = 0$ and $\omega \in [-\pi, \pi]$, this gives the variance-covariance matrix of X_t,

$$\gamma_X(0) = \int_{-\pi}^{\pi} f_X(\omega)d\omega.$$

Thus, the spectrum decomposes the variance of X into components of different frequencies. More generally, for any ω between 0 and π, the expression

$$\int_{-w}^{w} f_X(\omega)d\omega$$

corresponds to the fluctuations of X_t associated with the frequency band $[-w; w]$. This allows to examine the fluctuations of the series for frequency intervals of interest as business cycle, medium-term or low frequencies.

Periodogram For a given time series $\{x_t, t = 0, \cdots, T-1\}$ of finite length T, the estimate of the periodogram is computed in a three-step procedure.[41] First, we compute the discrete Fourier transform $\{X_k, k = 1, \cdots, T-1\}$, which stems from sampling the discrete time Fourier transform at frequency intervals $\Delta\omega = \frac{2\pi}{T} \in [-\pi; \pi)$:

$$X_k \equiv X\left(\exp\left(-i\frac{2\pi}{T}k\right)\right) = \sum_{t=0}^{T-1} x_t \exp\left(-i\frac{2\pi}{T}kt\right)$$

for $k = 1, \cdots, T-1$. Second, we compute samples of the sample spectral density, denoted by S_k, using the Schuster's periodogram I_k:

$$S_k = I_k - \frac{1}{T}\left|X_k\right|^2.$$

Said differently, we estimate the spectral density at T frequencies equally spaced between 0 and π. For each sample, the raw periodogram (spectrum) estimate) is the squared modulus of the discrete Fourier transform divided by the length of the data set. A third step makes use of a kernel-smoothing (e.g., with a Hamming window) estimate of the raw periodogram estimate.

Testing for a local peak Let $\omega_j = 2\pi j/T$, $j = 0, \cdots, T^*$ with $T^* = \lfloor T/2 \rfloor$ denote the Fourier transform. The test can be formally written as:

$$\begin{cases} H_0 : f(\omega) = \bar{f} \quad \forall\omega \in \Omega = [\underline{\omega}; \overline{\omega}] \\ H_a : f(\omega) \geq \overline{f} \end{cases}$$

where f is the spectral density of the variable of interest, $\bar{f}$ is some (nonnegative) value that reflects the "flat" null hypothesis, $\underline{\omega}$ is a nonzero lower bound of the frequency interval and the upper bound $\overline{\omega}$ cannot be equal to π. The test statistic given by

$$D = \frac{1}{m}\sum_{\omega_j \in \underline{\Omega}} R(\omega_j) \sim \frac{1}{m}\chi^2(2m),$$

where

$$R(\omega_j) = 2\frac{I(\omega_j)}{f(\omega_j)} \sim \texttt{i.i.d.}\chi^2(2m),$$

is asymptotically Chi-square distributed with $2m$ degrees of freedom, with m the number of (discrete) Fourier frequencies in a partition, denoted by $\underline{\Omega}$, of the frequency interval Ω.

[41] See [57] and [2].

9 Appendix 2

9.1 Proof of Proposition 1

Maximizing profit in each of the three sectors yields to the following first order conditions

$$\begin{aligned}
w &= (1-\alpha_1-\alpha_2)A_0(k_1^0)^{\alpha_1}(k_2^0)^{\alpha_2}(l^0)^{-\alpha_1-\alpha_2} \\
&= p_1(1-\beta_1-\beta_2)A_1(k_1^1)^{\beta_1}(k_2^1)^{\beta_2}(l^1)^{-\beta_1-\beta_2} \\
&= p_2(1-\gamma_1-\gamma_2)A_2(k_1^2)^{\gamma_1}(k_2^2)^{\gamma_2}(l^2)^{-\gamma_1-\gamma_2} \\
r_1 &= \alpha_1 A_0(k_1^0)^{\alpha_1-1}(k_2^0)^{\alpha_2}(l^0)^{1-\alpha_1-\alpha_2} \\
&= p_1\beta_1 A_1(k_1^1)^{\beta_1-1}(k_2^1)^{\beta_2}(l^1)^{1-\beta_1-\beta_2} \\
&= p_2\gamma_1 A_2(k_1^2)^{\gamma_1-1}(k_2^2)^{\gamma_2}(l^2)^{1-\gamma_1-\gamma_2} \\
r_2 &= \alpha_2 A_0(k_1^0)^{\alpha_1}(k_2^0)^{\alpha_2-1}(l^0)^{1-\alpha_1-\alpha_2} \\
&= p_1\beta_2 A_1(k_1^1)^{\beta_1}(k_2^1)^{\beta_2-1}(l^1)^{1-\beta_1-\beta_2} \\
&= p_2\gamma_2 A_2(k_1^2)^{\gamma_1}(k_2^2)^{\gamma_2-1}(l^2)^{1-\gamma_1-\gamma_2}
\end{aligned} \tag{31}$$

Let us define:

$$\begin{aligned}
\omega_{10} &= \frac{r_1}{w} = \frac{\alpha_1}{1-\alpha_1-\alpha_2}\frac{l^0}{k_1^0} = \frac{\beta_1}{1-\beta_1-\beta_2}\frac{l^1}{k_1^1} = \frac{\gamma_1}{1-\gamma_1-\gamma_2}\frac{l^2}{k_1^2} \\
\omega_{12} &= \frac{r_1}{r_2} = \frac{\alpha_1}{\alpha_2}\frac{k_2^0}{k_1^0} = \frac{\beta_1}{\beta_2}\frac{k_2^1}{k_1^1} = \frac{\gamma_1}{\gamma_2}\frac{k_2^2}{k_1^2}
\end{aligned} \tag{32}$$

Substituting these expressions into (31) and using at the steady state $r_1 = (\delta+g)p_1$ and $r_2 = (\delta+g)p_2$ yield

$$\begin{aligned}
r_1 &= p_1\beta_1 A_1\left(\frac{\beta_2}{\beta_1}\right)^{\beta_2}\left(\frac{1-\beta_1-\beta_2}{\beta_1}\right)^{1-\beta_1-\beta_2}\omega_{10}^{1-\beta_1-\beta_2}\omega_{12}^{\beta_2} = (\delta+g)p_1 \\
r_2 &= p_2\gamma_2 A_2\left(\frac{\gamma_2}{\gamma_1}\right)^{\gamma_2-1}\left(\frac{1-\gamma_1-\gamma_2}{\gamma_1}\right)^{1-\gamma_1-\gamma_2}\omega_{10}^{1-\gamma_1-\gamma_2}\omega_{12}^{\gamma_2-1} = (\delta+g)p_2
\end{aligned} \tag{33}$$

Taking logs allows to get in matrix form

$$\begin{pmatrix} \ln(\delta+g) - \ln\left[\beta_1 A_1\left(\frac{\beta_2}{\beta_1}\right)^{\beta_2}\left(\frac{1-\beta_1-\beta_2}{\beta_1}\right)^{1-\beta_1-\beta_2}\right] \\ \ln(\delta+g) - \ln\left[\gamma_2 A_2\left(\frac{\gamma_2}{\gamma_1}\right)^{\gamma_2-1}\left(\frac{1-\gamma_1-\gamma_2}{\gamma_1}\right)^{1-\gamma_1-\gamma_2}\right] \end{pmatrix} = \begin{pmatrix} 1-\beta_1-\beta_2 & \beta_2 \\ 1-\gamma_1-\gamma_2 & \gamma_2-1 \end{pmatrix}\begin{pmatrix} \ln\omega_{10} \\ \ln\omega_{12} \end{pmatrix}$$

Solving this system leads to the following steady state values for ω_{10} and ω_{12}:

$$\begin{aligned}
\omega_{10} &= (\delta+g)^{\frac{\gamma_2-1-\beta_2}{\beta_2\gamma_1-(1-\beta_1)(1-\gamma_2)}}\frac{\left[\gamma_2 A_2\left(\frac{\gamma_2}{\gamma_1}\right)^{\gamma_2-1}\left(\frac{1-\gamma_1-\gamma_2}{\gamma_1}\right)^{1-\gamma_1-\gamma_2}\right]^{\frac{\beta_2}{\beta_2\gamma_1-(1-\beta_1)(1-\gamma_2)}}}{\left[\beta_1 A_1\left(\frac{\beta_2}{\beta_1}\right)^{\beta_2}\left(\frac{1-\beta_1-\beta_2}{\beta_1}\right)^{1-\beta_1-\beta_2}\right]^{\frac{\gamma_2-1}{\beta_2\gamma_1-(1-\beta_1)(1-\gamma_2)}}} \\
\omega_{12} &= (\delta+g)^{\frac{\gamma_1+\gamma_2-\beta_1-\beta_2}{\beta_2\gamma_1-(1-\beta_1)(1-\gamma_2)}}\frac{\left[\beta_1 A_1\left(\frac{\beta_2}{\beta_1}\right)^{\beta_2}\left(\frac{1-\beta_1-\beta_2}{\beta_1}\right)^{1-\beta_1-\beta_2}\right]^{\frac{1-\gamma_1-\gamma_2}{\beta_2\gamma_1-(1-\beta_1)(1-\gamma_2)}}}{\left[\gamma_2 A_2\left(\frac{\gamma_2}{\gamma_1}\right)^{\gamma_2-1}\left(\frac{1-\gamma_1-\gamma_2}{\gamma_1}\right)^{1-\gamma_1-\gamma_2}\right]^{\frac{1-\beta_1-\beta_2}{\beta_2\gamma_1-(1-\beta_1)(1-\gamma_2)}}}
\end{aligned}$$

Consider now from (31):

$$\begin{aligned}
\alpha_1 A_0\left(\frac{k_2^0}{k_1^0}\right)^{\alpha_2}\left(\frac{l^0}{k_1^0}\right)^{1-\alpha_1-\alpha_2} &= p_1\beta_1 A_1\left(\frac{k_2^1}{k_1^1}\right)^{\beta_2}\left(\frac{l^1}{k_1^1}\right)^{1-\beta_1-\beta_2} \\
&= p_2\gamma_1 A_2\left(\frac{k_2^2}{k_1^2}\right)^{\gamma_2}\left(\frac{l^2}{k_1^2}\right)^{1-\gamma_1-\gamma_2}
\end{aligned} \tag{34}$$

which can be written using (32)

$$\alpha_1 A_0 \left(\frac{\alpha_2}{\alpha_1}\omega_{12}\right)^{\alpha_2}\left(\frac{1-\alpha_1-\alpha_2}{\alpha_1}\omega_{10}\right)^{1-\alpha_1-\alpha_2} = p_1\beta_1 A_1 \left(\frac{\beta_2}{\beta_1}\omega_{12}\right)^{\beta_2}\left(\frac{1-\beta_1-\beta_2}{\beta_1}\omega_{10}\right)^{1-\beta_1-\beta_2}$$
$$= p_2\gamma_1 A_2 \left(\frac{\gamma_2}{\gamma_1}\omega_{12}\right)^{\gamma_2}\left(\frac{1-\gamma_1-\gamma_2}{\gamma_1}\omega_{10}\right)^{1-\gamma_1-\gamma_2}$$

Solving these equations leads to the steady state values for p_1 and p_2:

$$p_1 = \frac{\alpha_1 A_0}{\beta_1 A_1}\left(\frac{\alpha_2}{\alpha_1}\right)^{\alpha_2}\left(\frac{1-\alpha_1-\alpha_2}{\alpha_1}\right)^{1-\alpha_1-\alpha_2}\left(\frac{\beta_2}{\beta_1}\right)^{-\beta_2}\left(\frac{1-\beta_1-\beta_2}{\beta_1}\right)^{\beta_1+\beta_2-1}\omega_{10}^{\beta_1+\beta_2-\alpha_1-\alpha_2}\omega_{12}^{\alpha_2-\beta_2}$$
$$p_2 = \frac{\alpha_1 A_0}{\gamma_1 A_2}\left(\frac{\alpha_2}{\alpha_1}\right)^{\alpha_2}\left(\frac{1-\alpha_1-\alpha_2}{\alpha_1}\right)^{1-\alpha_1-\alpha_2}\left(\frac{\gamma_2}{\gamma_1}\right)^{-\gamma_2}\left(\frac{1-\gamma_1-\gamma_2}{\gamma_1}\right)^{\gamma_1+\gamma_2-1}\omega_{10}^{\gamma_1+\gamma_2-\alpha_1-\alpha_2}\omega_{12}^{\alpha_2-\gamma_2}$$

and thus to the value of $w = (\delta+g)p_1/\omega_{10}$. Let us consider now Lemma 2. Solving the system $\mathcal{A}(\omega, p)y = x$ with respect to k_1 and k_2 using at the steady state $y_1 = gk_1 + c_1$ and $y_2 = gk_2$ yield

$$k_1 = \frac{a_{00}[a_{10}-g(a_{10}a_{22}-a_{12}a_{20})]}{[a_{00}-g(a_{22}a_{00}-ga_{20}a_{02})][a_{00}-g(a_{11}a_{00}-a_{10}a_{01})]-g^2(a_{12}a_{00}-a_{10}a_{02})(a_{21}a_{00}-a_{20}a_{01})}$$
$$+ \frac{c_1[(a_{11}a_{00}-a_{10}a_{01})[a_{00}-g(a_{22}a_{00}-ga_{20}a_{02})]+g(a_{12}a_{00}-a_{10}a_{02})(a_{21}a_{00}-a_{20}a_{01})]}{[a_{00}-g(a_{22}a_{00}-ga_{20}a_{02})][a_{00}-g(a_{11}a_{00}-a_{10}a_{01})]-g^2(a_{12}a_{00}-a_{10}a_{02})(a_{21}a_{00}-a_{20}a_{01})} \equiv \frac{M_1+c_1M_2}{M}$$
$$k_2 = \frac{a_{00}[a_{20}-g(a_{11}a_{20}-a_{21}a_{10})]+c_1a_{00}(a_{11}a_{00}-a_{10}a_{01})}{[a_{00}-g(a_{22}a_{00}-ga_{20}a_{02})][a_{00}-g(a_{11}a_{00}-a_{10}a_{01})]-g^2(a_{12}a_{00}-a_{10}a_{02})(a_{21}a_{00}-a_{20}a_{01})} \equiv \frac{M_3+c_1M_4}{M}$$

