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Public Goods in Networks: Comparative Statics Results*

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Abstract

We consider public goods games played on a potentially non-symmetric network and provide comparative statics results on individual and aggregate contributions, as well as on the effect of transfers between players. We show that, contrary to the case of the complete and symmetric network, a positive shock on a player can have adverse consequences. First, it could actually decrease this player's contribution, unless the interaction matrix is a P-matrix. Second, a positive shock on a contributing player increases aggregate contributions, but a positive shock on a non-contributing player will decrease aggregate contributions, even if the player who benefited from the positive shock increases his own contribution. In each case we provide simple conditions to determine whether a positive shock will have positive or negative consequences on contributions, by looking at the unconstrained solution of an alternative, associated game. The sign of the coordinates of this solution determines the effect of a shock. With this in hand, we further show that the aggregate neutrality result of Andreoni [1990] regarding transfers between players generally does not hold on non-symmetric networks and provide conditions for it to hold.

Finally, as an application of previous results, we consider introducing agents that follow Kantian moral principles and show that, depending on their position in the network, the presence of Kantian agents can, counter-intuitively, lead to a decrease in aggregate contributions.

Keywords: Public Goods, Network, Comparative Statics, Kantian players

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1 Introduction

The public goods provision game has been the subject of many papers in economics. In particular, several papers have attempted to derive comparative statics results on the equilibria of the game, seeking to clarify how individual and aggregate contributions change when one or more parameters of the game change.

This is the case, inter alia, of Corchón [1994] or Cornes and Hartley [2007], who show that a player undergoing a positive shock will increase his contribution and induce an increase in the aggregate contribution in a context where the equilibrium is unique. This work has been generalized by Acemoglu and Jensen [2013] to the case where there are multiple equilibria, or even a continuum of equilibria. Another comparative statics exercise is provided by Warr [1983], Bergstrom et al. [1986], and Andreoni [1990], who analyze the effect of a transfer of income between agents on the equilibrium of the game. The first two identify a neutrality property in a pure public good context, whereby a small redistribution of income between agents will change their contribution by precisely the amount of the transfer received, implying that the aggregate contribution will not change after the transfer. The third looks at the same problem in a context where individuals' provisions are imperfect substitutes, with possibly heterogeneous substitution rates, and shows that, while neutrality no longer holds, aggregate neutrality (i.e. the sum of contributions remains the same) still holds, if and only if the agents involved in the income transfer have the same substitution rates.

However, these comparative statics papers consider that the agents all interact with each other, i.e. on a network that is complete. Recently though, Bramoullé and Kranton [2007] introduced a model in which the provision of public goods is local instead of global, i.e. players only benefit from the contributions of their neighbors in a network of relationships. In that paper, the authors consider payoff functions with linear best-responses, show the existence of Nash equilibria, and analyze the stability of these equilibria, an analysis they extended in Bramoullé et al. [2014]. In parallel, Allouch [2015] considers local public good games with possibly non-linear best-responses, identifies conditions for the equilibrium to be unique, investigates whether the neutrality result found by Warr [1983] and Bergstrom et al. [1986] holds, and finds that it does not hold except for very specific network structures.

However, these papers restrict their attention to symmetric networks, and do not focus on comparative statics. Our aim in this paper is to provide comparative static results in the spirit of Corchón [1994] and Acemoglu and Jensen [2013], in a context where the public good is local instead of global, the pattern of interactions could be asymmetric, and there are possibly multiple equilibria. We also aim to verify whether Andreoni [1990]'s finding on aggregate neutrality holds on a network after a transfer between agents. Finally, we consider a third comparative statics exercise, which consists of replacing one player with a player of another type, who does not share the others' objective in the game. The general message being that results that hold on a complete network no longer hold on an arbitrary network, and that results that hold on symmetric networks do not hold on non-symmetric networks, we now detail each of them.

In section 2, we present a general public good model with linear best-response, allowing for as many sources of heterogeneity as possible. In particular, we allow for two unrelated sources of heterogeneity: in individual characteristics (benefit function, income, marginal cost, etc.), as well as in the interaction patterns between agents (existence and intensity of links, substitution rates). We then show (Proposition 1) that, although these heterogenities are different in nature, comparative statics on any dimension of the model can be captured through comparative statics on one parameter that we call the *needs* of agents. Needs are the amount of resources that an agent, if isolated, would choose to contribute to the public good. We also show (Proposition 2) that Nash equilibria of the game are solutions to a Linear Complementary Problem (LCP), and we make use of this LCP theory to provide an alternative proof of existence (Corollary 1) and a sufficient condition for uniqueness (Corollary 2) in non-symmetric networks, that encompasses the conditions in Bramoullé et al. [2014] and Allouch [2015] for symmetric network case. This sufficient condition states that the game's interaction matrix, which captures how players' actions influence each others' best-responses, should be a P-matrix[1].

In section 3, our first comparative statics result (Theorem 1) shows that a positive shock on a player will induce an increase in his contribution if the interaction matrix is a P-matrix. Thus, results in Corchón [1994] and Acemoglu and Jensen [2013], restricted to the linear case, are corollaries of this theorem since the complete network's interaction matrix, even with heterogeneous substitution rates, is a P-matrix (Proposition 3). More importantly, we also show that a positive shock can in fact decrease a player's contribution if the interaction matrix is not a P-matrix.

In section 4, we investigate how aggregate contribution changes with a positive shock on an agent. Here again, we show that aggregate contribution can increase as expected, but does not always do so. In particular, when players are homogeneous in terms of resources and substitution rates and interact on a undirected network, whether contributions increase or decrease depends on the status of the agent, i.e. contributor or free-rider, who is subject to the shock. If this agent was active (contributor) before the shock, then aggregate contributions will increase, even though his own contribution might decrease as noted earlier (Proposition 4); while if the agent was strictly inactive (free-rider) before the shock, then aggregate contributions will actually decrease (Proposition 5). This is surprising, being counter-intuitive. A policy maker seeking to increase aggregate contri-

[1]A square matrix is a P-matrix if and only if its principal minors are strictly positive. When the matrix is symmetric, this is equivalent to being positive definite.

butions would, in principle, target free-riders rather than already contributing players. However as we show, he will actually decrease aggregate contributions by doing so. The intuition behind this result will become clear once we look into the case of heterogeneous players.

Before stating the conditions that reveal whether a shock to a player will increase or decrease aggregate contributions when players are heterogeneous, it is important to note that the solution of an LCP problem, such as the one we analyze in this paper, is a constrained solution of a linear system. This system admits a unique unconstrained solution, but there is no known connection between this unconstrained solution (which may therefore have negative coordinates) and the set of constrained solutions (which have only positive coordinates) of this problem[2]. As we show, we can construct a problem associated with every equilibrium of the initial game, search for the unconstrained solution to this problem, and deduce from the sign of each coordinate of this solution the impact that a shock on the corresponding player will have on aggregate contributions in the initial game. If the player getting the shock has a positive coordinate, the aggregate contribution will increase; if the coordinate is negative, it will decrease (Proposition 6).

To see this, we notice that the unconstrained solution of a player in the associated problem represents the aggregate outgoing effect of this player. If a player has a positive outgoing effect, increasing his needs will increase aggregate contributions, whereas a player with negative outgoing effect will decrease aggregate contributions.

Coming back to the case of homogeneous players on an undirected network, we observe that the outgoing effects and the incoming effects are the same for each player because of the symmetry of the network. In addition, the incoming effect for a player is precisely the difference between this player's needs and how much public goods he receives from his neighbors. This is positive if a player is active, and negative if a player is strictly inactive. This is why we can identify active players as having positive outgoing effects (i.e. players who increase aggregate contributions) and strictly inactive players as having negative outgoing effects (i.e. players who decrease aggregate contributions) as in Propositions 4 and 5.

Incidentally, we notice that players whose coordinate in the unconstrained solution is zero have no effect whatsoever on aggregate contributions, whatever the shock they receive. To the best of our knowledge this is the first paper to identify these players, that we call *neutral players*. They occupy a position in the network such that if they suffer a shock (positive or negative), the contributions of all players including themselves will change, but the sum of these contributions will remain constant. These players are placed

[2] It would be tempting to think that coordinates that are positive in the unconstrained solution correspond to active players, while coordinates that are negative in the unconstrained solution correspond to inactive players. However this is not true and there is no such identification rule regarding who is active and who is not, based on these coordinates.

in positions such that the positive effects precisely compensate for the negative effects.

Next, in sections 5 and 6 we apply these results to analyze respectively transfers between players and replacement of standard maximizers by Kantian agents. These two exercises are different in nature, but both can be analyzed through changes in needs.

In section 5 we look at transfers between agents. In the same way that Allouch [2015] analyzed whether Bergstrom et al. [1986]'s results were true on a symmetric network other than the complete one, we want to know whether Andreoni [1990]'s results are true on an arbitrary non-symmetric network. A transfer between two players being an increase of needs of one player and an equivalent decrease of needs of another player, we can use results of section 4. We find that the original result on aggregate neutrality no longer holds (Proposition 8), in two distinct senses. First, it is possible that the agents involved in the income transfer have the same rates of substitution, but the aggregate contribution changes; second, it is possible that the agents do not have the same rates of substitution, yet the aggregate contribution remains constant before and after the transfer. In particular, any transfer between neutral players, despite affecting the entire equilibrium contributions, will leave the aggregate unchanged.

Finally, in section 6, we introduce Kantian agents into the local public good game, by considering that society is formed of both Kantian agents and Nash maximizers[3]. Following the literature and in particular Laffont [1975], we assume that *"A typical agent assumes (according to Kant's moral) that the other agents will act as he does, and he maximizes his utility function under this new constraint"*. This is also the definition adopted in Alger and Weibull [2013], and the definition that we adopt here.

Kantian agents were introduced in economics as a way of considering that *"in some economic circumstances, agents are capable of behavior other than the selfish pattern imputed to the "homo oeconomicus" by economic theory"* (Laffont [1975]). It was argued that the problem of under-provision of public goods is not as acute as suggested by the theory, and one possible explanation is that agents do not behave as standard maximizers, but adopt a Kantian morality when it comes to public goods. This view was later taken up by Sugden [1984], Bilodeau and Gravel [2004], Roemer [2010], Roemer [2015] or Alger and Weibull [2013], Alger and Weibull [2016].

[3] Other recent papers look at societies formed by both Kantian agents and Nash players, either on the complete network or on specific networks. Van Long [2016] and Grafton et al. [2017] discuss definitions of a Nash-Kant equilibrium and show that Kantian agents help decrease the inefficiency of Nash players in a context of oligopoly for the first, and production of a public bad for the second. Pitsuwan [2017] provides a uniqueness condition in the public good game played on the complete network, where players act as Kantian only towards a subset of the population. Finally, the closest to our setting is Mohanty et al. [2021], who investigate the same question but restrict the analysis to regular networks and perfect substitutability. They show that free-riding decreases when Kantian agents are linked together in the network, but this could come at the cost of a reduction in welfare.

The fact that Kantian agents would reduce the under-provision problem seems intuitive: when a standard maximizer decides whether to increase his contribution, he ignores the externalities he exerts on others by doing so. In contrast, when a Kantian agent increases his contribution, he assumes that everyone else will also do so. He thus suffers a direct increase in his cost, but benefits from all the positive externalities exerted by the other agents.

This intuition, which is consistent with Bilodeau and Gravel [2004] for instance, is actually confirmed on a complete network. In this section we check whether it still holds on an arbitrary network. To do that, we replace a standard maximizer by a Kantian agent, and compare aggregate contribution before and after replacement. As it turns out, replacing a standard maximizer by a Kantian agent amounts to increasing this player's needs and removing all his incoming links at the same time. Again, we can use results of section 4 and we show that the presence of one Kantian agent can in fact be detrimental to the aggregate contribution, depending on which player is replaced by a Kantian agent (Proposition 10). If this player was active, then the replacement will increase contributions, while if this player was inactive, the replacement will decrease contributions. Thus if a social planner wanted to replace one standard maximizer with one Kantian agent, he would have to choose a player already contributing to the public good, not a free-rider as intuition would recommend.

Finally, given that the presence of a Kantian agent could decrease contributions, we analyze whether it is always possible to find a player to replace so as to guarantee that contributions will increase, including if there are already other Kantian agents in the network. We show (Proposition 11) the conditions to be checked at each replacement to guarantee this increase. As in the previous sections, these conditions relate to the sign of the coordinates of the solution to the unconstrained problem associated with the initial equilibrium of the game.

2 Model, Existence, Multiplicity

2.1 Model

Consider a game $\mathcal{G} = (N, (X_i)_{i=1,\ldots,n}, u)$, where $N = \{1, \ldots, n\}$ is the set of players and $X_i = [0, +\infty[$ is the action space of player i, from which he chooses x_i, his contribution to the public good. We denote by X the sum of individual contributions to the public good, i.e. $X = \sum_{i \in N} x_i$. Finally $u = (u_i)_{i=1,\ldots,n}$ is the vector of payoff functions.

Agents are placed on a network represented by a graph G. By convention, we also denote by $G = \{g_{ij}, i \in N, j \in N\}$ the adjacency matrix of the graph with elements g_{ij}. We assume that $g_{ii} = 0$ for all i.

We say that the network is unweighted and undirected when $g_{ij} = g_{ji} \in \{0, 1\}$. We say that the network is weighted if $g_{ij} \in \mathbb{R}_+^*$ if i is linked to j and $g_{ij} = 0$ otherwise. We say it is directed if g_{ij} can be different from g_{ji}. The set of neighbors of player i is $N_i(G) := \{j \in N, g_{ij} > 0\}$. When positive, we refer to g_{ij} as an *incoming* link for player i and an *outgoing* link of player j. We also call g_{ij} the *incoming link intensity* of i from j.

We will sometimes only consider incoming links of a subset S of players, and delete the incoming links of players in $N \setminus S$. We denote that network by G_S, which is constructed from G, where for player $i \in S$, we set $g_{S_{ij}} = g_{ij}$ for all j, while $g_{S_{ij}} = 0$ for all $i \notin S$. Notice that this network is in general non-symmetric.

Best-responses

We consider games with payoff functions $u(.)$ that have unique best-responses of the following form

$$\forall i \in N, \;\; Br_i(x_{-i}) = \max\left\{ q_i - \delta_i \sum_{j=1}^{n} g_{ij} x_j, 0 \right\}. \tag{1}$$

where $\delta_i \in [0, 1]$ represents the rate of substitutability between agent i's neighbors' actions and agent i's action, and $q_i \in \mathbb{R}_+$ represents player i's demand for the public good. It is the level of contribution that player i would provide if he was isolated. In the remainder of this paper, we call q_i the *needs* of player i. Substitution rates are collected into the matrix $\Delta = diag(\delta_i)_{i \in N}$ and needs are collected into vector q.

A prominent example of such a game is provided in Bramoullé and Kranton [2007], where the payoff function is

$$u_i(x) = b\left(x_i + \sum_{j \in N_i(G)} x_j \right) - cx_i \tag{2}$$

where $c > 0$ is the marginal cost of effort and $b(.)$ is a differentiable, strictly increasing concave function normalized so that $b'(1) = c$. In that case, the best-response is given by

$$\forall i \in N, \;\; Br_i(x_{-i}) = \max\left\{ 1 - \sum_{j=1}^{n} g_{ij} x_j, 0 \right\}. \tag{3}$$

where $\delta_i = 1$ and $q_i = 1$ for all i.

Another prominent example is the quasi-linear version of the game of private provision of a public good from Bergstrom et al. [1986], adapted to networks in Allouch [2015] and with potentially imperfect substitutes: Assume player i has wealth $w_i \in \mathbb{R}_+$ that he allocates to the consumption of a private good and a public good, with payoff function

$$u_i = log\left(x_i + \delta_i \sum_{j=1}^{n} g_{i,j} x_j \right) + log(w_i - x_i)$$

This game produces the best-response (1) by setting $q_i = \frac{1}{2}w_i$ and $g_{ij} = \frac{1}{2}$ if i and j are linked and 0 otherwise.

2.2 Needs

In this paper, we are interested in the effects of changes in the parameters (q, Δ, G) - and, when relevant, of changes in wealth w - on equilibrium actions x_i^* and on $X^* = \sum_{i \in N} x_i^*$, the aggregate contribution level of players at equilibrium x^*.

Changes in q capture changes in individual characteristics of the players. Indeed, note that q_i is the contribution that a player i would choose if he were in autarky (i.e. linked to no-one). Thus changes in q can result from changes in costs, in wealth, in the concavity of the benefit function ($b(.)$ or $log(.)$ in the above examples), or from any change that would modify the preferred level of public good consumption of that player.

In turn, changes in Δ or in G capture changes in the way individuals interact together. These changes result from modifications of the network such as cutting out some links or changing link intensities, from changes in levels of substitution, or from the modification of player's type, as will be illustrated in section 6 where some players will follow a Kantian morality instead of being standard maximizers. We consider the following changes:

- changes in needs, with player 1's needs increasing from q_1 to q_1'
- changes in the substitution rate of player 1, due to change δ_1 to δ_1'
- changes in the incoming link intensity of player 1, due to changing g_{1j} to g_{1j}', with potentially different changes for neighbors j_1 and j_2. This includes cutting all the incoming links of player 1, cutting just one link, decreasing the weight of every link by the same factor, or decreasing each weight by a different factor.

Although each parameter plays a different role in the model, we show that the effects of changes in Δ and G can be captured by equivalent changes in needs. First, a change from δ_1 to δ_1' is observed to be equivalent to a change from g_{1j} to $g_{1j}' = \frac{\delta_1'}{\delta_1} g_{1j}$, so that changes in substitution rates can be expressed as changes in incoming link intensities. Next, we show that decreasing incoming link intensities can be captured as an increase of needs.

Proposition 1. *Assume we start with the game with parameters (q, Δ, G). Let x^ϵ be an equilibrium of the game with modified parameters (q, Δ, G^ϵ), where $g_{1j}^\epsilon = (1 - \epsilon_j) g_{1j}$ for $j \in N_1(G)$, with $\epsilon_j \in [0, 1]$ for all j. Then, there exists $\beta > 0$ such that x^ϵ is an equilibrium of the game (q^β, Δ, G) where $q^\beta = (q_1 + \beta, q_2, \ldots, q_n)^T$.*

All the proofs are relegated to the Appendix. Since all changes in individual characteristics or in the way players interact can be subsumed under changes in needs[4], we will

[4] In a recent paper, Sun et al. [2023] show the same kind of equivalence result between changes in the

only focus on changes in q in what follows.

2.3 Nash equilibria: Existence

Given a vector of needs q, and $\Delta = diag(\delta_i)_{i\in N}$, we call the matrix $(I+\Delta G)$ the *interaction matrix*, and we denote by $x^{unc}(\Delta G, q)$ the (unique) unconstrained solution to the system[5]

$$(I + \Delta G)x = q,$$

that is

$$x^{unc}(\Delta G, q) = (I + \Delta G)^{-1}.q$$

Of course, if $x_i^{unc}(\Delta G, q) \geq 0$ for all i, then $x^{unc}(\Delta G, q)$ is an interior Nash equilibrium of the game. However this will not generally be the case.

