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Flip-flopping and Endogenous Turnout

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Abstract

I consider an electoral competition model where each candidate is associated with an exogenous initial position from which she can deviate to maximize her vote share, a strategy known as flip-flopping. Citizens have an intrinsic preference for consistent candidates, and abstain due to alienation, i.e. when their utility from their preferred candidate falls below a common exogenous threshold (termed the alienation threshold). I show how the alienation threshold shapes candidates' flip-flopping strategy. When the alienation threshold is high, i.e. when citizens are reluctant to vote, there is no flip-flopping at equilibrium. When the alienation threshold is low, candidates flip-flop toward the center of the policy space. Surprisingly, I find a positive correlation between flip-flopping and voter turnout at equilibrium, despite voters' preference for consistent candidates. Finally, I explore alternative models in which candidates' objective function differs from vote share. I show that electoral competition can lead to polarization when candidates maximize their number of votes.

Keywords: flip-flopping, turnout, electoral competition, alienation, polarization.
JEL Classification: D72, C72

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1 Introduction

In electoral competition, citizens evaluate candidates based on their platform as well as on other characteristics, such as consistency. Yet between two elections or even two temporal periods, a candidate can indeed change her public stance on a specific issue, or modify her political position. Termed «flip-flopping», this can be a beneficial political strategy: for example, a candidate may flip-flop to get closer to the median voter. However, flip-flopping is costly. Citizens may draw negative inferences about the character of the politician, like a perception of incompetence (Tomz and Van Houweling, 2012), and thus may have an intrinsic preference for candidates who do not flip-flop. Campaigning candidates therefore face a tradeoff between the costs and the benefits of flip-flopping (Hummel, 2010).

When voting is compulsory, candidates compare the benefits of getting closer to the median voter to the costs of flip-flopping. However, when voting is voluntary, candidates' tradeoff is more complex. The negative perception of flip-flopping may lead some citizens to abstain. On the other hand, a candidate may have incentives to flip-flop to convince possible abstainers (or those who might vote for her opponent) to vote for her. Thus, while flip-flopping can attract some abstainers, it can also induce abstention.

In this paper, I design an electoral competition model where two candidates maximize their vote share. Citizens associate each candidate with an exogenous initial platform, and each candidate is able to flip-flop. This model integrates abstention due to alienation. Citizens vote if their utility from their favorite candidate exceeds an exogenous threshold, defined within the paper as the alienation threshold. This parameter represents citizens' propensity to abstain. The utility a citizen derives from a candidate depends negatively on the distance between the citizen's ideal policy and the platform promoted by the candidate, but also on the extent to which the candidate diverges from her initial position (i.e. flip-flopping).

At equilibrium, flip-flopping is negatively correlated with the alienation threshold. A low alienation threshold means that « extreme voters » in each electorate would vote for their preferred candidate if she stuck to her initial position. Consequently, candidates have incentives to flip-flop toward the center of the policy space so as to increase their vote share without losing the adhesion of the extreme part of their electorate. When the alienation threshold is high, candidates do not flip-flop, since this would lose them more voters than they would gain.

I obtain divergence at equilibrium in most cases, but do not find polarization (which I define as flip-flopping toward the extremes). Candidates either stick to their initial position or flip-flop toward the center. Consequently, the threat of voter abstention hinders the convergence of platforms, consistent with the analysis of Oprea et al. (2024). I also find that candidates are more likely to flip-flop when initial positions are extreme.

Additionally, this paper shows that flip-flopping is positively correlated with voter turnout, involving two opposite effects. A higher alienation threshold makes citizens more likely to abstain (direct effect), but also reduces flip-flopping, thereby increasing citizens' utility and positively impacting turnout (indirect effect). I find that the direct effect dominates the indirect effect. Consequently, both flip-flopping and voter turnout are negatively correlated with the alienation threshold at equilibrium. Moreover, as candidates only flip-flop toward the center, there is also a negative correlation between divergence and turnout.

Finally, I consider alternative objective functions for candidates. The equilibrium remains unchanged when candidates maximize either their plurality or their probability of winning. However, in a less competitive environment where candidates maximize their number of votes, I obtain cases where candidates have incentives to polarize. In this context, the competition for the votes of centrist citizens is less intense, which can induce candidates to flip-flop toward the extremes in order to consolidate their electorate.

The paper is organized as follows. Section 2 is devoted to a review of the literature. In section 3, I introduce the main model, where candidates maximize their vote share. Results are presented in section 4. In section 5, I consider various objective functions for candidates, and section 6 concludes.

2 Related Literature

Endogenous Turnout. The introduction of abstention in the classical electoral competition framework established by Downs (1957) has been shown to have little effect on candidates' strategic behavior (Hinich and Ordeshook, 1969). However, numerous papers show that when abstention is introduced alongside other departures from the Downsian setting, we can observe divergence at equilibrium. This divergence can arise due to the threat of entry of a third candidate (Callander and Wilson, 2007), heterogeneity on non-policy characteristics (Adams and Merrill, 2003), or asymmetry of information between citizens on candidates' platform (Glaeser et al., 2005). My analysis will yield divergent equilibria without incorporating these features.

An ongoing issue in the literature is how polarization relates to voter turnout. According to the analysis of Oprea et al. (2024), the ability of extreme citizens to threaten abstention induces polarization. This finding aligns with Grillo (2023). In Grillo's work, citizens derive a single-peaked and convex utility function (with respect to candidates' platform) and turnout is determined at the group level. The analysis reveals a positive correlation between polarization and citizens' propensity to abstain. I find a similar result in my paper within a different setting. However, I use the term "divergence" instead of "polarization", which I reserve for flip-flopping toward the extremes.

Additionally, Grillo (2023) reports ambiguous findings regarding the relationship between divergence and turnout, whereas my results demonstrate a positive correlation between the two.

Flip-flopping. Empirical and experimental studies in the literature have examined flip-flopping. DeBacker (2015) provides evidence that this strategy is electorally costly, estimating a dynamic model of candidate positioning. This electoral cost can be explained by citizens' negative perception of flip-flopping. Tomz and Van Houweling (2012) use survey-based experiments to show that citizens draw negative inferences about a flip-flopping politician's character. Other experimental work indicates that citizens react differently to flip-flopping depending on the degree of complexity attributed to the issue (Doherty et al., 2016) and on elite communications (Robison, 2017). Moreover, Tavits (2007) shows that policy shifts on principled issues are more costly than flip-flopping on pragmatic issues.

Flip-flopping can also take different forms, as shown by Tella et al. (2023) from U.S and French data. Between the two rounds of an election or between primaries and a general election, candidates can actually adjust their position toward that of their opponent. However, they also adapt the level of complexity of the language they use and diversify the set of topics they cover.

Theoretically, some papers have studied electoral competition when flip-flopping is costly. Hummel (2010) highlights a post-primary moderation effect in a dynamic model with both primaries and a general election. During primaries, voters elect a candidate who closely aligns with their preferences (sufficiently extreme) without compromising the candidate's chances of winning the general election[1]. Then, candidates flip-flop toward the center, solving a tradeoff between the costs and the benefits of flip-flopping. This post-primary moderation effect also appears in Agranov (2016) and in Fournier et al. (2023). In the first paper, voters infer the candidates' ideology through their campaign, while in the second, flip-flopping also incurs a financial cost for the candidates.

Finally, Aragonès and Xefteris (2022) study flip-flopping in a static model of electoral competition (i.e. initial positions are exogenous), where candidates do not have the same valence. When valence asymmetry is not too large, the candidate with the highest valence flip-flops more than her opponent. Otherwise, it is the disadvantaged candidate who flip-flops more in order to survive. Like these authors, I study flip-flopping in a static model, adding a new feature likely to have an impact on candidates' strategy (abstention due to alienation). However, I do not consider valence asymmetry.

3 Model

Two candidates $j \in \{1, 2\}$ compete on a uni-dimensional policy space $X = [0, 1]$. Each candidate j is associated with an exogenous platform $q_j \in [0, 1]$, which can be interpreted as the candidate's initial position (the one promoted during primaries for instance). I assume that initial positions are symmetric with respect to $\frac{1}{2}$, so that no candidate is disadvantaged *ex ante*: $q_1 = 1 - q_2$. Without loss of generality, let $q_1 < q_2$. Candidates can adjust their position during the campaign, choosing a platform $p_j \in [0, 1]$.

Definition 1. *Flip-flopping is the distance between p_j and q_j, i.e. $|p_j - q_j|$.*

Preferences. I consider a unit mass of citizens, identified with their preferred policy θ uniformly distributed on X. The utility of citizen θ for candidate j is negatively affected by the distance between the citizen's ideal platform and that promoted by the candidate, as well as by flip-flopping:

$$U_\theta^j(p_j) = -|p_j - \theta| - \frac{\gamma}{2}(p_j - q_j)^2.$$

The cost of flip-flopping is an increasing and convex function $\frac{\gamma}{2}(p_j - q_j)^2$, where $\gamma > 0$

[1]Note that flip-flopping can also be studied as a reaction to a change in the state of the world, looking at the tradeoff of an incumbent both policy-motivated and concerned for her reputation (Andreottola, 2021).

represents the relative weight of flip-flopping in citizens' utility. Each additional unit of flip-flopping increases the disutility to citizens.

Citizens are perfectly informed about the initial position (q_j) and the platform (p_j) of each candidate. Moreover, I assume full commitment.

Endogenous Turnout. The model incorporates abstention due to alienation. A citizen prefers to vote rather than to abstain if and only if her utility from her preferred candidate is higher than an exogenous alienation threshold denoted by $\overline{u} < 0$. In this case, she votes for her preferred candidate and randomizes if she is indifferent between the two candidates.

Each candidate has an attraction interval I_j, which is the set of citizens that prefer to vote for candidate j rather than to abstain (i.e $U_\theta^j(p_j) \geq \overline{u}$).[2] It follows that a citizen θ prefers to vote rather than to abstain if and only if $\theta \in I_1 \cup I_2$.