Recall that the input coefficients a_{ij} are functions of ω_{10}, p_1 and p_2 as follows

$$a_{00} = \frac{(1-\alpha_1-\alpha_2)\omega_{10}}{(\delta+g)p_1}, \quad a_{10} = \frac{\alpha_1}{(\delta+g)p_1}, \quad a_{20} = \frac{\alpha_2}{(\delta+g)p_2}$$
$$a_{01} = \frac{(1-\beta_1-\beta_2)\omega_{10}}{\delta+g}, \quad a_{11} = \frac{\beta_1}{\delta+g}, \quad a_{21} = \frac{p_1\beta_2}{(\delta+g)p_2}$$
$$a_{02} = \frac{p_2(1-\gamma_1-\gamma_2)\omega_{10}}{(\delta+g)p_1}, \quad a_{12} = \frac{p_2\gamma_1}{(\delta+g)p_1}, \quad a_{22} = \frac{\gamma_2}{\delta+g}$$

and thus do not depend on the capital stocks k_1 and k_2. We also get the expression of c_0:

$$c_0 = y_0 = \frac{1-a_{01}(gk_1+c_1)-a_{02}gk_2}{a_{00}}$$

Recall then that $c_1 = c_0(1-\theta)/\theta p_1$. Substituting the expressions of k_1, k_2 and c_1 into c_0 and solving for c_0 gives a unique solution:

$$c_0 = \frac{1-a_{01}g\frac{M_1}{M}-a_{02}g\frac{M_3}{M}}{a_{00}+\frac{1-\theta}{\theta p_1}\left[a_{01}\left(1+g\frac{M_2}{M}\right)+a_{02}g\frac{M_4}{M}\right]}$$

Substituting this expression into c_1, k_1 and k_2 allows then to prove the existence and uniqueness of a steady state. □

9.2 Proof of Proposition 2

Linearizing the dynamical system around $(c_1^*, k_1^*, k_2^*, p_1^*, p_2^*)$ gives the Jacobian matrix $\mathcal{J}$:

$$\mathcal{J} = \begin{pmatrix} \frac{\partial y_1}{\partial k_1} - g - \frac{\partial c_1}{\partial k_1} & \frac{\partial y_1}{\partial k_2} - \frac{\partial c_1}{\partial k_2} & \frac{\partial y_1}{\partial p_1} - \frac{\partial c_1}{\partial p_1} & \frac{\partial y_1}{\partial p_2} - \frac{\partial c_1}{\partial p_2} \\ \frac{\partial y_2}{\partial k_1} & \frac{\partial y_2}{\partial k_2} - g & \frac{\partial y_2}{\partial p_1} & \frac{\partial y_2}{\partial p_2} \\ \frac{\sigma}{E_\sigma^\theta}\frac{p_1}{c}\mathcal{B}_1 & \frac{\sigma}{E_\sigma^\theta}\frac{p_1}{c}\mathcal{B}_2 & J_{33} & J_{34} \\ \frac{\sigma}{E_\sigma^\theta}\frac{p_2}{c}\mathcal{B}_1 & \frac{\sigma}{E_\sigma^\theta}\frac{p_2}{c}\mathcal{B}_2 & J_{43} & J_{44} \end{pmatrix} \equiv \begin{pmatrix} \mathcal{J}_1 & \mathcal{J}_2 \\ \mathcal{J}_3 & \mathcal{J}_4 \end{pmatrix} \quad (35)$$

with

$$J_{33} = \frac{1}{E_\sigma^\theta}\left[\left(\delta + g - \frac{\partial y_1}{\partial k_1}\right)E_\sigma^\theta - \sigma\frac{p_1\left(\mathcal{A}_1 + \frac{\partial c_0}{\partial k_1}\frac{\partial c_1}{\partial p_1}\right)}{c_0} + (1-\theta)(1-\sigma)\left(\delta + g - \frac{\partial y_1}{\partial k_1}\right)\right]$$

$$J_{44} = \frac{1}{E_\sigma^\theta}\left[\left(\delta + g - \frac{\partial y_2}{\partial k_2}\right)E_\sigma^\theta - \sigma\frac{p_2\left(\mathcal{A}_2 + \frac{\partial c_0}{\partial k_1}\frac{\partial c_1}{\partial p_2}\right)}{c_0} - (1-\theta)(1-\sigma)\frac{p_2}{p_1}\frac{\partial y_2}{\partial k_1}\right]$$

$$J_{34} = -\frac{1}{E_\sigma^\theta}\left[\frac{\partial y_2}{\partial k_1}E_\sigma^\theta + \sigma\frac{p_1\left(\mathcal{A}_2 + \frac{\partial c_0}{\partial k_1}\frac{\partial c_1}{\partial p_2}\right)}{c_0} + (1-\theta)(1-\sigma)\frac{\partial y_2}{\partial k_1}\right]$$

$$J_{43} = -\frac{1}{E_\sigma^\theta}\left[\frac{\partial y_1}{\partial k_2}E_\sigma^\theta + \sigma\frac{p_2\left(\mathcal{A}_1 + \frac{\partial c_0}{\partial k_1}\frac{\partial c_1}{\partial p_1}\right)}{c_0} - (1-\theta)(1-\sigma)\frac{p_2}{p_1}\left(\delta + g - \frac{\partial y_1}{\partial k_1}\right)\right]$$

and

$$\mathcal{A}_1 = \frac{\partial c_0}{\partial p_1}\left(\frac{\partial y_1}{\partial k_1} - g - \delta\right) + \frac{\partial c_0}{\partial p_2}\frac{\partial y_1}{\partial k_2} - \frac{\partial c_0}{\partial k_1}\frac{\partial y_1}{\partial p_1} - \frac{\partial c_0}{\partial k_2}\frac{\partial y_2}{\partial p_1}$$

$$\mathcal{A}_2 = \frac{\partial c_0}{\partial p_1}\frac{\partial y_2}{\partial k_1} + \frac{\partial c_0}{\partial p_2}\left(\frac{\partial y_2}{\partial k_2} - g - \delta\right) - \frac{\partial c_0}{\partial k_1}\frac{\partial y_1}{\partial p_2} - \frac{\partial c_0}{\partial k_2}\frac{\partial y_2}{\partial p_2}$$

$$\mathcal{B}_1 = \frac{\partial c_0}{\partial k_1}\left(\frac{\partial y_1}{\partial k_1} - g - \frac{\partial c_1}{\partial k_1}\right) + \frac{\partial c_0}{\partial k_2}\frac{\partial y_2}{\partial k_1},\ \mathcal{B}_2 = \frac{\partial c_0}{\partial k_1}\left(\frac{\partial y_1}{\partial k_2} - \frac{\partial c_1}{\partial k_2}\right) + \frac{\partial c_0}{\partial k_2}\left(\frac{\partial y_2}{\partial k_2} - g\right)$$

Since the optimization program (5) has an Hamiltonian structure, and as initially proved by Kurz [41] and Levhari and Liviatan [43], if λ is a characteristic root then $\bar{\lambda}$, $\delta - \lambda$ and $\delta - \bar{\lambda}$ are also characteristic roots. This is confirmed by showing that $\mathcal{T} = 2\delta$. Consider indeed the fact that by definition

$$c_0(k_1, k_2, p_1, p_2) = T(k_1, k_2, y_1(k_1, k_2, p_1, p_2), y_2(k_1, k_2, p_1, p_2)) \tag{36}$$

It follows therefore

$$\begin{aligned} \frac{\partial c_0}{\partial k_1} &= T_{k_1} + T_{y_1}\frac{\partial y_1}{\partial k_1} + T_{y_2}\frac{\partial y_2}{\partial k_1} = r_1 - p_1\frac{\partial y_1}{\partial k_1} - p_2\frac{\partial y_2}{\partial k_1} \\ \frac{\partial c_0}{\partial k_2} &= T_{k_2} + T_{y_1}\frac{\partial y_1}{\partial k_2} + T_{y_2}\frac{\partial y_2}{\partial k_2} = r_2 - p_1\frac{\partial y_1}{\partial k_2} - p_2\frac{\partial y_2}{\partial k_2} \end{aligned} \tag{37}$$

and

$$\begin{aligned} \frac{\partial c_0}{\partial p_1} &= T_{y_1}\frac{\partial y_1}{\partial p_1} + T_{y_2}\frac{\partial y_2}{\partial p_1} = -p_1\frac{\partial y_1}{\partial p_1} - p_2\frac{\partial y_2}{\partial p_1} = -p_1\frac{\partial y_1}{\partial p_1} - p_2\frac{\partial y_1}{\partial p_2} \\ \frac{\partial c_0}{\partial p_2} &= T_{y_1}\frac{\partial y_1}{\partial p_2} + T_{y_2}\frac{\partial y_2}{\partial p_2} = -p_1\frac{\partial y_1}{\partial p_2} - p_2\frac{\partial y_2}{\partial p_2} = -p_1\frac{\partial y_2}{\partial p_1} - p_2\frac{\partial y_2}{\partial p_2} \end{aligned} \tag{38}$$

Evaluated at the steady state expressions (37) become:

$$\begin{aligned} \frac{\partial c_0}{\partial k_1} &= -\left[p_1\left(\frac{\partial y_1}{\partial k_1} - \delta - g\right) + p_2\frac{\partial y_2}{\partial k_1}\right] \\ \frac{\partial c_0}{\partial k_2} &= -\left[p_1\frac{\partial y_1}{\partial k_2} + p_2\left(\frac{\partial y_2}{\partial k_2} - \delta - g\right)\right] \end{aligned} \tag{39}$$

From the Jacobian matrix (35), we derive

$$\mathcal{T} = 2\delta - \sigma\frac{p_1\mathcal{A}_1 + p_2\mathcal{A}_2}{E_\sigma^\theta c_0} - \frac{\partial c_0}{\partial k_1}\frac{\left[c_1 E_\sigma^\theta + \sigma\left(p_1\frac{\partial c_1}{\partial p_1} + p_2\frac{\partial c_1}{\partial p_2}\right)\right]}{E_\sigma^\theta c_0} + \frac{(1-\theta)(1-\sigma)\left[\delta + g - \frac{\partial y_1}{\partial k_1} - \frac{p_2}{p_1}\frac{\partial y_2}{\partial k_1}\right]}{E_\sigma^\theta}$$

From (38) and (39) we derive that $p_1\mathcal{A}_1 + p_2\mathcal{A}_2 = 0$ and using (14)-(15) we conclude that

$$\frac{\partial c_0}{\partial k_1}\frac{\left[c_1 E_\sigma^\theta + \sigma\left(p_1\frac{\partial c_1}{\partial p_1}+p_2\frac{\partial c_1}{\partial p_2}\right)\right]}{E_\sigma^\theta c_0} = \frac{(1-\theta)(1-\sigma)\left[\delta + g - \frac{\partial y_1}{\partial k_1} - \frac{p_2}{p_1}\frac{\partial y_2}{\partial k_1}\right]}{E_\sigma^\theta}$$

It follows therefore that $\mathcal{T} = 2\delta$. Tedious but straightforward computations also allow to compute the Determinant of the Jacobian matrix as:

$$\mathcal{D}_\sigma^\theta = \frac{\left[\left(\frac{\partial y_1}{\partial k_1} - g - \frac{\partial c_1}{\partial k_1}\right)\left(\frac{\partial y_2}{\partial k_2} - g\right) - \left(\frac{\partial y_1}{\partial k_2} - \frac{\partial c_1}{\partial k_2}\right)\frac{\partial y_2}{\partial k_1}\right]\left[\left(\delta + g - \frac{\partial y_1}{\partial k_1}\right)\left(\delta + g - \frac{\partial y_2}{\partial k_2}\right) - \frac{\partial y_1}{\partial k_2}\frac{\partial y_2}{\partial k_1}\right]}{E_\sigma^\theta}$$

Denoting Σ_σ^θ the sum of minors of order three and $\mathcal{S}_\sigma^\theta$ the sum of minors of order two, let us consider the fact that if λ is a characteristic root of (18), then $\bar{\lambda}$, $\delta - \lambda$ and $\delta - \bar{\lambda}$ are also characteristic roots. In the case where the roots are complex, let us denote $\lambda = a + ib$. We easily derive that

$$\begin{aligned}
\mathcal{S}_\sigma^\theta &= \lambda\bar{\lambda} + \lambda(\delta-\lambda) + \lambda(\delta-\bar{\lambda}) + \bar{\lambda}(\delta-\lambda) + \bar{\lambda}(\delta-\bar{\lambda}) + (\delta-\lambda)(\delta-\bar{\lambda}) \\
&= \delta^2 + 2\left[b^2 - a^2 + \delta a\right] \\
\Sigma_\sigma^\theta &= \lambda\bar{\lambda}(\delta-\lambda) + \lambda(\delta-\lambda)(\delta-\bar{\lambda}) + \bar{\lambda}(\delta-\lambda)(\delta-\bar{\lambda}) + \lambda\bar{\lambda}(\delta-\bar{\lambda}) \\
&= 2\delta\left[b^2 - a^2 + \delta a\right] = \mathcal{T}\frac{S_\sigma^\theta - \delta^2}{2} = \delta\left(S_\sigma^\theta - \delta^2\right) \\
\mathcal{D}_\sigma^\theta &= \lambda\bar{\lambda}(\delta-\lambda)(\delta-\bar{\lambda}) \\
&= (a^2+b^2)\left[(\delta-a)^2 + b^2\right] = \left(\frac{S_\sigma - \delta^2}{2}\right)^2 + b^2\left[a - (\delta - a)\right]^2
\end{aligned} \tag{40}$$