The set of Nash equilibria of games with best-responses (1) is described by the set of all profiles x^* such that:

$$\begin{array}{rlcl} \text{(A)} & q_i - \delta_i \sum_{j\in N} g_{ij}x_j^* \geq 0 & \Longrightarrow & x_i^* = q_i - \delta_i \sum_{j\in N} g_{ij}x_j^* \\ \text{(SI)} & q_i - \delta_i \sum_{j\in N} g_{ij}x_j^* < 0 & \Longrightarrow & x_i^* = 0 \end{array}$$

Players of type (A) are active players, while players of type (SI) are strictly inactive players, or free-riders. The set of players of type (A) at equilibrium x^* is denoted by $A(x^*) \subseteq N$, while the set of (SI) players at x^* is denoted by $SI(x^*) \subset N$. It might be tempting to think that active players are those whose unconstrained solution components are positive, while the strictly inactive players are those with negative components. However, this is not true, as the many examples provided below will make clear[6].

Note that players such that $q_i - \delta_i \sum_{j\in N} g_{ij}x_j^* = 0$ are considered active players. Finally, for SI players, we call the quantity $\delta_i \sum_{j\in N} g_{ij}x_j^* - q_i > 0$ the excess public good of player i.

Existence of a Nash equilibrium is guaranteed by Brouwer's fixed point theorem, for any vector of needs q, any vector δ, and any network G. However, we present another proof of existence which uses results from the Linear Complementarity Problem (LCP) theory.[7]

network structure and changes in individual characteristics, in a setting with linear best-response and strategic complements.

[5]Throughout the paper we assume that the matrix $(I + \Delta G)$ is non-degenerate, i.e. 0 is not an eigenvalue.

[6]A recent paper by Zheng et al. [2016], adapted to our framework, proves the following: if $(I + \Delta G)$ is positive definite, then $x_i^{unc} > 0 \Longrightarrow i \in A(x^*)$ and $i \in SI(x^*) \Longrightarrow x_i^{unc} < 0$. This is, to the best of our knowledge, the only established relationship between x^{unc} and an equilibrium x^*.

[7]See for instance Murty and Yu [1988] for an overview of the literature.

Problem $LCP(b, A)$ consists of finding vectors $x \in \mathbb{R}^n$ and $w \in \mathbb{R}^n$ such that

$$\begin{aligned} &Ax + b = w, \\ &w \geq 0, \ x \geq 0, \ w^T x = 0 \end{aligned}$$

where $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$ are given. This class of linear problems are called complementary because of the constraint $w^T x = 0$, which implies that if one of the variables is strictly positive then the other is necessarily 0. As we show now, finding the Nash equilibria of games with best-responses (1) amounts to finding solutions to a linear complementary problem.

Proposition 2. *Let $A = I + \Delta G$ and $b = -q$. Then x is a solution to $LCP(-q, I + \Delta G)$ if and only if x is a Nash equilibrium of games with best-responses (1). Vector w represents the excess public good of players.*

The proof of this proposition is immediate. It relies on the fact that for every player i at a Nash equilibrium, either the excess public good w_i is 0 (and $i \in A(x)$ since i is active iff $w_i = 0$), or $w_i > 0$ and $x_i = 0$ (and $i \in SI(x)$).

Matrices that guarantee existence of at least one solution for each q are called $Q-$matrices.

Theorem 5.2 (Murty [1972]). *Let $A \geq 0$. A is a Q-matrix if and only if $a_{ii} > 0$ for each $i = 1, .., n$*

Corollary 1. *Games with best-responses (1) have at least one Nash equilibrium.*

This corollary is straightforward, since all terms of $(I + \Delta G)$ are positive and the diagonal terms are all equal to 1.

Now that existence is established, we define the *equilibrium interaction matrix*, noting the following: if x^* is a Nash equilibrium with sets $A(x^*)$ and $SI(x^*)$, and if $(I + \Delta G_{A(x^*)})$ is non-degenerate, then x^* is the unique equilibrium with sets $A(x^*)$ and $SI(x^*)$. This equilibrium x^* is the (unique) unconstrained solution to

$$(I + \Delta G_{A(x^*)})x = q_{A(x^*)} \tag{4}$$

where $q_{i,A(x^*)} = q_i$ if $i \in A(x^*)$, and $q_{i,A(x^*)} = 0$ is $i \in N \setminus A(x^*)$. In other words, this solution is found by deleting incoming links of SI players and by setting their needs to 0. Of course the set $A(x^*)$ is usually not known when searching for x^*, which is why solving an LCP is difficult.

Definition 1. *Let x^* be a Nash equilibrium of the game with interaction matrix $(I + \Delta G)$, with sets of active and strictly inactive players $A(x^*)$ and $SI(x^*)$. We call matrix $(I + \Delta G_{A(x^*)})$ the equilibrium interaction matrix of x^*.*

2.4 Nash equilibrium: Condition for Uniqueness

Although our comparative static results do not rely on uniqueness of the equilibrium, the conditions for uniqueness have so far only been established for symmetric interaction matrices. Here we extend these conditions to the case of non-symmetric matrices.

When network G is symmetric and when $\delta_i = \delta$ for all players, a sufficient condition for uniqueness is presented in Bramoullé et al. [2014]. It requires that $|\lambda_{min}(G)| < 1/\delta$ where λ_{min} denotes the lowest eigenvalue of the matrix[8]. However, this condition is not sufficient in the general non-symmetric case with potentially different substitution rates across players. Here again we take the LCP theory approach to provide a sufficient condition for uniqueness, which encompasses the condition for the symmetric case.

Definition 2. *A square matrix is a P-matrix if all its principal minors are strictly positive.*

Theorem 4.2 (Murty [1972])**.** *Let A be a square matrix of order n. The $LCP(b, A)$ has a unique solution for each $b \in \mathbb{R}^n$ if and only if A is a P-matrix.*

Corollary 2. *All games with best-responses* (1) *have a unique Nash equilibrium for each needs vector q if and only if $(I + \Delta G)$ is a P-matrix.*

A symmetric matrix is a P-matrix if and only if it is positive definite[9]. A symmetric matrix is positive definite if and only if all its eigenvalues are strictly positive. We thus recover the result in Bramoullé et al. [2014]:

Corollary 3. *If the interaction matrix $(I + \Delta G)$ is symmetric with $\delta_i = \delta$ for all i, then there is a unique Nash equilibrium for each needs vector q if and only if $|\lambda_{min}(G)| < 1/\delta$.*

Remark 1. *Note that even if matrix G is symmetric, this does not imply that $(I + \Delta G)$ is symmetric. For that to happen, it must be that $\delta_i g_{ij} = \delta_j g_{ji}$ for every pair of players (i, j). In that case, the corresponding uniqueness condition is $\lambda_{min}(I + \Delta G) > 0$. Except for section 4.1, we will be dealing with non-symmetric interaction matrices.*

Three remarks are in order. First, we do not need positive definiteness to guarantee uniqueness. It is necessary and sufficient in the symmetric case, but not necessary in the non-symmetric case. When the matrix is non-symmetric, as in most of this paper, the lowest eigenvalue condition does not necessarily work. A non-symmetric matrix can have

[8] This condition also coincides with the condition in Allouch [2015] for linear best-responses.

[9] A real matrix F, whether symmetric or not, is positive definite if $y^T F y > 0$ for all $y \in \mathbb{R}^n, y \neq 0$. If F is symmetric - and only then - it is positive definite if and only if all its eigenvalues are strictly positive. Most matrices we consider in this paper will not be symmetric.

only positive eigenvalues and yet not be positive definite[10], as illustrated in example 1. However, although even non-symmetric positive definite matrices are always P-matrices, some non-positive definite matrices are also P-matrices. This is true of the interaction matrix in example 1, which is not positive definite but is a P-matrix.

Example 1. Consider the following non-symmetric interaction matrix $(I + \Delta G)$ where G is the complete network, $\delta_1 = 1$ and $\delta_i = 0.1$ otherwise.

$$(I + \Delta G) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0.1 & 1 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 & 1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 1 \end{pmatrix}$$

Matrix $(I + \Delta G)$ has only positive eigenvalues. Yet, it is not positive definite. To see that, notice that $x^T(I+\Delta G)x = x^T \frac{1}{2}\left[(I + \Delta G) + (I + \Delta G)^T\right] x$, and $\frac{1}{2}\left[(I + \Delta G) + (I + \Delta G)^T\right]$ is symmetric but has a negative eigenvalue $\lambda_{min} = -0.046$, associated with the eigenvector x_{min}. Therefore $x_{min}^T(I + \Delta G)\, x_{min} = \lambda_{min} < 0$. However, $(I + \Delta G)$ is a P-matrix, as will be proved in Proposition 3.

Second, corollary 2 guarantees that comparative statics on q can be performed without losing the uniqueness of the Nash equilibrium. Indeed, if $(I+\Delta G)$ is a P-matrix, then the vector of needs q can be changed without altering the number of equilibria. Also, if some players are eliminated from the game, then the P-matrix property carries through to the resulting principal submatrix and uniqueness is preserved (every principal submatrix P' of a P-matrix is also a P-matrix, since all the principal minors of P' are principal minors of the original matrix[11]). This will be particularly useful in section 6.

Third, the necessary part does not rule out the possibility that a given $LCP(-q, I + \Delta G)$ will have a unique solution even though the matrix $(I + \Delta G)$ is not a P-matrix. Indeed, the necessary condition states that the solution should be unique for *each* q. This is illustrated in Figure 1, where G represents a star network and the game is played with $\delta_i = \frac{1}{2}$ for all i, and $q_i = \frac{1}{2}$ for all i. Then the equilibrium is unique, given by $x^* = (0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, although $(I + \frac{1}{2}G)$ is not a P-matrix (the interaction matrix is symmetric and $\lambda_{min} \approx -0.118 < 0$). However, Theorem 4.2 (Murty [1972]) tells us that

[10] A non-symmetric positive definite matrix has every real eigenvalue strictly positive, and every complex eigenvalue has a strictly positive real part. This is necessary but not sufficient, unlike the symmetric case, where all eigenvalues are real and the positivity condition is necessary and sufficient.

[11] This property carries through to positive definite matrices, since they are special instances of P-matrices.

for at least some vectors of needs q there will be more than one solution[12]. Take for instance needs such that $q_i = \frac{1}{2}$ for all peripherals and $q_j = 1$ for the center. Then there are two Nash equilibria: $x_1^* = (0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and $x_2^* = (1, 0, 0, 0, 0, 0)$. To the best of our knowledge, no one has identified a sufficient condition to guarantee uniqueness for a given q when the interaction matrix is not a P-matrix.

Incidentally, note that in the previous example, increasing a player's needs increases the number of equilibria. The reverse can also happen. Returning to the star network, consider $q_i = \frac{1}{2}$ for all peripherals and $q_j = 3$ for the center. Then the number of Nash equilibrium goes back down to 1: $x^* = (3, 0, 0, 0, 0, 0)$.

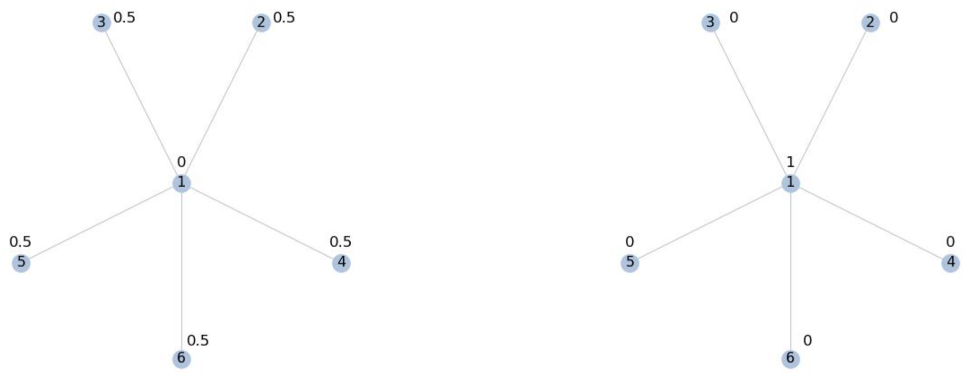


Figure 1: In the star with five peripherals, there is a unique equilibrium when $\delta_i = 0.5$ and $q_i = 0.5$ for all i, although the interaction matrix $(I + \Delta G)$ has a negative eigenvalue. This equilibrium is represented on the left panel. When the needs of the center are increased to $q_c = 1$, the profile on he left panel is still an equilibrium. However, a new equilibrium appears, represented on the right.

Remark 2. *The case of the complete network has been extensively analyzed, as pointed out in the introduction. In what follows, we will often refer to the complete network to derive previously established results as corollaries of our results. The complete network is defined as $g_{ij} = 1$ for all pairs of players (i, j), and $g_{ii} = 0$ for all i. In this case, the matrix $(I + \Delta G)$ is degenerate if (and only if) there are at least two players i and j such that $\delta_i = \delta_j = 1$. Then the two players are perfectly substitutable for all others and between themselves, inducing a continuum of Nash equilibria*[13]*. Therefore, when we refer to the complete network in what follows, we always assume that at most one player has $\delta_i = 1$.*

[12] We can actually go farther than that: if the number of solutions is a constant for every q, then this constant is 1 and $I + \Delta G$ is a P-matrix (Murty [1972], 7.2). Otherwise, even though there is a unique solution for some q, there is always some q' for which there is more than one solution.

[13] This actually happens often, even if the network is not complete, when $\delta = 1$ for several players. See for instance Bervoets and Faure [2019].

Proposition 3. *Let G be the complete network, and let $\Delta = diag(\delta_i)_{i=1,\ldots,n}$ with $\delta_i \in]0,1]$ for all i, and $\delta_i = 1$ for at most one player i. Then $(I + \Delta G)$ is a P-matrix.*

The proof, which is not straightforward, relies on proving that the determinant of any possibly non-symmetric interaction matrix associated with the complete network is strictly positive. This is done by using the *matrix determinant lemma.*

This proposition yields the following:

Corollary 4. *If network G is complete, there is a unique Nash equilibrium for each needs vector q whether the interaction matrix is symmetric or not.*

3 Effect of needs on own contribution

Here we analyze how a player's contribution is affected by an increase in his needs. We emphasize the following counter-intuitive observation: in the public good game with global and symmetric interactions (i.e. played on a complete network with the same substitution rates), it was established earlier that increasing the needs of one player always induces an increase in this player's contribution. However, this ceases to be true once a network structure is introduced, as illustrated in Figure 2, where an increase in player 1's needs induces a decrease in his contribution. This may seem counter-intuitive, but it is due to the complex pattern of interactions and substitutions between players.

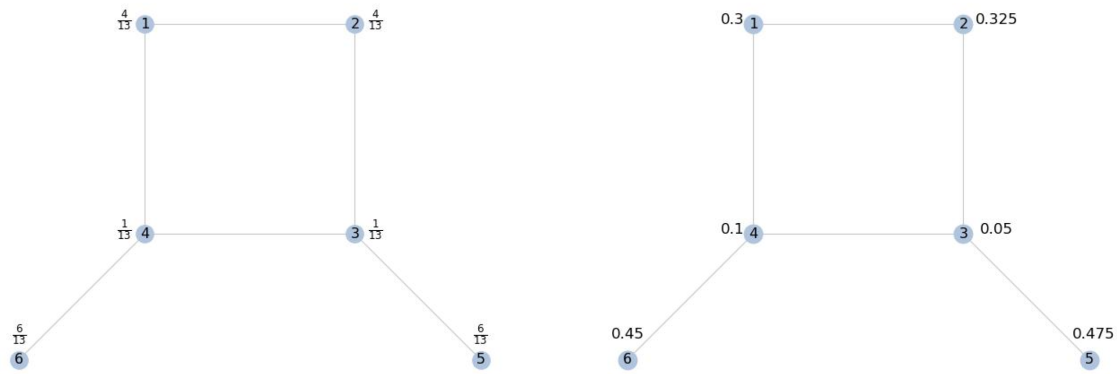


Figure 2: In the left panel, a Nash equilibrium x^* with homogeneous needs and substitution rates ($q_i = 0.5$ and $\delta_i = 0.5$ for all i). In the right panel, the needs of player 1 are increased to 0.5125 and in the new equilibrium the contribution of player 1 decreases from $\frac{4}{13}$ to 0.3

However, once the matrix $(I + \Delta G)$ is a P-matrix, we can guarantee that a player's contribution will increase when his needs increase:

Theorem 1. *Let $(I+\Delta G)$ be a P-matrix, and let x be the unique Nash equilibrium when needs are q. Consider $q' = (q_1+\beta, q_2, ..., q_n)^T$, the vector of needs where the needs of player 1 are increased by any amount $\beta > 0$, and let x' be the unique Nash equilibrium with needs q'. Then*

$$x'_1 \geq x_1 \text{ if } x_1 = 0$$
$$x'_1 > x_1 \text{ if } x_1 > 0$$

If in addition $(I+\Delta G)$ is symmetric, then

$$x'_1 - x_1 > \beta \text{ when } x_1 > 0$$

Remark 3. *The set of active and strictly inactive players at equilibrium might change after an increase in needs of a player. Theorem 1 holds regardless of these changes.*

We detail here the main steps of the proof of Theorem 1, since several ideas from the paper are used to prove it. Obviously, if $x_1 = 0$ then $x'_1 \geq x_1$. Now assume that $x_1 > 0$. The easy case is the following: Assume the two equilibria, before and after the increase in needs, are interior, i.e. everyone is active. Then the solutions are given by

$$(I+\Delta G)x = q \text{ and } (I+\Delta G)x' = q'$$

Letting M denote $(I+\Delta G)^{-1}$, we have

$$x = Mq \text{ and } x' = Mq'$$

and since $q_j = q'_j$ for all $j \neq 1$ and $q'_1 = q_1 + \beta$, we get

$$x' - x = M(q'-q) = M(\beta, 0, ..., 0)^T = \beta(m_{11}, ..., m_{n1})^T$$

Hence $x'_1 - x_1 = \beta m_{11}$. In the appendix, we show - lemma 1 - that the inverse of a P-matrix is also a P-matrix. Therefore M is a P-matrix. Since all principal minors of a P-matrix are strictly positive, it follows that $m_{11} > 0$. Therefore $x'_1 > x_1$[14]. For the second part of the theorem, notice that if $(I+\Delta G)$ is symmetric then it is positive definite. This implies that $m_{ii} > 1$ for all i [15]. Thus, we have $x'_1 > x_1 + \beta$, and any increase in a player's needs will be amplified through the network structure and will result in an even larger increase in action.

This specific case is easy to deal with, for two reasons: there are no SI players in x, and sets A and SI remain unchanged between x and x'. In the general case, $SI(x)$ could be non-empty, and $SI(x')$ could be different from $SI(x)$, so that some active players in x become strictly inactive in x' and conversely, some strictly inactive players become active in x'. This is illustrated in Figure 3.