Lemma 1. *If $(p_j - q_j)^2 \geq -\frac{2\overline{u}}{\gamma}$, then $I_j = \emptyset$. Otherwise, $I_j = [\theta_j^{inf}(p_j), \theta_j^{sup}(p_j)]$, with*

$$\begin{cases} \theta_j^{inf}(p_j) = p_j - f(p_j - q_j), \\ \theta_j^{sup}(p_j) = p_j + f(p_j - q_j), \end{cases}$$

where $f(x) = -\overline{u} - \frac{\gamma}{2}x^2$.

Proof. See Appendix A.1 □

The attraction interval I_j is symmetric around p_j, and its length is denoted by $\ell(I_j)$.

$$\ell(I_j) = 2f(p_j - q_j). \quad (1)$$

Both flip-flopping and the alienation threshold $\overline{u}$ negatively affect the length of I_j. Citizens are more likely to abstain when their alienation threshold is high and when flip-flopping increases.

Note that $\theta_j^{sup}(p_j)$ is increasing in p_j when $p_j \leq q_j + \frac{1}{\gamma}$, but becomes decreasing when $p_j \geq q_j + \frac{1}{\gamma}$. This means that, starting from the initial position, when a candidate adjusts her position in one direction, she attracts more voters in that direction, unless she moves too much. In the latter case, the candidate becomes less attractive to the voters she is getting closer to, as the quadratic electoral cost of flip-flopping outweighs the linear policy gains.

[2] Note that I_j is not necessarily a subset of the interval $[0, 1]$. Consequently, there can be some subsets of I_j where there is no citizen.

Electorates. Having determined which citizens will turn out to vote, we now focus on their voting choices.

Lemma 2. *Candidate $j \in \{1,2\}$ is unanimously preferred by citizens if and only if* $\phi_j(p_1,p_2) := \frac{\gamma}{2}\Big((p_{-j}-q_{-j})^2-(p_j-q_j)^2\Big) > |p_2-p_1|$.

Proof. See Appendix A.2. □

The result identifies a situation where one candidate j is unanimously preferred, which occurs when she flip-flops significantly less than her opponent.

There is also a particular case where a non-empty interval of citizens are indifferent between the two candidates. If those citizens vote, they are equally likely to vote for candidate 1 or for candidate 2, each with a probability of $\frac{1}{2}$.

Lemma 3. *If $\phi_j(p_1,p_2) = p_1 - p_2$ or $\phi_j(p_1,p_2) = p_2 - p_1$, then there is a non-empty interval of citizens θ such that $U^1_\theta(p_1) = U^2_\theta(p_2)$.*

Proof. See Appendix A.3. □

The following condition ensures that neither Lemma 2 nor Lemma 3 applies, so that electorates are separated by a single indifferent citizen.

Condition 1. *$\phi_j(p_1,p_2) \in (p_j - p_{-j}, p_{-j} - p_j)$, with j such that $p_j \leq p_{-j}$.*

I show in Appendix A.5 and A.8 that Condition 1 holds at equilibrium, except in the case of full convergence toward the center, i.e. $p_1^* = p_2^* = \frac{1}{2}$, where every citizen is indifferent between the two candidates.

Also, I show that at equilibrium, $p_1 \leq p_2$. I hereby state the following result in this specific case.

Proposition 1. *If $p_1 \leq p_2$ and if Condition 1 holds, a citizen θ:*

- *votes for candidate 1 if and only if $\theta \in I_1$ and $\theta < \overline{\theta}(p_1,p_2)$,*
- *votes for candidate 2 if and only if $\theta \in I_2$ and $\theta > \overline{\theta}(p_1,p_2)$,*
- *abstains otherwise,*

with $\overline{\theta}(p_1,p_2) := \dfrac{\theta_1^{sup}(p_1) + \theta_2^{inf}(p_2)}{2} \in (p_1,p_2)$.

Proof. See Appendix A.4 □

Citizens are divided into two groups by a single indifferent citizen, denoted by $\overline{\theta}(p_1, p_2)$. Every citizen θ such that $\theta < \overline{\theta}(p_1, p_2)$ prefers candidate 1 over candidate 2, i.e. $U^1_\theta(p_1) > U^2_\theta(p_2)$. Similarly, every citizen $\theta > \overline{\theta}(p_1, p_2)$ prefers candidate 2.

We can thus deduce the quantity of votes for each candidate denoted by $V_j(p_1, p_2)$:

$$\begin{cases} V_1(p_1, p_2) = \ell(I_1 \cap [0, \overline{\theta}(p_1, p_2)]), \\ V_2(p_1, p_2) = \ell(I_2 \cap [\overline{\theta}(p_1, p_2), 1]). \end{cases}$$

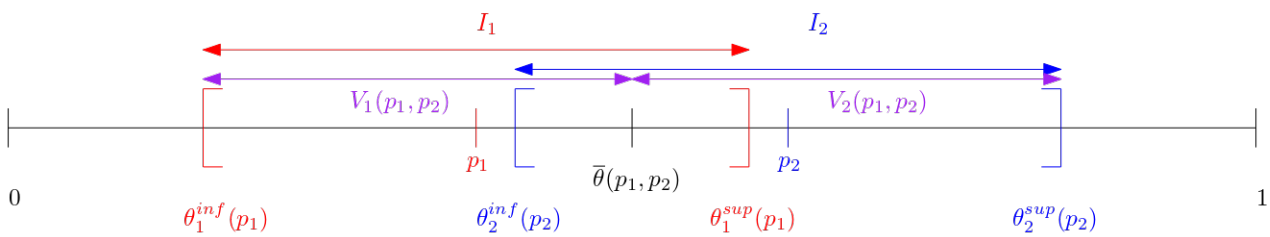

Figure 1: Example of policy space

Candidates' behaviour. Candidates maximize their vote share denoted by $VS_j(p_1, p_2)$.

$$VS_j(p_1, p_2) = \frac{V_j(p_1, p_2)}{V_j(p_1, p_2) + V_{-j}(p_1, p_2)}.$$

A best response p_j^* against p_{-j} is:

$$p_j^* \in \arg\max_{p_j \in [0,1]} \{VS_j(p_1, p_2)\}.$$

This objective function is justified in many contexts. For instance, in the case of elections using proportional representation, the parties' number of seats depends crucially on their vote share. Also, in single-winner elections, candidates have incentives to win with a higher vote share because this will increase their legitimacy.[3] Finally, note that, as argued in Callander and Carbajal (2022), vote share maximization can be micro-founded by assuming some citizens vote randomly. Alternative objective functions will be studied in Section 5.

[3] As mentioned in Herrera et al. (2014), even in non-proportional systems, vote share influences the majority party's power.

In the following section, I determine the pure strategy Nash equilibria of this game and then discuss the qualitative implications.

4 Results

4.1 Equilibrium

The following theorem fully characterizes the unique equilibrium of the game.

Theorem 1. *The game admits a unique equilibrium, which is symmetric and denoted by* $(p_1^*, 1-p_1^*)$ *with:*

$$p_1^* = \begin{cases} \min\{q_1 + \frac{1}{\gamma}, \frac{1}{2}\} & \text{if } \overline{u} < \overline{u}_0, \\ q_1 + \frac{-1+\sqrt{1-2\gamma(q_1+\overline{u})}}{\gamma} & \text{if } \overline{u} \in [\overline{u}_0, -q_1), \\ q_1 & \text{if } \overline{u} \geq -q_1, \end{cases} \tag{2}$$

with $\overline{u}_0 := \max\{-q_1 - \frac{3}{2\gamma}, -\frac{\gamma}{2}(\frac{1}{2} - q_1)^2 - \frac{1}{2}\}$.

Proof. See Appendix A.5 □

Figure 2 illustrates Theorem 1, graphically representing equilibrium policies and of attraction intervals for each value of the alienation threshold, given γ and q_1.[4] Three different regions of the alienation threshold appear, each associated with a particular equilibrium configuration. I will analyze this graph from right to left.

[4] I let $q_1 \in (\frac{1}{4}, \frac{1}{2})$, so that the case where $I_1 \cap I_2 \neq \emptyset$, $I_1 \subset X$ and $I_2 \subset X$ is represented in the figure. Moreover, to ensure readability, Figure 2 represents the case where $-q_1 - \frac{3}{2\gamma} > -\frac{\gamma}{2}(\frac{1}{2} - q_1)^2 - \frac{1}{2}$, which is equivalent to $\gamma > \frac{2}{1-2q_1}$, so that $p_1^* < \frac{1}{2}$ for all $\overline{u}$. Finally, $\theta_1^{inf}(p_1^*)$ and $\theta_2^{sup}(p_2^*)$ curves are dashed when they attain values not included in X, while $\theta_1^{sup}(p_1^*)$ and $\theta_2^{inf}(p_2^*)$ are dashed when the former becomes greater than the latter.