In the case where the roots are real, let us denote $\lambda_1 = a_1$ and $\lambda_2 = a_2$. We get now:

$$\begin{aligned}
\mathcal{S}_\sigma^\theta &= \lambda_1\lambda_2 + \lambda_1(\delta-\lambda_1) + \lambda_1(\delta-\lambda_2) + \lambda_2(\delta-\lambda_1) + \lambda_2(\delta-\lambda_2) + (\delta-\lambda_1)(\delta-\lambda_2) \\
&= \delta^2 + a_1(\delta-a_1) + a_2(\delta-a_2) \\
\Sigma_\sigma^\theta &= \lambda_1\lambda_2(\delta-\lambda_1) + \lambda_1(\delta-\lambda_1)(\delta-\lambda_2) + \lambda_2(\delta-\lambda_1)(\delta-\lambda_2) + \lambda_1\lambda_2(\delta-\lambda_2) \\
&= \delta\left[a_1(\delta-a_1) + a_2(\delta-a_2)\right] = \mathcal{T}\frac{S_\sigma^\theta - \delta^2}{2} = \delta\left(S_\sigma^\theta - \delta^2\right) \\
\mathcal{D}_\sigma^\theta &= \lambda_1\lambda_2(\delta-\lambda_1)(\delta-\lambda_2) \\
&= a_1 a_2(\delta-a_1)(\delta-a_2) = \left(\frac{S_\sigma - \delta^2}{2}\right)^2 - \left[\frac{a_1(\delta-a_1) - a_2(\delta-a_2)}{2}\right]^2
\end{aligned}$$

It follows therefore that for any set of characteristic roots we have

$$\Sigma_\sigma^\theta = \mathcal{T}\frac{S_\sigma^\theta \delta^2}{2} = \delta\left(S_\sigma^\theta - \delta^2\right)$$

Moreover, using the fact that $\mathcal{T} = 2\delta$, we can also show that

$$\frac{\partial c_1}{\partial k_1} = \frac{1-\theta}{\theta}\left(\delta + g - \frac{\partial y_1}{\partial k_1} - \frac{p_2}{p_1}\frac{dy_2}{dk_1}\right)$$

or equivalently using (13) and (14)

$$\left(\delta + g - \frac{\partial y_1}{\partial k_1} - \frac{p_2}{p_1}\frac{dy_2}{dk_1}\right) = \frac{\partial c_0}{\partial k_1}\frac{1}{p_1}$$

The degree-4 characteristic polynomial can then be written as

$$\mathcal{P}_\sigma^\theta(\lambda) = \lambda^4 - \lambda^3 2\delta + \lambda^2\mathcal{S}_\sigma^\theta - \lambda\delta\left(S_\sigma^\theta - \delta^2\right) + \mathcal{D}_\sigma^\theta \tag{41}$$

with

$$\mathcal{S}_\sigma^\theta = \mathcal{S}_0^\theta - \sigma\frac{\Gamma_\theta\left[1 - \frac{1}{c_0}\left(p_1\frac{\partial c_0}{\partial p_1} + p_2\frac{\partial c_0}{\partial p_2}\right)\right] + \theta\Theta_\theta}{\theta E_\sigma^\theta c_0}$$

and

$$
\begin{aligned}
\mathcal{S}_0^\theta &= \left(\frac{\partial y_1}{\partial k_1} - g - \frac{\partial c_1}{\partial k_1}\right)\left(\frac{\partial y_2}{\partial k_2} - g\right) - \left(\frac{\partial y_1}{\partial k_2} - \frac{\partial c_1}{\partial k_2}\right)\frac{\partial y_2}{\partial k_1} + \frac{\left(\delta+g-\frac{\partial y_1}{\partial k_1}\right)\left(\delta+g-\frac{\partial y_2}{\partial k_2}\right) - \frac{\partial y_1}{\partial k_2}\frac{\partial y_2}{\partial k_1}}{\theta}\\
&+ \left(\frac{\partial y_1}{\partial k_1} - g - \frac{\partial c_1}{\partial k_1} + \frac{\partial y_2}{\partial k_2} - g\right)\left[\delta + g - \frac{\partial y_1}{\partial k_1} + \delta + g - \frac{\partial y_2}{\partial k_2} + \frac{\partial c_1}{\partial k_1}\right]\\
\Gamma_\theta &= c_0(1-\theta)\left[\left(\delta + g - \frac{\partial y_1}{\partial k_1}\right)\left(\delta + g - \frac{\partial y_2}{\partial k_2}\right) - \frac{\partial y_1}{\partial k_2}\frac{\partial y_2}{\partial k_1} + \frac{\left(\frac{\partial y_1}{\partial k_1} - g - \frac{\partial c_1}{\partial k_1} + \frac{\partial y_2}{\partial k_2} - g\right)\frac{\partial c_0}{\partial k_1}}{p_1}\right]\\
\Theta_\theta &= \left[\frac{\partial c_0}{\partial k_1}\left(\frac{\partial y_1}{\partial k_1} - g - \frac{\partial c_1}{\partial k_1}\right) + \frac{\partial c_0}{\partial k_2}\frac{\partial y_2}{\partial k_1}\right]\left[p_1\left(\frac{\partial y_1}{\partial p_1} - \frac{\partial c_1}{\partial p_1}\right) + p_2\left(\frac{\partial y_1}{\partial p_2} - \frac{\partial c_1}{\partial p_2}\right)\right]\\
&+ \left[\frac{\partial c_0}{\partial k_1}\left(\frac{\partial y_1}{\partial k_2} - \frac{\partial c_1}{\partial k_2}\right) + \frac{\partial c_0}{\partial k_2}\left(\frac{\partial y_2}{\partial k_2} - g\right)\right]\left[p_1\frac{\partial y_2}{\partial p_1} + p_2\frac{\partial y_2}{\partial p_2}\right]\\
&+ \mathcal{A}_1\left[p_1\left(\frac{\partial y_1}{\partial k_1} - g - \delta\right) + p_2\frac{\partial y_2}{\partial k_1}\right] + \mathcal{A}_2\left[p_1\frac{\partial y_1}{\partial k_2} + p_2\left(\frac{\partial y_2}{\partial k_2} - g - \delta\right)\right]\\
&+ \frac{\partial c_0}{\partial k_1}\left\{\frac{\partial c_1}{\partial p_1}\left[p_1\left(\frac{\partial y_1}{\partial k_1} - g + \delta - \frac{\partial c_1}{\partial k_1}\right) + p_2\frac{\partial y_2}{\partial k_1}\right] + \frac{\partial c_1}{\partial p_2}\left[p_1\frac{\partial y_1}{\partial k_2} + p_2\left(\frac{\partial y_2}{\partial k_2} - g + \delta - \frac{\partial c_1}{\partial k_1}\right)\right]\right\}
\end{aligned}
$$

We finally conclude that, because of the structure of the characteristic roots, one of the following cases necessarily hold:

i) the four roots are real and distincts,

ii) the four roots are given by two pairs of non-real complex conjugates,

iii) there are two real double roots. □

9.3 Proof of Proposition 3

Using Lemmas 1 and 2, and recalling (3), (13) and (14), we need first to compute explicitly all the partial derivatives that will affect the linearized dynamical system around the steady state:

Lemma 9.1. *At the steady state we have*

$$
\begin{aligned}
\frac{\partial r_1}{\partial p_1} &= \frac{\partial y_1}{\partial k_1} = \frac{(\delta+g)[\alpha_2(1-\gamma_1)-\gamma_2(1-\alpha_1)]}{\beta_1(\alpha_2-\gamma_2)+\beta_2(\gamma_1-\alpha_1)+\gamma_2\alpha_1-\gamma_1\alpha_2}, & \frac{\partial r_1}{\partial p_2} &= \frac{\partial y_2}{\partial k_1} = \frac{(\delta+g)p_1[\beta_2(1-\alpha_1)-\alpha_2(1-\beta_1)]}{p_2[\beta_1(\alpha_2-\gamma_2)+\beta_2(\gamma_1-\alpha_1)+\gamma_2\alpha_1-\gamma_1\alpha_2]},\\
\frac{\partial r_2}{\partial p_1} &= \frac{\partial y_1}{\partial k_2} = \frac{(\delta+g)p_2[\gamma_1(1-\alpha_2)-\alpha_1(1-\gamma_2)]}{p_1[\beta_1(\alpha_2-\gamma_2)+\beta_2(\gamma_1-\alpha_1)+\gamma_2\alpha_1-\gamma_1\alpha_2]}, & \frac{\partial r_2}{\partial p_2} &= \frac{\partial y_2}{\partial k_2} = \frac{(\delta+g)[\alpha_1(1-\beta_2)-\beta_1(1-\alpha_2)]}{\beta_1(\alpha_2-\gamma_2)+\beta_2(\gamma_1-\alpha_1)+\gamma_2\alpha_1-\gamma_1\alpha_2},\\
\frac{\partial c_0}{\partial k_1} &= \frac{(\delta+g)p_1[\gamma_2(1-\beta_1)-\beta_2(1-\gamma_1)]}{\beta_1(\alpha_2-\gamma_2)+\beta_2(\gamma_1-\alpha_1)+\gamma_2\alpha_1-\gamma_1\alpha_2}, & \frac{\partial c_0}{\partial k_2} &= \frac{(\delta+g)p_2[\beta_1(1-\gamma_2)-\gamma_1(1-\beta_2)]}{\beta_1(\alpha_2-\gamma_2)+\beta_2(\gamma_1-\alpha_1)+\gamma_2\alpha_1-\gamma_1\alpha_2},\\
\frac{\partial c_1}{\partial k_1} &= \frac{1-\theta}{\theta}\frac{(\delta+g)[\gamma_2(1-\beta_1)-\beta_2(1-\gamma_1)]}{\beta_1(\alpha_2-\gamma_2)+\beta_2(\gamma_1-\alpha_1)+\gamma_2\alpha_1-\gamma_1\alpha_2}, & \frac{\partial c_1}{\partial k_2} &= \frac{1-\theta}{\theta}\frac{p_2}{p_1}\frac{(\delta+g)[\beta_1(1-\gamma_2)-\gamma_1(1-\beta_2)]}{\beta_1(\alpha_2-\gamma_2)+\beta_2(\gamma_1-\alpha_1)+\gamma_2\alpha_1-\gamma_1\alpha_2},\\
\frac{\partial w}{\partial p_1} &= \frac{w[\gamma_2\alpha_1-\gamma_1\alpha_2]}{p_1[\beta_1(\alpha_2-\gamma_2)+\beta_2(\gamma_1-\alpha_1)+\gamma_2\alpha_1-\gamma_1\alpha_2]}, & \frac{\partial w}{\partial p_2} &= \frac{w[\beta_1\alpha_2-\beta_2\alpha_1]}{p_2[\beta_1(\alpha_2-\gamma_2)+\beta_2(\gamma_1-\alpha_1)+\gamma_2\alpha_1-\gamma_1\alpha_2]},\\
\frac{\partial y_1}{\partial p_1} &= \left(\frac{\partial w}{\partial p_1}\right)^2\frac{1}{w} + \left(\frac{\partial y_1}{\partial k_1}\right)^2\frac{k_1}{r_1} + \left(\frac{\partial y_1}{\partial k_2}\right)^2\frac{k_2}{r_2} - \frac{y_1}{p_1}, & \frac{\partial y_2}{\partial p_2} &= \left(\frac{\partial w}{\partial p_2}\right)^2\frac{1}{w} + \left(\frac{\partial y_2}{\partial k_1}\right)^2\frac{k_1}{r_1} + \left(\frac{\partial y_2}{\partial k_2}\right)^2\frac{k_2}{r_2} - \frac{y_2}{p_2},\\
\frac{\partial y_2}{\partial p_1} &= \frac{\partial w}{\partial p_1}\frac{\partial w}{\partial p_2}\frac{1}{w} + \frac{\partial y_1}{\partial k_1}\frac{\partial y_2}{\partial k_1}\frac{k_1}{r_1} + \frac{\partial y_1}{\partial k_2}\frac{\partial y_2}{\partial k_2}\frac{k_2}{r_2}, & \frac{\partial y_1}{\partial p_2} &= \frac{\partial y_2}{\partial p_1},\\
\frac{\partial c_0}{\partial p_1} &= \frac{\partial w}{\partial p_1}\frac{\gamma_1\beta_2-\gamma_2\beta_1}{\beta_1(\alpha_2-\gamma_2)+\beta_2(\gamma_1-\alpha_1)+\gamma_2\alpha_1-\gamma_1\alpha_2} & \frac{\partial c_0}{\partial p_2} &= \frac{\partial w}{\partial p_2}\frac{\gamma_1\beta_2-\gamma_2\beta_1}{\beta_1(\alpha_2-\gamma_2)+\beta_2(\gamma_1-\alpha_1)+\gamma_2\alpha_1-\gamma_1\alpha_2}\\
&+ \frac{\partial c_0}{\partial k_1}\frac{\partial y_1}{\partial k_1}\frac{k_1}{r_1} + \frac{\partial c_0}{\partial k_2}\frac{\partial y_1}{\partial k_2}\frac{k_2}{r_2}, & &+ \frac{\partial c_0}{\partial k_1}\frac{\partial y_2}{\partial k_1}\frac{k_1}{r_1} + \frac{\partial c_0}{\partial k_2}\frac{\partial y_2}{\partial k_2}\frac{k_2}{r_2},\\
\frac{\partial c_1}{\partial p_1} &= \frac{1-\theta}{\theta p_1}\frac{\partial c_0}{\partial p_1} - \frac{c_0}{p_1}, & \frac{\partial c_1}{\partial p_2} &= \frac{1-\theta}{\theta p_1}\frac{\partial c_0}{\partial p_2}
\end{aligned}
$$