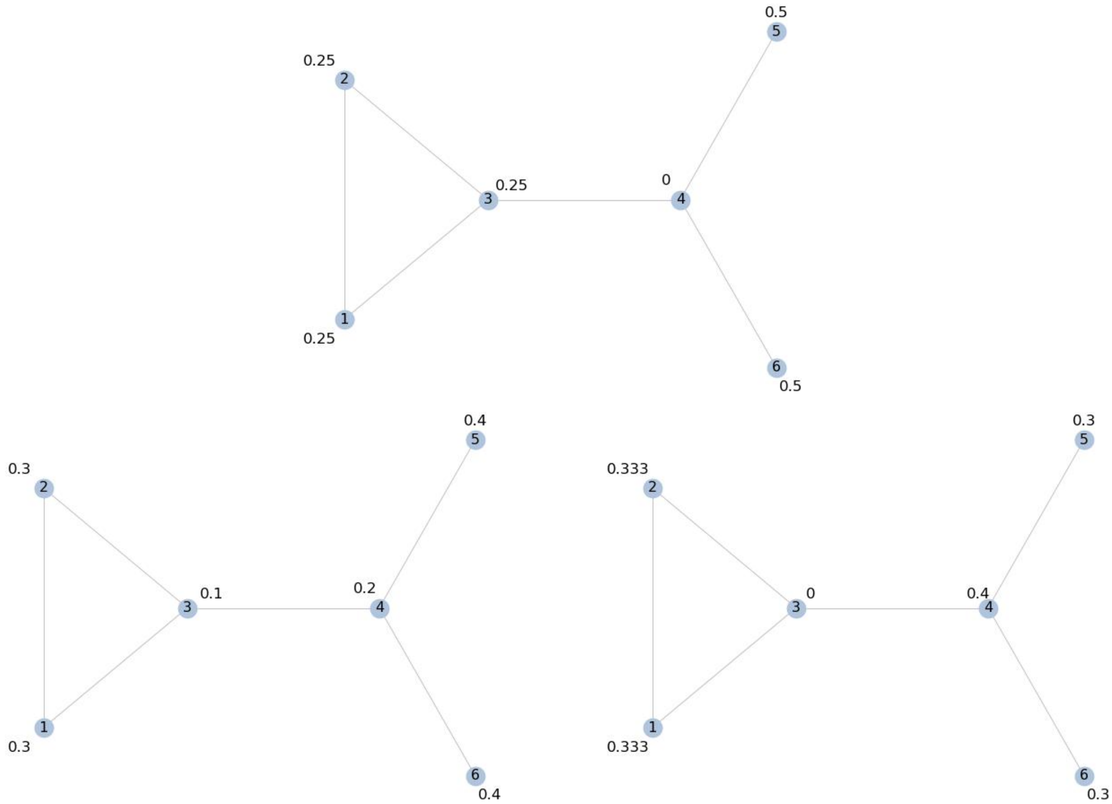


Figure 3: Upper panel: The unique Nash equilibrium x with homogeneous needs ($q_i = 0.5$ for all i) and $\delta_i = 0.5$ for all i, in a network such that the interaction matrix is a P-matrix. Here $A(x) = N \setminus \{4\}$ and $SI(x) = \{4\}$. Lower panels: On the left panel, the unique equilibrium where the needs of player 4 have been increased to 0.65 and to 0.7 on the right panel. On the left panel every player is active, while player 3 has become strictly inactive on the right panel. Despite both changes in the composition of types of players, the contribution of player 4 always increases (from 0 to 0.2 and 0.4) as his needs increase.

Here we illustrate why these situations are complex to deal with. Assume $SI(x) = \emptyset$ while $SI(x') = \{2\}$, i.e. player 2 becomes strictly inactive after the needs of player 1 increase. Then $x = Mq$ is still true, however, $x' \neq Mq'$ since, by equation (4), $x' = (I + \Delta G_{N\setminus\{2\}})^{-1} q_{N\setminus\{2\}}$, and thus operations like the above with only active players can no longer be performed.

However, we can write $G_{N\setminus\{2\}} = G - A$, where A is a matrix of 0's except for row 2. Then, $(I + \Delta G_{N\setminus\{2\}})x' = q_{N\setminus\{2\}} \implies (I + \Delta G)x' = q_{N\setminus\{2\}} + \delta_2 A x'$ and therefore $x' = Mq_{N\setminus\{2\}} + \delta_2 M A x'$. By developing, we finally get $x'_1 - x_1 = \beta m_{11} + (\delta_2 \sum_{i \in N} g_{2i} x'_i - q_2) m_{12}$.

It can be seen that the appearance of a new strictly inactive player adds the term $(\delta_2 \sum_{i \in N} g_{2i} x'_i - q_2) m_{12}$ to the previous interior situation. We know that $(\delta_2 \sum_{i \in N} g_{2i} x'_i - q_2) > 0$ since player 2 is strictly inactive in x', but the sign of m_{12} depends on the specific structure of the network and cannot be predicted by simple network statistics[16].

In the same way, the appearance of new active players will add other terms to the difference $x'_1 - x_1$. It is not possible, in general, to sign each of these terms, even less possible to sign the sum of these terms. However, when the interaction matrix is a P-matrix, we can show that the difference $x'_1 - x_1$ is always positive (see the details in the proof).

As a consequence of Theorem 1 and Proposition 3, we retrieve the standard result for the complete network, which we extend to the non-symmetric case:

Corollary 5. *If $(I + \Delta G)$ is the interaction matrix associated with the complete network, whether symmetric or not, increasing a player's needs results in an increase in his contribution.*

4 Effect of Needs on Aggregate Contributions

Here we analyze how the change in needs of one player affects aggregate contributions. We first present the case of identical players on a symmetric network because it lends itself to a nice interpretation of the results, which we then generalize.

[14] This simple case illustrates why the interaction matrix needs to be a $P - matrix$ for this monotony result to hold. Otherwise the term m_{11} could be negative, in which case $x'_1 < x_1$, as in Figure 2.

[15] See for instance Fiedler [1964], where it is shown that the product of a diagonal term of a positive definite matrix and the diagonal term of its inverse is greater than 1.

[16] The only networks for which the signs of the terms of the inverse of the interaction matrix are predetermined are the tree networks (see Roy and Xue [2021]).

4.1 Identical Players in Symmetric Networks

We start by analyzing the case of identical players, i.e. $q_i = 1$ for all i, who interact in an undirected and unweighted network G with identical substitution rates, i.e. $g_{ij} = g_{ji}$, $g_{ij} \in \{0, 1\}$ and $\delta_i = \delta$ for all pairs (ij). This is the most standard setting found in the literature on public good games played on networks. The best-response of any player i is thus:

$$Br_i(x_{-i}) = max\left\{1 - \delta \sum g_{ij}x_j, 0\right\} \tag{5}$$

We let x be a Nash equilibrium when needs are $q = (1, ..., 1)^T$ and consider $q' = (1 + \beta, 1, ..., 1)^T$, the vector of needs where the needs of player 1 are increased by an amount $\beta > 0$, and we call x' a Nash equilibrium with needs q'.

Although we can only guarantee that an increase in player 1's needs will increase his own contribution if the interaction matrix is a P-matrix, we prove that aggregate contributions will always increase if player 1 is active in x, even if the interaction matrix is not a P-matrix. This holds even if the set of active players changes after the increase.

Proposition 4. *Assume $1 \in A(x)$, and let x' be any equilibrium with needs q'. Then, if $A(x') \subseteq A(x)$, we have*

$$x_1 > 0 \Longrightarrow X' > X$$

If in addition $A(x') = A(x)$ or $A(x') = A(x) \setminus \{1\}$, then

$$x_1 = 0 \Longrightarrow X' = X$$

When x and x' are interior, the intuition is the following: increasing player 1's needs by β changes every player's contribution by an amount βm_{i1} (see the sketch of proof of theorem 1). When every player is active, the inverse matrix M is symmetric, guaranteeing that $m_{i1} = m_{1i}$.[17] Although these terms cannot be signed individually, the aggregate change is $\beta \sum_i m_{i1}$, and because needs are the same for every player, the interior equilibrium is $x_i = \sum_j m_{ij} = \sum_j m_{ji}$. Therefore $\sum_i m_{i1} = x_1 > 0$, so the change in aggregate contributions is proportional to x_1, which is positive.

Of course, the initial equilibrium is not always interior, but the result still holds. Also, if strictly inactive players appear after the increase in player 1's needs, we can show that aggregate contributions increase. Note that the result also holds if player 1 himself becomes strictly inactive after his needs increase.

The second point of the proposition says that if we increase the needs of an active player currently playing 0, the aggregate contribution will not change, whether he becomes strictly inactive or not. The fact that the aggregate contribution is preserved is surprising,

[17] This line of reasoning cannot be followed when some players are strictly inactive, since the matrix of interest is no longer the interaction matrix, but the equilibrium interaction matrix, which is not symmetric.

since all players' contributions will change, but these changes will somehow balance and cancel each other out. We discuss this further in section 4.3.

Corollary 6. *If G is the complete network and players are homogeneous, then increasing the needs of any player will increase aggregate contributions.*

This corollary holds since in the complete network with homogeneous players, every player is active and contributes the same amount. Therefore the set of active players can only shrink following an increase in needs.

Although Proposition 4 is not immediate, the result is in line with our intuition. The following proposition is more surprising:

Proposition 5. *Assume $1 \in SI(x)$, and let x' be any equilibrium with needs q'. Then if $SI(x') \subseteq SI(x)$, we have*

$$x'_1 > 0 \Longrightarrow X' < X$$
$$x'_1 = 0 \Longrightarrow X' \leq X$$

Although intuition suggests that free-riders are those driving contributions down and that these are the players that should be incited to contribute, Proposition 5 tells us precisely the opposite. Increasing the needs of a strictly inactive player until he becomes active will have a negative effect on aggregate contributions, despite the fact that this player is now contributing a positive amount.

The intuition for the first implication is the following. Suppose that player 1's needs are increased so that he contributes a positive amount ϵ at the new equilibrium. This is the same equilibrium as the one where player 1 is withdrawn from the game and the needs of all of player 1's neighbors have decreased by an amount $\delta\epsilon$. We show that when all neighbors of a player change their needs by the same amount $\delta\epsilon$, the aggregate effect is equal to $\delta\epsilon\bar{x}_1$, where $\bar{x}_1$ is the sum of contributions of player 1's neighbors in the initial equilibrium. If player 1 was SI, it must be that $\delta\bar{x}_1 > 1$. Therefore, the decrease in the aggregate contributions of the neighbors of player 1 is larger than ϵ. So an increase of ϵ for player 1 and an aggregate decrease larger than ϵ for other players result in a net decrease in aggregate contributions.

Remark 4. *As illustrated in Figure 1, increasing the needs of player 1 can either decrease or increase the number of equilibria. Propositions 4 and 5 hold true for every equilibrium.*

Propositions 4 and 5 are illustrated in Figures 4 and 5. We present an example with multiple equilibria to illustrate both the propositions and remark 4. In this network, with $\delta_i = 0.5$ and $q_i = 0.5$ for all i, there are three equilibria, an interior one and two with a strictly inactive player (only one is represented since the other is a permutation of the first). In the interior equilibrium, no player can become active by increasing anyone's needs, since everyone is already active. Thus Proposition 4 applies and aggregate contributions increase regardless of which player's needs are increased, and whether there is a new strictly inactive player or not. Notice that, in accordance with Figure 2, the contribution of player 1 decreases when his needs are increased; yet, aggregate contributions increase.

In the second equilibrium, an increase in the needs of active players still increases aggregate contributions; however, when the needs of the strictly inactive player are increased, this results in a decrease in aggregate contributions.

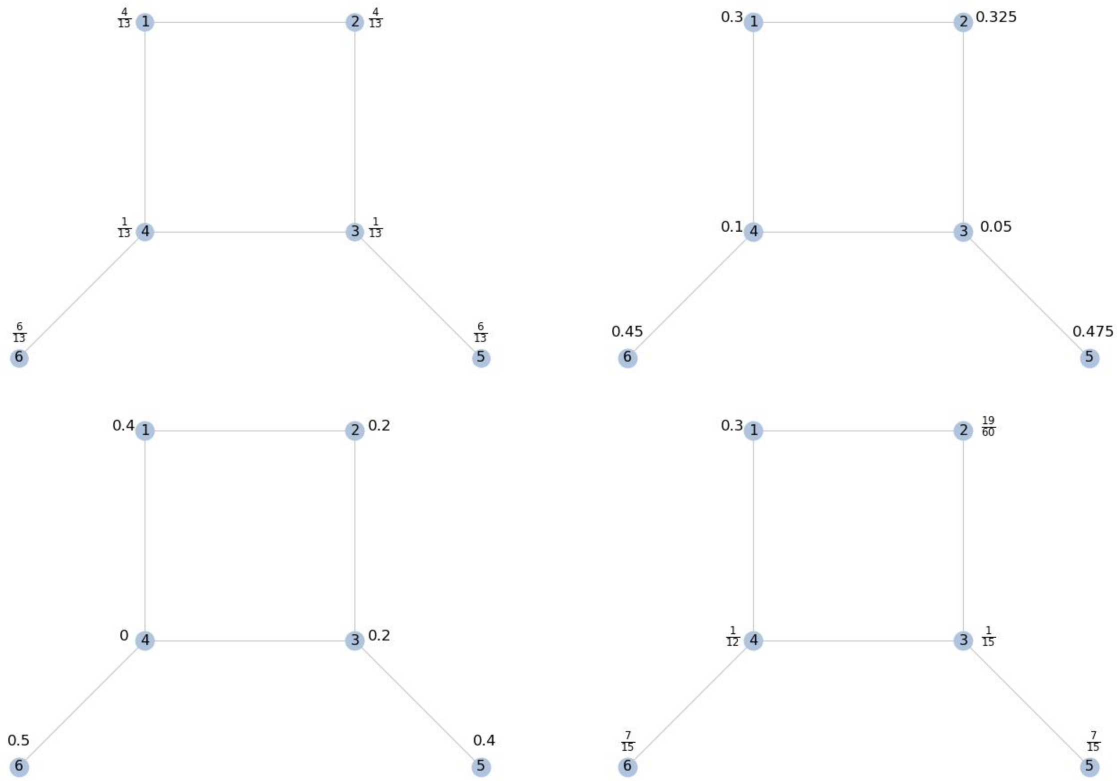


Figure 4: The upper left panel shows the interior equilibrium when $\delta_i = 0.5$ and $q_i = 0.5$ for all i. In the upper right panel, the equilibrium where needs of player 1 are increased to $\frac{41}{80}$; in the lower left panel, the equilibrium where needs of player 4 are increased to $\frac{11}{20}$; in the lower right panel, the equilibrium where needs of player 6 are increased to $\frac{61}{120}$. In all three cases, the aggregate contribution has risen from $\frac{22}{13}$ to 1.7.

4.2 Heterogeneous players in Non-Symmetric Networks

Here we relax the assumptions that $g_{ij} \in \{0, 1\}$, $q_i = 1$ and $\delta_i = \delta$ for all i. We let $g_{ij} = 0$ if player i is not connected to j, and $g_{ij} > 0$ otherwise, with possibly $g_{ij} \neq g_{ji}$. We also

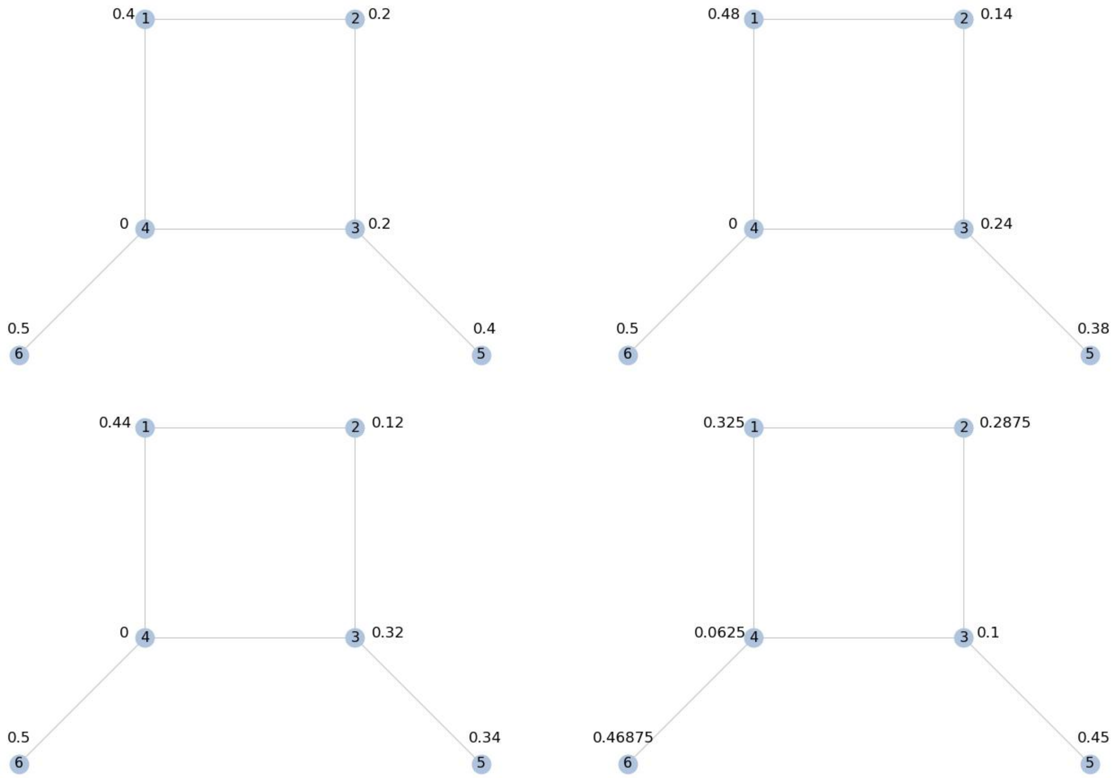


Figure 5: The upper left panel shows the corner equilibrium when $\delta_i = 0.5$ and $q_i = 0.5$ for all i. In the upper right panel, the equilibrium where needs of player 1 are increased to $\frac{11}{20}$; in the lower left panel, the equilibrium where needs of player 3 are increased to $\frac{11}{20}$; in the lower right panel, the equilibrium where needs of player 4 are increased to $\frac{12}{20}$ (he only becomes active when his needs are above $\frac{11}{20}$). In the first two cases, the aggregate contribution rises from 1.7 to 1.74 and 1.72, while in the third case, contributions decrease to 1.69375.

allow for $q_i \neq q_j$ and $\delta_i \neq \delta_j$. Recall, from Proposition 1, that changes in Δ or G are captured by changes in q. Therefore, here again, we focus on changes in q.

We start from an equilibrium x and we increase the needs of player 1 by an amount $\beta > 0$. We call x' an equilibrium with the new vector of needs. In the previous section, the determining factor in whether contributions increase or decrease was the status (active or strictly inactive) of player 1. In the general case, this is no longer true. What matters is whether player 1 plays a positive or a negative action in the unconstrained solution of an associated game: assume x is a Nash equilibrium of the game played on network G with needs $q = (q_1, ..., q_n)^T$, with set of active players $A(x)$. Recall that the vector $x^{unc}((\Delta G_{A(x)})^T, 1)$ denotes the unique unconstrained solution to the system with the transpose of the equilibrium interaction matrix, and homogeneous needs. It is the matrix in which players in $SI(x)$ have no outgoing links (instead of no incoming links). This solution being unconstrained, some coordinates will be positive, while others could be negative. The following result states that the effect of increasing needs depends on the sign of these coordinates, in particular of those of player 1, and of the players who change status before and after the increase.