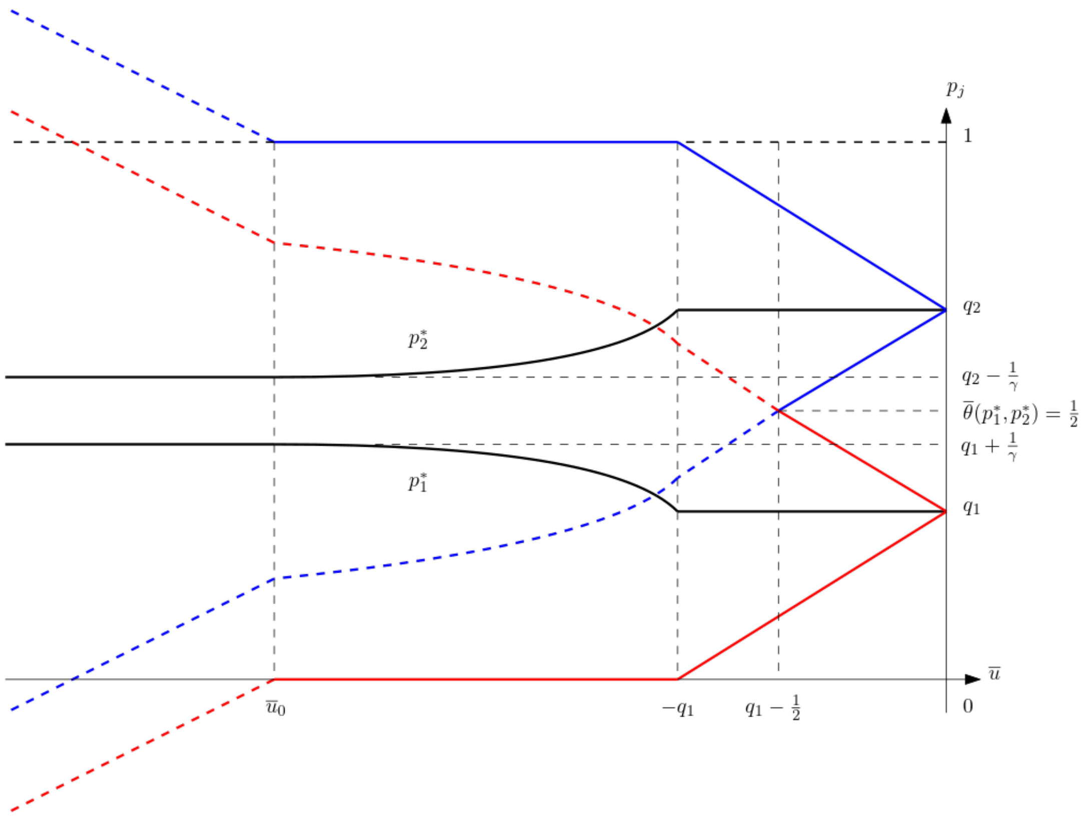

Figure 2: Equilibrium policies as functions of the alienation threshold.

For $\overline{u} \geq -q_1$, candidates have incentives not to deviate from their initial position. In this region of the alienation threshold, attraction intervals are subsets of the policy space. Therefore, flip-flopping would reduce candidates' attraction in the policy space, thus decreasing their vote share. Note that q_1 is the sole platform that guarantees candidate 1 a vote share of at least 1/2.

Then, if $\overline{u} \in [\overline{u}_0, -q_1)$, candidates flip-flop toward the center of the policy space at equilibrium. If they remained at their initial position, their attraction interval would fall partly outside of the policy space. In other words, the utility of citizens with an extreme ideal platform (0 or 1) would be higher than the alienation threshold. Candidates thus have incentives to flip-flop either to attract new voters at the center of the policy space, or to convince voters from the opposing electorate to change their vote, without losing the extreme part of their own electorate. They deviate from their initial position until their attraction interval falls entirely within the policy space. The platform $p_1^* = q_1 + \frac{-1+\sqrt{1-2\gamma(q_1+\overline{u})}}{\gamma}$ is the sole policy greater than q_1 such that $\theta_1^{inf}(p_1) = 0$. It follows from the previous paragraph that there is no additional profitable deviation.

If $\overline{u} < \overline{u}_0$, then candidates do maximum flip-flopping toward the center, and p_1^* is constant with respect to the alienation threshold. If the relative weight of flip-flopping in citizens' utility (represented by the parameter γ) is high enough[5], then $p_1^* = q_1 + \frac{1}{\gamma}$. Candidates do not flip-flop by more than $\frac{1}{\gamma}$, since this would deter the citizens they are targeting from voting for them, as explained in the previous section. If γ is low enough, then $p_1^* = p_2^* = \frac{1}{2}$. Candidate 1 does not have incentives to choose a strategy $p_1 > 1/2$, as citizens at the center of the policy space would change their vote, therefore diminishing candidate 1's vote share.

4.2 Qualitative Implications

I now discuss the implications of the equilibrium outcomes to investigate the interactions between the different features of the model.

Corollary 1. *For all* $\overline{u} < 0$ *and* $\gamma > 0$*, we have* $p_1^* \geq q_1$ *and* $p_2^* \leq q_2$*.*

Corollary 1 highlights the fact that there is no polarization. Candidates either stick to their initial platform, or flip-flop toward the center of the policy space. However, the following corollary indicates that in most cases, platforms diverge at equilibrium.

Corollary 2. *If* $\gamma > \frac{2}{1-2q_1}$*, then* $\forall \overline{u} < 0$, $p_1^* < \frac{1}{2}$ *(and* $p_2^* > \frac{1}{2}$*). Otherwise,* $p_1^* < \frac{1}{2}$ *for* $\overline{u} > -\frac{\gamma}{2}(\frac{1}{2} - q_1)^2 - \frac{1}{2}$*.*

If the relative weight of flip-flopping in citizens' utility is high enough, equilibrium platforms diverge for all values of the alienation threshold. Otherwise, equilibrium platforms converge only for citizens sufficiently inclined to vote.

Let $F(q_1) := \{\overline{u} \in \mathbb{R}_- \mid p_1^* \neq q_1\}$ be the set of values of the alienation threshold that leads candidates to flip-flop at equilibrium.

Corollary 3. *If* $q_1' < q_1$*, then* $F(q_1) \subset F(q_1')$*.*

This corollary indicates that when initial positions are more extreme, the range of alienation threshold values within which candidates flip-flop is larger. When q_1 is close[6] to 0, citizens who have an extreme ideal platform are highly satisfied with the initial

[5] If $\gamma > \frac{2}{1-2q_1}$, then $\overline{u}_0 = -q_1 - \frac{3}{2\gamma}$ and equivalently, $q_1 + \frac{1}{\gamma} < \frac{1}{2}$ as in Figure 2

[6] Symmetrically, q_2 is close to 1.

position of their favourite candidate. Candidates can thus flip-flop toward the center of the policy space without losing the votes of this electorate.

Corollary 4. *Flip-flopping is a monotonic non-increasing function of* $\overline{u}$ *and of* γ*.*

As figure 2 shows, flip-flopping is either decreasing or constant with respect to the alienation threshold. The more satisfied extreme voters are with the initial position of their preferred candidate relative to the alienation threshold, the more candidates can afford to flip-flop toward the center.

Flip-flopping follows a similar pattern with respect to γ, the relative weight of flip-flopping in citizens' utility. Candidates flip-flop less when it has a greater impact on citizens' preferences. However, note that when $\overline{u} \geq -q_1$, γ does not have any influence on flip-flopping as candidates do not deviate from their initial position.

We saw in Corollary 1 that candidates flip-flop only toward the center. Thus, there is a direct inverse relationship between flip-flopping and divergence, i.e. the distance between equilibrium platforms. I therefore obtain that divergence is a monotonic non-decreasing function of $\overline{u}$ and of γ. Both abstention due to alienation and the cost of flip-flopping hinder the convergence of platforms.

This result is in line with Grillo (2023) and Oprea et al. (2024). The ability of extreme citizens to threaten abstention induces platform divergence.[7] In my model, if citizens are reluctant to vote (i.e. $\overline{u}$ is high), then candidates do not have any margin to flip-flop toward the center (which would reduce divergence). However, if citizens are willing to vote (low $\overline{u}$), then candidates flip-flop toward the center, which makes divergence decrease. On a technical note, I differ from Grillo (2023) in that I do not require a polarized distribution of citizens' ideal platforms to obtain divergence at equilibrium.

Corollary 5. *Turnout is a monotonic non-increasing function of* $\overline{u}$ *and of* γ

Voter turnout, i.e. the total number of votes $V_1 + V_2$, is either decreasing or constant with respect to the alienation threshold. The parameter $\overline{u}$ has both a direct negative effect on the length of the attraction intervals and an indirect positive (or null) effect on citizens' utility through a negative (or null) effect on flip-flopping, as mentioned in Corollary 4.

When the alienation threshold is high enough ($\overline{u} \geq -q_1$), there is no flip-flopping at equilibrium. The effect of $\overline{u}$ on turnout is thus negative. Similarly, when the alienation threshold is low enough ($\overline{u} \leq \overline{u}_0$), this parameter has no impact on flip-flopping.

[7] Grillo (2023) and Oprea et al. (2024) mainly use "polarization" rather than "divergence".

Consequently, the effect on turnout is either negative, or null (in the case of full participation).

However, when the alienation threshold has an intermediate value ($\overline{u} \in (\overline{u}_0, -q_1)$), both the direct and the indirect effect exert influence. My result indicates that the direct effect dominates the indirect effect.

All this points to a positive correlation between flip-flopping and turnout, as they evolve in a similar way with respect to the alienation threshold. Regions of the parameters characterized by full participation at equilibrium are also associated with high levels of flip-flopping. This result differs from Grillo (2023), where numerical simulations reveal that the correlation between divergence and turnout can be either positive or negative, depending on the model's parameters.

Finally, turnout either decreases or remains constant with respect to γ, the relative weight of flip-flopping in citizens' utility. While an increase in γ has a direct negative effect on the utility function, it also has an indirect positive effect by making flip-flopping decrease (Corollary 4). Here again, my result indicates that the direct effect dominates the indirect effect.

5 Alternative Objective Functions

In this section, I analyze alternative objective functions for candidates. Many papers study electoral competition with endogenous turnout between two candidates that either maximize plurality or their number of votes (Hinich and Ordeshook, 1969, Hinich and Ordeshook, 1970, Anderson and Glomm, 1992). Plurality maximization leads to convergent equilibria, whereas maximizing the number of votes can induce divergence. In this section, I examine flip-flopping strategies under both objective functions.

I also explore an alternative model where candidates maximize their probability of winning. In this setting, candidates focus on winning but are indifferent about their margin of victory.