Proof. From Lemmas 1 and 2, we obtain the following derivatives:

$$
\begin{aligned}
\frac{\partial r_1}{\partial p_1} &= \frac{\partial y_1}{\partial k_1} = \frac{a_{02}a_{20}-a_{00}a_{22}}{X}, & \frac{\partial r_1}{\partial p_2} &= \frac{\partial y_2}{\partial k_1} = \frac{a_{00}a_{21}-a_{01}a_{20}}{X}, \\
\frac{\partial r_2}{\partial p_1} &= \frac{\partial y_1}{\partial k_2} = \frac{a_{00}a_{12}-a_{02}a_{10}}{X}, & \frac{\partial r_2}{\partial p_2} &= \frac{\partial y_2}{\partial k_2} = \frac{a_{01}a_{10}-a_{00}a_{11}}{X}, \\
\frac{\partial c_0}{\partial k_1} &= \frac{a_{01}a_{22}-a_{02}a_{21}}{X}, & \frac{\partial c_0}{\partial k_2} &= \frac{a_{02}a_{11}-a_{01}a_{12}}{X}, \\
\frac{\partial w}{\partial p_1} &= \frac{a_{10}a_{22}-a_{12}a_{20}}{X}, & \frac{\partial w}{\partial p_2} &= \frac{a_{11}a_{20}-a_{10}a_{21}}{X}, \\
\frac{\partial y_1}{\partial p_1} &= \left(\frac{\partial w}{\partial p_1}\right)^2 \frac{1}{w} + \left(\frac{\partial r_1}{\partial p_1}\right)^2 \frac{k_1}{r_1} + \left(\frac{\partial r_2}{\partial p_1}\right)^2 \frac{k_2}{r_2} - \frac{y_1}{p_1}, & \frac{\partial y_2}{\partial p_2} &= \left(\frac{\partial w}{\partial p_2}\right)^2 \frac{1}{w} + \left(\frac{\partial r_1}{\partial p_2}\right)^2 \frac{k_1}{r_1} + \left(\frac{\partial r_2}{\partial p_2}\right)^2 \frac{k_2}{r_2} - \frac{y_2}{p_2}, \\
\frac{\partial y_2}{\partial p_1} &= \frac{\partial w}{\partial p_1}\frac{\partial w}{\partial p_2}\frac{1}{w} + \frac{\partial r_1}{\partial p_1}\frac{\partial r_1}{\partial p_2}\frac{k_1}{r_1} + \frac{\partial r_2}{\partial p_1}\frac{\partial r_2}{\partial p_2}\frac{k_2}{r_2}, & \frac{\partial y_1}{\partial p_2} &= \frac{\partial y_2}{\partial p_1}, \\
\frac{\partial c_0}{\partial p_1} &= \frac{\partial w}{\partial p_1}\frac{1}{w}\frac{a_{12}a_{21}-a_{11}a_{22}}{X} + \frac{\partial c_0}{\partial r_1}\frac{\partial p_1}{\partial k_1}\frac{k_1}{r_1} + \frac{\partial c_0}{\partial k_2}\frac{\partial r_2}{\partial p_1}\frac{k_2}{r_2}, & \frac{\partial c_0}{\partial p_2} &= \frac{\partial w}{\partial p_2}\frac{1}{w}\frac{a_{12}a_{21}-a_{11}a_{22}}{X} + \frac{\partial c_0}{\partial k_1}\frac{\partial r_1}{\partial p_2}\frac{k_1}{r_1} + \frac{\partial c_0}{\partial k_2}\frac{\partial r_2}{\partial p_2}\frac{k_2}{r_2}
\end{aligned}
$$

with $X = a_{10}(a_{01}a_{22} - a_{02}a_{21}) + a_{11}(a_{02}a_{20} - a_{00}a_{22}) + a_{12}(a_{00}a_{21} - a_{01}a_{20})$. The final expressions are obtained from the input coefficient (2) and using (13) and (14). □

Using the expressions of the input coefficients (2) and Lemma 9.1, we easily derive that the discriminant (21) is negative if and only if condition (23) holds. Let us then assume that condition (23) holds. Using the expression (22), we derive the real parts of the complex characteristic roots such that

$$
\begin{aligned}
Re(\lambda_{1,2}) &= \frac{(\delta+g)}{2}\frac{\beta_2(\gamma_1-\alpha_1)-(1-\beta_1)(\alpha_2-\gamma_2)+(1-\gamma_2)(\beta_1-\alpha_1)-\gamma_1(\alpha_2-\beta_2)}{\beta_1(\alpha_2-\gamma_2)+\beta_2(\gamma_1-\alpha_1)+\gamma_2\alpha_1-\gamma_1\alpha_2} \\
&= -\frac{\delta+g}{2}\left[\frac{\alpha_2-\gamma_2+\alpha_1-\beta_1+\gamma_2\beta_1-\gamma_1\beta_2}{\beta_1(\alpha_2-\gamma_2)+\beta_2(\gamma_1-\alpha_1)+\gamma_2\alpha_1-\gamma_1\alpha_2} - 1\right] \equiv -\frac{(\delta+g)\mathcal{X}}{2} \\
Re(\lambda_{3,4}) &= \delta\frac{\alpha_2(1-\gamma_1)-\gamma_2(1-\alpha_1)+\alpha_1(1-\beta_2)-\beta_1(1-\alpha_2)}{\beta_1(\alpha_2-\gamma_2)+\beta_2(\gamma_1-\alpha_1)+\gamma_2\alpha_1-\gamma_1\alpha_2} \\
&+ g\frac{(1-\beta_1)(\alpha_2-\gamma_2)-\beta_2(\gamma_1-\alpha_1)+\gamma_1(\alpha_2-\beta_2)-(1-\gamma_2)(\beta_1-\alpha_1)}{\beta_1(\alpha_2-\gamma_2)+\beta_2(\gamma_1-\alpha_1)+\gamma_2\alpha_1-\gamma_1\alpha_2} \\
&= \frac{\delta}{2}\left[\frac{\alpha_2-\gamma_2+\alpha_1-\beta_1+\gamma_2\beta_1-\gamma_1\beta_2}{\beta_1(\alpha_2-\gamma_2)+\beta_2(\gamma_1-\alpha_1)+\gamma_2\alpha_1-\gamma_1\alpha_2} + 1\right] \\
&+ \frac{g}{2}\left[\frac{\alpha_2-\gamma_2+\alpha_1-\beta_1+\gamma_2\beta_1-\gamma_1\beta_2}{\beta_1(\alpha_2-\gamma_2)+\beta_2(\gamma_1-\alpha_1)+\gamma_2\alpha_1-\gamma_1\alpha_2} - 1\right] \equiv \frac{\delta\mathcal{Y}+g\mathcal{X}}{2}
\end{aligned}
$$

with $\mathcal{X} = \mathcal{Z} - 1$, $\mathcal{Y} = \mathcal{Z} + 1$ and

$$
\mathcal{Z} = \frac{\alpha_2-\gamma_2+\alpha_1-\beta_1+\gamma_2\beta_1-\gamma_1\beta_2}{\beta_1(\alpha_2-\gamma_2)+\beta_2(\gamma_1-\alpha_1)+\gamma_2\alpha_1-\gamma_1\alpha_2}
$$

Since $\mathcal{Y} > \mathcal{X}$, we easily derive the following results:

1- If $\mathcal{Y} < 0$ and thus $\mathcal{X} < 0$ we derive that $Re(\lambda_{1,2}) > 0$ and $Re(\lambda_{3,4}) < 0$ for any $\delta \geq 0$. Similarly, if $\mathcal{X} > 0$ and thus $\mathcal{Y} > 0$ we derive that $Re(\lambda_{1,2}) < 0$ and $Re(\lambda_{3,4}) > 0$ for any $\delta \geq 0$. Therefore, if $\mathcal{X}\mathcal{Y} > 0$, or equivalently $\mathcal{Z}^2 > 1$, then the steady state is saddle-point stable for any $\delta \geq 0$.

2- If $\mathcal{X} < 0$ and $\mathcal{Y} > 0$, or equivalently $-1 < \mathcal{Z} < 1$, then $Re(\lambda_{1,2}) > 0$ and $Re(\lambda_{3,4}) < 0$ for any $\delta \in [0, \delta^*)$ with $\delta^* = -g\mathcal{X}/\mathcal{Y}$ such that $Re(\lambda_{3,4})|_{\delta=\delta^*} = 0$. Moreover, we obviously derive that

$$
\left.\frac{dRe(\lambda_{3,4})}{d\delta}\right|_{\delta=\delta^*} > 0 \tag{42}
$$

with $Re(\lambda_{3,4}) > 0$ for any $\delta > \delta^*$. It follows that δ^* is a Hopf bifurcation value generically giving rise to non constant saddle-point stable (or unstable) closed orbits around the steady state in a right (or left) neighborhood of δ^*. The stability properties of the periodic orbit depend on the sign of a parameter which is obtained through a Taylor expansion of degree 3 of the dynamical system (16) on the dimension-2 center manifold associated to the two bifurcating eigenvalues. If this parameter is negative the periodic orbit occurs on the right neighborhood of δ^* and is stable on the dimension-2 center manifold, while if this parameter is positive the periodic orbit occurs on the left neighborhood of δ^* and is unstable on the dimension-2 center manifold (see Guckenheimer and Holmes [36], Theorem 3.4.2, p.151-152).

□

9.4 Proof of Corollary 1

Building on the results of Proposition 3, there is also the possibility to compute the periodicity on the cycle. Indeed, as shown initially by Hopf [38], and recalling that the bifurcating eigenvalues are such that $Im(\lambda_{3,4}) = \sqrt{-\Delta^1}|_{\delta=\delta^*}$, we derive from (42) that for every δ with $|\delta-\delta^*|$ sufficiently small, the periodic orbit is characterized by a period $T = T(\delta)$ such that

$$\lim_{|\delta-\delta^*|\to 0} T(\delta) = \frac{2\pi}{\sqrt{-\Delta^1}}\Big|_{\delta=\delta^*} = T^*$$

In other words, for δ very nearly equal to δ^*, the period of the (emergent) periodic orbits of (16) nearly equals the period of the concentric periodic orbits of the linearized system characterized by the Jacobiam matrix (35) with $\delta = \delta^*$, $\theta = 1$ and $\sigma = 0$.

□

9.5 Proof of Lemma 3

From (38) we derive that

$$\begin{aligned} p_1\frac{\partial c_0}{\partial p_1} + p_2\frac{\partial c_0}{\partial p_2} &= -\left(p_1^2\frac{\partial y_1}{\partial p_1} + p_1p_2\frac{\partial y_2}{\partial p_1} + p_1p_2\frac{\partial y_1}{\partial p_2} + p_2^2\frac{\partial y_2}{\partial p_2}\right) \\ &= -\begin{pmatrix} p_1 & p_2 \end{pmatrix}\begin{pmatrix} \frac{\partial y_1}{\partial p_1} & \frac{\partial y_1}{\partial p_2} \\ \frac{\partial y_2}{\partial p_1} & \frac{\partial y_2}{\partial p_2} \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \end{aligned}$$

Recalling that $c_0 = c_0(k_1, k_2, p_1, p_2)$, $c_1 = c_1(k_1, k_2, p_1, p_2)$ and $y_j = y_j(k_1, k_2, p_1, p_2)$, $j = 1, 2$, and using the first order conditions (6-12), we can define the maximized Hamiltonian $\bar{\mathcal{H}}$ as

$$\begin{aligned} \bar{\mathcal{H}}(k_1, k_2, p_1, p_2) &= \mathcal{H}(c_0, c_1, y_1, y_2, k_1, k_2, p_1, p_2) \\ &= \frac{(c_0^\theta c_1^{1-\theta})^{1-\sigma}-1}{1-\sigma} + c_0^{\theta(1-\sigma)-1}c_1^{(1-\theta)(1-\sigma)}p_1\left(y_1 - gk_1 - c_1\right) \\ &+ c_0^{\theta(1-\sigma)-1}c_1^{(1-\theta)(1-\sigma)}p_2\left(y_2 - gk_2\right) \end{aligned}$$

Considering that at the optimum we have $\frac{\partial \mathcal{H}}{\partial c_0} = \frac{\partial \mathcal{H}}{\partial c_1} = \frac{\partial \mathcal{H}}{\partial y_1} = \frac{\partial \mathcal{H}}{\partial y_2} = 0$, we find

$$\begin{aligned} \frac{\partial \bar{\mathcal{H}}}{\partial p_1} &= \frac{\partial \mathcal{H}}{\partial c_0}\frac{\partial c_0}{\partial p_1} + \frac{\partial \mathcal{H}}{\partial c_1}\frac{\partial c_1}{\partial p_1} + \frac{\partial \mathcal{H}}{\partial y_1}\frac{\partial y_1}{\partial p_1} + \frac{\partial \mathcal{H}}{\partial y_2}\frac{\partial y_2}{\partial p_1} + \frac{\partial \mathcal{H}}{\partial p_1} \\ &= \frac{\partial \mathcal{H}}{\partial p_1} = c_0^{\theta(1-\sigma)-1}c_1^{(1-\theta)(1-\sigma)}\left(y_1 - gk_1 - c_1\right) \\ \frac{\partial \bar{\mathcal{H}}}{\partial p_2} &= \frac{\partial \mathcal{H}}{\partial c_0}\frac{\partial c_0}{\partial p_2} + \frac{\partial \mathcal{H}}{\partial c_1}\frac{\partial c_1}{\partial p_2} + \frac{\partial \mathcal{H}}{\partial y_1}\frac{\partial y_1}{\partial p_2} + \frac{\partial \mathcal{H}}{\partial y_2}\frac{\partial y_2}{\partial p_2} + \frac{\partial \mathcal{H}}{\partial p_2} \\ &= \frac{\partial \mathcal{H}}{\partial p_2} = c_0^{\theta(1-\sigma)-1}c_1^{(1-\theta)(1-\sigma)}\left(y_2 - gk_2\right) \end{aligned}$$