We thus define the sets of players who change status after needs are increased. Let $a(x, x') := \{i \in N; i \in A(x) \cap SI(x')\}$ and $si(x, x') := \{i \in N; i \in SI(x) \cap A(x')\}$.

Proposition 6. *Let x be an equilibrium when needs are q. Let $q' = (q_1 + \beta, q_2, ..., q_n)^T$ with $\beta > 0$ and x' be any equilibrium with needs q'. Let $x^{unc} = x^{unc}((\Delta G_{A(x)})^T, 1)$. Then,*

$$\begin{cases} x_1^{unc} \geq 0, x_i^{unc} \geq 0 \text{ for all } i \in a(x, x') \cup si(x, x') & \Longrightarrow X' \geq X \\ x_1^{unc} \leq 0, x_i^{unc} \leq 0 \text{ for all } i \in a(x, x') \cup si(x, x') & \Longrightarrow X' \leq X \end{cases}$$

Note that the sets $a(x, x')$ and $si(x, x')$ can be empty, covering the case in which $A(x) = A(x')$. Note also that while Propositions 4 and 5 differed depending on the status of player 1 (active or not), this is no longer relevant in Proposition 6, which applies to both cases.

This provides a simple way to check whether contributions will increase or decrease, as illustrated in example 2. One noteworthy point is that there is no immediate relation between $x^{unc}((\Delta G_{A(x)})^T, 1)$ and x or x'. Both x and x' are constrained solutions to a problem with heterogeneous needs, while $x^{unc}((\Delta G_{A(x)})^T, 1)$ is the unconstrained solution to another problem with homogeneous needs. Moreover, the fact that $i \in A(x)$ (resp $i \in SI(x)$) in a Nash equilibrium x of the game with homogeneous needs does not guarantee that player i will be active (resp. strictly inactive) in the same game with heterogeneous needs. In the following example, we illustrate how Proposition 6 works on an arbitrary network.

Example 2. Consider the network of Figure 3, with $\delta_i = 0.5$ for all i. Since the matrix

$(I + \Delta G)$ is positive definite, and thus a P-matrix, there is a unique Nash equilibrium for any vector of needs. Take needs $q = (0.55, 0.25, 0.45, 0.65, 0.6, 0.4)^T$. The unique equilibrium x is $(0.5, 0, 0.1, 0.2, 0.5, 0.3)^T$, where $SI(x) = \{2\}$. To know whether increasing the needs of some players will increase or decrease aggregate contributions, we need to construct the interaction matrix $(I + (\Delta G_{A(x)})^T)$ from $(I + \Delta G)$ by taking out the incoming links of player 2 and transposing it:

$$(I+\Delta G) = \begin{pmatrix} 1 & 0.5 & 0.5 & 0 & 0 & 0 \\ 0.5 & 1 & 0.5 & 0 & 0 & 0 \\ 0.5 & 0.5 & 1 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 1 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0.5 & 1 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 1 \end{pmatrix} \qquad (I+(\Delta G_{A(x)})^T) = \begin{pmatrix} 1 & 0 & 0.5 & 0 & 0 & 0 \\ 0.5 & 1 & 0.5 & 0 & 0 & 0 \\ 0.5 & 0 & 1 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 1 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0.5 & 1 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 1 \end{pmatrix}$$

We now solve for $(I + (\Delta G_{A(x)})^T)x = 1$ and find $x^{unc} = (0, 0, 2, -2, 2, 2)^T$. This implies that increasing the needs of players $3, 5$ or 6 of any amount $\beta > 0$ will increase the aggregate contribution (since $x_3^{unc} = x_5^{unc} = x_6^{unc} = +2$), while increasing the needs of player 4 will decrease it (since $x_4^{unc} = -2$), as long as the set of strictly inactive players remains the same. Figure 6 illustrates several possibilities covered by Proposition 6.

The relation between Proposition 6 and Propositions 4 and 5 is the following: although there is no clear link between solution $x^{unc}((\Delta G_{A(x)})^T, 1)$ and the equilibrium of the game with heterogeneous agents in a non-symmetric network, there is one with the equilibrium of the game with identical players in a symmetric network. First, when players are identical, their needs are homogeneous, as in the unconstrained solution. Secondly, when the network is symmetric, the adjacency matrix is by definition equal to its transpose. Thus, we can show that $i \in A(x) \iff x_i^{unc}((\Delta G_{A(x)})^T, 1) \geq 0$, and therefore $i \in SI(x) \iff x_i^{unc}((\Delta G_{A(x)})^T, 1) < 0$. This is why the condition on $x^{unc}((\Delta G_{A(x)})^T, 1)$ can be expressed in terms of being active or strictly inactive in Propositions 4 and 5.

When the network is complete, we show that $x^{unc}((\Delta G_{A(x)})^T, 1)$ is positive for all players (Lemma 3 in the appendix). This yields the following:

Corollary 7. *If G is a complete network, then any increase in the needs of any player induces an increase in aggregate contribution.*

This corollary corresponds to the standard results obtained in the literature. It illustrates how the complete network is in fact a special case that does not generalize to arbitrary networks.

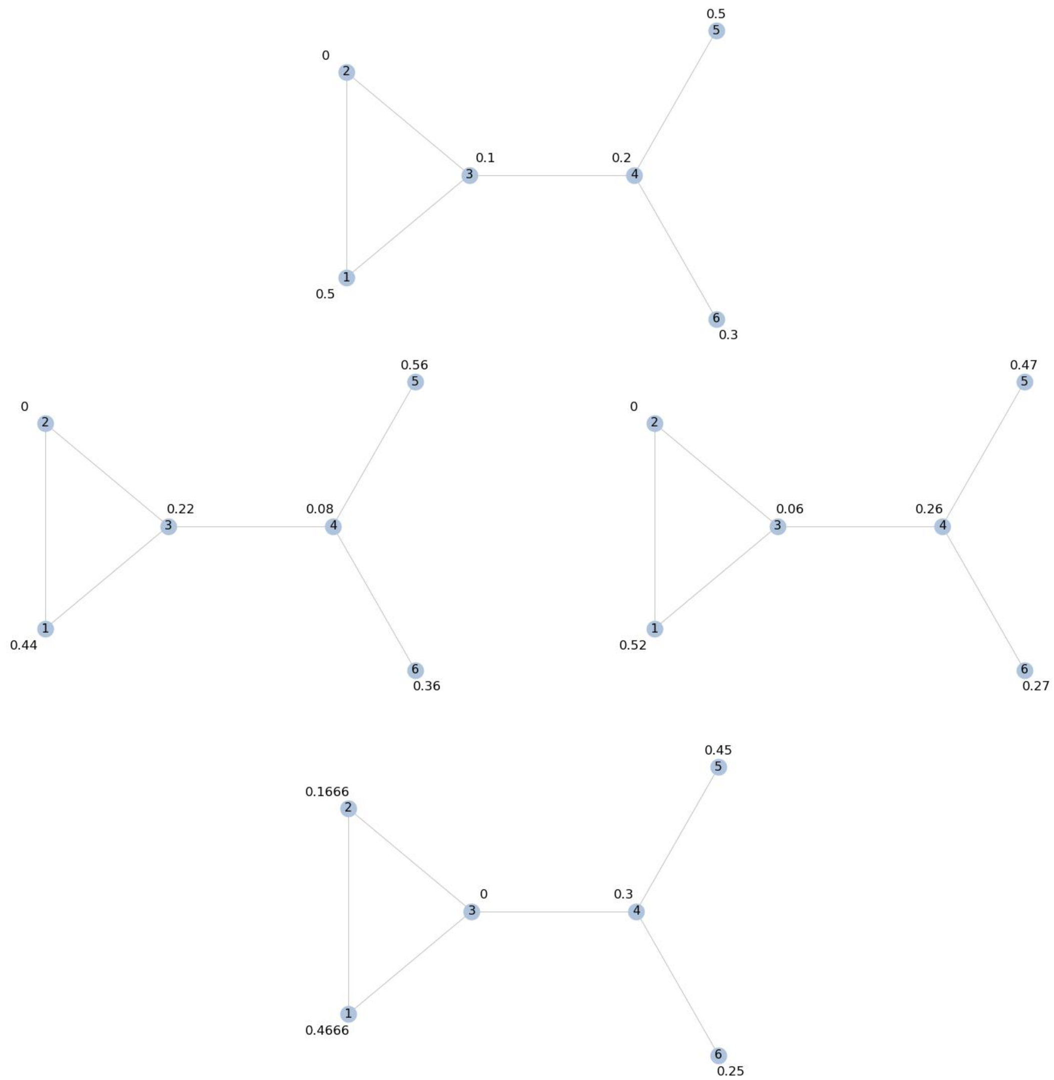


Figure 6: First panel, initial equilibrium with needs $q = (0.55, 0.25, 0.45, 0.65, 0.6, 0.4)^T$ and $\delta_i = 0.5$ for all i. The sum of contributions is equal to 1.6. In the second panel on the left, needs of player 3 are increased to 0.48 and since $x_3^{unc}((\Delta G_{A(x)})^T, 1) > 0$, the sum of contributions increases (to 1.66). On the right, needs of player 4 are increased to 0.66 and since $x_4^{unc}((\Delta G_{A(x)})^T, 1) < 0$, aggregate contributions decrease (to 1.58). Finally, in the third panel, needs of player 2 are increased to 0.4, and player 3 becomes strictly inactive. Still, since both $x_2^{unc}((\Delta G_{A(x)})^T, 1) \geq 0$ and $x_3^{unc}((\Delta G_{A(x)})^T, 1) \geq 0$, contributions increase (to 1.6333), as suggested by Proposition 6.

4.3 Neutral players

As a consequence of Propositions 4 and 6, changing the needs of an active player who plays 0 in the unconstrained solution of the transposed game played with homogeneous needs, will change the equilibrium contributions of potentially all players, including himself, but will leave the sum of contributions unchanged. These players can be called *neutral players*. They are special in that any change in their needs inducing a change in their contribution will be compensated for exactly by the adjustments in the network. We show examples in Figure 7.

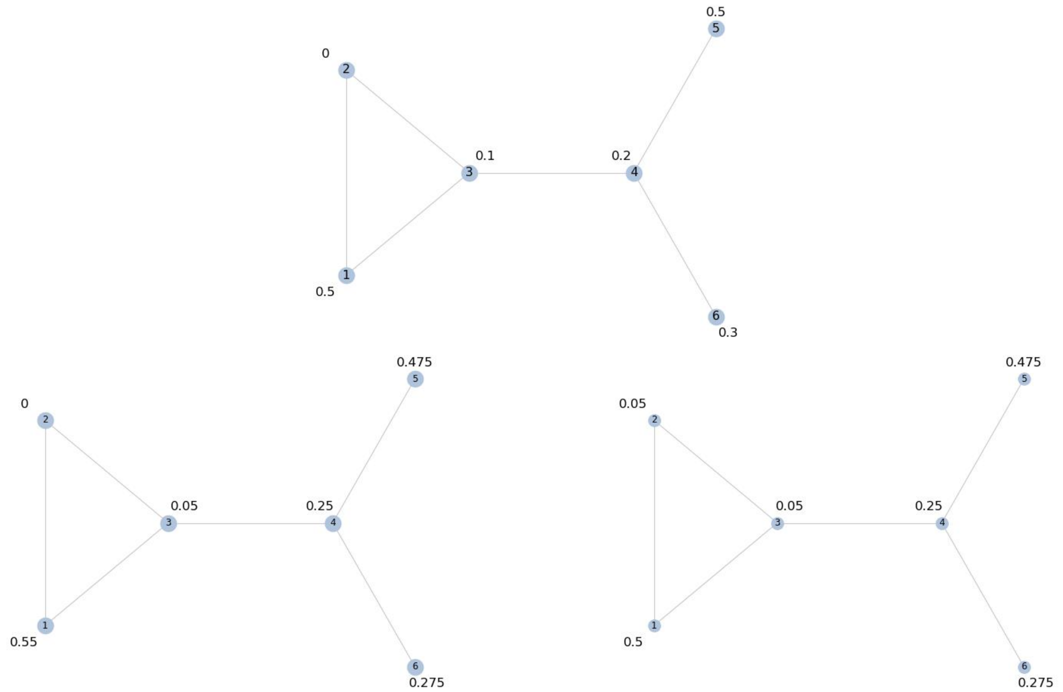
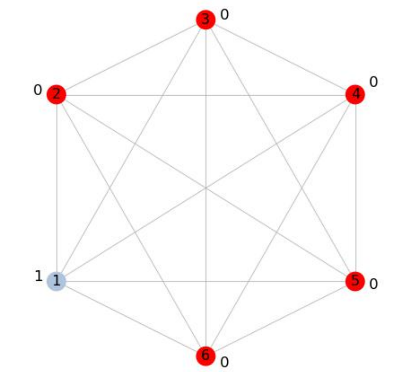


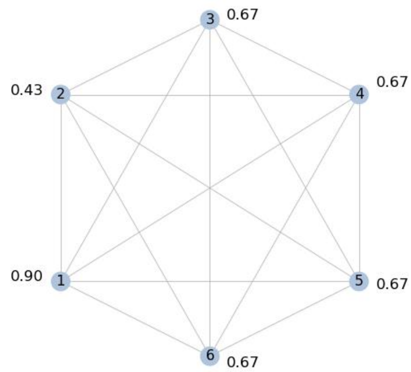
Figure 7: First panel, initial equilibrium with needs $q = (0.55, 0.25, 0.45, 0.65, 0.6, 0.4)^T$ and $\delta_i = 0.5$ for all i, where the sum of contributions is equal to 1.6. Recall, from example 2, that $x^{unc}((\Delta G_{A(x)})^T, 1) = (0, 0, 2, -2, 2, 2)^T$. Therefore, when needs of player 1 are increased to 0.575 the sum of contributions remains equal to 1.6 (second panel on the left). Second panel on the right, needs of player 2 are increased to 0.325, player 2 becomes active and since $x_2^{unc}((\Delta G_{A(x)})^T, 1) = 0$, the sum of contributions remains equal to 1.6. Both players are neutral.

Remark 5. *In the proof of lemma 2, we show that in a complete network with* $\delta_1 = 1$, *we have* $x_i^{unc} = 0$ *for all* $i \neq 1$. *Thus every player except player* 1 *is neutral, as illustrated in Figure 8.*

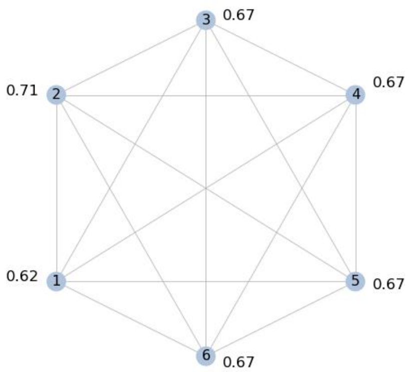
Notice that whether a player is neutral or not depends both on the network and on the substitution rates of other players. However, neutrality does not depend on the neutral player's substitution rate, as we show here:



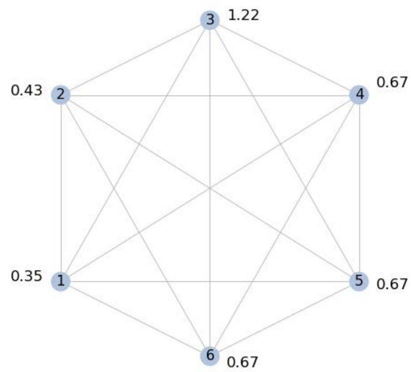
(a) Unconstrained solution



(b) $q = (4, 1.5, 1, 1, 1, 1)$



(c) $q = (4, 1.7, 1, 1, 1, 1)$



(d) $q = (4, 1.5, 1.5, 1, 1, 1)$

Figure 8: Substitution rates have been set to $\delta = (1, 0.3, 0.1, 0.1, 0.1, 0.1)$. Panel (a) is the solution of the unconstrained problem. We see that $x_i = 0$ for all $i \neq 1$, which implies that all players except player 1 are neutral. For panels (b)-(d), needs and equilibrium profiles are changed, but the aggregate contribution remains constant.

Proposition 7. *Assume player 1 is neutral with interaction matrix* $(I + \Delta G)$*. Consider* $\Delta' = diag(\delta'_i)_{i \in N}$ *such that* $\delta'_i = \delta_i$ *for all* $i \neq 1$ *and* $\delta'_1 \neq \delta_1$*, i.e. substitution rate of player 1 is changed. Then*

$$x^{unc}((\Delta G)^T, 1) = x^{unc}((\Delta' G)^T, 1) \tag{6}$$

In particular, player 1 remains neutral when his substitution rate changes.

This proposition implies that a neutral player remains neutral if his substitution rate changes, but it also implies that the effect of a shock on any player will always have the same effect (positive or negative) when the substitution rate of the neutral player changes, since x^{unc} does not change.

5 Transfers and Neutrality

An important literature has emerged on redistribution of wealth among individuals and the private provision of public goods since the work of Warr [1983], followed by i.a. Bergstrom et al. [1986] and Andreoni [1990]. The first two papers look at pure global public goods (i.e. the substitution rate δ_i is 1 for all i, and interactions take place on a complete network) , and obtain a neutrality result saying that small income redistribution among active players will change their contribution by exactly the amount of the transfer received, i.e. $y_i = x_i + \epsilon_i$, where x is a Nash equilibrium with the initial endowments and y is the Nash equilibrium after any small redistribution ϵ among players in $A(x)$, such that $A(y) = A(x)$. This implies that every player will consume exactly the same amount of private good and will enjoy the same amount of public good, since the network is complete and substitution is perfect.

Observing that empirical evidence seems to contradict neutrality, Andreoni [1990] considers possibly impure public goods (i.e. possibly $\delta_i \neq \delta_j$), still on complete networks, and shows that the previous neutrality result holds if and only if $\delta_i = 1\ \forall i$. However, he also shows that if $\delta_i = \delta_j$, not necessarily equal to 1, then we have an *aggregate neutrality* result telling us that the sum of contributions will remain constant if player i transfers wealth to player j. Of course, aggregate neutrality is weaker than neutrality. Nevertheless, it is still a strong result.

Allouch [2015] takes Bergstrom et al. [1986] to networks to check whether neutrality still holds on non-complete networks[18], and proves that it only holds on specific networks where all active players are linked together and where strictly inactive players are either

[18]Transposed into our setting, Allouch [2015] restricts the analysis to networks such that the interaction matrix is positive definite, thereby guaranteeing uniqueness of the solution, in a class of games that include some linear best-response games. Our analysis does not restrict to situations where equilibrium is unique, it also applies to all games with linear best-responses, but to those alone.

linked to every active player or to none. The general message is that neutrality tends to fail once some heterogeneity is introduced into the pattern of interactions. Allouch [2015] draws a parallel with Andreoni [1990], since Andreoni also explains failure of neutrality by (another sort of) heterogeneity. However, as for neutrality, aggregate neutrality results hold on a complete network. Here, we take Andreoni [1990] to networks to check under which conditions aggregate neutrality holds[19].

For ease of exposition, we assume that transfers only take place between two players, player 1 and player 2. Like other authors on this topic, we restrict to transfers small enough that the set of active players does not change after the transfer.

