Working Papers / Documents de travail

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WP 2015 - Nr 01

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January 12, 2015

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Abstract

A principal offers bilateral contracts to a set of agents organized in a network conveying synergies, in a context where agents' efforts are observable and where the principal's objective increases with the sum of efforts. We characterize optimal contracts as a function of agents' positions on the network. The analysis shows that contract enforceability is key to understand optimality. We also examine linear contracting and we analyze the situation where the principal is constrained to contract with a single agent on the network. Last, we extend this setting to network entry.

Keywords: Optimal Contracting, Multi-agency, Network, Strategic Complementarity, Enforceability
JEL: C72, D85

*We would like to thank Matt Jackson, Yann Bramoullé, Nizar Allouch, Luca Merlino, as well as participants in the 2013 UECE conference in Lisbon, in the 19^{th} CTN conference in Bruxels, the 2014 CEAFE conference (Agadir), and in the Workshop on Dynamic networks at PSE (december 2014). Financial support of AMSE is strongly acknowledged.
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1 Introduction

We model a situation where a principal contracts with agents organized in a network conveying synergies. In many applications, this network aspect impacts principal-agent relationships. To cite a few: innovation-oriented public funds when firms are organized in R&D networks, monopoly pricing in presence of consumer externalities, cash transfers contingent on school attendance in presence of networked peer effects in the school, the distribution of bonus in firms when there are local synergies between employees, taxation in shopping malls.

Because of interdependencies between agents' utilities, the principal should take into account how complementarities between neighbors propagate throughout the network. Taking care of these ripple effects should entail that more central agents be offered contracts where they exert a higher effort (and thus with the largest rewards). These general considerations suggest that the network structure can substantially affect contracting, which opens the following questions. How does the network shape optimal contracts? In particular, should the principal concentrate rewards on a subset of agents, and should the principal tax agents to sponsor other agents? Which network structures maximize contract performance? How does contract enforceability affect optimality? Do linear contracts, which are not optimal but simple to set up, perform well?

We study how the network structure affects optimal contracts in a static framework with observable efforts and linear quadratic utilities. In this setting, we are able to quantify the ripple effect as induced by the variation of individual efforts, which is crucial to provide tractable analytical results.

First, we study enforceable contracts. We show that the optimal contracts are contingent, in the sense that contract acceptance is conditional on the acceptance of other contracts. These contracts are desirable for the principal because they maximize the opportunity cost of contracting agents. We fully characterize optimal contracts. Our analysis shows that effort is increasing in bonacich centrality, but reward is increasing with centrality only for sufficiently large budget. We also identify conditions under which the principal optimally taxes part of the society. Taxation emerges for either low budgets or high intensity of interaction.

Second, we consider non enforceable contracts (still the principal commits to her promisse). Here contracts are simple options left to agents. By complementarities, optimal contracts are such that offers are made to all agents. Moreover, taking account of the heterogenous influence power of agents on the network, rewards are related to the square of Bonacich centralities, while efforts to weighted centralities where the weights are the centralities themselves. In terms of contract performance, two messages emerge. On the one hand, denser networks are more performant, and thus, contemplating all possible networks, the most performant is the complete network. On the othed hand, among all networks with same density (i.e. same number of links), the networks with maximal impact are Nested-Split Graphs. These graphs are such that all neighborhoods are nested. This means that performing networks have a high level of asymmetry and hierarchy.

Third, we study a linear contract where the principal rewards agents' excess efforts with respect to their initial effort. This constitutes a realistic setting guaranteeing full implementation. We show that the principal should optimally propose an homogenous excess-effort per-unit return to every agent, and a payment proportional to the *relative* centralities of agents (i.e., the ratio of individual centrality over the sum of centralities). We also show that denser networks enhance the performance of these contracts.

Fourth, we study the case where the principal is constrained to contract with a single agent, i.e. he undertakes a key-player intervention. With linear contracting, the targetted agent maximizes an inter-centrality index (which is known to play a crucial role in key-player analyzes). A principal setting up an optimal non enforceable contract has to target the agent maximizing the simple bonacich centrality. Last, with enforceable contracts, the optimal target has a more complicated index, and the main message is that this index is budget-dependent.

Last, we extend this model to network entry. A contract should guarantee that the agent is willing to enter the network. In this respect, the principal has to take care that it can be more costly to guarantee the participation of least central agents. We characterize optimal (enforceable) contracts. Essentially, the principal identifies the targetted agents by selecting the subnetwork that maximizes aggregate bonacich centrality. We mainly show that the nework participation cost motive can lead to exclude a subset of the population from network participation, and that contractual arrangements can result in larger rewards to least central agents. Moreover, conform to the previous model with fixed network, among all networks with same density (i.e. same number of links), the networks supporting maximal performance are Nested-Split Graphs.

Related literature. Our model bridges two literatures, multi-agent contract theory and network games.

This model is strongly connected with the literature on multi-agent contracting. In the context of teams, and related free riding problem, Holmstrom (1982) shows how a principal can restore efficiency by conditioning workers' payments on team performance. More recently, Segal (1999) discusses at a high level of generality the efficiency implications of contracting in context where agents' utilities are interdependent and when discrimination is possible. In the opposite, Segal (2003) examines the implications of imposing a non-discrimination close. Like the former paper, our analysis assumes that the principal can discriminate between agents. In regard to this literature, we introduce a network of local complementarities between agents. This echoes the recent paper of Bernstein and Winter (2012), who extend one model of Segal (1999) to a context where externalities between utilities are positive and heterogenous. Some of our results are qualitatively close to theirs, but the models are actually rather distinct. In particular, in contract with Segal's framework, the sum of agents utilities and principal objective is not a function of the sum of agents' efforts in our model (thus our hypotheses violate condition W as exposed in Segal [1999]). In the context of employment relationships, Levin [2002] model multilateral contracting between a firm owner and employees in a dynamical setting. This paper investigates the trade-off firms face between making commitments to their workforce as a whole (multilateral relational contracts), and making more limited commitments to individuals or smaller groups of employees (bilateral relational contracts). One main message is that bilateral contracts are easier to adjust in response to changes in the environment, which may help to explain why firms rely on temporary employees, and the adoption of two-tier workforces. One important departure with regard to this model is static incentives. In Levin's environment, the the firm cannot commit to reward performance, and thus static equilibrium is such that workers will do no more than the minimum, and the firm will do best not to produce at all. In contrast, our model builds on a credible commitment assumption.

Our model is also inserted in the network games literature with local complementarities. For instance, Goyal and Moraga (2001) introduce a model a R&D networks in which parter firms share R&D knowledge, and this generates local complementarities. Calvo and Zenou (2004) consider a model of crime economics, where criminal benefit from local information sharing. Ballester, Calvo and Zenou (2006) introduce models of linear interaction on networks, including strategic complementarity[1] and link agents' equilibrium play to their Bonacich centrality. Recently, Belhaj, Bramoullé and Deroïan (2014) consider more general games, where player's actions can be bounded from above. In this papier, we consider the linear interaction setting and we suppose that the intensity of interaction is low enough to guarantee equilibrium existence (and uniqueness).

Some few papers tackle policy or principal intervention on networks.

[1] See for instance Topkis (1998) for an overview of games with strategic complementarities.

Ballester et al. (2006) ask about key player policy. In short, in a context of strategic complementarities, which agent should be dropped out of the network so as to maximize or minimize aggregate efforts on the network? The authors show that the good target is an agent maximizing a specific centrality measure (Ballester, Calvo and Zenou [2010], Liu, Patacchini, Zenou and Lee [2014], and Konig, Liu and Zenou [2014] elaborate on this seminal paper). Our paper also addresses key-player analysis, but the principal subsidizes the targetted agent. Some recent alternative network policies in the context of linear network games can be mentioned. Allouch (2012) considers a model of local public good under linear interaction, and ask about optimal transfers so as to improve aggregate effort (the problem is equivalent to a zero-budget setting). The author shows that network structure decisively shapes optimal transfers. Zhou and Chen (2013) examine the benefits of sequentiality in the same game as ours. In their setting, one (forward looking) agent plays in a first stage and the others in the second stage. A network designer has to find the best agent to play first in order to increase aggregate efforts or utilities. They find in particular that, due to sequentiality, the key leader is not a simultaneous-move key player. In the context of monopoly pricing in presence of externalities between consumers, Bloch and Quérou (2013), as well as Candogan, Bimpikis and Ozdaglar (2012), study models where a monopolist charges a prices to interdependent consumers. These models exhibit an interesting tradeoff for the firm regarding pricing policy: central agents have increased demand for products, but they also originate large influence effects on other consumers. One interesting result is that with linear pricing, these effects compensate excatly, in such a way that, in the end, positions on the network do not affect prices. In our paper, the linear contract is formally isomorphic to linear pricing by monopolies as exposed in Bloch and Querou (2013). Our work complements their view by showing that the network matters under optimal contracting.

In various applications, the principal is a policymaker aiming at fostering agents' efforts by subsidizing agents directly. In the drop out game of Calvò-Armengol and Jackson (2004), in labor market context, the authors examine a binary game with local complementaries. A planner can suibisize agents entry, and see the consequence on the number of agents dropping in. Importantly, in this context the agents' consent is not an issue, while it is crucial in our framework. In Svetovic and Steiner (2012), agents face a coordination problem akin to the adoption of a network technology. A principal announces investment subsidies which, at minimal cost, attain a given likelihood of successful coordination. Optimal subsidies target agents who impose high externalities on others and on whom others impose low externalities. This result echoes our findings showing that rewards take into account the position of agents in the network. However, this latter paper is focused on the role of strategic uncertainty in coordination processes, which is not a concern in our paper.

The paper is organized as follows. Section 2 presents the model. Section 3 presents optimal contracting in both enforceable and non enforceable contexts, as well as linear contracting and key-player analysis. Section 4 extend the model to network entry. Section 5 concludes. All proofs are given in section 6.