We then easily derive at the steady state with $y_1 - gk_1 - c_1 = 0$ and $y_2 - gk_2 = 0$,

$$\begin{pmatrix} \frac{\partial^2 \bar{\mathcal{H}}}{\partial p_1^2} & \frac{\partial^2 \bar{\mathcal{H}}}{\partial p_1 \partial p_2} \\ \frac{\partial^2 \bar{\mathcal{H}}}{\partial p_2 \partial p_1} & \frac{\partial^2 \bar{\mathcal{H}}}{\partial p_2^2} \end{pmatrix} = c_0^{\theta(1-\sigma)-1} c_1^{(1-\theta)(1-\sigma)} \begin{pmatrix} \frac{\partial y_1}{\partial p_1} - \frac{\partial c_1}{\partial p_1} & \frac{\partial y_1}{\partial p_2} - \frac{\partial c_1}{\partial p_2} \\ \frac{\partial y_2}{\partial p_1} & \frac{\partial y_2}{\partial p_2} \end{pmatrix}$$

$$= c_0^{\theta(1-\sigma)-1} c_1^{(1-\theta)(1-\sigma)} \left(\left[\frac{\partial y}{\partial p}\right] - \begin{pmatrix} \frac{\partial c_1}{\partial p_1} & \frac{\partial c_1}{\partial p_2} \\ 0 & 0 \end{pmatrix} \right)$$

Since the maximized Hamiltonian is convex in the prices, the matrix on the right-hand-side is quasi-positive definite. Using (14), we know that this property must be true for any $\theta \in (0,1]$. It follows that the matrix $\left[\frac{\partial y}{\partial p}\right]$ must quasi-positive definite. This implies therefore that $p_1 \frac{\partial c_0}{\partial p_1} + p_2 \frac{\partial c_0}{\partial p_2} \leq 0$ and thus that $E_\sigma^\theta > 0$ is an increasing function of σ. □

9.6 Proof of Proposition 4

Let $\theta = 1$ and condition (23) hold. Consider then the expression of the Determinant as given by (19). We know that $\mathcal{D}_0^1 > 0$. We derive from Lemma 3 that $\mathcal{D}_\sigma^1$ is a monotonous decreasing function of σ with

$$\lim_{\sigma \to +\infty} \mathcal{D}_\sigma^1 = 0$$

It follows that at least one characteristic root is equal to zero which implies that all characteristic roots are real. We then derive that there exists $\bar{\sigma} > 0$ such that the characteristic roots are complex if and only if $\sigma \in [0, \bar{\sigma})$. To prove such a claim, let us first study the sign of Θ_1. Using (38) and (39) into Θ_1 allows to simplify its expression as follows:

$$\begin{aligned} \Theta_1 &= -\frac{\partial c_0}{\partial p_1} \left[\frac{\partial c_0}{\partial k_1} \left(\frac{\partial y_1}{\partial k_1} - g \right) + \frac{\partial c_0}{\partial k_2} \frac{\partial y_2}{\partial k_1} \right] - \frac{\partial c_0}{\partial p_2} \left[\frac{\partial c_0}{\partial k_1} \frac{\partial y_1}{\partial k_2} + \frac{\partial c_0}{\partial k_2} \left(\frac{\partial y_2}{\partial k_2} - g \right) \right] \\ &- \mathcal{A}_1 \frac{\partial c_0}{\partial k_1} - \mathcal{A}_2 \frac{\partial c_0}{\partial k_2} \end{aligned}$$

Straightforward computations using the expressions of $\mathcal{A}_1$ and $\mathcal{A}_2$ then give

$$\begin{aligned} \Theta_1 = &- \begin{pmatrix} p_1 & p_2 \end{pmatrix} \begin{pmatrix} \frac{\partial y_1}{\partial k_1} - g - \delta & \frac{\partial y_1}{\partial k_2} \\ \frac{\partial y_2}{\partial k_1} & \frac{\partial y_2}{\partial k_2} - g - \delta \end{pmatrix} \begin{pmatrix} \frac{\partial y_1}{\partial k_1} - g & \frac{\partial y_1}{\partial k_2} \\ \frac{\partial y_2}{\partial k_1} & \frac{\partial y_2}{\partial k_2} - g \end{pmatrix} \begin{pmatrix} \frac{\partial y_1}{\partial p_1} & \frac{\partial y_1}{\partial p_2} \\ \frac{\partial y_2}{\partial p_1} & \frac{\partial y_2}{\partial p_2} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \\ &- \begin{pmatrix} p_1 & p_2 \end{pmatrix} \begin{pmatrix} \frac{\partial y_1}{\partial k_1} - g - \delta & \frac{\partial y_1}{\partial k_2} \\ \frac{\partial y_2}{\partial k_1} & \frac{\partial y_2}{\partial k_2} - g - \delta \end{pmatrix}^2 \begin{pmatrix} \frac{\partial y_1}{\partial p_1} & \frac{\partial y_1}{\partial p_2} \\ \frac{\partial y_2}{\partial p_1} & \frac{\partial y_2}{\partial p_2} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \\ &+ \begin{pmatrix} p_1 & p_2 \end{pmatrix} \begin{pmatrix} \frac{\partial y_1}{\partial k_1} - g - \delta & \frac{\partial y_1}{\partial k_2} \\ \frac{\partial y_2}{\partial k_1} & \frac{\partial y_2}{\partial k_2} - g - \delta \end{pmatrix} \begin{pmatrix} \frac{\partial y_1}{\partial p_1} & \frac{\partial y_1}{\partial p_2} \\ \frac{\partial y_2}{\partial p_1} & \frac{\partial y_2}{\partial p_2} \end{pmatrix} \begin{pmatrix} \frac{\partial y_1}{\partial k_1} - g - \delta & \frac{\partial y_2}{\partial k_1} \\ \frac{\partial y_1}{\partial k_2} & \frac{\partial y_2}{\partial k_2} - g - \delta \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \end{aligned}$$

Denoting the vector $P = (p_1 \;\; p_2)$ and P^t its transpose, we can reformulate Θ_1 as a sum of three quadratic forms:

$$\begin{aligned} \Theta_1 &= -P \left[\frac{\partial y}{\partial k} - (g+\delta)I\right] \left[\frac{\partial y}{\partial k} - gI\right] \left[\frac{\partial y}{\partial p}\right] P^t - P \left[\frac{\partial y}{\partial k} - (g+\delta)I\right]^2 \left[\frac{\partial y}{\partial p}\right] P^t \\ &+ P \left[\frac{\partial y}{\partial k} - (g+\delta)I\right] \left[\frac{\partial y}{\partial p}\right] \left[\frac{\partial y}{\partial k} - (g+\delta)I\right]^t P^t \end{aligned}$$

Since the matrix $\left[\frac{\partial y}{\partial p}\right]$ is positive definite, the third quadratic form is necessarily positive.

Consider now the matrix $\left[\frac{\partial y}{\partial k} - (g+\delta)I\right]$. Under conditions (23) and (24) we get

$$\mathcal{D}et\left[\frac{\partial y}{\partial k}-(g+\delta)I\right] = \left(\frac{\partial y_1}{\partial k_1}-g-\delta\right)\left(\frac{\partial y_2}{\partial k_2}-g-\delta\right)-\frac{\partial y_1}{\partial k_2}\frac{\partial y_2}{\partial k_1} \geq 0$$
$$\mathcal{T}r\left[\frac{\partial y}{\partial k}-(g+\delta)I\right] = \frac{\partial y_1}{\partial k_1}-g-\delta+\frac{\partial y_2}{\partial k_2}-g-\delta < 0$$

It follows that the matrix $\left[\frac{\partial y}{\partial k}-(g+\delta)I\right]$ is quasi-negative definite with complex roots. Consider finally the matrix $\left[\frac{\partial y}{\partial k}-(g+\delta)I\right]^2$. We easily get

$$\left[\frac{\partial y}{\partial k}-(g+\delta)I\right]^2 = \begin{pmatrix} \left(\frac{\partial y_1}{\partial k_1}-g-\delta\right)^2+\frac{\partial y_1}{\partial k_2}\frac{\partial y_2}{\partial k_1} & \frac{\partial y_1}{\partial k_2}\left(\frac{\partial y_1}{\partial k_1}-g-\delta+\frac{\partial y_2}{\partial k_2}-g-\delta\right) \\ \frac{\partial y_2}{\partial k_1}\left(\frac{\partial y_1}{\partial k_1}-g-\delta+\frac{\partial y_2}{\partial k_2}-g-\delta\right) & \left(\frac{\partial y_2}{\partial k_2}-g-\delta\right)^2+\frac{\partial y_1}{\partial k_2}\frac{\partial y_2}{\partial k_1} \end{pmatrix}$$

and thus

$$\mathcal{D}et\left[\frac{\partial y}{\partial k}-(g+\delta)I\right]^2 = \left[\left(\frac{\partial y_1}{\partial k_1}-g-\delta\right)\left(\frac{\partial y_2}{\partial k_2}-g-\delta\right)-\frac{\partial y_1}{\partial k_2}\frac{\partial y_2}{\partial k_1}\right]^2 \geq 0$$
$$\mathcal{T}r\left[\frac{\partial y}{\partial k}-(g+\delta)I\right]^2 = \left(\frac{\partial y_1}{\partial k_1}-g-\delta\right)^2+\left(\frac{\partial y_2}{\partial k_2}-g-\delta\right)^2+2\frac{\partial y_1}{\partial k_2}\frac{\partial y_2}{\partial k_1}$$

We then have $\mathcal{T}r\left[\frac{\partial y}{\partial k}-(g+\delta)I\right]^2 < 0$ if and only if condition (26) holds. Moreover, considering the expression of the discriminant Δ^1 as given by (21), we easily show that

$$\Delta^1 = \left[2(\delta+g)-\frac{\partial y_1}{\partial k_1}-\frac{\partial y_2}{\partial k_2}\right]^2-4\left[\left(\delta+g-\frac{\partial y_1}{\partial k_1}\right)\left(\delta+g-\frac{\partial y_2}{\partial k_2}\right)-\frac{\partial y_1}{\partial k_2}\frac{\partial y_2}{\partial k_1}\right]$$
$$= \mathcal{T}r\left[\frac{\partial y}{\partial k}-(g+\delta)I\right]^2-2\mathcal{D}et\left[\frac{\partial y}{\partial k}-(g+\delta)I\right]$$

Therefore, under conditions (24) and (26), we have $\Delta^1 < 0$. It follows that under conditions (24) and (26), the matrix $\left[\frac{\partial y}{\partial k}-(g+\delta)I\right]^2$ is quasi-negative definite with complex roots. But then tedious but straightforward computations allow to show that under conditions (24) and (26), the matrices $\left[\frac{\partial y}{\partial k}-(g+\delta)I\right]\left[\frac{\partial y}{\partial k}-gI\right]\left[\frac{\partial y}{\partial p}\right]$ and $\left[\frac{\partial y}{\partial k}-(g+\delta)I\right]^2\left[\frac{\partial y}{\partial p}\right]$ are definite-negative with real roots. It follows that the first and second quadratic forms are necessarily negative and thus $\Theta_1 > 0$.

Recall now that when $\theta = 1$, the characteristic polynomial can be written as

$$\mathcal{P}^1_\sigma(\lambda) = \mathcal{P}^0_1(\lambda)-\frac{\sigma}{E^1_\sigma c_0}\tilde{\mathcal{P}}(\lambda) \tag{43}$$

with

$$\tilde{\mathcal{P}}(\lambda) = \lambda^2\Theta_1-\lambda\delta\Theta_1-\mathcal{D}^1_\sigma E^1_\sigma\left(p_1\frac{\partial c_0}{\partial p_1}+p_2\frac{\partial c_0}{\partial p_2}\right) \tag{44}$$

which does not depend on σ. Equivalently, any characteristic root λ must be a solution of

$$\mathcal{P}^1_0(\lambda) = \frac{\sigma}{E^1_\sigma c_0}\tilde{\mathcal{P}}(\lambda) \tag{45}$$

We know that $\mathcal{P}^1_0(\lambda)$ is a degree-four polynomial with $\mathcal{P}^1_0(0) = \mathcal{D}^1_0 > 0$ and $\lim_{\lambda\to\pm\infty}\mathcal{P}^1_0(\lambda) = +\infty$. Using Lemma 3, we also know, since $\Theta_1 > 0$, that $\tilde{\mathcal{P}}(\lambda)$ is a degree-two polynomial with $\tilde{\mathcal{P}}(0) > 0$ and $\lim_{\lambda\to\pm\infty}\tilde{\mathcal{P}}(\lambda) = +\infty$. We conclude that starting from the case $\sigma = 0$ in which the characteristic roots are all complex, if σ is increased, the degree-two polynomial $\sigma\tilde{\mathcal{P}}(\lambda)/(E^1_\sigma c_0)$, which is initially located below $\mathcal{P}^1_0(\lambda)$, is going up monotonically and comes closer and closer to $\mathcal{P}^1_0(\lambda)$ until $\sigma = \bar{\sigma}$ where it will be characterized by two tangency points with $\mathcal{P}_0^1(\lambda)$ where the imaginary part of the four roots is equal to zero and there exist two pairs of double real roots. Actually, when $\sigma \in (0, \bar{\sigma})$, the size of the imaginary part of the roots is proportional to the distance between the two polynomials and is then decreasing as σ is increased. Under conditions (24) and (26), it follows therefore that when $\sigma \in [0, \bar{\sigma})$, the characteristic roots are complex.