5.1 Plurality and Probability of Winning

5.1.1 Plurality

In this variant of the model, each candidate j maximizes her plurality $\mathcal{P}_j(p_1, p_2)$, defined by:

$$\mathcal{P}_j(p_1, p_2) = V_j(p_1, p_2) - V_{-j}(p_1, p_2).$$

A best response p_j^* against p_{-j} is:

$$p_j^* \in \arg\max_{p_j \in [0,1]} \{\mathcal{P}_j(p_1, p_2)\}.$$

Proposition 2. *The pair $(p_1^*, 1 - p_1^*)$ defined in (2) is the unique equilibrium of the plurality maximization game.*

Proof. See Appendix A.6 □

Proposition 2 indicates that the equilibrium of the game is the same when candidates maximize their plurality rather than their vote share. The intuition behind this result is that if a candidate j chooses a platform $p_j \in [0, 1]$, her opponent can in any case obtain a vote share of $\frac{1}{2}$ and a plurality of 0 by playing the symmetric strategy $p_{-j} = 1 - p_j$. Consequently, if (p_1, p_2) is an equilibrium, then for all j, $VS_j(p_1, p_2) = \frac{1}{2}$ and $\mathcal{P}_j(p_1, p_2) = 0$. In section 4, I proved that there is a unique pair (p_1, p_2) such that candidates cannot unilaterally increase their number of votes beyond the number of votes of their opponent (i.e. there is no p_1' such that $VS_1(p_1', p_2) > \frac{1}{2}$ and there is no p_2' such that $VS_2(p_1, p_2') > \frac{1}{2}$). It follows that (p_1, p_2) is also the unique equilibrium of the plurality maximization game.

5.1.2 Probability of Winning

I assume that each candidate j maximizes her probability of winning $\pi_j(p_1, p_2)$.

$$\pi_j(p_1, p_2) = \begin{cases} 1 & \text{if } VS_j(p_1, p_2) > \frac{1}{2} \\ \frac{1}{2} & \text{if } VS_j(p_1, p_2) = \frac{1}{2} \\ 0 & \text{if } VS_j(p_1, p_2) < \frac{1}{2}. \end{cases}$$

A best response p_j^* against p_{-j} is:

$$p_j^* \in \arg\max_{p_j \in [0,1]} \{\pi_j(p_1, p_2)\}.$$

Proposition 3. *The pair* $(p_1^*, 1 - p_1^*)$ *defined in (2) is the unique equilibrium of the probability of winning maximization game.*

Proof. See Appendix A.7 □

The intuition behind this proposition is similar to Proposition 2. At equilibrium, we know that for all j, $\pi_j(p_1, p_2) = \frac{1}{2}$. I proved in section 4 that there is a unique pair (p_1, p_2) such that candidates cannot unilaterally deviate to increase their vote share. Consequently, it can be deduced that (p_1, p_2) is also the unique equilibrium of the probability of winning maximization game.

5.2 Number of Votes

I now turn to the case where the objective function is the number of votes. In most elections, number of votes maximization is not fully consistent with office-motivated candidates. However, there are contexts where candidates have incentives to partially sacrifice their vote share to increase their number of votes. For instance, the French parliament is elected via a two-round election where the number of votes matters for office-motivated candidates. A candidate getting a vote share higher than one half and the votes of at least 25% of registered voters during the first round wins the election. Otherwise, a second round is organized, where the competitors are the two candidates with the highest vote share from the first round together with any candidate who gets the votes of at least 12.5% of registered voters.

Another interpretation of the number of votes maximization framework is that we are examining a less competitive environment, where candidates aim to get votes but are not focused on outperforming their opponent.

Formally, in this specification of the model, a best response p_j^* against p_{-j} is:

$$p_j^* \in \arg\max_{p_j \in [0,1]} \{V_j(p_1, p_2)\}.$$

We thus study an alternative version of the model which is no longer a zero-sum game. Nevertheless, the following theorem indicates that if initial positions are extreme enough, the equilibrium of the number of votes maximization game is unique and is the same as the equilibrium of the vote share maximization game.

Theorem 2. *If* $q_1 \leq \frac{1}{4}$, *then* $(p_1^*, 1 - p_1^*)$ *defined in (2) is the unique equilibrium of the number of votes maximization game.*

Proof. See Appendix A.8. □

However, when initial positions are moderate (i.e., close to the center), then the equilibrium of the number of votes maximization game is no longer unique and shows differences from the vote share maximization game. For ease of presentation of the results, the following theorem addresses symmetric equilibria in the case where initial positions are not too moderate (i.e. $q_1 \in (\frac{1}{4}, \frac{1}{4} + \frac{1}{3\gamma}]$). Interested readers can find the full characterization of the equilibria in the Appendix (Section A.8.2).

Theorem 3. *If* $q_1 \in (\frac{1}{4}, \frac{1}{4} + \frac{1}{3\gamma}]$, *there is a unique symmetric equilibrium* $(p_1^*, 1 - p_1^*)$ *with:*

$$p_1^* = \begin{cases} \min\{q_1 + \frac{1}{\gamma}, \frac{1}{2}\} & \text{if } \overline{u} < \overline{u}_0 \\ q_1 + \frac{-1+\sqrt{1-2\gamma(q_1+\overline{u})}}{\gamma} & \text{if } \overline{u} \in [\overline{u}_0, -\frac{\gamma}{2}(q_1 - \frac{1}{4})^2 - \frac{1}{4}) \\ q_1 + \frac{1-\sqrt{1+2\gamma(q_1-\overline{u}-\frac{1}{2})}}{\gamma} & \text{if } \overline{u} \in [-\frac{\gamma}{2}(q_1 - \frac{1}{4})^2 - \frac{1}{4}, q_1 - \frac{1}{2}) \\ q_1 & \text{if } \overline{u} \geq q_1 - \frac{1}{2}. \end{cases}$$

Proof. See Appendix A.8. □

Note that asymmetric equilibria exist for $\overline{u} \in (-\frac{\gamma}{2}(q_1 - \frac{1}{4})^2 - \frac{1}{4}, q_1 - \frac{1}{2})$.[8] Figure 3 graphically represents the symmetric equilibrium described in Theorem 3.

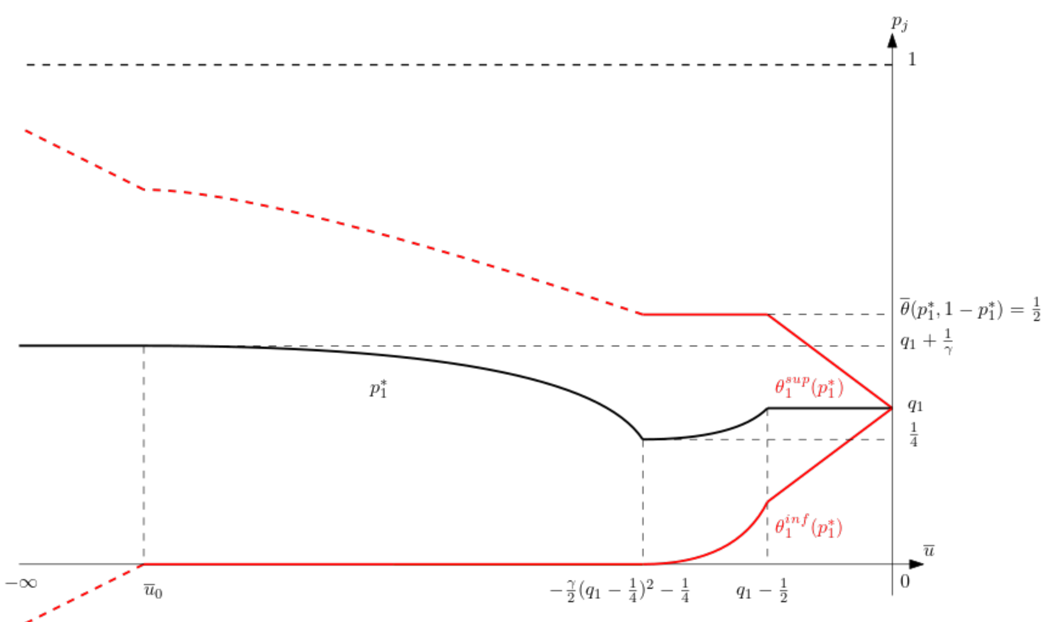

Figure 3: Symmetric equilibrium policy p_1^* as a function of the alienation threshold.

[8] Those equilibria are characterized by flip-flopping toward the extremes, either by both candidates or by only one candidate whose opponent does not flip-flop. See the Appendix section A.8.2 for more details.

When $q_1 \in (\frac{1}{4}, \frac{1}{4} + \frac{1}{3\gamma}]$ and $\overline{u} \in (-q_1, q_1 - \frac{1}{2})$, the equilibrium differs from that of the vote share maximization problem.

Corollary 6. *If $q_1 \in (\frac{1}{4}, \frac{1}{2})$ and $\overline{u} \in (-q_1, q_1 - \frac{1}{2})$, then $p_1^* < q_1$.*[9]

Corollary 6 indicates that it is not only divergence of platforms that is observed at equilibrium, but also polarization, which occurs under two jointly necessary and sufficient conditions. First, initial positions must be sufficiently moderate. Then, the alienation threshold must take an intermediate value such that at initial positions, there are no abstainers at the center, but some at the two extremes of the policy space.

In this parameter region, as explained in section 4.1, candidates do not flip-flop when they maximize their vote share. However, candidates can increase their number of votes by deviating toward the extremes of the policy space. Consider the case of candidate 1, she has incentives to flip-flop toward the left. By doing so, she convinces some left-wing citizens to vote for her rather than abstain, as she is moving closer to their ideal platform. The indifferent voter's location is also moving toward the left, but to a lesser extent, so that the deviation is profitable.