2 A model of contracts on a fixed network

A principal contracts with a finite set of agents organized in a fixed network of local complementarities. In our setting, agents exert effort, and the principal's objective is to maximize a function of the sum of agents' efforts net of transfers. We propose a three-stage game. In the first stage, the principal proposes contracts. In the second stage, agents simultaneously decide whether to accept or reject their respective offers. In the third stage, agents exert effort and transfers are realized. Both efforts and the network are assumed to be observable. We study Subgame Perfect Nash Equilibrium (SPNE). In this model, coordination is a matter and equilibrium multiplicity may arise. We analyze in this paper the SPNE of the game which maximizes the principal's objective; that is, we focus on the equilibrium such that all proposed offers are accepted[2].

Notations. Vectors and matrices are in capital letter, e.g. effort profile X, of network G, while matrix components are in lower case. Agents are indexed with a subscript, e.g. x_i will quote for the effort level of agent i. We let symbol $\|\cdot\|$ represent the vectorial 2-norm, e.g. $\|X\| = \sqrt{\sum_{k\in N} x_k^2}$. The sum of the components of a profile is written in lower case, e.g. $x = \sum_{k\in N} x_k$. Symbol '$\geq$' applied to vectors means the weak inequality on each component. We let denote $\mathbf{1}$ the profile of ones (its dimensionality will not necessarily be n, but we keep implicit this aspect for convenience when there is no confusion).

Network. We let $N = \{1, 2, \cdots, n\}$ be a set of agents organized in a network of bilateral relationship. The network is formally represented by a symmetric[3] adjacency matrix $G = [g_{ij}]$, with binary[4] element $g_{ij} \in \{0, 1\}$.

[2] The optimal contract guaranteeing a unique equilibrium in which all agents accept the principal's offer is of *divide and conquer* sort - see Seagal (2003). Its study is out of the scope of the present paper.

[3] All results, except linear contracting and in particular Proposition 4, are robust to non binary and asymmetric relationships, i.e. $G^T \neq G$, where centralities are replaced by outward centralities.

[4] Results are robust to weighted relationships. They are also robust to the introduction of a low level

The link between agents i and j exists whenever $g_{ij} = 1$. In that case, we shall say that agents i and j are neighbors. By convention, $g_{ii} = 0$ for all i. We may abuse the language and speak about network G. We let $\mu(G)$ denote the index of the adjacency matrix G, which is by symmetry its largest eigvenvalue.

Agents' utilities. We label by $x_i \in \mathbb{R}^+$ agent i's effort and by $X = \{x_1, x_2, \cdots, x_n\}$ the profile of efforts. When agent i exerts effort x_i on network G, she derives some utility from own effort as well as the sum of efforts of neighbors. We consider the simple benchmark of linear quadratic utilities which are frequently used in the network game literature (see Ballester et al. [2006]). Let $a \in \mathbb{R}^+$, $\delta \in [0, \frac{1}{\mu(G)}[$, $v_i \in \mathbb{R}$, and let $y_i = \sum_{j \in N} g_{ij} x_j$. When agent i exerts effort x_i, agent i's utility is written:

$$u_i(x_i, y_i) = v_i + a_i x_i - \frac{1}{2} x_i^2 + \delta x_i y_i \tag{1}$$

Here parameter a_i measures some private return to own effort, while parameter δ measures the strength of complementarities, or intensity of interaction, between neighbors. To isolate pure network effects and keep things simple, agents have homogenous characteristics $a_i = a$ and $v_i = 0$ for all i, but the analysis is easily extended to heterogenous agents' characteristics. With this specification, agents' utilities depend positively on neighbors' efforts, and efforts between neighbors are strategic complements.

When an agent does not accept the contract proposed by the principal, she plays a best-response to others' efforts as derived from utility (1), that is $x_i^{BR} = a + \delta y_i$. Agent i's utility level under best-response play is written:

$$u_i^{BR}(y_i) = \frac{1}{2}(a + \delta y_i)^2 \tag{2}$$

Contracts. The principal commits to a set of publicly observable contractual offers. Contracts are proposed to every agent in the society at no cost, and the principal can discriminate between agents.

We define s_i as agent i's acceptance strategy: $s_i = 1$ (resp. $s_i = 0$) means that agent i accepts (resp. rejects) the principal's offer. We let $S \in \{0, 1\}^N$ represent the profile of acceptance strategies of agents in stage 2. Generally speaking, a contract between the principal and agent i is a pair of functions $\big(x_i(s_{-i}), t_i(s_{-i})\big)$. The quantity $t_i \in \mathbb{R}$ can be interpreted as a monetary transfer from the principal to the agent (this transfer can be either positive or negative). Both transfer function and effort function are contingent on acceptance status of others. We consider two alternative economic contexts,

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of strategic substitutability.

depending on contract enforceability. When contracts are enforceable, agents are tied to the effort level proposed in the contract. When contracts are not enforceable, an agent behaves opportunistically, and she exerts the desired level of effort if she obtains at least the utility derived from her best-response effort. This means that contracting is a simple option, such that the agent receives a transfer in case she complies with a certain effort level.

The principal's payoff $\Pi_P(X,T)$ (where subscript P stands for 'Principal') under efforts X and realized transfers T, is increasing and concave in the sum of agents' efforts through function $F()$, strictly increasing ($F' > 0$), differentiable, and concave ($F'' < 0$):

$$\Pi_P(X,T) = F\left(\sum_{i\in N} x_i\right) - \sum_{i\in N} t_i \tag{3}$$

To guarantee that the principal's payoff is positive at optimum, we also assume $F(0) \geq 0, F'(0) > 1, lim_{z\to+\infty} F'(z) < 1$. The optimal set of contracts guarantees contract acceptance, and thus solves:

$$\max_{\{(x_i(s_{-i}),t_i(s_{-i}))\}_{i\in N}} \quad F\left(\sum_{i\in N} x_i\right) - \sum_{i\in N} t_i \tag{4}$$

$$\text{s.t.} \quad \forall i \in N, u_i(x_i, y_i) + t_i = u_i^{BR}(y_i)$$

The constraint in the above program represents an individual participation constraint[5]. Importantly, at the equilibrium agent i's reservation utility $u_i^{BR}(y_i)$, i.e. utility under offer rejection, depends on others' efforts. The level of effort exerted by others at optimum may vary according to the context, as we will detail further.

To identify optimal contracts, the principal can, in a first step, solve the sub-problem associated with a fixed budget, and obtaining the optimal set of contracts contingent on budget t:

$$\max_{\{(x_i(s_{-i}),t_i(s_{-i}))\}_{i\in N}} \quad \sum_{i\in N} x_i \tag{5}$$

$$\text{s.t.} \begin{cases} \forall i \in N, u_i(x_i, y_i) + t_i = u_i^{BR}(y_i) \\ \sum_{i\in N} t_i = t \end{cases}$$

[5]Agents are assumed to accept the offer under indifference, so the principal will necessarily extract agents' utility surplus until indifference; this is without loss of generality to restrict attention the program with binding participation constraints.

This subprogram generates an optimal effort profile $\hat{X}(t)$, with corresponding aggregate effort $\hat{x}(t)$. In a second step, the principal compares the performance of each budget, and determines the optimal budget. The optimal budget $\hat{t}$ solves[6]:

$$\frac{\partial \hat{x}(\hat{t})}{\partial t} F'(\hat{x}(\hat{t})) = 1 \tag{6}$$

Note that the conditions that we impose on function F imply that the profit of the principal is positive for all positive budget $t \in [0, \hat{t}]$.

The principal thus proposes contractual offers in order to maximize her payoff under agents' participation constraint. To identify optimal contracts, the principal can, in a first step, solve the sub-problem associated with a fixed budget t, and obtain the optimal set of contracts contingent on this budget. In a second step, the principal compares the performance of each budget, and determines the optimal budget. The shape of function $F()$ is crucial to understand budget selection. In concrete applications, the budget t is endogenous and related to agents' efforts through function $F()$, as for shopping malls or monopoly pricing. In other applications, like the funding of R&D by an international institution, the budget can be considered as exogenous. Throughout the paper, we will mainly abstract from this discussion and focus on the first step, where the budget is kept fixed. Technically, this means that it is sufficient to focus on the sub-problem in which the principal's objective is to maximize the sum of efforts.

This model fits with various applications.

Production in firms. We enrich the classical team production model (Holmstrom [1982]), where the firm is composed of n workers and a firm owner distinct from workers, by adding a network aspect as follows. Workers are organized in a network G describing local complementarities. In particular, a worker's effort generates quadratic costs but synergies with neighbors on network G contribute to reduce effort cost as follows:

$$c_i(x_i, y_i) = \frac{x_i^2}{2} - \delta x_i \left(\sum_{j \in N} g_{ij} x_j \right)$$

The larger the sum of efforts of agent i's neighbors, the lower the cost. Furthermore, the cost function exhibits complementarities in neighbors' efforts: for a fixed level of neighbors' efforts, a higher level of own effort entails a

[6] In the above program, it is immediate that the optimal budget $\hat{t} \geq 0$. Furthermore, it is also immediate that the principal spends the whole budget.

larger impact of neighbors' efforts on own effort cost. Workers are payed a wage from the firm owner. This wage is given by $v + ax_i$, $v, a > 0$, where v is a fixed common fee, and ax_i is a linear compensation in individual output - which is assumed for simplicity exactly reflected by effort. The term ax_i is not linked to the firm's profits. When firm's profits are uncertain, this part constitutes a kind of insurance against that risk. The value of the firm's output is an increasing function of the sum of efforts of employees. Due to diverse production costs, the value function is in general concave. To fit with our model, on top of initial wages, the firm owner can distribute to each agent i an individual bonus $t_i(x_i) \geq 0$ conditional on increased effort level. The firm owner wants to maximize output less wages, and the budget t is here endogenous to workers' efforts.