When $\sigma = \bar{\sigma}$, the four characteristic roots are given by two pairs of real double roots. In this case, we also derive from (40) with $b = 0$ that

$$\begin{aligned} \mathcal{S}_{\bar{\sigma}}^1 &= \delta^2 + 2a(\delta - a) \\ \Sigma_{\bar{\sigma}}^1 &= 2\delta a(\delta - a) = \mathcal{T}\tfrac{S_{\bar{\sigma}}^1 - \delta^2}{2} = \delta\left(S_{\bar{\sigma}}^1 - \delta^2\right) \\ \mathcal{D}_{\bar{\sigma}}^1 &= a^2(\delta - a)^2 = \left(\tfrac{S_{\bar{\sigma}}^1 - \delta^2}{2}\right)^2 \end{aligned} \tag{46}$$

We can therefore write the characteristic polynomial as follows

$$\mathcal{P}_{\bar{\sigma}}^1(\lambda) = \lambda^4 - \lambda^3\mathcal{T} + \lambda^2\mathcal{S}_{\bar{\sigma}}^1 - \lambda\mathcal{T}\left(\tfrac{S_{\bar{\sigma}}^1 - \delta^2}{2}\right) + \left(\tfrac{S_{\bar{\sigma}}^1 - \delta^2}{2}\right)^2 \tag{47}$$

We then observe that $\mathcal{P}_{\bar{\sigma}}^1(\lambda)$ is a quasi-palindromic polynomial such that

$$\mathcal{P}_{\bar{\sigma}}^1\left(\tfrac{S_{\bar{\sigma}}^1 - \delta^2}{2}\lambda\right) = \tfrac{4\lambda^4}{(S_{\bar{\sigma}}^1 - \delta^2)^2}\mathcal{P}_{\bar{\sigma}}^1\left(\tfrac{S_{\bar{\sigma}}^1 - \delta^2}{2\lambda}\right)$$

Building on this property, let us consider the change of variable

$$z = \lambda + \tfrac{S_{\bar{\sigma}}^1 - \delta^2}{2\lambda} \tag{48}$$

We then derive from (47)

$$\begin{aligned} \tfrac{\mathcal{P}_{\bar{\sigma}}^1(\lambda)}{\lambda^2} &= \lambda^2 - \mathcal{T}\left(\lambda + \tfrac{S_{\bar{\sigma}}^1 - \delta^2}{2\lambda}\right) + \mathcal{S}_{\bar{\sigma}}^1 + \left(\tfrac{S_{\bar{\sigma}}^1 - \delta^2}{2\lambda}\right)^2 \\ &= z^2 - \mathcal{T}z + \delta^2 = (z - \delta)^2 = 0 \end{aligned} \tag{49}$$

We can thus explicitly compute the two double real characteristic roots. Starting from $z = \delta$ we obtain indeed from (48) that the roots are solutions of the following polynomial:

$$\lambda^2 - \delta\lambda + \tfrac{S_{\bar{\sigma}}^1 - \delta^2}{2} = 0$$

namely

$$\lambda_1 = \tfrac{\delta + \sqrt{3\delta^2 - 2S_{\bar{\sigma}}^1}}{2} \text{ and } \lambda_2 = \tfrac{\delta - \sqrt{3\delta^2 - 2S_{\bar{\sigma}}^1}}{2} = \delta - \lambda_1$$

Using (46), we derive $3\delta^2 - 2S_{\bar{\sigma}}^1 = (\delta - 2a)^2 \geq 0$ and thus $\lambda_1 = \delta - a > 0$ and $\lambda_2 = a$. It follows that there exists $\bar{\delta} > 0$ such that when $\delta \in [0, \bar{\delta})$ the two roots λ_1 and λ_2 have opposite sign. We need now to locate $\bar{\delta}$ with respect to the bound δ^* as given in Proposition 3.

From Proposition 2 with $\theta = 1$, we derive that

$$\begin{aligned} \mathcal{S}_\sigma^1 &= \left(\tfrac{\partial y_1}{\partial k_1} - g\right)\left(\tfrac{\partial y_2}{\partial k_2} - g\right) - \tfrac{\partial y_1}{\partial k_2}\tfrac{\partial y_2}{\partial k_1} + \left(\delta + g - \tfrac{\partial y_1}{\partial k_1}\right)\left(\delta + g - \tfrac{\partial y_2}{\partial k_2}\right) - \tfrac{\partial y_1}{\partial k_2}\tfrac{\partial y_2}{\partial k_1} \\ &+ \left(\delta + g - \tfrac{\partial y_1}{\partial k_1} + \delta + g - \tfrac{\partial y_2}{\partial k_2}\right)\left(\tfrac{\partial y_1}{\partial k_1} - g + \tfrac{\partial y_2}{\partial k_2} - g\right) - \tfrac{\sigma\Theta_1}{E_\sigma^1 c_0} \end{aligned}$$

with $\partial\mathcal{S}_\sigma^1/\partial\sigma = -\Theta_1/(E_\sigma^1 c_0) < 0$, implying that $\mathcal{S}_\sigma^1$ is a monotone decreasing function of σ. The value δ^* is such that $Re(\lambda_{3,4}) = 0$, i.e.

$$\tfrac{\partial y_1}{\partial k_1} + \tfrac{\partial y_2}{\partial k_2} - 2g = 0$$

We then get

$$\mathcal{S}^1_\sigma\Big|_{\delta=\delta^*} = \delta^{*2} - 2\left[\left(\frac{\partial y_1}{\partial k_1} - g\right)^2 + \frac{\partial y_1}{\partial k_2}\frac{\partial y_2}{\partial k_1}\right] - \frac{\sigma\Theta_1}{E^1_\sigma c_0}$$

and under conditions (24) and (26)

$$\Delta^1 = \left(\frac{\partial y_1}{\partial k_1} - g\right)^2 + \frac{\partial y_1}{\partial k_2}\frac{\partial y_2}{\partial k_1} < 0$$

It follows therefore that $\mathcal{S}^1_0|_{\delta=\delta^*} > 0$. We then conclude that there exists $\tilde{\delta} > 0$ such that $\mathcal{S}^1_\sigma|_{\delta=\delta^*} = 0$. If we can show that when $\delta = \delta^*$ and $\sigma = \tilde{\sigma}$, the characteristic roots are still complex, then we can conclude that $\bar{\delta} > \delta^*$ and $\tilde{\sigma} < \bar{\sigma}$. We derive from (40) that $\mathcal{D}_{\tilde{\sigma}}|_{\delta=\delta^*} > (\delta^{*2}/2)^2$ and thus that

$$\mathcal{P}^\theta_{\tilde{\sigma}}(\lambda)\Big|_{\delta=\delta^*} > \lambda^4 - \lambda^3 2\delta^* - \lambda\delta^{*3} + \frac{\delta^{*4}}{4} = \frac{\left(\delta^{*2} + 2\lambda\delta^* - 2\lambda^2\right)^2}{4} \geq 0 \tag{50}$$

It is easy to see that the polynomial on the rhs of equation (50) is characterized by two double real roots of opposite sign. We then conclude that the polynomial $\mathcal{P}^\theta_{\tilde{\sigma}}(\lambda)$ has four complex roots and thus that when $\delta = \delta^*$, $\tilde{\sigma} < \bar{\sigma}$. Therefore, we derive that $\mathcal{S}^1_{\tilde{\sigma}} < 0$ and the roots λ_1 and λ_2 have opposite sign as long as $\delta < \bar{\delta}$ with $\bar{\delta} > \delta^*$. □

9.7 Proof of Theorem 1

Recall from (43)-(45) that any characteristic root λ must be a solution of

$$\mathcal{P}^1_0(\lambda) = \frac{\sigma}{E^1_\sigma c_0}\tilde{\mathcal{P}}(\lambda)$$

As explained in the Proof of Proposition 4, starting from the case $\sigma = 0$ in which the characteristic roots are all complex, if σ is increased, $\sigma\tilde{\mathcal{P}}(\lambda)/(E^1_\sigma c_0)$, which is initially located below $\mathcal{P}^1_0(\lambda)$, is going up monotonically and comes closer and closer to $\mathcal{P}^1_0(\lambda)$ until it will be characterized by two tangency points with $\mathcal{P}^1_0(\lambda)$ where the imaginary part of the four roots is equal to zero and there exist two pairs of double real roots. When $\sigma \in (0, \bar{\sigma})$, the size of the real and imaginary parts of the roots are proportional to the distance between the two polynomials and are then decreasing as σ is increased.

Consider first the case where condition (24) of Proposition 3 holds with $\delta < \delta^*$. When $\sigma = 0$ the steady state is saddle-point stable with one pair of complex conjugate characteristic roots having positive real parts and one pair of complex conjugate characteristic roots having negative real parts. When σ is increased, the negative real part is decreasing and remains thus negative for any $\sigma < \bar{\sigma}$, since when $\sigma = \bar{\sigma}$, there is one double negative root and one double positive root. There is no Hopf bifucation and the steady state remains saddle-point stable for any $\sigma \in (0, \bar{\sigma})$.

Consider now the case where condition (24) of Proposition 3 holds with $\delta > \delta^*$. When $\sigma = 0$ the steady state is totally unstable with two pairs of complex conjugate characteristic roots having positive real parts. When σ is increased, the real part of one pair is decreasing until it reaches zero leading thus to a pair of purely imaginary roots. This case is obtained for a value $\sigma^* < \bar{\sigma}$ and implies a Hopf bifurcation. When $\sigma \in (\sigma^*, \bar{\sigma})$, the steady state is saddle-point stable. □

9.8 Proof of Proposition 5

Using the expressions of the input coefficients (2) and Lemma 9.1, we easily derive that the discriminant Δ^θ as given by (27) is negative if and only if

$$\begin{aligned}
&\Big[\theta\left[\alpha_2(1-\gamma_1)-\gamma_2(1-\alpha_1)+\beta_1(1-\alpha_2)-\alpha_1(1-\beta_2)\right] \\
- &\ (1-\theta)[\gamma_2(1-\beta_1)-\beta_2(1-\gamma_1)]\Big]^2 \\
< &\ 4\theta[\alpha_2(1-\beta_1)-\beta_2(1-\alpha_1)]\Big[\theta[\gamma_1(1-\alpha_2)-\alpha_1(1-\gamma_2)] \\
- &\ (1-\theta)[\beta_1(1-\gamma_2)-\gamma_1(1-\beta_2)]\Big]
\end{aligned}$$

Since this inequality cannot be satisfied when $\theta = 0$, we derive that there exists $\underline{\theta} \in (0,1)$ such that complex characteristic roots are obtained if and only if condition (23) holds and $\theta \in (\underline{\theta}, 1]$. We also have $\Delta^{\underline{\theta}} = 0$.

We need now to check whether there may exist a value $\theta^* \in (\underline{\theta}, 1)$ such that the real part of a pair of characteristic roots is equal to zero. We know that under condition (23) and $\theta \in (\underline{\theta}, 1]$, we have $\Delta^\theta < 0$ or equivalently

$$\begin{aligned}
0 < &\left[\theta\left(\delta+g-\tfrac{\partial y_1}{\partial k_1}+\delta+g-\tfrac{\partial y_2}{\partial k_2}\right)+(1-\theta)\left(\delta+g-\tfrac{\partial y_1}{\partial k_1}-\tfrac{p_2}{p_1}\tfrac{\partial y_2}{\partial k_1}\right))\right]^2 \\
&< 4\left[\left(\delta+g-\tfrac{\partial y_1}{\partial k_1}\right)\left(\delta+g-\tfrac{\partial y_2}{\partial k_2}\right)-\tfrac{\partial y_1}{\partial k_2}\tfrac{\partial y_2}{\partial k_1}\right]
\end{aligned}$$

Considering $\lambda_{1,2}$ as given by (28), we derive that $Re(\lambda_{1,2}) = 0$ if and only if there is a value θ^* such that

$$\theta^* = -\frac{\delta+g-\frac{\partial y_1}{\partial k_1}-\frac{p_2}{p_1}\frac{\partial y_2}{\partial k_1}}{\delta+g-\frac{\partial y_2}{\partial k_2}+\frac{p_2}{p_1}\frac{\partial y_2}{\partial k_1}} < 1$$

If such a value exists, then it must be larger than $\underline{\theta}$ since under condition (23) we have

$$\Delta^{\theta^*} = -4\left[\left(\delta+g-\tfrac{\partial y_1}{\partial k_1}\right)\left(\delta+g-\tfrac{\partial y_2}{\partial k_2}\right)-\tfrac{\partial y_1}{\partial k_2}\tfrac{\partial y_2}{\partial k_1}\right] < 0$$