More formally, when $\overline{u} \in [-\frac{\gamma}{2}(q_1 - \frac{1}{4})^2 - \frac{1}{4}, q_1 - \frac{1}{2})$, candidates flip-flop to convince as many abstainers as possible without causing abstention at the center of the policy space. The platform $p_1^* = q_1 + \frac{1 - \sqrt{1 + 2\gamma(q_1 - \overline{u} - \frac{1}{2})}}{\gamma}$ is the unique policy such that $p_1 < q_1$ and $\theta_1^{sup}(p_1) = \theta_2^{inf}(1 - p_1)$. Then, when the alienation threshold is lower (i.e. $\overline{u} \in [\overline{u}_0, -\frac{\gamma}{2}(q_1 - \frac{1}{4})^2 - \frac{1}{4})$), candidates can capture all abstainers at the extremes of the policy space without causing abstention at the center. Candidate 1 chooses the platform $p_1^* = q_1 + \frac{-1 + \sqrt{1 - 2\gamma(q_1 + \overline{u})}}{\gamma}$, the unique equilibrium policy such that $\theta_1^{inf}(p_1) = 0$.

To sum up, when candidates maximize their total number of votes rather than their vote share, competition for the center becomes less intense. This incentivizes candidates to flip-flop toward the extremes to convince some abstainers to vote for them, thereby increasing their electoral base.

6 Conclusion

In this paper, I develop an electoral competition model with both a flip-flopping cost and endogenous turnout. I assume that citizens abstain due to alienation, i.e. they abstain

[9] In this corollary, the condition on initial positions is $q_1 \in (\frac{1}{4}, \frac{1}{2})$, since the statement is also true for $q_1 \in (\frac{1}{4} + \frac{1}{3\gamma}, \frac{1}{2})$. The corollary thus follows from the complete characterization of the equilibria (Appendix, section A.8.2), and not only from Theorem 3.

if their utility from their preferred candidate is lower than an exogenous alienation threshold. In the main model, candidates maximize their vote share, but I also explore alternative models with different objective functions for candidates.

I fully characterize equilibrium platforms as functions of the alienation threshold. When citizens are reluctant to vote, i.e., when the alienation threshold is high, then candidates do not flip-flop. When citizens are willing to vote, then extreme citizens' utility from their favorite candidate is relatively high compared to the alienation threshold. Candidates thus flip-flop toward the center, seeking to attract citizens at the center of the policy space without losing the adhesion of the extreme part of their electorate. The ability of citizens to threaten abstention hinders convergence of platforms and is thus a factor in divergence, in line with Grillo (2023) and Oprea et al. (2024).

I obtain divergence in most cases, but I do not find polarization (i.e flip-flopping toward the extremes) at equilibrium. If candidates flip-flop, it is always toward the center of the policy space. Moreover, I show that flip-flopping is positively correlated with voter turnout, implying a negative correlation between turnout and divergence of platforms.

My results also indicate that candidates are more likely to flip-flop when initial positions are closer to the extremes. More precisely, in such cases, there is a larger range of values for the alienation threshold within which candidates flip-flop.

Finally, I study alternative models where candidates' objective functions differ from vote share. Specifically, when candidates aim to maximize their number of votes, I obtain polarization when citizens are moderately able to threaten abstention and candidates' initial positions are sufficiently moderate. In this framework, competition for the center is less intense, leading candidates to flip-flop toward the extremes to increase their electoral base.

In this study, I assume for simplicity that citizens are uniformly distributed over the policy space. In the literature, it is also standard to consider a symmetric single- peaked distribution, which would involve accounting not only for the length of the intervals but also for the density of citizens within those intervals. Consequently, the electoral benefits of flip-flopping toward the center would increase, and candidates would still not polarize in the vote share maximization setting. Moreover, candidates would still tend to flip-flop more toward the center when the alienation threshold is low. For instance, if candidate 1's attraction interval is large, then flip-flopping toward the center is less costly because she would be sacrificing a low-density part of the policy space.

This work provides some predictions that would be interesting to study empirically. One result that could be empirically verified is the positive correlation between candidates' propensity to flip-flop and the extremeness of their initial position (Corollary 3). It would also be relevant to investigate whether a positive correlation between flip-flopping and voter turnout is observed in the data, as well as a negative correlation between divergence and turnout.

References

J. Adams and S. Merrill. Voter turnout and candidate strategies in american elections. *The Journal of Politics*, 65(1):161–189, 2003.

M. Agranov. Flip-flopping, primary visibility, and the selection of candidates. *American Economic Journal: Microeconomics*, 8(2):61–85, 2016.

S. P. Anderson and G. Glomm. Alienation, indifference and the choice of ideological position. *Social Choice and Welfare*, 9(1):17–31, 1992.

G. Andreottola. Flip-flopping and electoral concerns. *The Journal of Politics*, 83(4): 1669–1680, 2021.

E. Aragonès and D. Xefteris. Ideological Consistency and Valence. Working Papers 1383, Barcelona School of Economics, 2022.

S. Callander and J. C. Carbajal. Cause and effect in political polarization: A dynamic analysis. *Journal of Political Economy*, 130(4):825–880, 2022.

S. Callander and C. H. Wilson. Turnout, polarization, and duverger's law. *The Journal of Politics*, 69(4):1047–1056, 2007.

J. M. DeBacker. Flip-Flopping: Ideological Adjustment Costs In The United States Senate. *Economic Inquiry*, 53(1):108–128, 2015.

D. Doherty, C. M. Dowling, and M. G. Miller. When is Changing Policy Positions Costly for Politicians? Experimental Evidence. *Political Behavior*, 38(2):455–484, 2016.

A. Downs. An economic theory of democracy. *Harper and Row*, 28, 1957.

G. Fournier, A. Grillo, and Y. Tsodikovich. Strategic flip-flopping in political competition. *arXiv preprint arXiv:2305.02834*, 2023.

E. L. Glaeser, G. A. Ponzetto, and J. M. Shapiro. Strategic extremism: Why republicans and democrats divide on religious values. *The Quarterly journal of economics*, 120 (4):1283–1330, 2005.

A. Grillo. Political alienation and voter mobilization in elections. *Journal of public economic theory*, 25(3):515–531, 2023.

H. Herrera, M. Morelli, and T. Palfrey. Turnout and power sharing. *The Economic Journal*, 124(574):F131–F162, 2014.

M. J. Hinich and P. C. Ordeshook. Abstentions and equilibrium in the electoral process. *Public Choice*, pages 81–106, 1969.

M. J. Hinich and P. C. Ordeshook. Plurality maximization vs vote maximization: A spatial analysis with variable participation. *American Political Science Review*, 64 (3):772–791, 1970.

P. Hummel. Flip-flopping from primaries to general elections. *Journal of Public Economics*, 94(11):1020–1027, 2010.

A. Oprea, L. Martin, and G. H. Brennan. Moving toward the median: Compulsory voting and political polarization. *American Political Science Review*, pages 1–15, 2024.

J. Robison. The role of elite accounts in mitigating the negative effects of repositioning. *Political Behavior*, 39, 09 2017.

M. Tavits. Principle vs. pragmatism: Policy shifts and political competition. *American Journal of Political Science*, 51(1):151–165, 2007.

R. D. Tella, R. Kotti, C. L. Pennec, and V. Pons. Keep your enemies closer: Strategic platform adjustments during u.s. and french elections. NBER Working Papers 31503, National Bureau of Economic Research, Inc, 2023.

M. Tomz and R. Van Houweling. Candidate repositioning. *Unpublished Manuscript, Stanford University*, 2012.

A Proofs

A.1 Proof of Lemma 1

Citizen θ prefers to vote for candidate j rather than abstain if and only if $U_\theta^j(p_j) \geq \overline{u}$. For $\theta \leq p_j$, it gives:

$$\theta \geq \theta_j^{inf}(p_j) := p_j + \overline{u} + \frac{\gamma}{2}(p_j - q_j)^2.$$

For $\theta \geq p_j$, it gives:

$$\theta \leq \theta_j^{sup}(p_j) := p_j - \overline{u} - \frac{\gamma}{2}(p_j - q_j)^2.$$

Consequently, citizen θ prefers to vote for candidate j rather than abstain if and only if $\theta \in I_j := [\theta_j^{inf}(p_j), \theta_j^{sup}(p_j)]$. □

A.2 Proof of Lemma 2

Suppose first that $p_j \leq p_{-j}$. Candidate j is unanimously preferred by citizens if $\forall \theta \in [0,1]$, $U_\theta^j(p_j) > U_\theta^{-j}(p_{-j})$. For $\theta \leq p_j$, this condition writes:

$$\theta - p_j - \frac{\gamma}{2}(p_j - q_j)^2 > \theta - p_{-j} - \frac{\gamma}{2}(p_{-j} - q_{-j})^2.$$

It is equivalent to:

$$\phi_j(p_1, p_2) := \frac{\gamma}{2}\Big((p_{-j} - q_{-j})^2 - (p_j - q_j)^2\Big) > p_j - p_{-j}.$$

For $\theta \geq p_{-j}$, $U_\theta^j(p_j) > U_\theta^{-j}(p_{-j})$ can be written as:

$$\phi_j(p_1, p_2) > p_{-j} - p_j.$$

Finally, citizens $\theta \in [p_j, p_{-j}]$ prefer candidate j if:

$$\theta < \frac{1}{2}(p_j + p_{-j} + \phi_j(p_1, p_2)).$$

This condition holds for all $\theta \in [p_j, p_{-j}]$ if $\phi_j(p_1, p_2) > p_{-j} - p_j$.

Conclusion: When $p_j \leq p_{-j}$, candidate j is unanimously preferred by citizens if and only if $\phi_j(p_1, p_2) > p_{-j} - p_j$. When $p_{-j} \leq p_j$, we can show that candidate j is unanimously preferred by citizens if and only if $\phi_j(p_1, p_2) > p_j - p_{-j}$. These two conditions can be summed up as follows: $\phi_j(p_1,p_2) > |p_j - p_{-j}|$. $\square$

Remark 1. *$\phi_j(p_1,p_2) > |p_j - p_{-j}|$ is equivalent to $I_{-j} \subset I_j$.*

It is the case as $\theta^{inf}_{-j}(p_{-j}) > \theta^{inf}_j(p_j)$ is equivalent to $\phi_j(p_1,p_2) > p_j - p_{-j}$, while $\theta^{sup}_{-j}(p_{-j}) < \theta^{sup}_j(p_j)$ is equivalent to $\phi_j(p_1,p_2) > p_{-j} - p_j$.