Research activity and science parks. The world of research is a world of synergies. A huge literature documents the role played by research collaborative activities between firms or academics (see Goyal and Van der Leij (2006) for an empirical analysis of the properties of the networks of collaboration between academics; see for instance Hagedoorn (2002) for a description of R&D networks between firms). As in Goyal and Moraga (2001), consider independent markets with linear demand $d - p$, $d > 0$, and x_i as firm i's R&D effort. There is no fixed cost, and marginal costs are related to partners' efforts through the equation $c_i = c - x_i - y_i$. We thus have

$$\pi_i(x_i, y_i) = (d - c + x_i + y_i)^2 - \gamma x_i^2$$

where γx_i^2 is the cost of R&D effort and c the constant marginal production cost, $\gamma > 1$. This profit function corresponds to a modified version of utility (1), where agent i's utility is of type $u_i(x_i, y_i) + V_i(y_i)$, with $V()$ a non decreasing function. In this context, the principal can be a public institution, like a regional institution, a state or a state union, wanting to foster the amount of research. To proceed, such institutions frequently reward agents and some dots can be conditional on performance. In this frame, the budget of the public institution's fund can be exogenous (then the principal solves program (5) with a given budget t in hand), or partly related to the industry's aggregate profit.[7]

Conditional cash transfers in education. It is often argued that peer effects are prevalent at school. Some recent papers have identified friendship networks at school. For instance, Calvo, Patacchini and Zenou (2009) identify peer effects in frienship networks at school, and they use utility function in

[7] We note that the public authority can also be interested in minimizing the sum of industry costs. In that case, the objective is a weighted sum of efforts where weight are proportional to agent's degree. Even if we present the simple case of equal weights for clarity, our model is easily extended to the setting where the principal's objective is an increasing function of the sum of weighted efforts (with fixed weights).

equation (1) as teenagers' utilities. Grants are often conditional to students performance. For example, there are two influential programs (PROGRESA-Opportunidades in Mexico and Bolsa-Familia in Brazil) of cash transfers conditional on school attendance (for a review, see Fiszbein, Schady, Ferreira, Grosh, Keleher, Olinto and Skoufias [2009], see also Dieye, Djebbari and Barrera-Osorio [2014]). In this application, one may consider the principal's budget as exogenous to students performance.

Monopoly pricing with interdependent consumers. In many cases, consumptions of neighborhood or friends affects own consumption. Bloch and Quérou (2013) consider monopoly pricing in a model of network externalities where consumers only care about the consumption of a subset of agents determined by an exogenous social network. Their setting is extended to divisible goods in Candogan et al. (2012), whose consider utility (1) with $a = d_i - p_i$; here d_i measures private preference parameter and p_i represents the per unit price charged by the monopolist. As shown by Candogan et al., the optimal price profile $\{p_i^*\}$ charged by the monopolist is homogenous, i.e. $p_i^* = p^*$ for all i. Assume that the monopolist has already set up the optimal price p^* in the past. On top of this price, suppose the monopolist can propose to each consumer an idiosyncratic discount conditional on some quantity sold. One important message with regard to this literature is that, with regard to linear pricing, the existence of discounts-like contracts allows the monopolist to significantly enhance the demand and thus to increase her profit by charging heterogenous optimal prices.
One interesting related application concerns shopping malls. Shop owners borrow space to a mall developer. Typically stores are placed together in close proximity within a large shopping mall. Shops are organized in the mall space, and there is positive externalities between demands for adjacent shops in the space: first, customer traffic in likely to be affected by the upkeep of adjacent stores; second, anchor stores and national name-brand stores, generate positive externalities by drawing customer traffic not only to their own store, but also to other stores (the network of local externalities is likely to be asymmetric). Here, x_i denotes firm i's effort to increase its sells, and the network G represents the physical mall space, where adjacent locations correspond to a link. In shopping malls, mall store contracts are rather sophisticated and partly depend on the amount of sells of the shop in the mall[8]. Interestingly, a part of the payoff of the mall owner can result from externalities related to shops efforts. For instance, in case the mall owner installs a costly car park, a larger demand in the mall increases car park revenue.

[8] Gould et al. (2006) provide a detailed empirical analysis based on a well documented dataset.

3 Optimal contracts

In this section, we examine diverse contractual arrangements in a situation where the principal contracts with agents organized in a fixed network. In this context, we study the impact of the network structure on contractual design in various situations. We examine in the order optimal contingent contracting (the first-best in our context), then optimal bilateral contracting, then linear contracts, and last we undertake a key-player analysis, which corresponds to the situation where the principal is constrained to contract with a single agent on the network.

3.1 Optimal enforceable contracts

In this subsection, we consider that contracts are enforceable, i.e. agents do not behave opportunistically. This economic environment is in general favorable to the principal's objective.

Here, we are interested in the first-best contract which can be reached by the principal. First note that, when all agents play their best-response, this originates the unique equilibrium effort profile X^0 and the corresponding equilibrium utility profile U^0 as follows. Let matrix $M = (I - \delta G)^{-1} \geq 0$[9] and $B = M\mathbf{1}$. Equilibrium efforts can be interpreted in terms of network Bonacich centrality. For any profile $Z \geq 0$, we let $B_Z = MZ$ denote agent i's *Bonacich centrality* (weighted by Z). The quantity $b_{Z,i}$ is the number of paths from agent i to others, where paths are weighted as follows: the weight of a path of length k from agent i to agent j is $\delta^k z_j$. We have:

Result 1 (**Ballester et al. (2006)**). *An equilibrium effort profile exists if and only if $\delta\mu(G) < 1$. In this case,*

$$x_i^0 = aB \tag{7}$$

and

$$u_i^0 = \frac{a^2}{2} b_i^2 \tag{8}$$

The first task for the principal is to determine the minimum agents' outside opportunity, this level being crucially tied to the effort level he can ask agents to exert. Basically, the lowest reservation utilities agents may consent

[9]The condition $\delta\mu(G) < 1$ guarantees $M \geq 0$; otherwise there is no equilibrium: efforts would escalate to infinity.

to obtain under the principal's offer are exactly the utilities they would obtain in the absence of principal's intervention. This would induce a maximum surplus extraction from agents' utilities. Can the principal design contracts such that an agent refusing the offer would obtain this minimal reservation utility?

The point is that, by complementarities and positive externalities, utilities under offer rejection are increasing with others' efforts, so simple bilateral contracts, for sure, would not allow the principal to extract this maximum utility surplus from agents[10]. It is straightforward to observe that the principal should raise contingent contracts. A contingent contract is a *take-it-or-leave-it* offer such that, if one agent rejects the offer, no contract is accepted, the consequence of which is that agents play the equilibrium X^0.

Contingent contracts can be written as follows. We note first that we are interested in the first-best optimum, and in particular we do not examine how agents behave out of the optimal contract. It is thus without loss of generality to restrict attention to the simplified setting in which the principal proposes a transfer against some effort level[11]. Second, we have to take into account that offers are conditional on other offers' acceptance. Define for convenience the profile $\mathbf{1}_{-i} = (1, \cdots, 1)$ which contains $n-1$ ones. A contingent contract with agent i is a pair $(x_i(s_{-i} = \mathbf{1}_{-i}), t_i(s_{-i} = \mathbf{1}_{-i}))$. This contract means that agent i should both exert effort x_i and receive transfer t_i if all other agents accept the offer. The key implication of this formulation is that if agent i does not accept her offer, other agents are not tied to the effort level prescribed in their offer.

Given the above discussion, we can state:

Proposition 1. *Under enforceable contracts, the optimal contract is a contingent contract. It guarantees that agent i's utility under offer rejection is equal to* $\frac{a^2}{2}b_i^2$.

The next question is to characterize optimal efforts and transfers, given that agent i's participation constraint is written at the first-best:

$$u_i(x_i, y_i) + t_i = \frac{a^2}{2}b_i^2$$

Note that the principal can always propose to any agent i the *default* contract $(x_i^{def}, 0)$, such that the agent is indifferent between accepting the

[10]This statement is valid even with contracts à la *Mirrlees* where agents are payed conditionally on reaching a joint performance. Indeed, even in this case, an agent refusing the offer would benefit from neighbors' increased efforts.

[11]Since we are only interested in the first-best contract, and given that there is no uncertainty in the game, this formulation is sufficient here: the principal is able to compute exactly the optimal contract that guarantees agents' acceptance, and agents exert the exact effort level prescribed in the contract at optimum.

offer or not (status quo offer). The proposed effort level x_i^{def} corresponds to agent i's best-response given others efforts. Also, in this game of complementarities, the principal never finds profitable to propose to agent i an effort level $x_i < x_i^{BR}$.

In particular, two questions need to be addressed. First, should more central agents be proposed higher rewards and/or efforts? The answer is far from being obvious. Indeed, if central agents generate a huge amount of externalities on other, they also receive more externalities; thus their effort level is initially higher, and with convex effort cost, increasing their effort is budget-demanding. Second, it is interesting to know whether or not this maximum surplus extraction requires to tax some agents in order to increase rewards to the most influencial agents on the network.

To guarantee a well-defined problem, we assume that $\delta < \frac{1}{2\mu(G)}$. We allocation maximizing utilities $X^e = aB'$ satisfies $u_i^e = \frac{a^2}{2}b'_i$. For convenience, we write $B'(G) = B(G, 2\delta)$ and $b'(G) = b(G, 2\delta)$. We obtain:

Proposition 2. *The optimal contingent contract, for given budget t, is written for all i:*

$$\hat{t}_i(t) = \frac{1}{2}\left[a^2 b_i^2 + \left(\frac{2t - a^2\|B\|^2}{b'}\right) b'_i\right]$$

$$\hat{x}_i(t) = \left(a + \sqrt{a^2 + \frac{2t - a^2\|B\|^2}{b'}}\right) b'_i$$

Optimal efforts are always well-defined. The positiveness of the member under the square root in the latter equation is equivalent to budget t being larger than the difference between aggregate initial utilities and the aggregate utility of the allocation X^e. This effort profile maximizing precisely aggregate utilities, and the budget being nonnegative, this constraint always holds.