Let x be an equilibrium of the game played on G with needs q. As mentioned earlier, changing players' wealth is equivalent to changing their needs in terms of comparative statics on equilibria. Thus, considering transfers of wealth between players is equivalent to considering transfers of needs between players. Let $t = (-\epsilon, +\epsilon, 0, ..., 0)^T$ be a vector of transfers of needs between players 1 and 2, and let x' be an equilibrium of the game played on G with needs $q + t$.

Let $\Delta = diag(\delta_i)_{i \in N}$ be the diagonal matrix of the substitution terms. Now let $\delta'_2 = \delta_1$ and $\delta'_i = \delta_i$ otherwise, and $\Delta' = diag(\delta'_i)_{i \in N}$. Then $(I + \Delta' G)$ is the interaction matrix in the game where player 2 has the same substitution rate as player 1. Finally, we denote $x^{unc} = x^{unc}((\Delta' G)^T, 1)$ the unconstrained solution to the homogeneous needs transposed problem:

$$(I + (\Delta' G)^T)x = 1$$

We have the following result:

Proposition 8. *Let x be an equilibrium of the game played on G with needs q and let x' be an equilibrium of the game played on G with needs $q + t$. Then*

$$X' - X = \epsilon(x_2^{unc} - x_1^{unc}) + (\delta_1 - \delta_2)x_2^{unc} \sum_j g_{2j}(x'_j - x_j) \tag{7}$$

In the Appendix we prove that on the complete network, $x_2^{unc} = x_1^{unc}$, and $x_2^{unc} \sum_j g_{2j}(x'_j - x_j) < 0$, so we have the following:

[19] In Allouch [2015], the class of games under consideration includes some games with linear best-responses and heterogeneous substitution rates, as illustrated in Corollary 2. Note, however, that in order to exhibit such best-responses, it is necessary to relate players' wealth to substitution rates in a specific way. In Andreoni [1990]'s setting, all the games in the class of Allouch [2015] are such that players are pure altruists, which implies that the altruism coefficients α_i defined in Andreoni [1990] are all equal to 1.

In turn, in our setting where there is no a priori relation between wealth and substitution rates, we can easily check that the altruism coefficients α_i in Andreoni [1990] are related to our substitution rates δ_i, according to the relation $\alpha_i = \frac{1}{2-\delta_i}$. Thus $\alpha_1 > \alpha_2 \iff \delta_1 > \delta_2$.

Corollary 8 (Andreoni (1990)). *When network G is complete, then*

$$Sign(X' - X) = Sign(\delta_2 - \delta_1)$$

When $\delta_1 = \delta_2$, $X' = X$ and aggregate neutrality holds. When $\delta_1 \neq \delta_2$, then transferring wealth from player 1 to player 2 increases aggregate contributions if and only if player 2's rate of substitution is greater than player 1's, when the network is complete. This is the main result in Andreoni [1990]. To see why it holds on the complete network but not on arbitrary networks, let us focus on the case where the equilibrium interaction matrix is of spectral radius smaller than 1. In that case, we can use the power series development:

$$(I + \Delta G)^{-1} = \sum_{k=0}^{\infty} (-1)^k (\Delta G)^k,$$

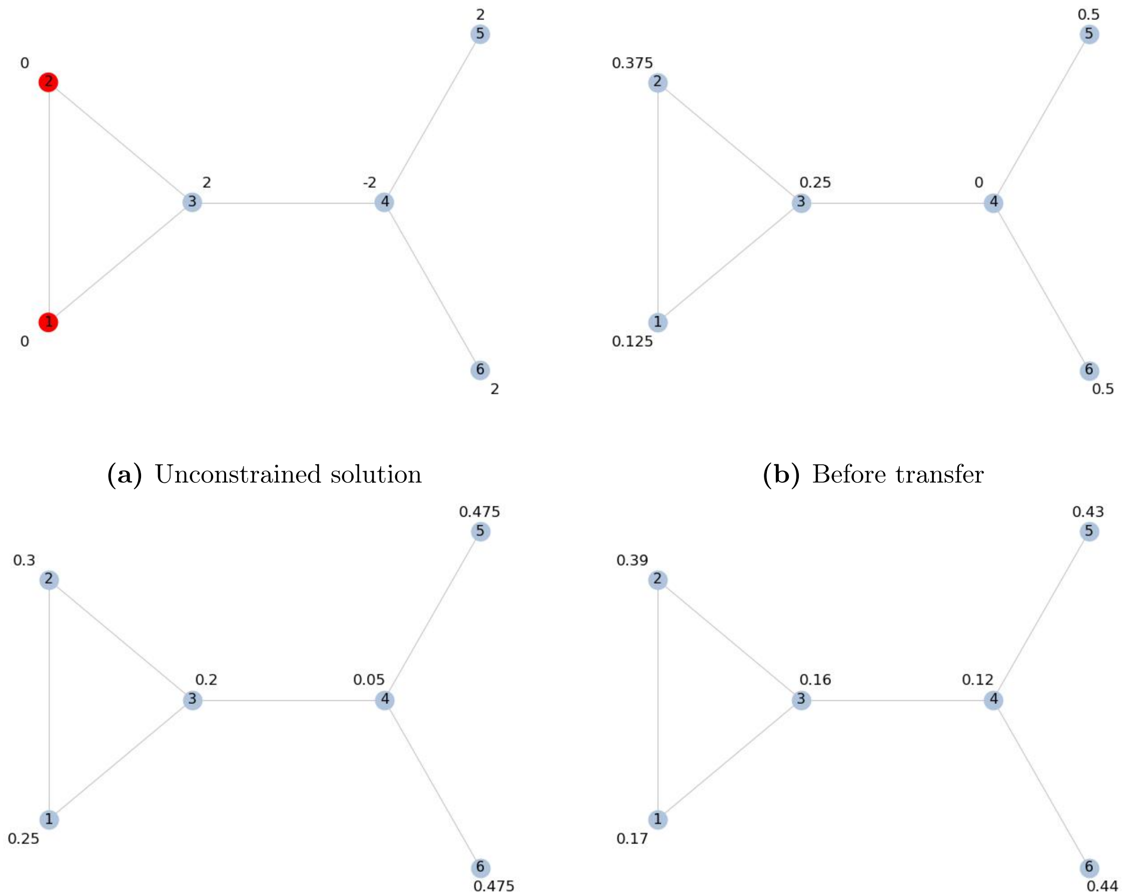
where the term $(\Delta G)^k_{ij}$ is the number of paths of length k going from player i to player j, where each link between any player l and any other player is discounted by δ_l. When a transfer takes place between player 1 and 2, it is easy to show that the net effect only depends on paths leaving from player 1 and player 2, the former being discounted by δ_1, while the latter are discounted by δ_2. Since the network is complete, to each path leaving from 1 and reaching any other player i in k steps, we can associate a path leaving from 2, reaching i in k steps, and going through the same set of players[20]. The aggregate effect of the transfer will thus be captured by the different effects of all these paths in the network. Therefore, the (positive) effect stemming from player 2 getting richer will be discounted by δ_2, while the (negative) effect stemming from player 1 getting poorer will be discounted by δ_1. Hence the proportionality with $(\delta_2 - \delta_1)$ of the aggregate effect.

However, once the network is not complete, paths leaving from 1 and reaching any other player i in k steps cannot be associated to an equivalent path leaving from 2 and reaching i in k steps. This asymmetry in the network explains why aggregate effects cannot be captured as simply as with the complete network.

Besides its corollary, Proposition 8 tells us on the one hand that aggregate neutrality can hold even though substitution rates among players involved in the transfer differ, and on the other hand that it can fail even though substitution rates among players involved in the transfer are equal. This is illustrated in Figure 9, where these two types of failure of Andreoni's result on non-complete networks are illustrated.

Remark 6. *As explained in section 4.3, changing needs of neutral players keeps the aggregate constant. Since a transfer is a positive change in needs for one player and a corresponding negative change in needs for another player, any transfer between neutral players will also leave the aggregate constant, as it is shown in figure 9c.*

[20] It is in fact a bit more subtle, since player 1 can reach player 2 in one step while player 2 cannot. We leave these subtleties out since they do not change the intuition.



(a) Unconstrained solution

(b) Before transfer

(c) transfer between player 1 and 2

(d) transfer between player 4 and 5

Figure 9: Substitution rates have been set to $\delta = (0.6, \frac{1}{3}, 0.5, 0.5, 0.5, 0.5)$. The unconstrained solution is given in panel (a). We observe that $x_1^{unc}((\Delta G_{A(x)})^T, 1) = x_2^{unc}((\Delta G_{A(x)})^T, 1) = 0$, implying that player 1 and 2 are neutral. In panel (b), the needs of players are $q_i = 0.5$ for all players except $q_4 = 0.625$. The sum of contributions is 1.75. In panel (c), a transfer has been made between players 1 and 2, the needs of player 1 have increased to 0.55 while those of player 2 have decreased to 0.45. We observe that the sum of contributions remains constant, equal to 1.75, despite the fact that $\delta_1 \neq \delta_2$, and every player has changed his contribution after the transfer. In panel (d), a transfer has been made between players 4 and 5, the needs of player 4 have increased to 0.635 while those of player 5 have decreased to 0.44. This time the sum of contributions has decreased to 1.71, although $\delta_4 = \delta_5$.

6 Introducing Kantian agents: an application

In this section we introduce Kantian agents into the local public good game, considering that society is formed of both Kantian agents and Nash maximizers. As mentioned in the introduction, the observed over-provision of public goods with respect to theoretical predictions is sometimes explained by the fact that agents might show other kinds of behavior than simply maximizing their own payoff.

Definition 3. *A Kantian agent is an agent maximizing $u_i(x_i, x_{-i})$ with $x_j = x_i$ for all j*

As mentioned in the introduction, the literature suggests that considering Kantian agents could reduce the under-provision of public goods. However, these papers assume that the entire population is formed of Kantian agents, and that all these agents interact with everyone else. The main difference from our analysis is that we allow Nash players and Kantian agents to coexist, and to interact on an incomplete network. We ask the following question: under what conditions does the introduction of Kantian agents reduce the under-provision of public goods? As it turns out, we can answer this by applying the results of the previous sections.

Contrary to Nash players, the action chosen by a Kantian agent does not depend only on the best-response function, it depends on the payoff function itself. For instance, if payoffs are given by

$$u_i(x_i, x_{-i}) = log\left(x_i + \delta \sum_{j=1}^{n} g_{i,j} x_j\right) + log(w_i - x_i) \tag{8}$$

then Kantian agents are maximizing[21]

$$u_i^K(x_i) = log\left(x_i + \delta \sum_{j=1}^{n} g_{i,j} x_i\right) + log(w_i - x_i)$$

which results in setting

$$x_i^K = \frac{w_i}{2}$$

for all Kantian agents.

If payoffs are now given by

$$u_i(x_i, x_{-i}) = b(x_i + \delta \sum_{j=1}^{n} g_{i,j} x_j) - c x_i$$

as in Bramoullé and Kranton [2007], then Kantian agents are maximizing

$$u_i^K(x_i) = b(x_i + \delta_i \sum_{j=1}^{n} g_{i,j} x_i) - c x_i = b(x_i(1 + \delta_i n_i)) - c x_i$$

[21] In what follows, variables with superscript K designate variables related to Kantian agents, while those without superscript designate variables related to Nash players.

where n_i is the number of neighbors of player i. This results in setting

$$x_i^K = \frac{a_i}{1 + \delta_i n_i}$$

where $a_i = b'^{-1}(\frac{c}{1+\delta_i n_i})$.

In the first case, the action chosen by the Kantian agents is independent of the network, while in the second, it depends on the number of neighbors. In what follows, we consider general payoff functions with linear best-responses (1), and we thus allow for Kantian agents whose actions might depend on the network structure.

Remember that needs q_i are player i's autarkic contribution. Our results hold under the following (mild) condition:

Assumption 1. *The payoff functions giving best-responses* (1) *are such that a Kantian agent contributes at least his autarkic contribution:* $x_i^K \geq q_i$

This condition is satisfied by many standard payoff functions. In particular, with the log-log payoff function, a Kantian agent contributes $\frac{w_i}{2}$, which is equal to his needs. In the setting of Bramoullé and Kranton [2007], the condition holds as long as $b'^{-1}(c)(1+\delta_i n_i) \leq b'^{-1}\left(\frac{c}{1+\delta_i n_i}\right)$, which is usually satisfied if $b(.)$ is not too concave[22].

In what follows, our aim is to examine the effect of Kantian behavior on aggregate contributions. We can do that either by directly replacing a Nash maximizer by a Kantian agent, or by raising the "kantianness" of players:

Definition 4. *An α-Nash-Kant player is a player playing action*

$$\alpha BR(x_{-i}) + (1-\alpha)x_i^K$$

for some $\alpha \in [0, 1]$

Nash players and Kantian agents are special cases with $\alpha = 1$ or $\alpha = 0$. In fact, in the analysis that follows, the effects of replacing a Nash player by a Kantian agent (i.e. increasing α_i from 0 to 1) are qualitatively equivalent to the effects of raising α_i to $\alpha_i + \epsilon$ (as long as $\alpha_i + \epsilon \leq 1$). For clarity of exposition, we will only consider replacements of Nash players by Kantian agents, but the reader should keep in mind that the results hold for arbitrary increases in the kantianness of agents.

We denote by $\mathcal{G}^K = ((N \setminus K, K), (X_i)_{i=1,\dots,n}, (u_i)_{i \in N \setminus K}, (u_i^K)_{i \in K})$ the game in which players in the set $N \setminus K$ are Nash maximizers, while agents in K are Kantian agents.

[22]For instance, with a CRRA benefit function $b(x) = \frac{1}{1-\alpha}x^{1-\alpha}$ with $\alpha > 0$ and $\alpha \neq 1$, our assumption is satisfied as long as $0 < \alpha < 1$.

Definition 5 (Nash-Kant equilibrium)**.** *A profile x is a Nash-Kant equilibrium of the game $\mathcal{G}^K$ if:*
- *for any player $i \in N \setminus K$, $x_i \in BR(x_{-i})$*
- *for any player $i \in K, x_i = x_i^K$*

Proposition 9. *For any $K \subseteq N$, a Nash-Kant equilibrium of $\mathcal{G}^K$ exists. Furthermore, if $(I + \Delta G)$ is a P-matrix, then this equilibrium is unique.*

Existence relies on the same argument as for when there are only Nash players. For uniqueness, the proof works as follows: once the action of all Kantian agents is fixed, the action of all Nash players is the same as the action they would choose if the Kantian agents were removed from the network, and the needs of the Nash players were reduced by the amount of the Kantian agent's contribution. Also, a submatrix of a P-matrix is a P-matrix. Thus, uniqueness will be preserved when removing players from the game. This is illustrated in Figure 10. In that figure, as well as the next ones in this section, we assume that Nash players have a payoff function as in (8), with $w_i = 1$ and $\delta_i = 0.5$, so that needs are $\frac{1}{2}$ and Kantian agents play $\frac{1}{2}$.

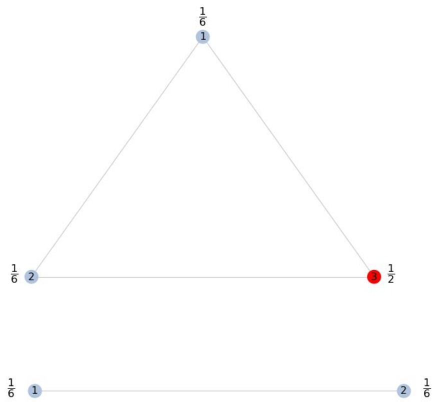
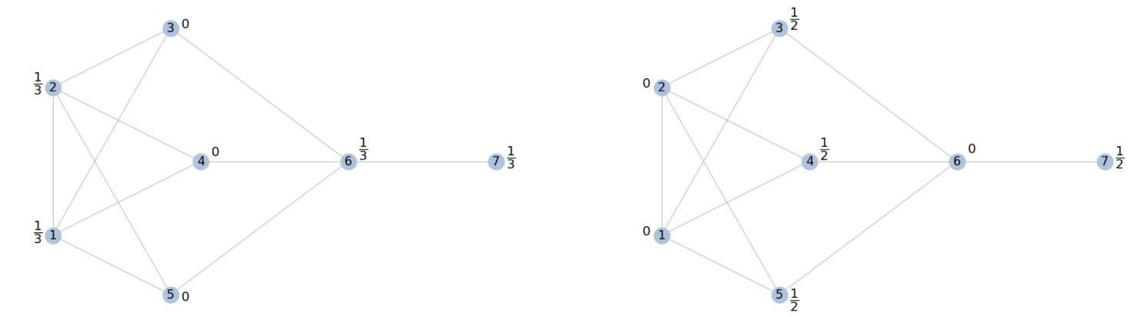


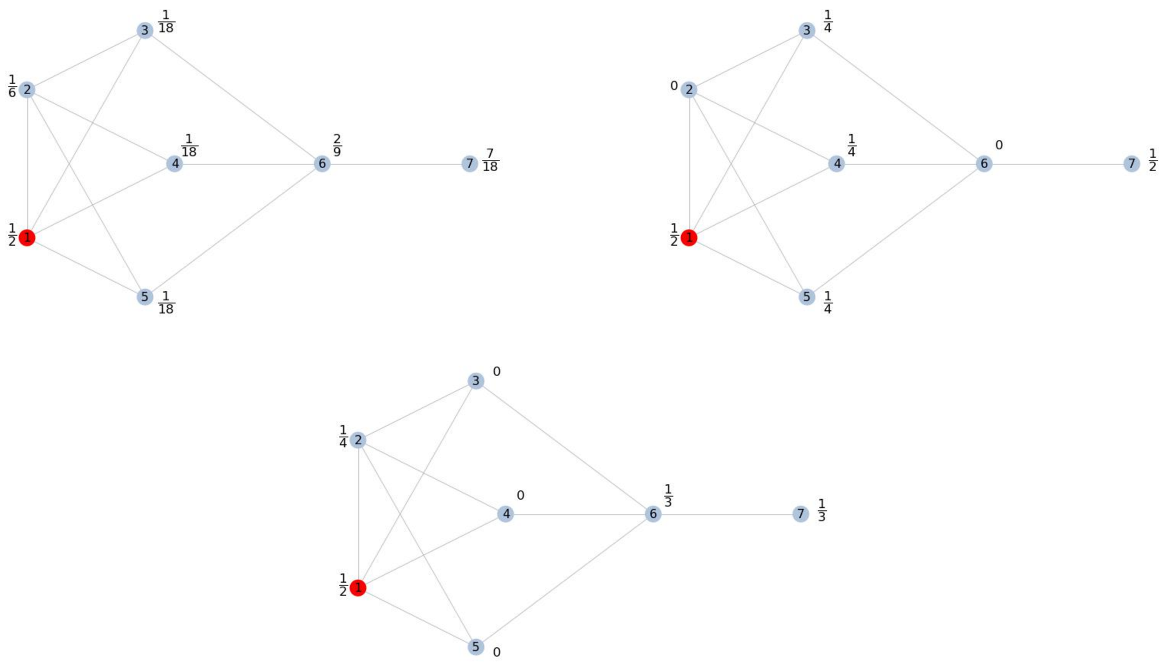
Figure 10: In the initial network, a triangle where two agents are Nash players while the third - in red - is Kantian. The Kantian agent plays $\frac{1}{2}$ and the other two play their mutual best-responses, $\frac{1}{6}$. This equilibrium can also be found by removing the Kantian agent from the network as in the lower panel, decreasing the needs of the Nash players who are connected to the Kantian agent by an amount of $\delta_i \times \frac{1}{2} = \frac{1}{4}$, and looking for a Nash equilibrium of this game.