We need now to prove that such a value θ^* exists. Using the expression (28), we derive the real parts of the complex characteristic roots such that

$$\begin{aligned}
Re(\lambda_{1,2}) &= -\tfrac{\delta+g}{2\theta}\Bigg\{\tfrac{\alpha_2-\gamma_2+\alpha_1-\beta_1+\gamma_2\beta_1-\gamma_1\beta_2}{\beta_1(\alpha_2-\gamma_2)+\beta_2(\gamma_1-\alpha_1)+\gamma_2\alpha_1-\gamma_1\alpha_2}-1 \\
&+\ (1-\theta)\left[\tfrac{\beta_2-\alpha_2+\beta_1-\alpha_1}{\beta_1(\alpha_2-\gamma_2)+\beta_2(\gamma_1-\alpha_1)+\gamma_2\alpha_1-\gamma_1\alpha_2}+1\right]\Bigg\} \\
&\equiv -\tfrac{(\delta+g)[\mathcal{X}+(1-\theta)\mathcal{V}]}{2\theta} = -\tfrac{(\delta+g)[\mathcal{X}+\mathcal{V}-\theta\mathcal{V}]}{2\theta} \\
Re(\lambda_{3,4}) &= \tfrac{1}{2\theta}\Bigg\{\theta\delta\left[\tfrac{\alpha_2-\gamma_2+\alpha_1-\beta_1+\gamma_2\beta_1-\gamma_1\beta_2}{\beta_1(\alpha_2-\gamma_2)+\beta_2(\gamma_1-\alpha_1)+\gamma_2\alpha_1-\gamma_1\alpha_2}+1\right] \\
&+\ \theta g\left[\tfrac{\alpha_2-\gamma_2+\alpha_1-\beta_1+\gamma_2\beta_1-\gamma_1\beta_2}{\beta_1(\alpha_2-\gamma_2)+\beta_2(\gamma_1-\alpha_1)+\gamma_2\alpha_1-\gamma_1\alpha_2}-1\right] \\
&-\ (1-\theta)(\delta+g)\tfrac{\gamma_2(1-\beta_1)-\beta_2(1-\gamma_1)}{\beta_1(\alpha_2-\gamma_2)+\beta_2(\gamma_1-\alpha_1)+\gamma_2\alpha_1-\gamma_1\alpha_2}\Bigg\} \\
&\equiv \tfrac{\theta(\delta\mathcal{Y}+g\mathcal{X})-(1-\theta)(\delta+g)\mathcal{W}}{2\theta} = \tfrac{\theta[\delta\mathcal{Y}+g\mathcal{X}+(\delta+g)\mathcal{W}]-(\delta+g)\mathcal{W}}{2\theta}
\end{aligned}$$

with

$$\mathcal{V} = \frac{\beta_2-\alpha_2+\beta_1-\alpha_1}{\beta_1(\alpha_2-\gamma_2)+\beta_2(\gamma_1-\alpha_1)+\gamma_2\alpha_1-\gamma_1\alpha_2} + 1 \text{ and } \mathcal{W} = \frac{\gamma_2(1-\beta_1)-\beta_2(1-\gamma_1)}{\beta_1(\alpha_2-\gamma_2)+\beta_2(\gamma_1-\alpha_1)+\gamma_2\alpha_1-\gamma_1\alpha_2}$$

Note that $\mathcal{W} = -(\mathcal{X}+\mathcal{V})$. Let us then assume that $\mathcal{X} < 0$ and $\mathcal{Y} > 0$. We immediately derive that $Re(\lambda_{1,2}) > 0$ when $\theta = 1$, while the sign of $Re(\lambda_{1,2})$ is given by the sign of $-(\mathcal{X}+\mathcal{V})$ when θ is close to zero. Therefore, as $Re(\lambda_{1,2})$ is a monotone function of θ, it follows that there exists θ^* such that $Re(\lambda_{1,2})|_{\theta=\theta^*} = 0$ if and only if $\mathcal{X}+\mathcal{V} > 0$ or equivalently if and only if condition (29) holds. It is also worthwhile to notice that condition (29) is equivalent to $\partial c_0/\partial k_1 < 0$ and thus $\partial c_1/\partial k_1 < 0$, and implies $\mathcal{V} > 0$. The bifurcation value θ^* is then given by

$$\theta^* = \frac{\mathcal{X}+\mathcal{V}}{\mathcal{V}} = \frac{\gamma_2(1-\beta_1)-\beta_2(1-\gamma_1)}{(\beta_1-\alpha_1)(1-\gamma_2)+(\beta_2-\alpha_2)(1-\gamma_1)+\beta_1\alpha_2-\alpha_1\beta_2} \in (\underline{\theta}, 1)$$

We then get $Re(\lambda_{1,2}) > 0$ when $\theta \in (\theta^*, 1]$ and $Re(\lambda_{1,2}) < 0$ when $\theta \in (\underline{\theta}, \theta^*)$, with obviously

$$\left.\frac{dRe(\lambda_{1,2})}{d\theta}\right|_{\theta=\theta^*} > 0 \tag{51}$$

As explained in the proof of Proposition 3, the stability properties of the periodic orbit depend on the sign of a parameter which is obtained through a Taylor expansion of degree 3 of the dynamical system (16) on the dimension-2 center manifold associated to the two bifurcating eigenvalues. If this parameter is negative the periodic orbit occurs on the right neighborhood of θ^* and is stable on the dimension-2 center manifold, while if this parameter is positive the periodic orbit occurs on the left neighborhood of δ^* and is unstable on the dimension-2 center manifold (see Guckenheimer and Holmes [36], Theorem 3.4.2, p.151-152).

Let us consider now $Re(\lambda_{3,4})$. As proved by Proposition 3, there exists $\delta^* > 0$ such that $\delta\mathcal{Y} + g\mathcal{X} < 0$ when $\delta < \delta^*$ and $\delta\mathcal{Y} + g\mathcal{X} > 0$ when $\delta > \delta^*$. Since

$$Re(\lambda_{3,4}) = \frac{\theta(\delta\mathcal{Y}+g\mathcal{X})+(1-\theta)(\delta+g)(\mathcal{X}+\mathcal{V})}{2\theta}$$

we immediately conclude that when $\delta > \delta^*$ and condition (29) holds, $Re(\lambda_{3,4}) > 0$ for any $\theta \in (0, 1]$. If we consider instead that $\delta < \delta^*$ and that condition (29) still holds, then we can also find a bifurcation value $\hat{\theta}$ such that $Re(\lambda_{3,4}) = 0$. Indeed, we have in this case that $Re(\lambda_{3,4}) < 0$ when $\theta = 1$ while $Re(\lambda_{3,4}) > 0$ when θ is close to zero. Since $Re(\lambda_{3,4})$ is also a monotone function of θ, we conclude that there exists $\hat{\theta}$ such that $Re(\lambda_{3,4})|_{\theta=\hat{\theta}} = 0$ with

$$\hat{\theta} = \frac{(\delta+g)(\mathcal{X}+\mathcal{V})}{(\delta+g)(\mathcal{X}+\mathcal{V})-(\delta\mathcal{Y}+g\mathcal{X})}$$

and

$$\left.\frac{dRe(\lambda_{3,4})}{d\theta}\right|_{\theta=\hat{\theta}} < 0 \tag{52}$$

It is easy to derive that when $\delta < \delta^*$ and condition (29) holds, $\hat{\theta} \in (\theta^*, 1)$. As previously, if the parameter obtained through a Taylor expansion of degree 3 of the dynamical system (16) on the dimension-2 center manifold associated to the two bifurcating eigenvalues is negative the periodic orbit occurs on the left neighborhood of θ^* and is stable on the dimension-2 center manifold, while if it is positive the periodic orbit occurs on the right neighborhood of δ^* and is unstable on the dimension-2 center manifold (see Guckenheimer and Holmes [36], Theorem 3.4.2, p.151-152).

□

9.9 Proof of Corollary 2

In case 1- of Proposition 5, since the bifurcating eigenvalues are such that $Im(\lambda_{1,2}) = \sqrt{-\Delta^{\theta^*}}$, we derive from (51) that for every θ with $|\theta - \theta^*|$ sufficiently small, the periodic orbit is characterized by a period $T = T(\theta)$ such that

$$\lim_{|\theta-\theta^*|\to 0} T(\theta) = \frac{2\pi}{\sqrt{-\Delta^{\theta^*}}} = T^*$$

In other words, for θ very nearly equal to θ^*, the period of the (emergent) periodic orbits of (16) nearly equals the period of the concentric periodic orbits of the linearized system characterized by the Jacobian matrix (35) with $\theta = \theta^*$ and $\sigma = 0$.

In case 2- of Proposition 5, as the bifurcating eigenvalues are such that $Im(\lambda_{3,4}) = \sqrt{-\Delta^{\theta}}$ with $\theta = \theta^*$ or $\hat{\theta}$, we first derive the same result as in the previous case 1-. We also derive from (52) that for every θ with $|\theta - \hat{\theta}|$ sufficiently small, the periodic orbit is characterized by a period $T = T(\theta)$ such that

$$\lim_{|\theta-\hat{\theta}|\to 0} T(\theta) = \frac{2\pi}{\sqrt{-\Delta^{\hat{\theta}}}} = \hat{T}$$

In other words, for θ very nearly equal to $\hat{\theta}$, the period of the (emergent) periodic orbits of (16) nearly equals the period of the concentric periodic orbits of the linearized system characterized by the Jacobian matrix (35) with $\theta = \hat{\theta}$ and $\sigma = 0$. $\square$

9.10 Proof of Theorem 2

Tedious but straightforward computations using (38)-(39) and the expressions of $\mathcal{A}_1$ and $\mathcal{A}_2$ give

$$\begin{aligned}
\Theta_\theta \;=\; & -\begin{pmatrix} p_1 & p_2 \end{pmatrix} \begin{pmatrix} \frac{\partial y_1}{\partial k_1} - g - \delta & \frac{\partial y_1}{\partial k_2} \\ \frac{\partial y_2}{\partial k_1} & \frac{\partial y_2}{\partial k_2} - g - \delta \end{pmatrix} \begin{pmatrix} \frac{\partial y_1}{\partial k_1} - g - \frac{\partial c_1}{\partial k_1} & \frac{\partial y_1}{\partial k_2} - \frac{\partial c_1}{\partial k_2} \\ \frac{\partial y_2}{\partial k_1} & \frac{\partial y_2}{\partial k_2} - g \end{pmatrix} \\
\times & \begin{pmatrix} \frac{\partial y_1}{\partial p_1} - \frac{\partial c_1}{\partial p_1} & \frac{\partial y_1}{\partial p_2} - \frac{\partial c_1}{\partial p_2} \\ \frac{\partial y_2}{\partial p_1} & \frac{\partial y_2}{\partial p_2} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \\
- & \begin{pmatrix} p_1 & p_2 \end{pmatrix} \begin{pmatrix} \frac{\partial y_1}{\partial k_1} - g - \delta & \frac{\partial y_1}{\partial k_2} \\ \frac{\partial y_2}{\partial k_1} & \frac{\partial y_2}{\partial k_2} - g - \delta \end{pmatrix}^2 \begin{pmatrix} \frac{\partial y_1}{\partial p_1} & \frac{\partial y_1}{\partial p_2} \\ \frac{\partial y_2}{\partial p_1} & \frac{\partial y_2}{\partial p_2} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \\
+ & \begin{pmatrix} p_1 & p_2 \end{pmatrix} \begin{pmatrix} \frac{\partial y_1}{\partial k_1} - g - \delta & \frac{\partial y_1}{\partial k_2} \\ \frac{\partial y_2}{\partial k_1} & \frac{\partial y_2}{\partial k_2} - g - \delta \end{pmatrix} \begin{pmatrix} \frac{\partial y_1}{\partial p_1} & \frac{\partial y_1}{\partial p_2} \\ \frac{\partial y_2}{\partial p_1} & \frac{\partial y_2}{\partial p_2} \end{pmatrix} \begin{pmatrix} \frac{\partial y_1}{\partial k_1} - g - \delta & \frac{\partial y_2}{\partial k_1} \\ \frac{\partial y_1}{\partial k_2} & \frac{\partial y_2}{\partial k_2} - g - \delta \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \\
+ & \frac{\partial c_0}{\partial k_1} \left\{ \left(2\delta - \frac{\partial c_1}{\partial k_1}\right) \left(p_1 \frac{\partial c_1}{\partial p_1} + p_2 \frac{\partial c_1}{\partial p_2}\right) - \frac{\partial c_0}{\partial k_1}\frac{\partial c_1}{\partial p_1} - \frac{\partial c_0}{\partial k_2}\frac{\partial c_1}{\partial p_2} \right\}
\end{aligned}$$

Denoting the vector $P = (p_1 \;\; p_2)$ and P^t its transpose, we can reformulate Θ_θ as follows:

$$\begin{aligned}
\Theta_\theta \;=\; & -P\left[\frac{\partial y}{\partial k} - (g+\delta)I\right][\mathcal{J}_1][\mathcal{J}_2]P^t - P\left[\frac{\partial y}{\partial k} - (g+\delta)I\right]^2\left[\frac{\partial y}{\partial p}\right]P^t \\
+ & \; P\left[\frac{\partial y}{\partial k} - (g+\delta)I\right]\left[\frac{\partial y}{\partial p}\right]\left[\frac{\partial y}{\partial k} - (g+\delta)I\right]^t P^t \\
+ & \; \frac{\partial c_0}{\partial k_1}\left\{ \left(2\delta - \frac{\partial c_1}{\partial k_1}\right)\left(p_1\frac{\partial c_1}{\partial p_1} + p_2\frac{\partial c_1}{\partial p_2}\right) - P\left[\frac{\partial y}{\partial k} - (g+\delta)I\right]\left[\frac{\partial y}{\partial p}\right]P^t \right\}
\end{aligned}$$

with $[\mathcal{J}_1]$ and $[\mathcal{J}_2]$ the 2×2 matrices as defined in (35). We know that the matrices $\left[\frac{\partial y}{\partial p}\right]$ and $[\mathcal{J}_2]$ are quasi-positive definite. As shown before, under conditions (24) and (26), the matrices $\left[\frac{\partial y}{\partial k} - (g+\delta)I\right]$ and $\left[\frac{\partial y}{\partial k} - (g+\delta)I\right]^2$ are quasi-negative definite. Consider then matrix $[\mathcal{J}_1]$. We have

$$\mathcal{D}et\,[\mathcal{J}_1] \;=\; \left(\frac{\partial y_1}{\partial k_1} - g - \frac{\partial c_1}{\partial k_1}\right)\left(\frac{\partial y_2}{\partial k_2} - g\right) - \frac{\partial y_2}{\partial k_1}\left(\frac{\partial y_1}{\partial k_2} - \frac{\partial c_1}{\partial k_2}\right) \geq 0$$

and tedious computations show that under conditions (24) and (26) the matrices

$$\left[\frac{\partial y}{\partial k} - (g+\delta)I\right][\mathcal{J}_1]\,[\mathcal{J}_2] \text{ and } \left[\frac{\partial y}{\partial k} - (g+\delta)I\right]\left[\frac{\partial y}{\partial p}\right]$$

are respectively negative definite and positive definite with real roots. Recalling that condition (29) implies $\partial c_0/\partial k_1 < 0$ and thus $\partial c_1/\partial k_1 < 0$. It follows therefore that Θ_θ is positive for any $\theta \in (0,1]$.