We define the following centrality index $H = (h_1, h_2, \cdots, h_n)$:

$$h_i = \frac{b_i^2}{b'_i} \tag{9}$$

This index is the ratio $\frac{u_i^0}{u_i^e}$ of the (initial) agent i's equilibrium utility over the utility of the efficient allocation. For all $i \in N$, we define

$$\bar{t}_i = \frac{a^2}{2}\left(||B||^2 - b' h_i\right)$$

and

$$\bar{t} = \max_{i \in N} \ \bar{t}_i$$

For all regular networks G^k, $k = 0, 1, \cdots, n-1$, we have $\bar{t} = 0$ because the Bonacich centrality is homogenous across agents on regular networks. For other networks, we obtain

Lemma 1. *For any non regular network G, $\bar{t} > 0$.*

The positiveness of the quantity $\bar{t}$ is crucial to understand taxation (i.e. the set of agents offered a negative transfer). Exploiting Lemma 1, we are now in position to identify taxed agents. From Proposition 2, we deduce that agent i is taxed whenever $t < \bar{t}_i$. We deduce how to find taxed agents:

Corollary 1. *The principal taxes at least one agent on the network as soon as $t < \bar{t}$. Moreover, taxed agents have smaller the index h_i.*

Hence, taxing some agents allows the principal to contract with agents whose gap between equilibrium utility and efficient utility is the largest. The centrality index H and Bonacich centrality index are linked but they may have different ordinal rankings. To illustrate, for $\delta = 0.01$, this is the case on the four-player four-links network built as a four-player star complemented with a link between two peripherals. Here, Bonacich centralities are aligned with degrees while the ranking of the index H puts first the most connected agent, then the least connected, then the others.

Fix δ and vary t. The optimal contract satisfies that, whatever the network, all agents are rewarded for a large enough budget. Moreover, the condition $t < \bar{t}$ is always met for regular networks, meaning that, whatever the budget, every agent is rewarded for this class of graphs. In contrast, for a low budget, except on regular networks certain agents are taxed (with still an effort level higher than their best-response choice).

Fix t and vary δ. For δ close to 0 or equal to 0, the solution is distributed and agents receive approximately the same reward. When δ takes a large value, the network effects are stronger. In general, $\bar{t}$ is increasing with δ. For a given budget, this means that taxation can emerge for a high enough value of δ.

It is worth emphasizing that the network is crucial to understand taxation. Indeed, in the empty network all agents are isolated and thus every agent gets an equal sharing of the budget[12].

[12]This can be even seen when extending the model to heterogeneous agents' characteristics. Hence, the network effect inducing taxation is not reducible to an idiosyncratic heterogeneity effect.

Finally, the optimal contract performance is measured by

$$\hat{x}(t) = \left(a + \sqrt{a^2 + \frac{2t - a^2\|B\|^2}{b'}}\right)b' \tag{10}$$

We deduce from equation (10) that *aggregate effort at the optimum is increasing in aggregate centrality* b'. Making a comparative statics to assess the performance of various network structures, two messages emerge. First, performance is increased with link addition; not surprisingly, increasing synergies is a good news for the principal[13]. Second, less obvious is to compare network structures this same density. Belhaj et al. (2013) show that among all networks with same total number of links, a network maximizing the quantity b' is a Nested-Split GraphIn short, Nested-Split graphs are such that, for any pair of agents on the network, one neighborhood is nested in the other one[14]. Hence, we can state:

Corollary 2. *Among all networks with fixed number of links, a network maximizing contract performance is a Nested-Split Graph.*

Therefore, performant networks are highly hierarchical structures.

3.2 Optimal non enforceable contracts

With non enforceable contracts, agents accepting offers are not tied to their promisse. Therefore, conditioning individual contract on other agents' acceptance has no value for the principal. In particular, the effort profile corresponding to optimal enforceable contract cannot be sustained as a Nash equilibrium (with same budget). Indeed, each agent, by playing a best-response effort, obtains a utility level than u_i^0. Each agent will deviate from the prescribed level of effort of the contract.

Hence, the principal must resort to offer simple options, such that the principal commits to a transfer if some effort level is exerted. It is therefore without loss of generality to restrict attention to simple bilateral contracts. Under non enforceable contracts, an offer to agent i is thus a pair (x_i, t_i), meaning that if agent exert effort x_i the principal is tied to proceed to tranfer t_i (in our setting the principal does not defect to his promisse).

How does non enforceability affects the optimum? A first immediate remark is that the principal should propose offers with nonnegative transfers

[13]Similarly, contract performance increases with parameter δ.

[14]This class encompasses stars and quasi-star networks, complete and Quasi-complete networks, dominant groups, and also more complex architectures - see for instance Mahadev and Peled (1995) for more details.

(an agent's utility under negative transfer cannot exceed the reservation utility, which corresponds to a null transfer and a best-response), which means that the sole incentive compatible contract with null transfer is the default contract. Now, does the principal propose offers with positive transfers to every agent in the society, or in contrast should the principal concentrate rewards to a strict subset of the society?

This issue is inutitively unclear. Subsidizing one agent increases the reservation utilities of others, which de facto limits the efforts to be proposed by the principal. The next proposition characterizes optimal bilateral contracts as a function of agents' positions on the network. We define $E = (b_1^2, \cdots, b_n^2)$ as the profile of squares of non weighted centralities. Then:

Proposition 3. *Optimal contracts satisfy $\hat{t}_i > 0$ for all $i \in N$, and are written:*

$$\begin{cases} \hat{T}(t) = \frac{t}{\|B\|^2} \cdot E \\ \hat{X}(t) = X^0 + \frac{\sqrt{2t}}{\|B\|} \cdot B_B \end{cases}$$

Proposition 3 expresses that all transfers are positive and all efforts are larger than the best-response effort, as indicated by indexes E and B_B - in short the principal does not concentrate rewards to a subset of agents. We obsesrve that the sharing of the budget among agents is independent of the budget level, since $\frac{\hat{t}_i}{t} = \frac{b_i^2}{\|B\|}$. Furthermore, transfers are shaped by the square of the un-weighted Bonacich centralities (parameter a does not play any role here), thus the distribution of the budget is somehow concentrated in favor of the most central agents.

In terms of contract performance, Proposition 3 implies the sum of optimal efforts

$$\hat{x}(t) - x^0 = \sqrt{2t} \cdot \|B\| \tag{11}$$

The equation (11) is useful to undertake some comparative analysis of the performance of network structures. From the above formula, contract performance is clearly increasing with $\|B\|$. One immediate consequence is that contract performance is strictly increasing with link addition. Less obvious is the comparison of the performance of networks with same number of links. Belhaj et al. (2013) show that among all networks with fixed number of links, a network maximizing the quantity $\|B\|$ is a Nested-Split Graph. Thus, similar to the case of enforceable contracts, we can state:

Corollary 3. *When contracts are not enforceable, a network maximizing contract performance among all networks with same number of links is a Nested-Split Graph.*

3.3 Linear contracting

Linear contracts, by their simplicity, are often more realistic than optimal contracts. Moreover, with contracts rewarding effort only, there is no coordination concern because agents accept to contract independently of others' choices[15]. The main question is to know whether the principal takes agents' positions into account in her pricing strategy.

We examine the following excess-effort linear contract. With this contract, the principal offers in the first stage a function rewarding increased effort, letting agents choose in the second stage their optimal effort level. Formally, agent i is proposed a transfer function $t_i(x_i) = \gamma_i(x_i - x_i^0)$, with $\gamma_i \in \mathbb{R}_+$[16]. Note that, from the principal's view, this affine contract is clearly better than a linear contract $\gamma_i x_i$; indeed, with the above affine contract, the principal does not subsidize efforts lower than x_i^0. By complementarities, it can be seen that agents accept such an offer whatever the other's choices[17]. As a consequence, in contrast with optimal contracts, there is no coordination issue here.

We let $\Gamma = (\gamma_1, \gamma_2, \cdots, \gamma_n)$, and we call by $X^*(\Gamma)$ the equilibrium effort associated with the set of contracts Γ. Profile $X^*(\Gamma)$ takes into account both the variation in effort of the contracting agent and the induced variation in other efforts on the network. The equilibrium effort profile satisfies $X^*(\Gamma) = MA'$, with $a'_i = a + \gamma_i$ for all $i \in N$. It is worth noting that the presence of linear contracts does not affect the conditions of equilibrium existence and uniqueness, the condition being still $\delta\mu(G) < 1$.

As in preceding subsections, we solve the principal's program for a fixed budget t. We let $\breve{\Gamma}(t)$ denote the optimal excess-effort linear contract, and $\breve{X}(t) = X^*(\breve{\Gamma}(t))$ for convenience. The contract $\breve{\Gamma}(t)$ solves:

$$\max_{\substack{\Gamma \geq 0 \\ \text{s.t. } \sum_{i=1}^n \gamma_i \left(x_i^*(\Gamma) - x_i^0\right) = t}} \sum_{i=1}^n x_i^*(\Gamma)$$

[15]For instance, the empirical study of Gould et al. (2005) reports linear contracting in the context of shopping malls (the typical contracs described being a fixed charge plus a tax proportional to sells in excess of a threshold). As well, it is natural to consider linear pricing in the context of monopolies.

[16]In fact, imposing nonnegative per unit returns is without loss of generality; as we will see, this constraint is not binding at optimum.