A.3 Proof of Lemma 3

Proof. Let $p_j \leq p_{-j}$. For all $\theta \in [0,1]$, $U^j_\theta(p_j) = U^{-j}_\theta(p_{-j})$ is equivalent to $\phi_j(p_1,p_2) = |p_j - \theta| - |p_{-j} - \theta|$.

We can deduce that citizens $\theta \leq p_j$ are indifferent between the two candidates if $\phi_j(p_1,p_2) = p_j - p_{-j}$. Citizens $\theta \geq p_{-j}$ are indifferent if $\phi_j(p_1,p_2) = p_{-j} - p_j$. $\square$

A.4 Proof of Proposition 1

Proof. Suppose that $p_1 \leq p_2$ and that condition 1 holds. It follows from Condition 1 that $\phi_1(p_1,p_2) \in (p_1 - p_2, p_2 - p_1)$. Consequently, we obtain from Lemma 2's proof that citizens $\theta \leq p_1$ prefer candidate 1 rather than candidate 2, while citizens $\theta \geq p_2$ prefer candidate 2. Moreover, it follows from Lemma 2's proof that citizens $\theta \in [p_1, \frac{1}{2}(p_1 + p_2 + \phi_1(p_1,p_2)))$ prefer candidate 1 while citizens $\theta \in (\frac{1}{2}(p_1 + p_2 + \phi_1(p_1,p_2)), p_2]$ prefer candidate 2. There is a unique indifferent citizen $\overline{\theta}(p_1,p_2) := \frac{1}{2}(p_1 + p_2 + \phi_1(p_1,p_2)) \in (p_1,p_2)$.

Consequently, a citizen $\theta < \overline{\theta}(p_1,p_2)$ prefers and votes for candidate 1 if $U^1_\theta(p_1) \geq \overline{u}$, which is equivalent to $\theta \in I_1$. A citizen $\theta > \overline{\theta}(p_1,p_2)$ prefers and votes for candidate 2 if $\theta \in I_2$. $\square$

The following notations will be used in the subsequent proofs.

Definition 2. *The maximum of flip-flopping toward the center for candidate j is denoted by $\overline{p}_j$, with $\overline{p}_1 := \min\{q_1 + \frac{1}{\gamma}, \frac{1}{2}\}$ and $\overline{p}_2 := \max\{q_2 - \frac{1}{\gamma}, \frac{1}{2}\}$.*

Platforms $\overline{p}_1$ and $\overline{p}_2$ are crucial, as they bound flip-flopping toward the center. As explained in section 3, $\theta^{sup}_1(p_1)$ is a decreasing function with respect to $p_1 > q_1 + \frac{1}{\gamma}$. Consequently, additional flip-flopping toward the center would decrease the utility of the citizens from which candidate 1 gets closer from. However, if $q_1 + \frac{1}{\gamma} > \frac{1}{2}$, flip-flopping toward the center is upper-bounded by $\frac{1}{2}$. We will show in Claim 1 (section A.5) that $p_1 \leq \frac{1}{2} \leq p_2$ at equilibrium.

A.5 Proof of Theorem 1

Claim 1. *If* (p_1^*, p_2^*) *is an equilibrium, then* $p_1^* \leq \frac{1}{2} \leq p_2^*$

Proof. Let's consider a pair (p_1, p_2) such that $p_1 > \frac{1}{2}$. To show that (p_1, p_2) cannot be an equilibrium, we consider two cases: $VS_2(p_1, p_2) < 1$ and $VS_2(p_1, p_2) = 1$. If $VS_2(p_1, p_2) < 1$, then $p_2' = p_1$ is a profitable deviation for candidate 2. Indeed, as $\phi_2(p_1, p_1) = \frac{\gamma}{2}\Big((p_1 - q_1)^2 - (p_1 - q_2)^2\Big) > |p_1 - p_1|$, it follows from Lemma 2 that $VS_2(p_1, p_1) = 1$. The deviation is thus profitable for candidate 2. If $VS_2(p_1, p_2) = 1$, then $VS_1(p_1, p_2) = 0$. It follows from Lemma 2 that $p_1' = q_1$ is a profitable deviation for candidate 1. Indeed, since we have $\phi_2(q_1, p_2) = -\frac{\gamma}{2}(p_2 - q_2)^2 < |p_2 - q_1|$, candidate 2 is not unanimously preferred by citizens. Consequently, candidate 1 gets votes, and $VS_1(q_1, p_2) > 0$.

We have shown that a pair (p_1, p_2) with $p_1 > \frac{1}{2}$ cannot be an equilibrium. Consequently, if (p_1, p_2) is an equilibrium, then $p_1 \leq \frac{1}{2}$. A similar (and thus omitted) argument can be used to prove that if (p_1, p_2) is an equilibrium, then $p_2 \geq \frac{1}{2}$. □

Claim 2. *If* (p_1^*, p_2^*) *is an equilibrium, then* $VS_1(p_1^*, p_2^*) = VS_2(p_1^*, p_2^*) = \frac{1}{2}$.

Proof. By symmetry, playing $p_2 = 1 - p_1$ gives candidate 2 a vote share equal to $\frac{1}{2}$. This strategy is therefore a profitable deviation from any p_2 such that $VS_2(p_1, p_2) < \frac{1}{2}$. Symmetrically for candidate 1, $p_1 = 1 - p_2$ is a profitable deviation from any p_1 such that $VS_1(p_1, p_2) < \frac{1}{2}$. □

Claim 3. *If* (p_1^*, p_2^*) *is an equilibrium, then* $p_1^* = 1 - p_2^*$.

Proof. Let (p_1, p_2) with $p_1 \neq 1 - p_2$ be an equilibrium. I consider three cases:

1. $\forall j$, $I_j \subseteq X$
2. For some $j \in \{1, 2\}$, $I_j \subseteq X$ and $I_{-j} \cap X \neq I_{-j}$
3. $\forall j$, $I_j \cap X \neq I_j$

I will show that we will obtain a contradiction.

Case 1. Let's assume that for all j, $I_j \subseteq X$. It follows from Claim 2 that $VS_1(p_1, p_2) = VS_2(p_1, p_2)$ (which is equivalent to $V_1(p_1, p_2) = V_2(p_1, p_2)$). We now prove that it is equivalent to $\ell(I_1) = \ell(I_2)$ (defined in equation (1)). If $I_1 \cap I_2 = \emptyset$, for all j, $V_j(p_1, p_2) = \ell(I_j)$, the equality thus follows. If $I_1 \cap I_2 \neq \emptyset$, $V_1(p_1, p_2) = \overline{\theta}(p_1, p_2) - \theta_1^{inf}(p_1)$

and $V_2(p_1,p_2) = \theta_2^{sup}(p_2) - \overline{\theta}(p_1,p_2)$. It follows from the definition of $\overline{\theta}(p_1,p_2)$ that the equality $V_1(p_1,p_2) = V_2(p_1,p_2)$ can be written as:

$$\frac{1}{2}(\theta_1^{sup}(p_1) + \theta_2^{inf}(p_2)) - \theta_1^{inf}(p_1) = \theta_2^{sup}(p_2) - \frac{1}{2}(\theta_1^{sup}(p_1) + \theta_2^{inf}(p_2)),$$

which simplifies to:

$$\ell(I_1) = \theta_1^{sup}(p_1) - \theta_1^{inf}(p_1) = \theta_2^{sup}(p_2) - \theta_2^{inf}(p_2) = \ell(I_2).$$

Using Lemma 1, we find $-2\overline{u} - \gamma(p_1-q_1)^2 = -2\overline{u} - \gamma(p_2-q_2)^2$, which implies $|p_1-q_1| = |p_2-q_2|$. Both candidates flip-flop by the same magnitude and therefore, (p_1,p_2) is either of the form $(q_1+\delta, q_2+\delta)$ or $(q_1+\delta, q_2-\delta)$ for $\delta \in \mathbb{R}$. We are left to prove that the first form can not be an equilibrium if $\delta \neq 0$.

Indeed, for $\delta > 0$, since $\theta_2^{sup}(q_2+\delta) \leq 1$, then $\theta_1^{inf}(q_1+\delta) > 0$. Consequently, candidate 1 has a profitable marginal negative deviation p_1' (i.e toward the left). Marginally reducing her flip-flopping, she would increase the length of her attraction interval. We would have $\ell(I_1) > \ell(I_2)$, and therefore $VS_1(p_1', q_2+\delta) > \frac{1}{2}$.

A similar argument holds for $\delta < 0$, candidate 2 has a profitable marginal positive deviation (i.e toward the right). Consequently, (p_1,p_2) is not an equilibrium, we obtain a contradiction.

Case 2. Suppose that for some $j \in \{1,2\}$, $I_j \subseteq X$ and $I_{-j} \cap X \neq I_{-j}$. We consider $j=1$ without loss of generality: $I_1 \subseteq X$ and $I_2 \cap X \neq I_2$. We first prove that p_2 must be equal to $\overline{p}_2$ as defined in definition 2. We then show that we obtain a contradiction. We consider separately the cases where $q_2 - \frac{1}{\gamma} > \frac{1}{2}$ and $q_2 - \frac{1}{\gamma} \leq \frac{1}{2}$.

- We consider first the case where $q_2 - \frac{1}{\gamma} > \frac{1}{2}$. If $p_2 > q_2 - \frac{1}{\gamma}$ (resp. $p_2 < q_2 - \frac{1}{\gamma}$), a marginal negative (resp. positive) deviation of candidate 2 implies that $\theta_2^{inf}(p_2)$ and $\overline{\theta}(p_1,p_2)$ would decrease while $\theta_2^{sup}(p_2)$ would remain greater than 1. V_2 thus increases while V_1 decreases or remains stable. Consequently, V_2 becomes greater than V_1, which contradicts Claim 2.