Surprisingly however, note that replacing a Nash player by a Kantian agent does not necessarily reduce the multiplicity problem. This is illustrated in Figure 11.

One Kantian agent



(a) 2 Nash equilibria



(b) 3 Nash equilibria

Figure 11: In this network with needs equal to 0.5 and $\delta_i = 0.5$ for all i, there are two Nash equilibria. However, when Nash player 1 is replaced by a Kantian agent, there are three Nash-Kant equilibria

Intuitively, contributions should increase once one Nash player is replaced by a Kantian agent, since Kantian agents produce a large quantity of public goods regardless of strategic considerations. This intuition is consistent with Bilodeau and Gravel [2004] for instance, who show that, in their context and on a complete network, the Kantian equilibrium is in fact a Lindahl equilibrium. Thus Kantian agents enhance aggregate contributions with respect to the Nash equilibrium.

However, we show that this is not always true. Replacing a Nash maximizer by a Kantian agent in order to reduce the inefficiencies due to free-riding could turn out to be detrimental to aggregate contributions.

Proposition 10 (One Kantian agent)**.** *Consider any network G and assume x is a Nash equilibrium on G with all players having the same needs. Let $K = \{1\}$ and consider the game $\mathcal{G}^{\{1\}}$, i.e. agent 1 is the only Kantian agent. Then,*

- *if $1 \in A(x)$ and if $A(x^{\{1\}}) \subseteq A(x)$, we have*

$$x_1 > 0 \Longrightarrow X^{\{1\}} > X$$

- *if $1 \in A(x)$ and if $A(x^{\{1\}}) = A(x)$, we have*

$$x_1 = 0 \Longrightarrow X^{\{1\}} = X$$

- *if $1 \in SI(x)$ and if $SI(x^{\{1\}}) \subseteq SI(x)$, we have*

$$X^{\{1\}} < X$$

We see that, surprisingly, the replacement of a free-rider by a high contributor does not increase aggregate contributions, it actually decreases them. In fact, if we wish to increase aggregate contributions, it is an already active agent that should be replaced. This proposition can be derived from propositions 4 and 5, changing the status of the agent to Kantian. Figure 12 illustrates proposition 10.

Several Kantian agents

Of all the Nash-Kant equilibria, the one with the largest aggregate contributions is the one associated with $\mathcal{G}^N$, where every agent is Kantian. Therefore going from playing $\mathcal{G}$ to playing $\mathcal{G}^N$ necessarily increases contributions, but as we have just seen, replacing one Nash player by a Kantian agent might decrease contributions. We are thus interested in whether there is always an improving sequence of single replacements (of Nash players by Kantian agents) such that contributions increase along that sequence. Here again, it is the unconstrained solution to a modified problem that provides the answer.

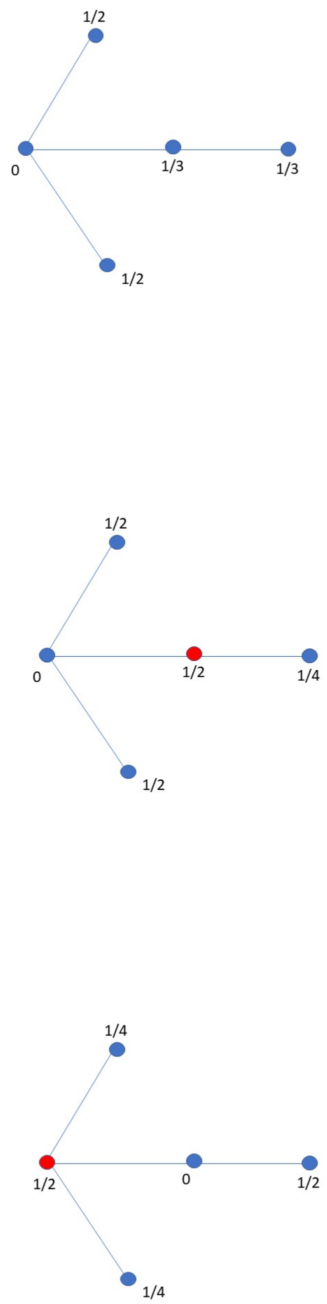


Figure 12: In the top panel, the unique Nash equilibrium of the game with $\delta_i = 0.5$ and $q_i = 0.5$ for all i. All players are active except the player playing 0, who is strictly inactive. The sum of contributions is 1.666. In the middle panel, an active player playing 1/3 is replaced by a Kantian agent playing 1/2. This increases total contributions, reaching 1.75. In the lower panel, the strictly inactive player is replaced by a Kantian agent playing 1/2. This decreases total contributions, down to 1.5.

Given a network G, a set of Kantian agents K, and a Nash-Kant equilibrium x^K, we denote by $\left(I + \Delta G_{A(x^K)}\right)_{N \setminus K}$ the submatrix of the equilibrium interaction matrix $\left(I + G_{A(x^K)}\right)$ obtained by removing Kantian agents from the game, and we call $x^{K,unc}$ the unconstrained solution to the transposed homogeneous problem $(I+\Delta G_{A(x^K)})^T_{N\setminus K}.x = 1$. Note that $x^{K,unc}$ is an $(\#N - \#K)$-size vector.

As in proposition 6, let $a(x^K, x^{K\cup\{1\}}) := \{i \in N; i \in A(x^K) \cap SI(x^{K\cup\{1\}})\}$ and $si(x^K, x^{K\cup\{1\}}) := \{i \in N; i \in SI(x^K) \cap A(x^{K\cup\{1\}})\}$.

Proposition 11. *Consider any network G and assume x^K is a Kant-Nash equilibrium on G with K the set of Kantian agents. Assume that we replace player $1 \in N \setminus K$ by a Kantian agent. Let $x^{K\cup\{1\}}$ be a Kant-Nash equilibrium on G with $K \cup \{1\}$ the set of Kantian agents. Then,*

$$\begin{cases} x_1^{K,unc} \geq 0, x_i^{K,unc} \geq 0 \text{ for all } i \in a(x^K, x^{K\cup\{1\}}) \cup si(x^K, x^{K\cup\{1\}}) & \Longrightarrow X' \geq X \\ x_1^{K,unc} \leq 0, x_i^{Kunc} \leq 0 \text{ for all } i \in a(x^K, x^{K\cup\{1\}}) \cup si(x^K, x^{K\cup\{1\}}) & \Longrightarrow X' \leq X \end{cases}$$

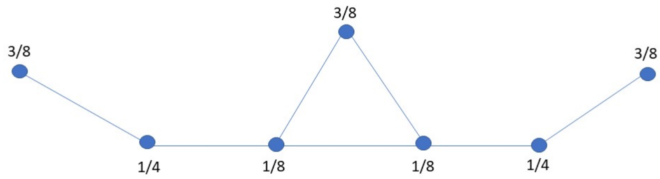
Proposition 11 provides an algorithm to identify whether the replacement of a specific Nash player by a Kantian agent will increase or decrease this sum and therefore to find a path of replacements such that contributions increase along the path. We illustrate how to implement this algorithm in the following example.

Example 3. Consider the network represented on Figure 13 and payoff function (8) with $\delta = \frac{1}{2}$ and $w_i = 1$ for all i. In the first graph, we represent the unique Nash equilibrium of the game with only Nash players. All players are active and the sum of contributions is 1.875. In the second graph, an active player is replaced by a Kantian and, as predicted by proposition 10, contributions increase to 1.9285. For the next replacement, we look at the unconstrained solution of the homogeneous problem where the Kantian agent is removed. This solution is represented in the third graph.

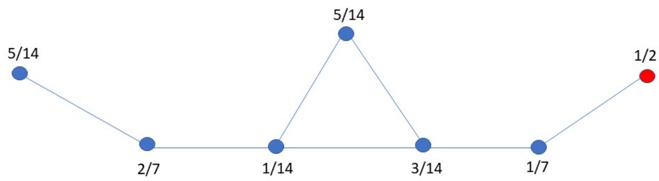
In the fourth graph, we replace the player playing $-1/7$ in the unconstrained solution by a Kantian agent, and the sum of contributions goes down to 1.9166, as predicted by proposition 11. Now we replace instead the player at the extreme left of the network playing $3/7$ in the unconstrained solution and, as expected, the sum of contributions goes up to 2. We then remove the second Kantian agent to proceed with the next replacement.

In the seventh graph, we replace a player playing 0 in the unconstrained problem (i.e. a neutral player), and as predicted, the sum of contributions does not change although the contribution of every remaining Nash player has changed. This sum is 2. We then take out the third Kantian agent and show the unconstrained solution of this further reduced problem. We see that an isolated player appears, his unconstrained solution is therefore positive, and when we replace him by a Kantian agent the sum of contributions increases to 2.5. The algorithm continues until we reach the profile with only Kantian agents, where the sum of contributions reaches a maximum of 3.5.

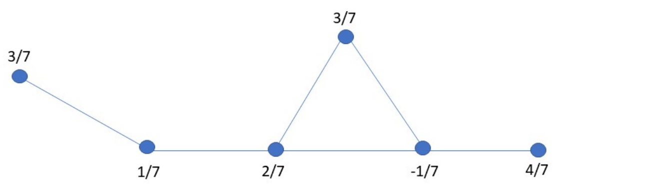
Note that Proposition 10 is a special case of Proposition 11, where $K = \emptyset$. Its statement is neater though, and this comes from the fact that when there are only Nash players, $i \in A(x) \Rightarrow x_i^{unc}((G_{A(x)})^T, 1) \geq 0$. This is no longer true once there is at least one Kantian agent in the game.



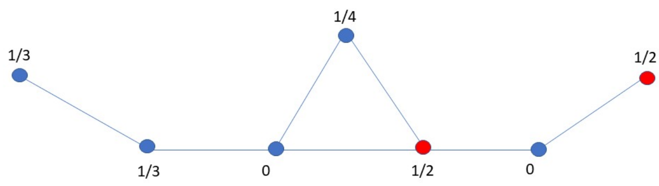
(a) Sum = 1.875



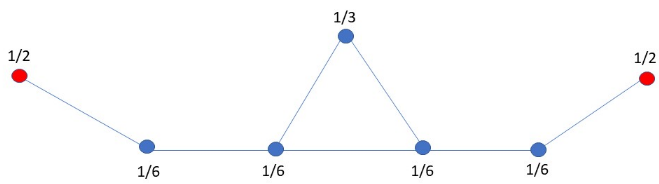
(b) Sum = 1.9285



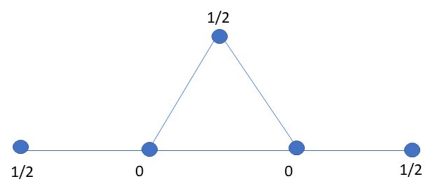
(c) Unconstrained solution without the first Kantian



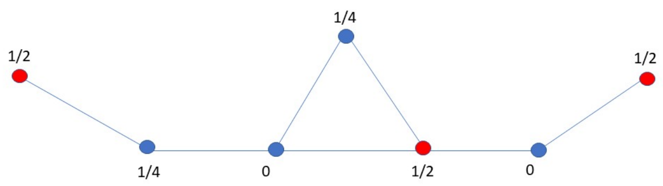
(d) Sum = 1.9166



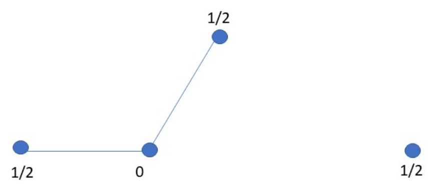
(e) Sum = 2



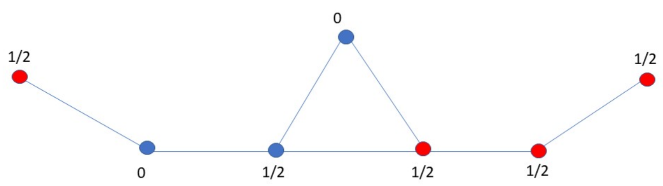
(f) Unconstrained solution without the first and the second Kantian



(g) Sum = 2



(h) unconstrained solution without the 1st, 2nd and 3rd Kantian



(i) Sum = 2.5

Figure 13: An example of successive replacements guaranteeing that aggregate contributions increase at each step.

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A Appendix: Proofs

Proof of Proposition 1: The equilibrium profile x^ϵ of the game with parameters (q, Δ, G^ϵ) satisfies the following conditions.

$$\begin{cases} q_i - \delta_i \sum_{j \in N} g^\epsilon_{ij} x^\epsilon_j \geq 0 & \Longrightarrow \quad x^\epsilon_i = q_i - \delta_i \sum_{j \in N} g^\epsilon_{ij} x^\epsilon_j \\ q_i - \delta_i \sum_{j \in N} g^\epsilon_{ij} x^\epsilon_j < 0 & \Longrightarrow \quad x^\epsilon_i = 0 \end{cases}$$

By definition of G^ϵ, we obtain

$$q_1 - \delta_1 \sum_{j \in N} g^\epsilon_{1j} x^\epsilon_j = (q_1 + \delta_1 \sum_{j \in N} \epsilon_j g_{1j} x^\epsilon_j) - \delta_1 \sum_{j \in N} g_{1j} x^\epsilon_j$$

Let $q^\beta_1 = q_1 + \beta = q_1 + \delta_1 \sum_{j \in N} \epsilon_j g_{1j} x^\epsilon_j$, then we have

$$\begin{cases} q^\beta_i - \delta_i \sum_{j \in N} g_{ij} x^\epsilon_j \geq 0 & \Longrightarrow \quad x^\epsilon_i = q^\beta_i - \delta_i \sum_{j \in N} g_{ij} x^\epsilon_j \\ q^\beta_i - \delta_i \sum_{j \in N} g_{ij} x^\epsilon_j < 0 & \Longrightarrow \quad x^\epsilon_i = 0 \end{cases}$$

which are the conditions for x^ϵ to be an equilibrium profile of the game with parameters (q^β, Δ, G). $\square$

Proof of Proposition 3: When G is the complete network, the interaction matrix is

$$I + \Delta G = \begin{bmatrix} 1 & \delta_1 & \dots & \delta_1 \\ \delta_2 & 1 & \dots & \delta_2 \\ \vdots & \ddots & \ddots & \vdots \\ \delta_n & \dots & \delta_n & 1 \end{bmatrix}$$

To show this is a P-matrix we need to show that every principal minor is strictly positive. It is sufficient to prove that $det(I + \Delta G) > 0$, since any principal minor can be seen as the determinant of a smaller complete network. Let us then prove that $det(I + \Delta G) > 0$.

This matrix can be decomposed as

$$I + \Delta G = \begin{bmatrix} 1-\delta_1 & 0 & \dots & 0 \\ 0 & 1-\delta_2 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1-\delta_n \end{bmatrix} + \begin{bmatrix} \delta_1 & \delta_1 & \dots & \delta_1 \\ \delta_2 & \delta_2 & \dots & \delta_2 \\ \vdots & \ddots & \ddots & \vdots \\ \delta_n & \dots & \delta_n & \delta_n \end{bmatrix}$$

which is equal to $I - \Delta + u.v^T$, where $u = (\delta_1, ..., \delta_n)^T$ and $v^T = (1, ..., 1)$ and $u.v^T$ is the outer product of u and v. By calling $A = I - \Delta$, we have $I + \Delta G = A + u.v^T$, and we can use the *matrix determinant lemma*, which states the following:

- If A is an invertible $n \times n$ matrix, and u and v are two n-dimensional column vectors, then $det(A + uv^T) = (1 + v^T A^{-1} u) det(A)$.
- Whether A is invertible or not, $det(A+uv^T) = v^T adj(A) u$, where $adj(A)$ is the adjugate matrix of A.

Let us start with case $\delta_i \in (0,1)$ for all i. Then A is diagonal and invertible, we thus use the first statement of the matrix determinant lemma, with $det(A)$ the product of all diagonal terms. This leads to

$$det(A + uv^T) = \left(1 + \sum_i^n \frac{\delta_i}{1-\delta_i}\right) \prod_i^n (1-\delta_i)$$

which is strictly positive.

Now assume that $\delta_1 = 1$ while $\delta_i \in (0,1)$ for all $i \neq 1$. Then A is no longer invertible since the first row is a row vector of 0's. We therefore use the second statement of the matrix determinant lemma. We get

$$adj(A) = \begin{bmatrix} \prod_{i\neq 1}(1-\delta_i) & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

from which we get

$$det(A + uv^T) = \prod_{i \neq 1} (1-\delta_i)$$

which is strictly positive.