[17]Whatever the profile $\Gamma \geq 0$ chosen by the principal, agent i's individual participation constraint at equilibrium $X^*(\Gamma)$ is written:

$$\frac{1}{2}(a + \gamma_i + \delta y_i^*)^2 - \frac{1}{2}(a + \delta y_i^0)^2$$

For any $\Gamma \geq 0$ and by complementarities, $X^*(\Gamma) \geq X^0$. Hence, the participation constraint is satisfied.

where the budget constraint in the above program expresses that the sum of rewards is equal to the budget. We obtain:

Proposition 4. *For all $i \in N$, the optimal linear contract is:*

$$\sqrt{\frac{t}{b}}\,(x_i^* - x_i^0)$$

All transfers are positive[18] and the return per unit of effort is homogenous across agents. Moreover, agents are payed proportionally to their relative centralities[19]. Indeed, taking account of the optimal efforts selected by the agents, agent i's payment can be computed as

$$\breve{t}_i(t) = \frac{b_i}{b} \cdot t$$

Also, the variation of aggregate effort is equal to

$$\breve{x}(t) - x^0 = \sqrt{tb}$$

Hence, the networks maximizing the impact of the principal's intervention are also those maximizing the sum of centralities (whatever the magnitude of the budget). As for optimal contracts, this means that, on the one hand, adding links will increase the impact of the principal's intervention, and on the other hand, among all networks with same number of links the networks with maximal impact are Nested-Split Graphs (see Belhaj et al. [2013]).

3.4 Key-player analysis

In many situations, the principal can be constrained to contract with a restricted number of agents in the society. This can result for instance from contracting costs. We study the polar case where the principal is constrained to choose a single agent to contract with. The question for the principal reduces to find the optimal agent to target.

We examine linear contracts, optimal non enforceable contract, and optimal enforceable contract. Of course, the most performing contract is the enforceable contract, then the non enforceable contract, then the linear. Whatever the case studied, one single agent is proposed an offer, thus all other agents play their best-response. The principal subsidizes an increased effort

[18] It should be noted that the symmetry assumption $G^T = G$ is crucial here.

[19] This is not a surprise. This result is known from the literature on linear pricing in monopolies (see Bloch and Quérou [2013]).

for the targetted agent until a point which is compatible with the agent's incentive.

Following the same logic as in the benchmark case where the principal is not contrained to select a unique agent, at the optimal enforceable contract, the selected agent obtains in the end her initial utility, i.e. the principal extracts all her utility surplus; in the opposite, at the optimal non enforceable contract, the principal has to take care that the reservation utility of the targetted agent at the optimum is larger than her initial utility, because others' efforts are increased by complementarities; i.e. the principal cannot extract the full utility surplus of the targetted agent.

We introduce the *inter-centrality index* of agent i:

$$I_i = \frac{(b_i)^2}{m_{ii}}$$

This index plays an important role in problems where the objective is to drop out the agent which has the maximum (resp. minimum) impact on others' efforts, and it turns out that the optimal target, so as referred to the *key player*, is the agent with maximal (resp. minimal) inter-centrality index (see Ballester et al. [2006], Bellester et al. [2010] or Liu et al. [2014]). This index plays a key role for the linear contract:

Proposition 5. *With the linear contract* Γ*, the agent to target has maximal inter-centrality index* I_i*.*

More precisely, let agent l be the targetted agent. The optimal reward per unit of excess effort of the targetted agent is $\breve{\gamma}_l(t) = \sqrt{\frac{t}{m_{ll}}}$, her effort is $\breve{x}_l(t) = x_l^0 + \sqrt{t\, m_{ll}}$, and aggregate effort variation is $\breve{x}(t) - x^0 = \sqrt{t\, I_l}$.

The optimal non enfocreable contract selects an other agent:

Proposition 6. *At the optimal non enforceable contracting, the targetted agent maximizes the un-weighted centrality index.*

Let agent z be the targetted agent. Her effort is $\hat{x}_z(t) = (a + \sqrt{2t})\, m_{zz}$ and the aggregate effort variation is $\hat{x}(t) - x^0 = \sqrt{2t}\, b_z$.

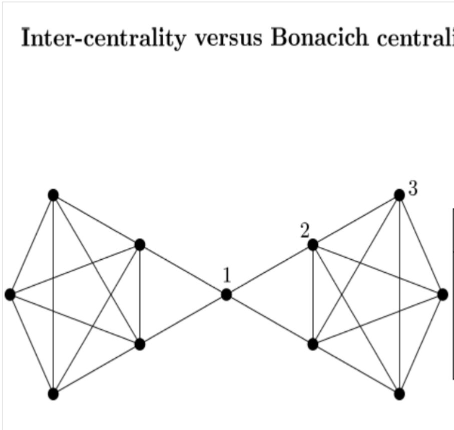
To finish, we turn to the optimal enforceable contract. The selected agents exerts an effort level that guarantees her to obtain her initial utility. Her effort is written $b_i + \alpha_i(t)$, where the quantity $\alpha_i(t)$ is as follows. Define $A_i = 1 - \frac{2\delta(GM)_{ii}}{m_{ii}}$, $B_i = b_i - 1 - \delta(GB)_i - \delta(GM)_{ii}\frac{b_i}{m_{ii}}$, $C_i = -2(t - b_i^2) - 2b_i(1 + \delta(GB)_i)$. Then

$$\alpha_i(t, b_i, m_{ii}, (GB)_i, (GM)_{ii}) = \frac{1}{A_i} \max\left(- B_i - \sqrt{B_i^2 - A_iC_i}, -B_i + \sqrt{B_i^2 - A_iC_i}\right)$$

Hence,

Proposition 7. *At the optimal enforceable contract, the targetted agent maximizes the budget-dependent quantity* $\alpha_i(t)\frac{b_i}{m_{ii}}$.

Propositions 5, 6 and 7 have deep implication for the principal. For instance, the ordinal ranking of un-weighted centralities can be distinct from the ranking of inter-centrality indexes, and notably the agent with maximal index. As an illustration, consider the 11-agent example of Ballester et al. (2006), which is presented in figure 3.4:



agent	b_i	c_i
1	8.33	41.67*
2	9.17*	40.33
3	7.78	32.67

Figure 1: For $\delta = 0.2$, agent 1 maximizes centrality, agent 2 maximizes inter-centrality.

When $\delta = 0.2$, the agent with the highest centrality is agent 2, while the agent with higher inter-centrality is agent 1. Hence, under optimal contracting, the preferred target is agent 2, while with linear contracting the principal should select agent 1.

We note that the ratio of aggregate effort reached at the optimal contract over aggregate effort reached at the linear contract, $\sqrt{2\, m_{ll}}\frac{b_z}{b_l}$, is independent of budget t. Also, since diagonal elements of matrix M exceed 1, the ratio exceeds $\sqrt{2}$. To illustrate the performance gap between linear and optimal contract, in the example given in Figure 3.4, we find $l = 1$ and $z = 2$ and the ratio is around 2.008, thus targetting adequately optimal contract doubles contract performance with regard to linear contracting.

4 Extension: contracting with network entry

In this section, we explore enforceable optimal contracting in a model with network entry. In this extension, we assume that agents are initially not on the network. In the context of malls, the network is a fixed network with physical nodes, and the principal can then offer a location (on top of proposing an effort and a transfer). In the previous section, shops were initially occupying the mall space and the principal task was to contract with every shop in the mall. However, when a developer acquires a new mall, shops are initially out of the mall space. Facing both this physical network and the pool of agents, the principal has to select a set of shops to contract with and to propose to each selected shop an offer of the following kind: the contract specifies a node of the network to be occupied by the shop, an effort level and a transfer. Hence, the principal has to deal with network entry (a non-contracting shop does not enter the network). As another concrete application with network entry, consider a firm manager who can select among its workers a pool of workers to set up a specific project; here the network describes the potential synergies between all pair of workers.

Formally, there is initially a pool of agents and a physical network of possible locations for agents (we suppose with loss of generality that the number of possible locations is equal to the number of agents). The principal selects a set of agents to contract with. As shown in the section on enforceable contracts on a fixed network, the optimal contract is contingent, thus utilities under offer rejection are given by some exogenous value. However, the principal should determine not only the optimal efforts and transfers, but also a location on the network for each selected agent. In practise, since agents are homogenous and anonymous, the selection problem of the principal consists in choosing the best subnetwork of network G and locate arbitrary agents on this subnetwork, irrespective of the agents' labels. Figure 4 illustrates the actions of the principal (in the picture, all links between agents in S are active; this is of course conditional on agents acceptance).

For convenience, the model can be treated as if all agents where occupying all nodes, and the principal can either propose to each agent i a contract of the type (x_i, t_i) or can unilaterally exclude agent i from the network (for convenience we omit the term K_i in the contract, as we know that $\hat{K}_i = N$). We call by $(0, 0)$ the default point consisting in the *unilateral* exclusion of the agent (hence, the default point is not *stricto sensu* a contract). With this conventional trick, the pool of possible contracts $\mathcal{C} = [0, +\infty[\times\mathbb{R}$ includes the default point. Under the offer (X, T), the set $\{i \in N, (x_i, t_i) \neq (0, 0)\}$ corresponds to the set of agents who are not unilaterally excluded from the

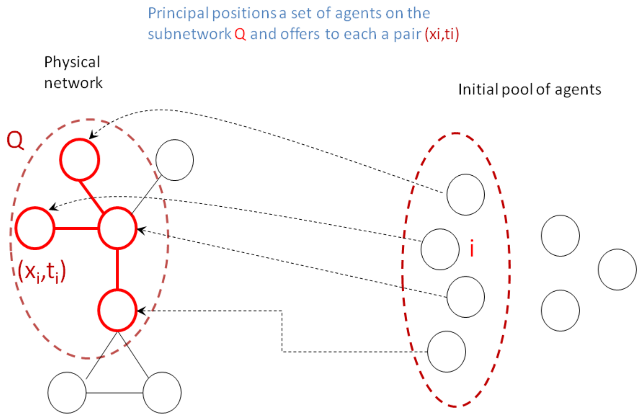


Figure 2: Contracting with network entry

network. An agent rejecting the offer or being proposed the default point exerts a null effort and obtains an exogenous and homogenous reservation utility equal to u. Hence in opposite with the model with fixed network, non contracting agents do not affect the utility of agents joining the network. This means that in network G active links are only those involving the agents accepting the offer, who are located on subnetwork Q (active links are depicted in thick draw in figure 4.