- Then we study the case where $q_2 - \frac{1}{\gamma} \leq \frac{1}{2}$. It follows from Claim 1 that at equilibrium, $p_2 \geq \frac{1}{2}$. If $p_2 > \frac{1}{2}$, then candidate 2 has a profitable marginal negative deviation, as $\theta_2^{inf}(p_2)$ and $\overline{\theta}(p_1,p_2)$ would decrease while $\theta_2^{sup}(p_2)$ would remain greater than 1.

We proved that p_2 must be equal to $\overline{p}_2$. We then show that we obtain a contradiction.

From Claim 2, we have $V_1(p_1, \overline{p}_2) = V_2(p_1, \overline{p}_2)$. Moreover, it follows from the definition of $\overline{\theta}(p_1, p_2)$ that half of the citizens included in $I_1 \cap I_2$ vote for candidate 1, while the other half vote for candidate 2. Consequently, since $I_1 \subseteq X$ and $I_2 \cap X \neq I_2$, we must have $\ell(I_1) = \ell(I_2 \cap X) < \ell(I_2)$. We are now ready to contradict that $(p_1, \overline{p}_2)$ is an equilibrium, considering separately the cases where $\overline{p}_2 = q_2 - \frac{1}{\gamma}$ and $\overline{p}_2 = \frac{1}{2}$.

- If $\overline{p}_2 = q_2 - \frac{1}{\gamma}$, then it follows from $\ell(I_1) < \ell(I_2)$ that we must have either $p_1 < q_1 - \frac{1}{\gamma}$ or $p_1 > q_1 + \frac{1}{\gamma}$. In the first (resp. second) case, candidate 1 has a positive (resp. negative) marginal profitable deviation, as $\theta_1^{inf}(p_1)$ would decrease while $\theta_1^{sup}(p_1)$ and $\overline{\theta}(p_1, p_2)$ would increase. V_1 thus increases and V_2 decreases or remains constant.

- If $\overline{p}_2 = \frac{1}{2}$, then we have $X \subset I_2$. Consequently, it follows from $I_1 \subseteq X$ that $I_1 \subset I_2$. We can deduce from Lemma 2 and Remark 1 that $VS_1(p_1, \overline{p}_2) = 0$, which contradicts Claim 2.

We obtained a contradiction.

Case 3. Suppose that $\forall j,\ I_j \cap X \neq I_j$. Following the argument of the previous case, we must have $p_1 = \overline{p}_1$ and $p_2 = \overline{p}_2$. Consequently, two equilibrium candidates are $(q_1 + \frac{1}{\gamma}, q_2 - \frac{1}{\gamma})$ and $(\frac{1}{2}, \frac{1}{2})$. In both cases, we have $p_1 = 1 - p_2$. □

I denote by $br_j(p_{-j}) = \underset{p_j \in [0,1]}{\arg\max}\, VS_j(p_1, p_2)$, the set of best responses of candidate j against a platform x. In the following claim, we discard the study of the special case mentioned in Lemma 3 (except in the case of full convergence, i.e $p_1 = p_2 = \frac{1}{2}$).

Claim 4. *If $p_{-j} \neq \frac{1}{2}$, then the platform p_j such that $\phi_j(p_1, p_2) = p_1 - p_2$ or $\phi_j(p_1, p_2) = p_2 - p_1$ does not belong to $br_j(p_{-j})$.*

Proof. Let $j = 1$ without loss of generality. I first consider $p_1 < p_2$, studying both the case where $\phi_1(p_1, p_2) = p_1 - p_2$ and $\phi_1(p_1, p_2) = p_2 - p_1$. Then, I let $p_1 > p_2$ and I also investigate the two following cases: $\phi_1(p_1, p_2) = p_1 - p_2$ and $\phi_1(p_1, p_2) = p_2 - p_1$. Finally, I study the case where $p_1 = p_2 \neq \frac{1}{2}$. Consequently, I will prove that p_1 is not a best response to p_2 in five distinct cases.

Case 1. Consider first $p_1 < p_2$ such that $\phi_1(p_1, p_2) = p_1 - p_2$. With some algebra, we can show that this equality is equivalent to $\theta_1^{inf}(p_1) = \theta_2^{inf}(p_2)$. We also obtain from Lemma 2's proof (Appendix A.2) that for $\theta \in [0, p_1]$, we have $U^1_\theta(p_1) = U^2_\theta(p_2)$. Finally, it follows from $p_1 < p_2$ that $\phi_1(p_1, p_2) < p_2 - p_1$. We can deduce from Lemma 2's proof that for $\theta \in (p_1, 1]$, $U^1_\theta(p_1) < U^2_\theta(p_2)$. With some algebra, we can also show that $\theta^{sup}_1(p_1) < \theta^{sup}_2(p_2)$.

Consequently, we have $V_1(p_1, p_2) = \frac{1}{2}(p_1 - \max\{\theta^{inf}_1(p_1), 0\})$ and $V_2(p_1, p_2) = \frac{1}{2}(p_1 - \max\{\theta^{inf}_1(p_1), 0\}) + \min\{\theta^{sup}_2(p_2), 1\} - p_1$. Candidate 1 has a profitable infinitesimal deviation p'_1 (toward the left if $p_1 < q_1 - \frac{1}{\gamma}$ and toward the right otherwise), so that $\theta^{inf}_1(p'_1) < \theta^{inf}_2(p_2)$. We obtain a unique indifferent citizen $\overline{\theta}(p'_1, p_2) > p'_1$. We also have $V_1(p'_1, p_2) = \overline{\theta}(p'_1, p_2) - \max\{\theta^{inf}_1(p'_1), 0\}$ and $V_2(p'_1, p_2) = \min\{\theta^{sup}_2(p_2), 1\} - \overline{\theta}(p'_1, p_2)$. As the deviation is infinitesimal, it follows that V_1 increases while V_2 decreases. Consequently, VS_1 increases, the deviation is thus profitable, and we can conclude that $p_1 \notin br_1(p_2)$.

Case 2. Consider now $p_1 < p_2$ such that $\phi_1(p_1, p_2) = p_2 - p_1$. It follows from $p_1 < p_2$ that $\phi_1(p_1, p_2) > p_1 - p_2$. We can thus show with some algebra that $\theta^{inf}_1(p_1) < \theta^{inf}_2(p_2)$ and that $\theta^{sup}_1(p_1) = \theta^{sup}_2(p_2)$. Finally, we can deduce from Lemma 2's proof that $U^1_\theta(p_1) > U^2_\theta(p_2)$ for $\theta \in [0, p_2)$ and that $U^1_\theta(p_1) = U^2_\theta(p_2)$ for $\theta \in [p_2, 1]$.

We thus have $V_1(p_1, p_2) = p_2 - \max\{\theta^{inf}_1(p_1), 0\} + \frac{1}{2}(\min\{\theta^{sup}_2(p_2), 1\} - p_2)$ and $V_2(p_1, p_2) = \frac{1}{2}(\min\{\theta^{sup}_2(p_2), 1\} - p_2)$. Candidate 1 has an infinitesimal profitable deviation p'_1 (toward the right if $p_1 < q_1 + \frac{1}{\gamma}$, and toward the left otherwise), so that $\theta^{sup}_1(p'_1) > \theta^{sup}_2(p_2)$ and thus $I_2 \subset I_1$. Candidate 1's vote share would be equal to 1 and her number of votes would increase ($V_1(p'_1, p_2) = \min\{\theta^{sup}_1(p'_1), 1\} - \max\{\theta^{inf}_1(p'_1), 0\}$). We can conclude that $p_1 < p_2$ such that $\phi_1(p_1, p_2) = p_1 - p_2$ or $\phi_1(p_1, p_2) = p_2 - p_1$ is never a best response to p_2.

Case 3. We now focus on $p_1 > p_2$ such that $\phi_1(p_1, p_2) = p_1 - p_2$. It follows that $\phi_1(p_1, p_2) > p_2 - p_1$, we thus have $\theta^{inf}_1(p_1) = \theta^{inf}_2(p_2)$ and $\theta^{sup}_1(p_1) > \theta^{sup}_2(p_2)$. We can deduce from Lemma 2's proof that citizens $\theta \in [0, p_2]$ are indifferent between the two candidates, while the remaining citizens prefer candidate 1. Consequently, $V_1(p_1, p_2) = \min\{\theta^{sup}_1(p_1), 1\} - p_2 + \frac{1}{2}(p_2 - \max\{\theta^{inf}_1(p_1), 0\})$ and $V_2(p_1, p_2) = \frac{1}{2}(p_2 - \max\{\theta^{inf}_1(p_1), 0\})$. Following the same argument as in Case 2, candidate 1 has a infinitesimal profitable deviation p'_1 such that $\theta^{inf}_1(p'_1) < \theta^{inf}_2(p_2)$.

Case 4. Let $p_2 > p_1$ such that $\phi_1(p_1, p_2) = p_2 - p_1$. It follows that $\phi_1(p_1, p_2) < p_1 - p_2$, we thus have $\theta^{inf}_1(p_1) > \theta^{inf}_2(p_2)$ and $\theta^{sup}_1(p_1) = \theta^{sup}_2(p_2)$. We can deduce from Lemma 2's proof that citizens $\theta \in [0, p_1]$ prefer candidate 2 rather than candidate 1, while the remaining citizens are indifferent. Consequently, $V_1(p_1, p_2) = \frac{1}{2}(\min\{\theta_1^{sup}(p_1), 1\} - p_1)$ and $V_2(p_1, p_2) = \frac{1}{2}(\min\{\theta_1^{sup}(p_1), 1\} - p_1) + p_1 - \max\{\theta_2^{inf}(p_2), 0\}$. Following the same argument as in Case 1, candidate 1 has a infinitesimal profitable deviation p_1' such that $\theta_1^{sup}(p_1') > \theta_2^{sup}(p_2)$. We can conclude that $p_1 > p_2$ such that $\phi_1(p_1, p_2) = p_1 - p_2$ or $\phi_1(p_1, p_2) = p_2 - p_1$ is never a best response to p_2.