Following the same lines as in the proof of Proposition 4, for any given $\theta \in (0,1]$, there exists $\bar{\sigma}_\theta > 0$ such that the characteristic roots are complex if and only if $\sigma \in [0, \bar{\sigma}_\theta)$ with $\bar{\sigma}_\theta$ and increasing function of θ. Moreover, consider that the derivatives $\partial y_1/\partial k_1$, $\partial y_1/\partial k_2$, $\partial y_2/\partial k_1$, $\partial y_2/\partial k_2$ and $\partial c_0/\partial k_1$ do not depend on θ. Tedious but straightforward computations allow to show that under conditions (24), (26) and (29), Γ_θ as given by

$$\begin{aligned}\Gamma_\theta \;=\; & c_0(1-\theta)\Bigg[\left(\delta + g - \frac{\partial y_1}{\partial k_1}\right)\left(\delta + g - \frac{\partial y_2}{\partial k_2}\right) - \frac{\partial y_1}{\partial k_2}\frac{\partial y_2}{\partial k_1} + \frac{\left(\frac{\partial y_1}{\partial k_1} - g + \frac{\partial y_2}{\partial k_2} - g\right)\frac{\partial c_0}{\partial k_1}}{p_1} \\ & - \frac{1-\theta}{\theta p_1^2}\left(\frac{\partial c_0}{\partial k_1}\right)^2\Bigg]\end{aligned}$$

is a hump-shape function of θ over $[0,1]$ with $\Gamma_1 = 0$, $\partial\Gamma_\theta/\partial\theta|_{\theta=0} > 0$, $\lim_{\theta\to 0}\Gamma_\theta < 0$, and, when $\theta = \theta^*$,

$$\Gamma_{\theta^*} = c_0(1-\theta^*)\left[\left(\delta + g - \frac{\partial y_1}{\partial k_1}\right)\left(\delta + g - \frac{\partial y_2}{\partial k_2}\right) - \frac{\partial y_1}{\partial k_2}\frac{\partial y_2}{\partial k_1} + \frac{2\delta}{p_1}\frac{\partial c_0}{\partial k_1}\right] > 0$$

We conclude therefore that there exists $\tilde{\theta} \in [\underline{\theta}, \theta^*)$ such that $\Gamma_\theta \geq 0$ for any $\theta \in [\tilde{\theta}, 1]$. The rest of the proof, available upon request, follows exactly the same lines as the proof of Proposition 4 and Theorem 1.

□

10 Appendix 3: 3-sector decomposition

10.1 Investment-consumption decomposition

A mapping between industries and sectors can be created using the benchmark Input-Output (IO) tables by the Bureau of Economic Analysis (BEA). Whole industries are not exclusively consumption or investment industries, but produce a mixture of investment goods and consumption goods.[42] Yet the literature has either categorized entire industries into "consumption" or "investment" (e.g., Huffman and Wynne [39]) or considered five "aggregate" industries (agriculture, manufactured consumption, services, equipment, and construction) and then aggregated them within a two-sector model (e.g., Valentinyi and Herrendorf [63]) in

[42] For example, as pointed out by Valentinyi and Herrendorf [63], cars sold to consumers are counted as consumption whereas they are counted as investment when sold to firms.

which consumption includes manufactured consumption, agriculture, and services and investment includes construction and equipment. In the following, we use both approaches—the categorization-based approach and an "aggregate" industry-based approach. Starting from a different classification, we then proceed as in Valentinyi and Herrendorf [63] to estimate the capital shares in the investment and consumption sectors.

Categorization Following Baxter [4], the sectoral decomposition is first based on the final use of the output of each industry as consumption or investment goods. Using the two-digit Input-Output Table of the US Bureau of Economic Analysis, Table 1A provides a decomposition of sectoral output by final use in 2017. Entries in each row correspond to the percentage of total sectoral output allocated to personal consumption expenditures and private fixed investment.

Table 1A: Industries by final use in the US economy (2017)

Industry	Personal consumption expenditures (%)	Private fixed Investment (%)
Agriculture, forestry, fishing, and hunting	100%	0%
Mining	0%	100%
Utilities	100%	0%
Construction	0%	100%
Manufacturing	70%	30%
Wholesale trade	70%	30%
Retail trade	93%	7%
Transportation and warehousing	88%	12%
Information	70%	30%
Finance, insurance, real estate, rental, and leasing	95%	5%
Professional and business services	23%	77%
Educational services, health care, and social assistance	100%	0%
Arts, entertainment, recreation, accommodation, and food services	99%	1%
Other services, except government	100%	0%
Government	100%	0%

Source: 2017 Input-Output Table by the US Bureau of Economic Analysis.

Industries are then classified under the consumption sector (respectively, investment sector) when personal consumption expenditures (respectively, private fixed investment) has the predominant share, and otherwise under the consumption-investment sector. Accordingly, the *investment* sector includes the "Mining", "Construction", and "Professional and business services” industries, while the *consumption* sector includes the “Agriculture”, “Utilities”, “Manufacturing”, ‘Wholesale trade”, “Information”, “Retail trade”, “Transportation and warehousing”, “Educational services and health care”, “Culture, leisure and food services”, “other services”, and “Public administration” industries.

Table 2A: Industries by final use in the US economy

NAICS Classification	Commodity/Industry	Personal consumption expenditures (%)	Private fixed Investment (%)
111CA	Farms	100%	0%
113FF	Forestry, fishing, and related activities	100%	0%
211	Oil and gas extraction		
212	Mining, except oil and gas	54%	46%
213	Support activities for mining	0%	100%
22	Utilities	100%	0%
23	Construction	0%	100%
321	Wood products	42%	57%
327	Nonmetallic mineral products	100%	0%
331	Primary metals	100%	0%
332	Fabricated metal products	59%	41%
333	Machinery	3%	96%
334	Computer and electronic products	33%	67%
335	Electrical equipment, appliances, and components	54%	46%
3361MV	Motor vehicles, bodies and trailers, and parts	51%	49%
3364OT	Other transportation equipment	33%	67%
337	Furniture and related products	71%	29%
339	Miscellaneous manufacturing	64%	36%
311FT	Food and beverage and tobacco products	100%	0%
313TT	Textile mills and textile product mills	93%	7%
315AL	Apparel and leather and allied products	100%	0%
322	Paper products	100%	0%
323	Printing and related support activities	100%	0%
324	Petroleum and coal products	100%	0%
325	Chemical products	99%	1%
326	Plastics and rubber products	99%	1%
42	Wholesale trade	70%	30%
441	Motor vehicle and parts dealers	82%	18%
445	Food and beverage stores	100%	0%
452	General merchandise stores	98%	2%
4A0	Other retail	93%	7%
481	Air transportation	100%	0%
482	Rail transportation	74%	26%
483	Water transportation	100%	0%
484	Truck transportation	78%	22%
485	Transit and ground passenger transportation	100%	0%
486	Pipeline transportation	100%	0%
487OS	Other transportation and support activities	100%	0%
493	Warehousing and storage	100%	0%
511	Publishing industries, except internet (includes software)	42%	58%
512	Motion picture and sound recording industries	25%	75%
513	Broadcasting and telecommunications	90%	10%
514	Data processing, internet publishing, and other information services	88%	12%
521CI	Federal Reserve banks, credit intermediation, and related activities	100%	0%
523	Securities, commodity contracts, and investments	100%	0%
524	Insurance carriers and related activities	98%	2%
525	Funds, trusts, and other financial vehicles	100%	0%
HS	Housing	100%	0%
ORE	Other real estate	4%	96%
532RL	Rental and leasing services and lessors of intangible assets	100%	0%
5411	Legal services	78%	22%
5415	Computer systems design and related services	0%	100%
5412OP	Miscellaneous professional, scientific, and technical services	11%	89%
55	Management of companies and enterprises		
561	Administrative and support services	100%	0%
562	Waste management and remediation services	100%	0%
61	Educational services	100%	0%
621	Ambulatory health care services	100%	0%
622	Hospitals	100%	0%
623	Nursing and residential care facilities	100%	0%
624	Social assistance	100%	0%
711AS	Performing arts, spectator sports, museums, and related activities	88%	12%
713	Amusements, gambling, and recreation industries	100%	0%
721	Accommodation	100%	0%
722	Food services and drinking places	100%	0%
81	Other services, except government	100%	0%

As reported in Table 2A, using a 3-digit IO table (commodities and industries) provides a broader picture. It supports the view that there is some between- and within- heterogeneity regarding the final use of industry outputs, to some extent contradicting the assumption of mutually exclusive consumption-investment industries. Accordingly, we also consider two other classifications as a robustness check. The first rests on Table 2A and essentially uses a more granular decomposition of the manufacturing industry. The second classifies industries under the consumption sector (respectively, investment sector) when the personal consumption expenditures share (respectively, private fixed investment share) is higher than 80%, and otherwise under the consumption-investment sector. Accordingly, the *investment* sector includes the "Mining" and "Construction" industries, the *consumption* sector includes the "Agriculture", "Utilities", "Retail trade", "Transportation and warehousing", "Educational services and health care", "Culture, leisure and food services", "other services", and "Public administration" industries, and the *investment-consumption* sector includes the "Manufacturing", "Wholesale trade", "Information", and "Professional and Business services" industries.

"Aggregate" industries: Following Valentinyi and Herrendorf[63]), we consider five "aggregate" industries (agriculture, manufactured consumption, services, equipment, and construction) using Table 1A (or Table 2A). The investment sector then comprises construction and equipment whereas the consumption sector includes manufactured consumption, agriculture, and services.

10.2 Sectoral capital shares

The second step involves calibrating the technological parameters of the two-sector model $\{\vartheta_1, \vartheta_2, 1 - \vartheta_1 - \vartheta_2\}$ where $\vartheta = \alpha$ (consumption) and γ (investment). More specifically, we focus on the sector capital shares $\vartheta_1+\vartheta_2$. In so doing, we follow the Valentinyi and Herrendorf [63] method of measuring factor income shares at the sectoral level. Before the procedure is detailed, it should be borne in mind that we are assuming that production opportunities are specified with Cobb-Douglas technologies without intermediate goods. Said differently, the sector capital share $\vartheta_1 + \vartheta_2$ reflects both the capital inputs into the production of sector j's value added as well as into all intermediate inputs that are used directly or indirectly by sector j.

Let θ_j denote the sector capital share in sector j. As explained in Valentinyi and Herrendorf [63], θ_j is given by:

$$\theta_j = \frac{\alpha_k' W(I-BW)^{-1} y_j}{(\alpha_k+\alpha_\ell)' W(I-BW)^{-1} y_j}$$

where W is the Make matrix, B is the Use matrix, $W(I - BW)^{-1}$ is the Industry-by-Commodity Total Requirements matrix, vectors y_j record the final (US dollar) expenditures on the commodities that belong to sector j, and vectors α_k and α_ℓ measure respectively capital and labor income generated per unit of industry i's output g_i.[43] Given the BEA-produced

[43]The Make Matrix (or Make Table) shows the production of commodities by industries. More specifically, it provides the value (in producers' prices) of each commodity produced by each industry. Each entry (i,j) shows for industry i the (per one dollar) values of commodities j it produces as a primary producer (respectively,

matrix $W(I-BW)^{-1}$, computing the sector capital shares requires simply measuring α_k and α_ℓ.[44] Notably, the industry capital and labor shares can be written as:

$$\alpha_{k,i} = \left(\mathtt{gos}_i - \frac{\mathtt{com}_i}{\mathtt{com}_i + \mathtt{gos}_i - \mathtt{pro}_i}\mathtt{pro}_i\right)\frac{1}{g_i}$$
$$\alpha_{\ell,i} = \left(\mathtt{gos}_i + \frac{\mathtt{com}_i}{\mathtt{com}_i + \mathtt{gos}_i - \mathtt{pro}_i}\mathtt{pro}_i\right)\frac{1}{g_i}$$

where $\mathtt{gos}_i$ and $\mathtt{com}_i$ stand respectively for gross operating surplus and the compensation of employees in industry i, and $\mathtt{pro}_i$ is the (two-digit) proprietors' income corresponding to four-digit industry i. Both gross operating surplus and compensation of employees *per* industry are reported in IO tables, whereas proprietors' income is defined at the two-digit (SIC) level in the GDP-by-industry tables of the BEA and thus requires adjustment (see Valentinyi and Herrendorf [63], section 3). Table 1 in the main text displays the capital shares of the two-sector model.

secondary producer) when $i = j$ (respectively, $i \neq j$). The Use Matrix (or Use Table) shows the value (in producers' prices) of each commodity used by each industry/final use. It also provides details on the components of the value added and total intermediate inputs used by each industry to produce its output. Finally, total requirement matrices/tables show the inputs that are required directly and indirectly to deliver one US dollar of output to final uses. Each column (respectively, each row) of the industry-by-commodity table displays the commodity delivered to final uses (respectively, the required total production of each industry).

[44]This stage involves several steps that are described in Section 2 of Valentinyi and Herrendorf [63]. We provide a summary of the methodological steps in the Appendix and we refer to their paper for further detail.