Furthermore, we assume that $\delta < \frac{1}{2\mu(G)}$. This condition guarantees finiteness of efforts at the optimal contract. The (binding) individual participation constraint associated with agent i's contract with network entry is written at optimum:

$$u_i(x_i, y_i) + t_i = u$$

The principal's tasks can be decomposed into three nested optimization problems. First, considering any set of agents located on a given subnetwork say Q of network G, and set $N \setminus Q$ of excluded agents, the principal has to select the optimal contracts over the subnetwork Q, conditional on budget t. Since agents are anonymous, this task consists in defining optimal contracts on every subnetwork of network G. Formally, defining $\mathcal{G}$ as the set of subnetworks of network G. We define $(\hat{X}(t), \hat{T}(t))_Q$ as the optimal set of contracts associated with the subnetwork $Q \in \mathcal{G}$ and conditional on budget t. Second, the principal has to select the optimal subnetwork $\hat{Q}(t) = \arg\max_{Q\in\mathcal{G}} \Pi_P((\hat{X}(t), \hat{T}(t))_Q)$, given the optimized contracts on each subnetwork. Last, the principal should select the optimal budget $\hat{t} = \arg\max_{t \in \mathbb{R}_+} \Pi_P((\hat{X}(t), \hat{T}(t))_{\hat{Q}(t)})$, given that to each budget is associated an optimal subnetwork with optimized contracts on it.

We start with the first task. We thus consider an arbitrary subnetwork Q with $q = |Q|$ the number of its nodes. In the benchmark model where agents cannot be excluded from the network, there always exists an optimal contract on any subnetwork Q; even with zero budget, there exists an optimal contract in which the principal offers increased effort by taking advantage of complementarities. However, in the model with network entry, for large reservation utilities and limited budget, there is no warranty of beneficial trade for the agents, because there may be no way to manage a contract with increased efforts without violating at least one individual participation constraint. Precisely, the budget should be big enough to cover the difference between the sum of reservation utilities of the agents entering the network and the maximum utility that these agents could attain on the network. We denote for simplicity $B'_Q = B(Q, 2\delta)$ (which is a vector with q elements). The principal can raise an *optimal* set of contracts on Q if only if

$$t \geq qu - \frac{a^2}{2} b'_Q \tag{12}$$

The optimal set of contracts $(\hat{X}, \hat{T})_Q(t)$, conditional on subnetwork Q and budget t, has the following structure. We obtain:

Proposition 8. *Consider a fixed budget t and a subnetwork Q satisfying Condition (12). The optimal contract $(\hat{X}, \hat{T})_Q(t)$ is written:*

$$\hat{T}_Q(t) = u\boldsymbol{1} + \left(\frac{t - qu}{b'_S}\right) B'_Q \tag{13}$$

$$\hat{X}_Q(t) = \left(a + \sqrt{a^2 + 2\Big(\frac{t - qu}{b'_S}\Big)}\right) B'_Q \tag{14}$$

We observe that optimal offers, through both efforts and rewards, take into account centralities (to take into account of the impact of agents on neighbors' utilities, the relevant decay parameter is 2δ at optimum). Second, *rewards are increasing in centralities if an only if $t > qu$.* This apparently surprising result can be explained as follows. Two forces shape rewards. On the one hand, the optimal contract internalizes the influence that agents have on others; more central agents generating larger network effects, this force tends to align rewards with centrality. On the other hand, the optimal contract has to guarantee network participation to agents; this mechanism tends to puts greater reward to peripheral agents (their utility upon contract acceptance is lower whereas the exist option is homogenous). When the budget exceeds the sum of reservation utilities, network influence dominates and more central agents receive higher rewards. In the opposite, when the budget is lower than the sum of reservation utilities, network entry dominates and least central agents receive higher rewards.

Note also that transfers can be either positive or negative. A deep result is that the possibility of negative transfers is exclusively related to network exclusion considerations. Indeed, *negative transfers can exist only if $t < qu$, and taxed agents are those with the largest centralities.* Conversely, when the budget is large enough so that $t > qu$, more central agents get higher rewards. In this case, equation (13) shows that the principal never finds profitable to tax peripheral agents.

Regarding optimal efforts, letting $X^e_Q = aB'_Q$ denote the efficient allocation on subnetwork Q, equation (14) shows that $\hat{X}_Q(t) \geq X^e_Q$. This offer is optimal when the budget is equal to the efficient aggregate utility level, while for a larger budget, the principal can even sustain higher effort levels. Moreover, we observe that $\frac{x_i - u}{x_j - u} = \frac{b'_{Q,i}}{b'_{Q,j}}$, i.e. the ratio of excess effort with regard to exit option is equal to the ratio of centralities; in particular it is independent from the budget. Overall, Proposition 8 indicates that more central agents always exert a higher effort. Indeed, in this case the two forces, network entry and network influence, go in the same direction with efforts: lowering peripherals' (costly) effort favors their entry, and highering central agents favors their influence. One spectacular case is $t = qu$. In this case, the network does not affect rewards, while it still shapes efforts.

The optimal contract performance on subnetwork Q is measured by

$$\hat{x}_Q(t) = ab'_Q + \sqrt{b'_Q}\sqrt{a^2 b'_Q + 2(t - qu)} \tag{15}$$

We deduce from equation (15) that *aggregate effort at the optimum is increasing in aggregate centrality.* Moreover, not surprisingly, high reservation utilities dampen the aggregate at optimum since participation constraints are demanding.

We turn to the second principal's task. Equation (15) explains how the principal selects the optimal subnetwork $\hat{Q}(t)$. First, a direct implication is that, among all subnetworks associated with same cardinality, the preferred network for the principal maximizes the sum of centralities. Second, this equation enables to compare subnetworks with different cardinalities. In particular, there is network exclusion when the optimal set $q < n$. The reason for exclusion is linked to the cost of integrating agents on the network.

The impact of reservation utilities on exclusion can be observed under low level of interaction. When δ tends to 0, there is almost no interaction, and all offers contain approximately the same reward ($\simeq \frac{t}{q}$). Yet, the number $\hat{q}$ can be smaller than n when u is large enough. We illustrate further exclusion issue by examining two polar network structures. First, we consider G as a complete network with n agents. Any subnetwork Q of network G is a complete network of size q, and thus aggregate centrality on this subnetwork is equal to $b'_Q = \frac{q}{1-2\delta(q-1)}$. For low budget t and large reservation utility u there exist two values $\underline{q}, \bar{q}$ such that Condition (12) holds for any $q \in [1, \underline{q}[\cup]\bar{q}, +\infty[$ ($\underline{q}$ and $\bar{q}$ are the roots of the equation $4u\delta q^2 + (1 - 2u(1 + 2\delta) - 4t\delta)q + 2t(1 + 2\delta) = 0$). This shows possible network exclusion: it can be seen that for $q < \underline{q}$, $x(q)$ is either increasing or is single-peaked with a maximum, while for $q > \bar{q}$, $x(q)$ is either increasing or is single-peaked with a minimum. Take for instance $a = 1, u = 3, \delta = 0.03, t = 12$. Then we get $\underline{q} = 6, \bar{q} = 13$, and the maximal component size compatible with centrality existence is $q = 17$. Basically, the optimal size maximizes the

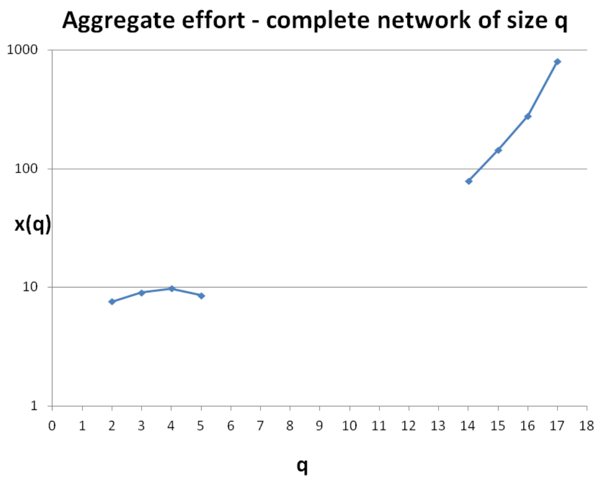


Figure 3: Finding the optimal subnetwork from the complete network.

depicted function under the constraint $\hat{q} \leq n$. When $n \in [1, 4] \cup [14, 17]$, we get $\hat{q} = n$, which means that the targetted group has maximal size. When $n \in [5, 13]$ we find $\hat{q} = 4$, meaning that netork exclusion obtains. Note that exclusion is not only due to the feasibility constraint, it also results from a strategic trade-off.

The above general picture is generic. To give a flavor, consider the star network. For $n = 4$, and G the four-star network with agent 1 as central, set $a = 1, t = 10, \delta = 0,1$. For $u \leq 3$, we find $q = n$, while for $u = 3.1$ we find $\hat{Q} = \{12, 13\}$. Second, for large stars, contract existence is also a matter, as confirmed in figure 4. As for the complete network, there exists an interval

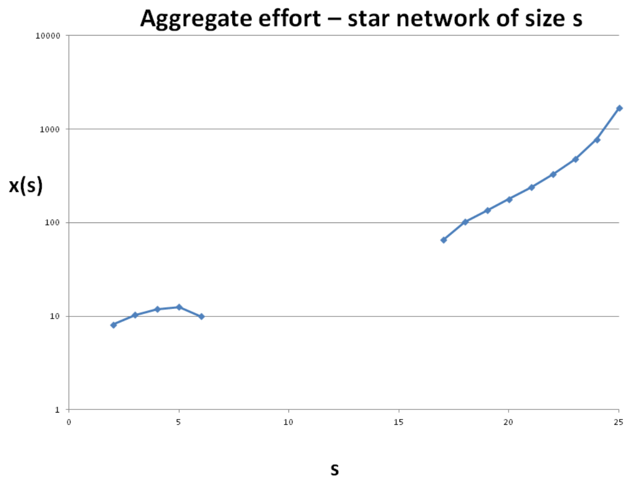


Figure 4: Finding the optimal group size on the star network.

where contract existence does not hold (the bounds of the interval, $\underline{q}$ and $\bar{q}$, are the roots of the equation $8u\delta^2 q^2 + (1 + 4\delta - 2u(1 + 4\delta^2) - 8t\delta^2)q + 2t(1 + 4\delta^2) = 0)$.