□

Proof of Theorem 1: It is straightforward for the case where $x_1 = 0$. If agent 1 is inactive, then he either remains inactive or become active in x'. Thus $x'_1 \geq x_1$ if $x_1 = 0$. Now assume $x_1 > 0$. We first prove the following lemma:

Lemma 1. *Let M be a P-matrix. Then, M^{-1} is also a P-matrix.*

Proof. We show that all principal minors of M^{-1} are positive. The Jacobi identity (Miao and Ben-Israel [1993]) relates the minors of M^{-1} to those of M. Let $\alpha \subseteq N$ and $\alpha' = N \backslash \alpha$, and let $M[\alpha]$ be the principal submatrix of M with rows and columns indexed by α. The Jacobi identity states that, for all α

$$det\, M^{-1}[\alpha] = (-1)^{2\sum_{i\in\alpha} i} \, \frac{det\, M[\alpha']}{det\, M}$$

Since M is a P-Matrix, $det\, M[\alpha']$ and $det\, M$ are both strictly positive, and $(-1)^{2\sum_{i\in\alpha} i} = 1$. Therefore, $det\, M^{-1}[\alpha] > 0$, which proves the lemma. □

Let us define $a(x,x') = \{i \in N : i \in A(x) \cap SI(x')\}$, and $si(x,x') = \{i \in N : i \in SI(x) \cap A(x')\}$.
By definition of the equilibrium interaction matrix, we have $(I + \Delta G_{A(x)})x = q_{A(x)}$ and $(I + \Delta G_{A(x')})x' = q'_{A(x')}$. From the first equation, we obtain

$$(I + \Delta G_{A(x)}) = \left(I + \Delta\left(G_{A(x)\cup si(x,x')} - B\right)\right) x = q_{A(x)}$$

where B is such that $b_{ij} = g_{ij}$ for $i \in si(x,x')$ and $j \in N$, and $b_{ij} = 0$ otherwise. B is a matrix obtained by setting the elements to g_{ij} on the rows which correspond to those who change from SI to active, and zero otherwise.
By doing so, we can rewrite the system as

$$\left(I + \Delta G_{A(x)\cup si(x,x')}\right) x = q_{A(x)} + b \tag{9}$$

where $b = \Delta Bx$, with $b_i = \delta_i \sum_{j\in N} g_{ij}x_j$ if $i \in si(x,x')$ and $b_i = 0$ otherwise. From the second equation, we obtain $(I + \Delta G_{A(x)\cup si(x,x')})x' = q'_{A(x')} + b'$, where $b'_i = \delta_i \sum_{j\in N} g_{ij}x'_j$ if $i \in a(x,x')$ and $b'_i = 0$ otherwise.
Let $M = (I + \Delta G_{A(x)\cup si(x,x')})^{-1}$.
Then, we have $x = M(q_{A(x)} + b)$ and $x' = M(q'_{A(x')} + b')$. Therefore, we obtain

$$x' - x = M(q'_{A(x')} - q_{A(x)} + b' - b).$$

From this equation, we obtain

$$x'_i - x_i = \beta m_{i1} + \sum_{j\in si(x,x')} \left[\left(\delta_j \sum_{k\in N} g_{jk}x'_k - q_j\right) m_{ij}\right] + \sum_{j\in a(x,x')} \left[\left(q_j - \delta_j \sum_{k\in N} g_{jk}x_k\right) m_{jk}\right]$$

Let $\beta_j = \delta_j \sum_{k\in N} g_{jk}x'_k - q_j$ for $j \in si(x,x')$ and $\beta_j = q_j - \delta_j \sum_{k\in N} g_{jk}x_k$ for $j \in a(x,x')$, so that

$$x'_i - x_i = \beta m_{i1} + \sum_{j\in si(x,x')} \beta_j m_{ij} + \sum_{j\in a(x,x')} \beta_j m_{ij}.$$

Notice that $\beta_j > 0$ for $j \in a(x,x')$, $\beta_k < 0$ for $j \in si(x,x')$ by the fact that $a(x,x') \subseteq SI(x')$ and $si(x,x') \subseteq SI(x)$. Since $x'_j = 0$ for $j \in a(x,x')$ and $x_j = 0$ for $j \in a(x,x')$, we have

$$\begin{cases} x'_1 - x_1 &= \beta m_{11} + \sum_{j\in si(x,x')} \beta_j m_{1j} + \sum_{j\in a(x,x')} \beta_j m_{1j} \\ x'_i - x_i &= \beta m_{1i} + \sum_{j\in si(x,x')} \beta_j m_{ij} + \sum_{j\in a(x,x')} \beta_j m_{ij} \le 0 \quad \text{for } i \in a(x,x') \\ x'_i - x_i &= \beta m_{1i} + \sum_{j\in si(x,x')} \beta_j m_{ij} + \sum_{j\in a(x,x')} \beta_j m_{ij} \ge 0 \quad \text{for } i \in si(x,x') \end{cases}$$

Let $M[\{1\} \cup a(x,x') \cup si(x,x')]$ be the submatrix of M constituted by rows and columns indexed by the elements of $\{1\} \cup a(x,x') \cup si(x,x')$. Since $(I + \Delta G)$ is a P-matrix, by lemma 1, its inverse M is also a P-matrix, and any submatrix is also, implying that

$M[\{1\} \cup a(x,x') \cup si(x,x')]$ is a P-matrix.
Finally, we use Theorem 2 of Gale and Nikaido [1965], which states that a matrix M is a P-matrix if and only if it reverses the sign of no vector except 0, i.e. [$Mx = y$ with $x_i y_i \leq 0$ for all i] is true only for $x = y = 0$. By calling $\beta^* = (\beta, (\beta_i)_{i \in a(x,x') \cup si(x,x')})^T$, we can rewrite the 3 equations above as the following linear system.

$$M[\{1\} \cup a(x,x') \cup si(x,x')]\beta^* = x' - x$$

Notice that for all $i \in a(x,x')$, $\beta_i > 0$ and $x_i' - x_i' \leq 0$ so that $\beta_i(x_i' - x_i') \leq 0$. Moreover, for all $i \in si(x,x')$, $\beta_i < 0$ and $x_i' - x_i' \geq 0$ so that $\beta_i(x_i' - x_i') \leq 0$. Therefore, Theorem 2 of Gale and Nikaido [1965] implies that $\beta(x_1' - x_1) > 0$ since $\beta^* \neq 0$. Because $\beta > 0$, necessarily $x_1' - x_1 > 0$. □

Proof of Proposition 4: Recall that $a(x,x') = \{i \in N : i \in A(x) \cap SI(x')\}$.
By definition of the equilibrium interaction matrix, we have $(I + \Delta G_{A(x)})x = q_{A(x)}$ and $(I + \Delta G_{A(x')})x' = q'_{A(x')}$. By the second equation, we obtain

$$(I + \Delta G_{A(x)})x' = q'_{A(x')} + b$$

where $b_i = \delta \sum_{j \in N} g_{ij} x_j'$ for $i \in a(x,x')$ and $b_i = 0$ otherwise.
By denoting $M = (I + \Delta G_{A(x)})^{-1}$, we write $x = Mq_{A(x)}$ and $x' = M(q'_{A(x')} + b)$. Therefore, we have

$$x' - x = M(q'_{A(x')} - q_{A(x)} + b).$$

Notice that $(q'_{A(x')} - q_{A(x)} + b)_1 = \beta$, $(q'_{A(x')} - q_{A(x)} + b)_i = -1 + \delta \sum_{j \in N} g_{ij} x_j' > 0$ for $i \in a(x,x')$, $(q'_{A(x')} - q_{A(x)} + b)_i = 0$ otherwise. Thus,

$$X' - X = \beta \sum_{j \in N} m_{j1} + \sum_{i \in a(x,x')} \left[\left(-1 + \delta \sum_{j \in N} g_{ij} x_j' \right) \sum_{j \in N} m_{ji} \right]$$

By rewriting $(I + \Delta G_{A(x)})$ in blocks, separating active and strictly inactive players, we have

$$(I + \Delta G_{A(x)}) = \left(\begin{array}{c|c} (I + \Delta G)[A(x)] & (I + \Delta G)[A(x), SI(x)] \\ \hline \mathbf{0}_{SI(x),A(x)} & I_{SI(x)} \end{array} \right)$$

Thus,

$$(I + \Delta G_{A(x)})M = \left(\begin{array}{c|c} (I + \Delta G)[A(x)] & (I + \Delta G)[A(x), SI(x)] \\ \hline \mathbf{0}_{SI(x),A(x)} & I_{SI(x)} \end{array} \right) \left(\begin{array}{c|c} M_1 & M_2 \\ \hline M_3 & M_4 \end{array} \right) = I$$

where M_1 is $|A(x)| \times |A(x)|$, M_2 is $|A(x)| \times |SI(x)|$, M_3 is $|SI(x)| \times |A(x)|$, M_4 is $|SI(x)| \times |SI(x)|$ matrix.

Therefore, $M_1 = [(I + \Delta G)[A(x)]]^{-1}$, $M_3 = \mathbf{0}$, and $M_4 = I_{SI(x)}$.
Thus, we have $\sum_{j\in N} m_{ji} = \sum_{j\in A(x)} m_{ji} = \sum_{j\in A(x)} m_{ij}$ for all $i \in A(x)$. The second equality comes from the fact that M_1 is symmetric. Moreover, by $x = Mq_{A(x)}$, we have $x_i = \sum_{j\in A(x)} m_{ij}$ since $q_{i,A(x)} = 0$ for all $i \in SI(x)$, so that $\sum_{j\in N} m_{ji} = \sum_{j\in A(x)} m_{ij} = x_i \geq 0$ for all $i \in A(x)$.
Finally,

$$\begin{aligned}
X' - X &= \beta \sum_{j\in N} m_{j1} + \sum_{i\in a(x,x')} \left[\left(-1 + \delta \sum_{j\in N} g_{ij} x'_j \right) \sum_{j\in N} m_{ji} \right] \\
&= \beta \sum_{j\in A(x)} m_{1j} + \sum_{i\in a(x,x')} \left[\left(-1 + \delta \sum_{j\in N} g_{ij} x'_j \right) \sum_{j\in A(x)} m_{ij} \right] \\
&= \beta x_1 + \sum_{i\in a(x,x')} \left[\left(-1 + \delta \sum_{j\in N} g_{ij} x'_j \right) x_i \right] \geq 0
\end{aligned}$$

For the second point of the proposition, notice that if $x_1 = 0$, and if no player changes status, except maybe player 1, then $X' - X = 0$. $\square$

Proof of Proposition 5: Assume that $x'_1 > 0$, and recall that $si(x, x') = \{i \in N : i \in SI(x) \cup A(x')\}$, so $1 \in si(a, a')$.
We have $(I + \Delta G_{A(x)})x = q_{A(x)}$ and $(I + \Delta G_{A(x')})x' = q'_{A(x')}$. The second equation can be written as

$$(I + \Delta G_{A(x)})x' = q'_{A(x')} + b$$

where $b_i = \delta \sum_{j\in N} g_{ij} x'_j$ for $i \in si(x, x')$ and $b_i = 0$ otherwise.
Let $M = (I + \Delta G_{A(x)})^{-1}$, so that $x' - x = M(q'_{A(x')} - q_{A(x)} + b)$, and thus

$$X' - X = \sum_{j\in si(x,x')} \left[\left(q'_j - \delta \sum_{i\in N} g_{ji} x'_i \right) \sum_{i\in N} m_{ij} \right].$$

Notice that $q'_i - \sum_{j\in N} g_{ij} x'_j \geq 0$ for $j \in si(x, x')$ since they are active in x'. Hence, it is sufficient to prove that $\sum_{i\in N} m_{ij} < 0$ for $j \in si(x, x')$.

Using the same matrix decomposition as in Proposition 4, we have $M_1 = I$, $M_2 = \mathbf{0}$, and $M_4 = [(I + \Delta G)[A(x)]]^{-1}$. By the fact that $M(I + \Delta G_{A(x)}) = I$, we obtain $m_{ij} + \sum_{k\in A(x)} m_{ik} \delta g_{kj} = 0$, for $i \in A(x)$ and $j \neq i$. Summing up over all $i \in A(x)$, we get:

$$\sum_{i\in A(x)} m_{ij} + \delta \sum_{i\in A(x)} \left[g_{ij} \sum_{k\in A(x)} m_{ki} \right] = 0$$

Recall that $x = Mq_{A(x)}$, and since M_1 is symmetric, $\sum_{j\in A(x)} m_{ji} = \sum_{j\in A(x)} m_{ij} = x_i$ for all $i \in A(x)$. Therefore,

$$\sum_{i\in A(x)} m_{ij} + \delta \sum_{i\in A(x)} \left(g_{ij} \sum_{k\in A(x)} m_{ki} \right) = \sum_{i\in A(x)} m_{ij} + \delta \sum_{i\in A(x)} g_{ij} x_i = 0.$$

Since $j \in SI(x)$, we know that $\delta \sum_{i\in A(x)} g_{ji} x_i > q_j = 1$. Therefore, $\sum_{i\in A(x)} m_{ij} + 1 < 0$. Since $M_4 = I$, $m_{jj} = 1$ and $m_{ij} = 0$ for $i \in SI(x)$. Hence, we have

$$\sum_{i\in N} m_{ij} = m_{jj} + \sum_{i\in A(x)} m_{ij} = 1 + \sum_{i\in A(x)} m_{ij} < 0$$

for all $j \in si(x, x')$. The statement is proven. □

Proof of Proposition 6: We have $(I + \Delta G_{A(x)})x = q_{A(x)}$, and $(I + \Delta G_{A(x')})x' = q'_{A(x')}$. Notice that $A(x') \cup a(x, x') \setminus si(x, x') = A(x)$. Thus, the second equation can be written as follows:

$$(I + \Delta G_{A(x)})x' = q'_{A(x')} + b$$

where $b_i = \delta_i \sum_{j\in N} g_{ij} x'_j$ for $i \in a(x, x')$, $b_i = -\delta_i \sum_{j\in N} g_{ij} x'_j$ for $i \in si(x, x')$ and $b_i = 0$ otherwise.

Let $M = (I+\Delta G_{A(x)})^{-1}$, so that we obtain $x = Mq_{A(x)}$ and $x' = M(q'_{A(x')}+b)$. Therefore, $x' - x = M(q'_{A(x')} - q_{A(x)} + b)$ and thus

$$X'-X = \beta \sum_{j\in N} m_{j1} + \sum_{i\in a(x,x')} \left[\left(-q_i + \delta_i \sum_{j\in N} g_{ij} x'_j \right) \sum_{j\in N} m_{ji} \right] + \sum_{i\in si(x,x')} \left[\left(q_i - \delta_i \sum_{j\in N} g_{ij} x'_j \right) \sum_{j\in N} m_{ji} \right]$$

Note that $-q_i + \delta_i \sum_{j\in N} g_{ij} x'_j > 0$ for $i \in a(x, x')$ since they are SI in x', and $q_i - \delta_i \sum_{j\in N} g_{ij} x'_j \geq 0$ for $i \in si(x, x')$ since they are active in x'.

Consider $x^{unc} = x^{unc}((\Delta G_{A(x)})^T, 1)$, the solution to the linear system $(I+(\Delta G_{A(x)})^T)x = 1$. Then $x^{unc} = M^T.1$, and thus

$$x_i^{unc} = \sum_{j\in N} m^T_{i,j} = \sum_{j\in N} m_{j,i}.$$

Therefore,

$$\begin{aligned} X' - X &= \beta \sum_{j\in N} m_{j1} + \sum_{i\in a(x,x')} \left[\left(-q_i + \delta_i \sum_{j\in N} g_{ij} x'_j \right) \sum_{j\in N} m_{ji} \right] + \sum_{i\in si(x,x')} \left[\left(q_i - \delta_i \sum_{j\in N} g_{ij} x'_j \right) \sum_{j\in N} m_{ji} \right] \\ &= \beta x_1^{unc} + \sum_{i\in a(x,x')} \left[\left(-q_i + \delta_i \sum_{j\in N} g_{ij} x'_j \right) x_i^{unc} \right] + \sum_{i\in si(x,x')} \left[\left(q_i - \delta_i \sum_{j\in N} g_{ij} x'_j \right) x_i^{unc} \right] \end{aligned}$$

Thus, if $x_1^{unc} \geq 0$ and $x_i^{unc} \geq 0$ for all $i \in a(x,x') \cup si(x,x') \Rightarrow X' - X \geq 0$. On the contrary, $x_1^{unc} \leq 0$ and $x_i^{unc} \leq 0$ for all $i \in a(x,x') \cup si(x,x') \Rightarrow X' - X \leq 0$. We can easily check that this also holds when $1 \in a(x,x') \cup si(x,x')$. □

Proof of Corollary 7: We first prove the following lemma.

Lemma 2. *Let G be the adjacency matrix of the complete network, let $\Delta = diag(\delta_i)_{i=1,\dots,n}$ with $\delta_i \in (0,1]$ for all i, and $\delta_i = 1$ for at most one player i, and call $M = (I + \Delta G)^{-1}$. Then $m_{ii} > 1$ for all i, $m_{ij} \leq 0$ for $j \neq i$ and $m_{ij} < 0$ for $j \neq i$ for at least one i, and $x_j^{unc} = \sum_i m_{ij} > 0$ for all j.*

Proof. Assume first that $\delta_i \in (0,1)$ for all i.
First we prove that all non-diagonal terms are negative. Observe that $M(I + \Delta G) = I$. Thus, for $i \neq j$ and $i \neq k$,

$$m_{ij} + \sum_{l \neq j} \delta_l m_{il} = 0 \tag{10}$$

$$m_{ik} + \sum_{l \neq k} \delta_l m_{il} = 0 \tag{11}$$

$$m_{ii} + \sum_{l \neq i} \delta_l m_{il} = 1 \tag{12}$$

By substracting (10) and (11), we get $m_{ij}(1-\delta_j) = m_{ik}(1-\delta_k)$, from which we conclude that $Sign(m_{ij}) = Sign(m_{ik})$, since $(1-\delta_j) > 0$ and $(1-\delta_k) > 0$. So, all non-diagonal terms have the same sign. Let us show that this sign has to be negative. By substracting (12) and (10), we get

$$m_{ii}(1-\delta_i) = 1 + m_{ij}(1-\delta_j)$$

Assuming $m_{ij} > 0$, then necessarily, $m_{ii} > 0$. Yet, if $m_{ii} > 0$ and all non-diagonal terms m_{ij} are also positive, it is not possible that equation (10) holds. Thus necessarily $m_{ij} < 0$, and by (10) we must have $m_{ii} > 0$.
Further, using equation (12), since $\sum_{l \neq i} \delta_l m_{il} < 0$, we obtain that $m_{ii} > 1$ for all i.

For the last claim of the lemma, summing terms of (10) over all i, and terms of (11) over all i, we get

$$\sum_i m_{ij} + \sum_{l \neq j} \delta_l \sum_i m_{il} = 1 \tag{13}$$

$$\sum_i m_{ik} + \sum_{l \neq k} \delta_l \sum_i m_{il} = 1 \tag{14}$$

and by subtracting (13) and (14), we get $\sum_i m_{ij} + \delta_k \sum_i m_{ik} = \sum_i m_{ik} + \delta_j \sum_i m_{ij}$, resulting in

$$(1-\delta_j) \sum_i m_{ij} = (1-\delta_k) \sum_i m_{ik}$$

from which $Sign(\sum_i m_{ij}) = Sign(\sum_i m_{ik})$. By plugging this into (13), necessarily $\sum_i m_{ij} > 0$ or one of the $\sum_i m_{il} > 0$ for some l. But they all have the same sign, so that $\sum_i m_{ij} > 0$ for all j. Thus $x_j^{unc} > 0$ for all j.