To finish, the principal has to determine the optimal budget $\hat{t}$ through equation (6). The optimal budget makes the balance between the rate at which increased budget improves objective function F through its induced effect on efforts and the direct cost effect. The optimal budget is increased for functions F with larger slopes and vice-versa. At the limit, when F is null, the optimal efforts are those minimizing the budget t. This is done by putting all efforts to the efficient allocation X^e, which entails a maximal ex post aggregate utility (net of transfers) equal to $\frac{a^2}{2} b'_{\hat{Q}}$, and thus establishing $\hat{t} = u^0 - \frac{a^2}{2} b'_{\hat{Q}}$.

Remark. In the present extension, the principal cannot modify the network. However, in science park development for instance, the park owner can design the location of firms as he wishes, and more generally invest in the general park ergonomy, thus indirectly impacting the intensity of communications and synergies between firms. This aspect calls naturally for network design, in a setting with costly link formation. Assume for simplicity that total link formation cost is an increasing function $c(l)$ where l is the number of existing links in the building network. As shown in equation (15), contract performance is increasing with b'_Q. From Belhaj et al. (2013), we deduce that when contracting with network entry, among all networks with fixed number of links, a network maximizing contract performance is a Nested-Split Graph. In particular, when $c()$ is concave, the principal will form either the empty or the complete network (still selecting a possible subgroup of agents to join the network, as shown before). Alternatively, when $c()$ is convex, more complex NSGs may be optimally designed (see the cited paper for more details).

5 Conclusion

This paper presented a situation where a principal contracts with a set of agents organized in a network of local complementarities, in the purpose of maximizing the sum of agents' efforts. We studied how the network structure affects optimal contracts in a static framework with observable efforts. Our study mainly stresses that optimal contracts do take into account agents' positions on the network. Optimal contracts reward all agents and, taking account of the heterogenous influence of agents on the network, rewards are aligned with agents' centralities. Our study also stresses that contract enforceability is key to understand trading. When contracts are enforceable, we identified conditions under which the principal, raising contingent contracts, would tax some agents, whom we identified through specific centrality index. Contrastingly, taxation is not optimal under non enforceable contracts, and we showed that all agents are proposed a positive transfer. We studied (excess-effort) linear contracts, and we found that per unit returns are homogenous across agents. We examined the case where the principal can establish a single contract among all agents. We showed that with linear contracting, the agent to target maximizes the inter-centrality index (which is known to play a crucial role in key-player analyzes); with non enforceable contracts, the principal should target the agent maximizing the un-weighted Bonacich centrality; with enforceable contracts, the optimal agent to target is budget-dependent. Last, we studied network entry under enforceable contracts, and showed that the principal has to take care that guaranteeing the participation of the least central agents could be more costly than that of central agents. This may lead first to exclude a subset of the population, and second to offer larger rewards to the least central agents.

This model raises other issues. First, it would be interesting to extend this analysis to substitute interaction. In particular, if our study formally holds for all types of interaction under low level of interaction, the general analysis is challenging. Notably one has to take into account that the principal may be willing to pay for setting effort to zero. Second, it would be interesting to understand full implementation. Under this requirement, the optimal contract is of *divide-and-conquer* type (Seagal [2003]), but not much is known about the impact of the network structure on its design. Third, it would be important to understand non-discriminating contracts. One possibility is to consider the case where the principal, knowing the network, proposes a menu of contracts and to examine conditions under which agents choose the contract corresponding to their positions on the network. Fourth, in many circumstances, the principal is imperfectly informed of either efforts and/or the detail of the network. This opens new insight. In particular, linked agents on the network may collude, which my complicate the task of the principal. Last, network design is also a means to increase the sum of agents' efforts, and it would therefore be interesting to know more on optimal network design in this context. This is left for future research.

6 Proofs

This section collects all proofs.

Proof of Proposition 2. See the proof of Proposition 8. ■

Proof of Proposition 3. We observe that when $F()$ is concave or linear, the objective function is linear while constraints are convex, so there is a single maximum to this problem. We examine the subproblem where t is kept fixed. Reminding that, for all k, $x_k^{BR} = a + \delta y_k$, we note that

$$u_k(x_k^{BR}, y_k) - u_k(x_k, y_k) = \frac{1}{2}\left(x_k^{BR}\right)^2 - \left(x_k x_k^{BR} - \frac{1}{2}x_k^2\right)$$

That is,

$$u_k(x_k^{BR}, y_k) - u_k(x_k, y_k) = \frac{1}{2}\left(x_k - x_k^{BR}\right)^2$$

Let $\psi_k = \sqrt{2t_k}$ for all k. The solution of the program is such that the incentive constraint is binding. Thus, for all k, all X on the incentive constraint,

$$x_k - x_k^{BR} = \psi_k$$

or in matrix form

$$X - X^{BR} = \Psi$$

That is, with $A = a\mathbf{1}$,

$$X = M(A + \Psi) \tag{16}$$

We note that

$$X - X^{BR} = (I - \delta G)(X - B_A) \tag{17}$$

The total budget constraint is written

$$(X - X^{BR})^T (X - X^{BR}) = 2t$$

That is, given equation (17), we have to solve

$$\max_{\substack{\{(t_i, x_i)\}_{i\in N} \\ \text{s.t. } (X-B_A)^T(I-\delta G)^T(I-\delta G)(X-B_A)=2t}} \mathbf{1}^T X$$

We define $Z = (I - \delta G)(X - B_A)$ (that is $X = MZ + B_A$). Since $Z = \Psi$, we have $Z > 0$ if and only if $T > 0$. Noting that $\mathbf{1}^T MZ = (M^T \mathbf{1})^T Z$, and reminding that $M^T = M$, we obtain (ignoring the term of the objective function which is independent of Z),

$$\max_{\substack{\{(t_i, x_i)\}_{i\in N} \\ \text{s.t. } Z^T Z=2t}} (M\mathbf{1})^T Z$$

Geometrically, we have select on the circle of radius $2t$ the point Z such that the projection on $M\mathbf{1}$ is maximal. This means that the optimal vector $\hat{Z}$ is co-linear to vector $M\mathbf{1}$, that is, there exists a real number $\beta > 0$ such that

$$\hat{Z} = \beta M\mathbf{1}$$

As $\Psi = Z$, we have $t_i = \frac{z_i^2}{2}$, we get

$$\hat{t}_i = \frac{\beta^2}{2} b_i^2$$

Reminding that $\|\hat{\Psi}\|^2 = 2t$, we deduce that

$$\beta^2 = \frac{2t}{\|B\|^2}$$

Thus,

$$\hat{t}_i = \frac{t}{\|B\|^2} b_i^2$$

Also, $X = B_A + MZ$, hence

$$\hat{X} = B_A + \beta M^2 \mathbf{1}$$

and plugging β, we get

$$\hat{X} = B_A + \frac{\sqrt{2t}}{\|B\|} MB$$

and

$$\hat{x} - x^0 = \sqrt{2t}\|B\|$$

and we are done. ■

Proof of Lemma 1. Consider any network G which is not regular. Suppose that the optimal allocation is such that all agents receive a positive reward for any arbitrarily low budget, i.e. $\bar{t}(G, \delta) \leq 0$. This means that for every agent i we have

$$\bar{t}_i(G, \delta) \leq 0 \tag{18}$$

However, summing all threshold budgets, we observe that

$$\sum_{i=1}^{n} b_i' \cdot \sum_{k=1}^{n} b_k^2 = \sum_{i=1}^{n} b_i^2 \cdot \sum_{k=1}^{n} b_k'$$

That is,

$$\sum_{i=1}^{n} \bar{t}_i(G, \delta) = 0 \tag{19}$$

From this latter global balanced condition (19), we conclude that $\bar{t}_i(G, \delta) = 0$ for all $i \in N$. Now, for any pair of agents (i, j) who get a different Bonacich centrality, we have

$$(b_i' - b_j') \cdot \sum_{k=1}^{n} b_k^2 \neq (b_i^2 - b_j^2) \cdot \sum_{k=1}^{n} b_k'$$

(except perhaps for values of δ of null measure). This shows that $\bar{t}(G,\delta) > 0$ if and only if G contains at least two classes of agents with distinct centralities. $\diamond$

Proof of Proposition 4. We have

$$x^*(\Gamma) - x^0 = \sum_{i=1}^{n} \gamma_i b_i = \Gamma^T B$$

where b_i is agent i's un-weighted centrality. The principal's problem is written as follows:

$$\max_{\substack{\Gamma \in \mathbb{R}^{+n} \\ \Gamma^T M \Gamma = t}} (M\mathbf{1})^T \Gamma$$

We define $\breve{S} = \{i \in N, \breve{\gamma}_i > 0\}$ as the set of agents with positive rewards. To explain the role played by the nonnegativity constraint, suppose that we do not take the condition $\Gamma \geq \mathbf{0}$ into account. Geometrically, we have then to maximize the projection of a vector on B over an ellipsoïd. However, with symmetric adjacency matrices $G^T = G$, we have $\breve{S} = N$. Indeed, as M is symmetric, it is definite positive and there exists a square root matrix of M (all the eigenvalues of a symmetric invert-matrix are real and positive - see for instance Poole and Boullion (1974), Theorem 2.1 p. 420 -, so M is definite positive, and there exists a matrix Q where $M = Q^T Q$), and the constraint can be written as a norm. That is, the ellipsoïd is a circle, so the optimal direction of Γ is that of $M\mathbf{1}$. This shows that the solution satisfies $\breve{S} = N$. Given this, the Lagrangian L is written:

$$L(\Gamma, \lambda) = \sum_{i=1}^{n} \gamma_i b_i + \lambda\left(t - \sum_{j=1}^{n}\sum_{k=1}^{n} \gamma_j \gamma_k m_{jk}\right) + \sum_{i=1}^{n} \nu_i \gamma_i$$

where $\nu_i = 0$ if and only if $\gamma_i > 0$. We apply the first order conditions w.r.t. γ_i for all $i \in N$. We get $\frac{\partial L}{\partial \gamma_i} = 0$, $i \in N$, if and only if

$$\frac{b_i}{\breve{\lambda}} = \frac{\partial}{\partial \gamma_i}\left(\breve{\gamma}_i^2 m_{ii} + \breve{\gamma}_i \sum_{k \neq i} m_{ik}\gamma_k + \breve{\gamma}_i \sum_{j \neq i} m_{ji}\gamma_j + \sum_{k \neq i}\sum_{j \neq i} m_{jk}\gamma_k\gamma_j\right)$$

This means

$$\frac{B}{\breve{\lambda}} = 2M\breve{\Gamma}$$

that is, since $B = M\mathbf{1}$,

$$\breve{\Gamma} = \frac{1}{2\breve{\lambda}}\mathbf{1}$$

To finish, given that $\breve{\Gamma}^T M \breve{\Gamma} = t$, and reminding that $\mathbf{1}^T M \mathbf{1} = b$, we derive

$$\breve{\lambda} = \frac{1}{2}\sqrt{\frac{b}{t}}$$

It follows that

$$\breve{\Gamma} = \sqrt{\frac{t}{b}} \cdot \mathbf{1} \tag{20}$$

Furthermore, the reward equation is written

$$\breve{t}_i = \breve{\gamma}_i \cdot (\sum_j m_{ij}\breve{\gamma}_j) \tag{21}$$

Replacing equation (20) into equation (21), we get

$$\breve{t}_i = \frac{b_i}{b} \cdot t$$

Last, regarding aggregate effort, we get

$$\breve{x} - x^0 = \sum_{i=1}^{n} \breve{\gamma}_i b_i \tag{22}$$

That is, as incorporating equation (20) into equation (22), we obtain

$$\breve{x} - x^0 = \sqrt{tb}$$

and we are done. ■

Proof of Proposition 5. Equilibrium efforts with linear contract are written

$$X = X^0 + M\Gamma \tag{23}$$

with $\Gamma = (0, \cdots, 0, \gamma_i, 0, \cdots, 0)$, agent i being the targetted agent; and the variation of aggregate effort is written:

$$x - x^0 = \sum_k \sum_j m_{kj}\gamma_j = \sum_j \gamma_j b_j = \gamma_i b_i \tag{24}$$

Also, the budget constraint is written

$$t = \gamma_i(x_i - x_i^0)$$

Suppose without loss of generality that agent i is the optimal target for fixed budget t, thus $\breve{\gamma}_i(t) > 0$. From equation (23) we obtain

$$\breve{x}_i(t) - x_i^0 = m_{ii}\breve{\gamma}_i(t) \tag{25}$$

Plugging equation (25) into the budget constraint, we find

$$\breve{\gamma}_i(t) = \sqrt{\frac{t}{m_{ii}}} \tag{26}$$

Plugging equation (26) into equation (25), we obtain

$$\breve{x}_i(t) - x_i^0 = \sqrt{t\, m_{ii}}$$

while plugging equation (26) into equation (24), we obtain

$$\breve{x}(t) - x^0 = \sqrt{t} \cdot \frac{b_i}{\sqrt{m_{ii}}} = \sqrt{t\, I_i}$$

and we are done. ■

Proof of Proposition 6. We consider the optimal non enforceable contract with fixed budget t. Suppose that agent i is proposed the principal's offer (x_i, t). Let $\mathbf{1}_i = (0, 0, ..., 0, 1, 0, ..., 0)$ at with 1 at position i. Agent i's participation constraint imposes

$$\hat{x}_i - \delta y_i = a + \sqrt{2t}$$

For convenience, let $\hat{X}(t)$ denote the profile containing the optimal effort $\hat{x}_i(t)$ proposed by the principal to agent i, and other coordinates represent the best-response efforts of other agents on the network. Since other agents play their best-response, this vector solves:

$$(I - \delta G)\hat{X}(t) = a\mathbf{1} + \sqrt{2t}\mathbf{1}_i$$

That is, letting M_i denote column i in matrix M,

$$\hat{X}(t) - X^0 = \sqrt{2t}M_i$$

and thus we find $\hat{x}_i(t) = (a + \sqrt{2t})\, m_{ii}$ and $\hat{x}(t) - x^0 = \sqrt{2t}\, b_i$. ■

Proof of Proposition 7. We consider the optimal enforceable contract with fixed budget t. Suppose that agent i is the target. The principal will be able to raise agent i's effort to level $x_i = b_i + \alpha$, $\alpha > 0$, with her budget until $u_i(x_i, y_i) = u_i^0$, and taking into account that other agents play their best-response. This problem is equivalently studied through the system

$$(I - \delta G)X = \mathbf{1} + \alpha E \tag{27}$$

where profile $E = (0, \cdots, 0, \frac{1}{m_{ii}}, 0, \cdots, 0)$. Indeed, system (27) is also written

$$X = B + \alpha M E$$

which entails $x_i = b_i + \alpha$. We note also that, when putting x_i to $b_i + \alpha$, we find $x_j = b_j + \alpha \frac{m_{ji}}{m_{ii}}$. Summing all efforts, we get

$$x = b + \alpha \frac{b_i}{m_{ii}}$$

Hence, the optimal target maximizes the quantity $\alpha \frac{b_i}{m_{ii}}$, where α solves $u_i(x_i, y_i) = u_i^0$; that is, $x_i^2 - 2x_i(a + \delta y_i) - 2(t - u_i^0) = 0$. Replacing x_j by $b_j + \alpha \frac{m_{ji}}{m_{ii}}$ for all j in u_i, α solves the second-order equation $A_i \alpha^2 + B_i \alpha + C_i = 0$, where we define $A_i = 1 - \frac{2\delta(GM)_{ii}}{m_{ii}}$, $B_i = b_i - 1 - \delta(GB)_i - \delta(GM)_{ii} \frac{b_i}{m_{ii}}$, $C_i = -2(t - b_i^2) - 2b_i(1 + \delta(GB)_i)$. The result follows. Note that by the presence of budget t in C_i, the optimal target is budget-dependent. ■

Proof of Proposition 8. The program of the optimal contingent contract on a fixed network and the program of optimal contract with network entry are isomorphic, both correspond to a model where non contracting agents have an exogenous reservation utility. We develop a unified proof with $u_i^0 \geq 0$ as agent i's reservation utility and $u^0 = \sum_{i \in N} u_i^0$. For Proposition 2, we take $u_i^0 = \frac{a^2}{2} b_i^2$, while for Proposition 8, we take $u_i^0 = u$ for all i.

By concavity of utilities, the problem admits a single maximum (the objective function is linear and constraints are convex). Consider a set of agents S compatible with contract sustainability. Given that the global maximum is unique, these conditions will be necessary and sufficient.

The Lagrangian associated with program (4) is written

$$L = \sum_{k \in N} x_k + \sum_{k \in N} \lambda_k \left(u_k(x_k, y_k) + t_k - u_k^0 \right) + \mu \left(t - \sum_{k \in N} t_k \right)$$

where the participation constraint of agent k is associated with weight λ_k. Suppose that every agent receives a positive reward. From derivatives of the

Lagrangian w.r.t. t_i, it follows that $\lambda_i = \lambda$ for all i. The derivative of the Lagrangian w.r.t. x_i entails (remind that $a_i = a$ for all i):

$$x_i - 2\delta y_i = a + \frac{1}{\lambda} \tag{28}$$

That is, defining $b'_i = b_i(G, 2\delta)$ for convenience (this centrality is well-defined as $2\delta < \mu(G)$), the equation (28) is also written

$$x_i = \left(\frac{1 + a\lambda}{\lambda}\right) b'_i \tag{29}$$

Agent i's participation constraint is written

$$2t_i = 2u_i^0 + \left(x_i - 2\delta y_i - 2a\right) x_i \tag{30}$$

Plugging equation (28) into (30), we get

$$2t_i = 2u_i^0 + \left(\frac{1 - a\lambda}{\lambda}\right) x_i \tag{31}$$

Plugging now equation (29) into equation (31), we find

$$t_i = u_i^0 + \frac{1}{2}\left[\frac{1 - a^2\lambda^2}{\lambda^2}\right] b'_i \tag{32}$$

We have now to explicit the expression $\frac{1-a^2\lambda^2}{\lambda^2}$. Summing rewards over all agents, we obtain

$$t = u^0 + \frac{1}{2}\left(\frac{1 - a^2\lambda^2}{\lambda^2}\right) b'$$

Rearranging, we obtain

$$\frac{1}{2}\frac{1 - a^2\lambda^2}{\lambda^2} = \frac{t - u^0}{b'} \tag{33}$$

Incorporating equation (33) into (32), we deduce that

$$\hat{t}_i = u_i^0 + \left(\frac{t - u^0}{b'}\right) b'_i$$

We define $\bar{t}_i = u^0 - \frac{b'}{b'_i} u_i^0$. Transfers are positive whenever $t > \bar{t}_i$. In total, defining $\bar{t} = \max_{i \in N} \bar{t}_i$, all rewards are positive if and only if $t > \bar{t}$. Finally, agent i's optimal effort $\hat{x}_i$ is easily characterized. From equation (33) we get λ, which we incorporate into equation (29) to obtain

$$\hat{x}_i = \left(a + \sqrt{a^2 + 2\Big(\frac{t - u^0}{b'}\Big)} \right) b'_i$$

and we are done. ■

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