Now assume that $\delta_1 = 1$ and $\delta_i \in (0,1)$ for all other i. Then, by taking $i = 1$ and substracting (12) and (10) we get

$$m_{1j} = -\frac{1}{1-\delta_j} < 0, \;\; j \neq 1$$

Plugging back into (12), we also get

$$m_{11} = 1 + \frac{\delta_2}{1-\delta_2} + ... + \frac{\delta_n}{1-\delta_n} > 1$$

By taking $i \neq 1$ and $j = 1$ and substracting (12) and (10) we get

$$m_{ii} = \frac{1}{1-\delta_i} > 1, \;\; i \neq 1$$

Next, by taking $i \neq 1$ and $k = 1$ and by substracting (10) and (11) we get $m_{ij}(1-\delta_j) = 0$ for all $j \neq 1, i$, thus

$$m_{ij} = 0, \;\; i, j \neq 1, i \neq j$$

Finally, by replacing this into (12) with $i \neq 1$, we get

$$m_{i1} = -\frac{\delta_i}{1-\delta_i} < 0, \;\; i \neq 1$$

and the lemma is proven. □

Lemma 3. *Suppose that G is complete. Take any set $A(x) \subseteq N$. Then, for all $i \in N$, $x_i^{unc} = x_i^{unc}((\Delta G_{A(x)})^T, 1) \geq 0$.*

Proof. Let $M := (I + \Delta G_{A(x)})^{-1}$. By using the matrix decomposition in the proof of proposition 4, we have

$$(I + \Delta G_{A(x)})M = \left(\begin{array}{c|c} (I+\Delta G)[A(x)] & (I+\Delta G)[A(x), SI(x)] \\ \hline \mathbf{0}_{SI(x),A(x)} & I_{SI(x)} \end{array}\right)\left(\begin{array}{c|c} M_1 & M_2 \\ \hline M_3 & M_4 \end{array}\right) = I$$

We know from the proof of proposition 4, that $M_1 = [(I + \Delta G)[A(x)]]^{-1}$, $M_3 = \mathbf{0}$, and $M_4 = I_{SI(x)}$. With G being complete, lemma 2 tells us that for all $i \in A(x)$, $x_i^{unc}((\Delta G_{A(x)})^T, 1) = x_i^{unc}((\Delta G[A(x)])^T, 1) \geq 0$ since $G[A(x)]$ is complete.
Therefore, we need to prove that for all $i \in SI(x)$, $x_i^{unc}((\Delta G_{A(x)})^T, 1) \geq 0$.
Notice that from the structure of M, we have for $j \in SI(x)$, $x_j^{unc}((\Delta G_{A(x)})^T, 1) = m_{jj} +$

$\sum_{k\in A(x)} m_{kj}$.
By taking the (i,j)-th element of $(I+\Delta G_{A(x)})M$ for $i \in A(x)$ and $j \in SI(x)$, we have

$$\delta_i m_{jj} + m_{ij} + \delta_i \sum_{k\in A(x)\backslash\{i\}} m_{kj} = 0$$

$$\delta_i \left(m_{jj} + \sum_{k\in A(x)} m_{kj} \right) + (1-\delta_i) m_{ij} = 0$$

Since m_{jj} is a diagonal element of M_4, $m_{jj} = 1$. Hence, we have

$$\delta_i \left(1 + \sum_{k\in A(x)} m_{kj} \right) + (1-\delta_i) m_{ij} = 0 \tag{15}$$

Suppose that for all $l \in A(x), m_{lj} \geq 0$. Then, if $m_{ij} \geq 0$, (15) could not hold. Therefore, for at least one $i \in A(x)$, $m_{ij} < 0$. Take such i, and we have $\delta_i \left(m_{jj} + \sum_{k\in A(x)} m_{kj} \right) = -(1-\delta_i)m_{ij} > 0$. This is true for all $j \in SI(x)$. The lemma is proven. □

With lemma 3, the proof of corollary 7 follows as a direct application of proposition 6. □

Proof of Proposition 7: To ease notations, we write $A = (I + \Delta G)$ and $B = A^{-1}$. We want to show that if player 1 is neutral with ΔG he will also be neutral with $\Delta' G$ where $\Delta' = diag(\delta'_i)_{i\in N}$ such that $\delta'_i = \delta_i$ for all $i \neq 1$ and $\delta'_1 \neq \delta_1$. To do that, we change element a_{12} of matrix A to $a_{12} + \epsilon$ and show that $x^{unc}((\Delta G)^T, 1) = x^{unc}((\Delta' G)^T, 1)$.

Using Sherman Morrison formula with $u = (1, 0, ..., 0)^T$ and $v^T = (0, 1, 0, ..., 0)$ we get

$$\begin{aligned}
(A + uv^T)^{-1} - A^{-1} &= -\frac{A^{-1}uv^TA^{-1}}{1+v^TA^{-1}u} \\
&= -\frac{\epsilon}{1+b_{21}} \begin{pmatrix} b_{12}b_{11} & \cdots & b_{n2}b_{11} \\ b_{12}b_{21} & \cdots & b_{n2}b_{21} \\ \vdots & \ddots & \vdots \\ b_{12}b_{n1} & \cdots & b_{n2}b_{n1} \end{pmatrix}
\end{aligned}$$

Therefore, the column sum vector is

$$\begin{aligned}
1^T \left[(A + uv^T)^{-1} - A^{-1} \right] &= -\frac{\epsilon}{1+b_{21}} (b_{12} \sum_j b_{j1}, ..., b_{n2} \sum_j b_{j1}) \\
&= -x_1^{unc}((\Delta G)^T, 1) \frac{\epsilon}{1+b_{21}} (b_{12}, ..., b_{n2})
\end{aligned}$$

Since $x_1^{unc}((\Delta G)^T, 1) = 0$ because player 1 is neutral, we have

$$1^T \left[(A + uv^T)^{-1} - A^{-1} \right] = 0$$

$\square$

Proof of Proposition 8: Since the set of active players does not change before and after the transfer, without loss of generality, we can assume that both equilibria are interior. Therefore, the equilibrium before the transfer satisfies $x_i + \delta_i \sum_{j \in N} g_{ij} x_j = q_i$ for all $i \in N$. We can write this as follows:

$$x_i + \delta_1 \sum_{j \in N} g_{ij} x_j = q_i + (\delta_1 - \delta_i) \sum_{j \in N} g_{ij} x_j$$

Let Δ' be such that $\delta'_2 = \delta_1$ and $\delta'_i = \delta_i$ otherwise, and $\Delta' = diag(\delta'_i)_{i \in N}$. Moreover, let q' be such that $q'_2 = q_2 + (\delta_1 - \delta_2) \sum_{j \in N} g_{2j} x_j$. The equilibrium before the transfer satisfies $(I + \Delta' G)x = q'$ so that $x = Mq'$ where $M = (I + \Delta' G)^{-1}$.

Accordingly, the equilibrium after the transfer satisfies $(I + \Delta' G)x' = q'' + t$ where $q''_2 = q_2 + (\delta_1 - \delta_2) \sum_{j \in N} g_{2j} x'_j$, so that $x = M(q'' + t)$. Thus,

$$x' - x = M(q'' - q' + t)$$

Notice that

$$\begin{aligned} q''_1 - q'_1 + t_1 &= -\epsilon \\ q''_2 - q'_2 + t_2 &= (\delta_1 - \delta_2) \sum_{j \in N} g_{2j}(x'_j - x_j) + \epsilon \end{aligned}$$

Therefore,

$$X' - X = \epsilon \left(\sum_{j \in N} m_{j2} - \sum_{j \in N} m_{j1} \right) + \left(\sum_{j \in N} m_{j2} \right) (\delta_1 - \delta_2) \sum_{j \in N} g_{2j}(x'_j - x_j)$$

Let us define $x^{unc} = x^{unc}((\Delta' G)^T, 1)$, the solution to the linear equation $(I + (\Delta' G)^T)x = \mathbf{1}$, so that $x^{unc} = M^T.\mathbf{1}$, and

$$\begin{aligned} x_1^{unc} &= \sum_{j \in N} m_{1j}^T = \sum_{j \in N} m_{j1} \\ x_2^{unc} &= \sum_{j \in N} m_{2j}^T = \sum_{j \in N} m_{j2} \end{aligned}$$

From which

$$X' - X = \epsilon \left(x_2^{unc} - x_1^{unc} \right) + x_2^{unc} (\delta_1 - \delta_2) \sum_{j \in N} g_{2j}(x'_j - x_j)$$

$\square$

Proof of corollary 8: First notice that if G is complete and Δ' is such that $\delta'_1 = \delta'_2$, then $x_2^{unc} = x_1^{unc}$. Thus the first term in equation (7) is 0. Next we show that $\sum_j g_{2j}(x'_j - x_j) < 0$ and $x_2^{unc} > 0$.

Let x be the equilibrium before transfer and x' the equilibrium after transfer, where transfer vector $t = (-\epsilon, +\epsilon, 0, ..., 0)^T$, and let $M = (I + \Delta G)^{-1}$. Since we restrict to only active players, we have $x = Mq$ and $x' = Mq'$ where $q'_1 = q_1 - \epsilon, q'_2 = q_2 + \epsilon$ and $q'_i = q_i$ otherwise. Thus $x'_2 - x_2 = \epsilon(m_{22} - m_{21})$. By lemma 2, $m_{21} \leq 0$ and $m_{22} > 1$ so $x'_2 - x_2 > \epsilon$. Since also $x_2 = q_2 - \delta_i \sum_j g_{2j} x_j$ and $x'_2 = q_2 + \epsilon - \delta_i \sum_j g_{2j} x'_j$, we get $x'_2 - x_2 = \epsilon - \delta_i \sum_j g_{2j}(x'_j - x_j)$, implying $\sum_j g_{2j}(x'_j - x_j) < 0$.
Also, $x_2^{unc} > 0$ according to lemma 2, so that $Sign(X' - X) = Sign(\delta_2 - \delta_1)$. $\square$

Proof of Proposition 9: Let K be the set of Kantian players. The Nash-Kant equilibrium x of the game $\mathcal{G}^K = ((N \setminus K, K), (X_i)_{i=1,\cdots,n}, (u_i)_{i \in N \setminus K}, (u_i^K)_{i \in K})$ satisfies the following conditions:

$$\begin{cases} x_i + \delta_i \sum_{j \in N} g_{ij} x_j = q_i \text{ for all } i \in (N \setminus K) \cap A(x) \\ \delta_i \sum_{j \in N} g_{ij} x_j \leq q_i \text{ for all } i \in (N \setminus K) \cap SI(x) \\ x_i = x_i^K \text{ for all } i \in K \end{cases}$$

For Nash players it implies

$$\begin{cases} x_i + \delta_i \sum_{j \in N \setminus K} g_{ij} x_j = q_i - \delta_i \sum_{j \in K} g_{ij} x_j \text{ for all } i \in (N \setminus K) \cap A(x) \\ \delta_i \sum_{j \in N \setminus K} g_{ij} x_j \leq q_i - \delta_i \sum_{j \in K} g_{ij} x_j \text{ for all } i \in (N \setminus K) \cap SI(x) \end{cases}$$

By proposition 2, this is the solution of the $LCP(-q'_{N \setminus K}, (I + \Delta G)_{N \setminus K})$, where $q'_{N \setminus K}$ is a vector of $|N \setminus K|$ dimensions such that $q'_{N \setminus K, i} = q_i - \delta_i \sum_{j \in K} g_{ij} x_j$, and $(I + \Delta G)_{N \setminus K}$ is the submatrix of $(I + \Delta G)$ constituted by the rows and columns indexed by the elements in $N \setminus K$. Since $(I + \Delta G)$ is a P-matrix, $(I + \Delta G)_{N \setminus K}$ is also. Therefore, this LCP has a unique solution. $\square$

Proof of proposition 10: By assumption 1, we have $x_i^K \geq q_i$. Assume $q_i = q$ for all i. Let $a(x, x^{\{1\}}) = \{i \in N : i \in A(x) \cap SI(x^{\{1\}})\}$. Both x and $x^{\{1\}}$ satisfy

$$(I + \Delta G_{A(x)})x = q_{A(x)}$$
$$(I + \Delta G'_{A(x^{\{1\}})})x^{\{1\}} = q'_{A(x^{\{1\}})}$$

where $g'_{1j} = 0$ for all j and $q'_1 = x_1^K$, $g'_{ij} = g_{ij}$ and $q'_i = q_i$ if $i \neq 1$. The second equation can be rewritten as:

$$(I + \Delta G_{A(x)})x^{\{1\}} = q_{A(x^{\{1\}})} + b$$

where $b_1 = \delta \sum_{j \in N} g_{1j} x_j^{\{1\}}$ and $b_i = \delta \sum_{j \in N} g_{ij} x_j^{\{1\}}$ for $i \in a(x, x^{\{1\}})$.
Let $M = (I + \Delta G_{A(x)})^{-1}$, so that $x = Mq_{A(x)}$ and $x^{\{1\}} = M(q'_{A(x^{\{1\}})} + b)$, and then

$x^{\{1\}} - x = M(q'_{A(x^{\{1\}})} - q_{A(x)} + b)$. Therefore,

$$X^{\{1\}} - X = \left(x_1^K - q\right) \sum_{j \in N} m_{j1} \sum_{j \in N} g_{1j} x_j^{\{1\}} + \sum_{i \in a(x, x^{\{1\}})} \left[\sum_{j \in N} m_{ji} \left(\delta \sum_{j \in N} g_{ij} x_j^{\{1\}} - q \right) \right]$$

Notice that $\delta \sum_{j \in N} g_{ij} x_j^{\{1\}} - q > 0$ for $i \in a(x, x^{\{1\}})$ since they are SI in $x^{\{1\}}$. Therefore, it is sufficient to prove that $\sum_{j \in N} m_{j1} > 0$ and $\sum_{j \in N} m_{ji} \geq 0$ for $i \in a(x, x^{\{1\}})$ if $x_1 > 0$.

Using the decomposition of M from the proof of the proposition 4, we have $\sum_{j \in N} m_{ji} = \sum_{j \in A(x)} m_{ji} = \sum_{j \in A(x)} m_{ij}$ for all $i \in A(x)$. Since $(I + \Delta G_{A(x)})x = q_{A(x)}$, $x_i = \sum_{j \in A(x)} q m_{ij} = q \sum_{j \in A(x)} m_{ij}$ for all $i \in A(x)$, and since $x_1 > 0$, $x_i \geq 0$ for $i \in a(x, x')$, we have

$$\sum_{j \in N} m_{j1} > 0$$
$$\sum_{j \in N} m_{ji} \geq 0 \text{ for } i \in a(x, x^{\{1\}})$$

which proves the first point.

For the second point, notice that $a(x, x^{\{1\}}) = \emptyset$, and $\sum_{j \in N} m_{j1} = 0$ when $x_1 = 0$ and $1 \in A(x)$.

For the third point, we replicate the same logic with the set $si(x, x^{\{1\}})$ instead of $a(x, x^{\{1\}})$. We find

$$X^{\{1\}} - X = x_1^K \sum_{j \in N} m_{j1} \sum_{j \in N} g_{1j} x_j^{\{1\}} + \sum_{i \in si(x, x')} \left[\sum_{j \in N} m_{jk} \left(q - \delta \sum_{j \in N} g_{2j} x_j^{\{1\}} \right) \right]$$

Notice that $q - \delta \sum_{j \in N} g_{ij} x_j^{\{1\}} > 0$ for $i \in si(x, x')$ since they are active in $x^{\{1\}}$. Therefore, it is sufficient to prove that $\sum_{j \in N} m_{j1} < 0$ and $\sum_{j \in N} m_{ji} < 0$ for $i \in si(x, x')$. With the decomposition of M from the proof of the proposition 5, we have $\sum_{j \in N} m_{j1} = m_{11} + \sum_{i \in A(x)} m_{i1} = 1 + \sum_{i \in A(x)} m_{i1} < 0$ and $\sum_{j \in N} m_{ji} = m_{ii} + \sum_{j \in A(x)} m_{ji} = 1 + \sum_{j \in A(x)} m_{ji} < 0$ for $i \in si(x, x')$, which proves the statement. □

Proof of Proposition 11: By assumption 1, we have $x_i^K \geq q_i$ for all $i \in K$. Moreover, let us define q^K such that for all $i \in N \setminus K$,

$$q_i^K = \begin{cases} q_i - \delta \sum_{j \in K} g_{ij} x_j^K & \text{if } \; q_i - \delta \sum_{j \in K} g_{ij} \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

q^K represents the needs of each player after taking into account the contributions of Kantian neighbors which are fixed. q^K is defined for $i \in N \setminus K$, so that this is a vector of

$|N \backslash K|$ dimensions. To simplify the notations, we denote $(I+\Delta G_{A(x)}) := (I+\Delta G_{A(x)})_{N\backslash K}$, $x := x^K_{N\backslash K}$, $x' := x^{K\cup\{1\}}_{N\backslash K}$.

We first prove (i). By the definition of an equilibrium interaction matrix, we have $(I+\Delta G_{A(x)})x = q^K_{A(x)}$. Recall that $q^K_{A(x)}$ is such that $q^K_{i,A(x)} = q^K_i$ if $i \in A(x)$, $q^K_{i,A(x)} = 0$ otherwise. Moreover, let us define G' such that $g_{1j} = 0$ for all $j \in N \setminus K$, $g'_{ij} = g_{ij}$ otherwise. We have $(I+\Delta G'_{A(x')})x' = q^{K'}_{A(x')}$, where $q^{K'}_1 = x^K_1$, $q^{K'}_i = q^K_i$ otherwise.

By the second equation, we obtain $(I+\Delta G_{A(x)})x' = q'_{A(x')} + b$, where

$$b_1 = \begin{cases} \delta \sum_{j\in N} g_{1j}x'_j & \text{if } 1 \in A(x) \\ 0 & \text{if } 1 \notin A(x) \end{cases}$$

Moreover, $b_i = \delta \sum_{j\in N\backslash K} g_{ij}x'_j$ for all $i \in a(x^K, x^{K\cup\{1\}})$, $b_i = -\delta \sum_{j\in N\backslash K} g_{ij}x'_j$ for all $i \in si(x^K, x^{K\cup\{1\}})$, and $b_i = 0$ otherwise. Let $M = (I+\Delta G_{A(x)})^{-1}$, so that we obtain $x = Mq_{A(x)}$ and $x' = M\left(q^{K'}_{A(x')} + b\right)$. Therefore,

$$x' - x = M\left(q^{K'}_{A(x')} - q^K_{A(x)} + b\right).$$

Note that

$$\left(q^{K'}_{A(x')} - q^K_{A(x)} + b\right)_i = \begin{cases} -q^K_i + b_i = -q^K_i + \delta \sum_{j\in N\backslash K} g_{ij}x'_j \geq 0 & (i \in a(x^K, x^{K\cup\{1\}})) \\ q^{K'}_i + b_i = q^K_i - \delta \sum_{j\in N\backslash K} g_{ij}x'_j \geq 0 & (i \in si(x^K, x^{K\cup\{1\}})) \end{cases}$$

The first inequality comes from the fact that they are SI in $x^{K\cup\{1\}}$, and the second inequality comes from the fact that they are active in $x^{K\cup\{1\}}$.
For player 1, we have

$$\left(q^{K'}_{A(x')} - q^K_{A(x)} + b\right)_1 = \begin{cases} x'_1 - q^K_1 + b_1 = x'_1 - q^K_1 + \delta \sum_{j\in N\backslash K} g_{1j}x'_j \geq 0 & (1 \in A(x)) \\ x'_1 + b_1 = x'_1 \geq 0 & (1 \notin A(x)) \end{cases}$$

The first inequality comes from assumption 1 that $x'_1 - q_1 \geq 0$. For all other players, $\left(q^{K'}_{A(x')} - q^K_{A(x)} + b\right)_i = 0$.

For the sake of simplicity, we let a_i denote $\left(q^{K'}_{A(x')} - q^K_{A(x)} + b\right)_i$. Therefore, we have the following:

$$X' - X = a_1 \sum_{j\in N\backslash K} m_{j1} + \sum_{i\in a(x^K,x^{K\cup\{1\}})\cup si(x^K,x^{K\cup\{1\}})} \left[a_i \sum_{j\in N\backslash K} m_{ji}\right]$$

Note that $a_i \geq 0$, as we confirmed. Thus, it is sufficient to prove that for all $i \in a(x^K, x^{K\cup\{1\}}) \cup si(x^K, x^{K\cup\{1\}})$,

$$\begin{cases} x^{K,unc}_1 \geq 0, x^{K,unc}_i \geq 0 \Rightarrow \sum_{j\in N} m_{j1} \geq 0, \sum_{j\in N} m_{ji} \geq 0 \\ x^{K,unc}_1 \leq 0, x^{K,unc}_i \leq 0 \Rightarrow \sum_{j\in N} m_{j1} \leq 0, \sum_{j\in N} m_{ji} \leq 0 \end{cases}$$

Take $x^{K,unc}$ as it is defined in the main text. This is the solution of the linear system $(I + (\Delta G_{A(x)})^T)x = \mathbf{1}$. By solving this, we obtain $x^{K,unc} = M^T.\mathbf{1}$, and thus $x_i^{K,unc} = \sum_{j \in N \setminus K} m_{i,j}^T = \sum_{j \in N \setminus K} m_{ji}$. This proves the statement. $\square$