Case 5. Finally, we consider $p_1 = p_2 \neq \frac{1}{2}$. In this case, we can prove with some algebra that $\phi_1(p_1, p_2) \neq p_2 - p_1$ and $\phi_1(p_1, p_2) \neq p_1 - p_2$.

□

Claim 5. *If $br_1(p_2) = \{p_1\}$ and $br_2(p_1) = \{p_2\}$, then (p_1, p_2) is the unique equilibrium.*

Proof. Let $br_1(p_2) = \{p_1\}$ and $br_2(p_1) = \{p_2\}$, then (p_1, p_2) is an equilibrium. Moreover, the pairs (p_1', p_2) and (p_1, p_2') with $p_1 \neq p_1'$ and $p_2 \neq p_2'$ are not equilibria.

Now, let (p_1', p_2') with $p_1' \neq p_1$ and $p_2' \neq p_2$ be an equilibrium, we will obtain a contradiction. We know from Claim 2 that for all j, $VS_j(p_1, p_2) = VS_j(p_1', p_2') = \frac{1}{2}$. Then, it follows from $br_1(p_2) = \{p_1\}$ that $VS_1(p_1', p_2) < \frac{1}{2}$. Consequently, we have $VS_2(p_1', p_2) > \frac{1}{2} = VS_2(p_1', p_2')$. We can deduce that candidate 2 has a profitable deviation p_2 from p_2', this is a contradiction. □

Claim 6. *If $\overline{u} \geq -q_1$, then $(p_1^*, 1 - p_1^*)$ with $p_1^* = q_1$ is the unique pure strategy equilibrium.*

Proof. Let $\overline{u} \geq -q_1$. If $p_2 = q_2$, then for any $p_1 \neq q_1$ we have $\ell(I_1) < \ell(I_2)$. Also, it follows from $\overline{u} \geq -q_1$ that $I_2 \subseteq X$. Consequently, $VS_1(p_1, q_2) < \frac{1}{2}$. As $VS_1(q_1, q_2) = \frac{1}{2}$, we can conclude that $br_1(q_2) = \{q_1\}$. By a symmetric argument, it can be deduced that $br_2(q_1) = \{q_2\}$. It follows from Claim 5 that $(q_1, q_2) = (q_1, 1 - q_1)$ is the unique equilibrium. □

Claim 7. *If $\overline{u} \in [\overline{u}_0, -q_1)$, then $(p_1^*, 1 - p_1^*)$ with $p_1^* = \hat{p}_1 := q_1 + \frac{-1+\sqrt{1-2\gamma(q_1+\overline{u})}}{\gamma}$ is the unique pure strategy equilibrium.*

Proof. Let $\overline{u} \in [\overline{u}_0, -q_1)$ and $\hat{p}_2 = 1 - \hat{p}_1$. We first prove that $br_1(\hat{p}_2) = \{\hat{p}_1\}$.

With some algebra, we can show that $\theta_1^{inf}(\hat{p}_1) = 0$, and symmetrically that $\theta_2^{sup}(\hat{p}_2) = 1$. We can also prove that $\hat{p}_1 \in (q_1, \overline{p}_1]$. Consequently, we have $\hat{p}_1 \leq \frac{1}{2}$, it follows that $I_1 \subseteq X$, and symmetrically that $I_2 \subseteq X$. We can deduce that if $p_2 = \hat{p}_2$, then for all $p_1 > \hat{p}_1$, we have $l(I_1) < l(I_2)$, which implies $VS_1(p_1, \hat{p}_2) < \frac{1}{2}$. As $VS_1(\hat{p}_1, \hat{p}_2) = \frac{1}{2}$, we have shown that there is no $p_1 > \hat{p}_1$ that belongs to $br_1(\hat{p}_2)$.

Consider now $p_1 < \hat{p}_1$. Since $\hat{p}_1 \leq \overline{p}_1$, then $\theta_1^{sup}(p_1) < \theta_1^{sup}(\hat{p}_1)$ and $\overline{\theta}(p_1, \hat{p}_2) < \overline{\theta}(\hat{p}_1, \hat{p}_2)$. It follows that $V_1(\hat{p}_1, 1-\hat{p}_1) > V_1(p_1, 1-\hat{p}_1)$ while $V_2(\hat{p}_1, 1-\hat{p}_1) \leq V_2(p_1, 1-\hat{p}_1)$. Consequently, we have $VS_1(p_1, \hat{p}_2) < \frac{1}{2}$, there is thus no $p_1 < \hat{p}_1$ that belongs to $br_1(\hat{p}_2)$. We can conclude that $br_1(\hat{p}_2) = \{\hat{p}_1\}$. By a symmetric argument, it can be deduced that $br_2(\hat{p}_1) = \{\hat{p}_2\}$. It follows from Claim 5 that $(\hat{p}_1, \hat{p}_2) = (\hat{p}_1, 1 - \hat{p}_1)$ is the unique equilibrium. □

Claim 8. *If $\overline{u} \leq \overline{u}_0$, then $(p_1^*, 1-p_1^*)$ with $p_1^* = \overline{p}_1$ is the unique pure strategy equilibrium.*

Proof. Let $\overline{u} \leq \overline{u}_0$. First note that $\overline{u}_0 = -q_1 - \frac{3}{2\gamma}$ is equivalent to $\overline{p}_1 = q_1 + \frac{1}{\gamma}$. We thus divide this proof in two parts. We will first prove that if $\overline{u}_0 = -q_1 - \frac{3}{2\gamma}$, then $(q_1 + \frac{1}{\gamma}, 1 - q_1 - \frac{1}{\gamma})$ is the unique equilibrium. Then, we will show that if $\overline{u}_0 = -\frac{\gamma}{2}(\frac{1}{2} - q_1)^2 - \frac{1}{2}$, the pair $(\frac{1}{2}, \frac{1}{2})$ is the unique equilibrium.

Case 1: $\overline{u}_0 = -q_1 - \frac{3}{2\gamma}$.

We first prove that $br_1(q_2 - \frac{1}{\gamma}) = \{q_1 + \frac{1}{\gamma}\}$. It follows from $\overline{u} \leq -q_1 - \frac{3}{2\gamma}$ that $\theta_1^{inf}(q_1 + \frac{1}{\gamma}) \leq 0$, and symmetrically that $\theta_2^{sup}(q_2 - \frac{1}{\gamma}) \geq 1$. If $p_1 \neq q_1 + \frac{1}{\gamma}$ and $p_1 \in [0, q_2 - \frac{1}{\gamma})$, then $\theta_1^{sup}(q_1 + \frac{1}{\gamma}) > \theta_1^{sup}(p_1)$ and $\overline{\theta}(q_1 + \frac{1}{\gamma}, q_2 - \frac{1}{\gamma}) > \overline{\theta}(p_1, q_2 - \frac{1}{\gamma})$. Consequently, $V_1(q_1 + \frac{1}{\gamma}, q_2 - \frac{1}{\gamma}) > V_1(p_1, q_2 - \frac{1}{\gamma})$ while $V_2(q_1 + \frac{1}{\gamma}, q_2 - \frac{1}{\gamma}) \leq V_2(p_1, q_2 - \frac{1}{\gamma})$. Finally, if $p_1 \geq q_2 - \frac{1}{\gamma}$, then $I_1 \subset I_2$ so that $VS_1(p_1, 1 - q_1 - \frac{1}{\gamma}) = 0$. We can conclude that $br_1(q_2 - \frac{1}{\gamma}) = \{q_1 + \frac{1}{\gamma}\}$. By a symmetric argument, it can be deduced that $br_2(q_1 + \frac{1}{\gamma}) = \{q_2 - \frac{1}{\gamma}\}$. It follows from Claim 5 that $(q_1 + \frac{1}{\gamma}, q_2 - \frac{1}{\gamma}) = (q_1 + \frac{1}{\gamma}, 1 - q_1 - \frac{1}{\gamma})$ is the unique equilibrium.

Case 2: $\overline{u}_0 = -\frac{\gamma}{2}(\frac{1}{2} - q_1)^2 - \frac{1}{2}$.

We first show that $br_1(\frac{1}{2}) = \{\frac{1}{2}\}$. Let $p_2 = \frac{1}{2}$, it follows from $\overline{u} \leq -\frac{\gamma}{2}(\frac{1}{2} - q_1)^2 - \frac{1}{2}$ that $X \subseteq I_2$. Consider $p_1 < \frac{1}{2}$, we can show that $\overline{\theta}(p_1, \frac{1}{2}) < \frac{1}{2}$. Consequently, $V_2(p_1, \frac{1}{2}) > \frac{1}{2}$ which implies $VS_1(p_1, \frac{1}{2}) < \frac{1}{2}$. As $VS_1(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2}$, we have shown that there is no $p_1 < \frac{1}{2}$ that belongs to $br_1(\frac{1}{2})$.

Consider now $p_1 > \frac{1}{2}$. I denote $\overline{\theta}_0(p_1, p_2) := \frac{1}{2}(\theta_1^{inf}(p_1) + \theta_2^{sup}(p_2))$, the indifferent citizen when $p_1 > p_2$. It follows that $V_2(p_1, \frac{1}{2}) = \overline{\theta}_0(p_1, \frac{1}{2})$. We can show that $\overline{\theta}_0(p_1, \frac{1}{2}) > \frac{1}{2}$. Consequently, we have $V_2(p_1, \frac{1}{2}) > \frac{1}{2}$, which implies $VS_1(p_1, \frac{1}{2}) < \frac{1}{2}$. We have shown that there is no $p_1 > \frac{1}{2}$ that belongs to $br_1(\frac{1}{2})$. We can conclude that $br_1(\frac{1}{2}) = \{\frac{1}{2}\}$. By a symmetric argument, it can be deduced that $br_2(\frac{1}{2}) = \{\frac{1}{2}\}$. It follows from Claim 5 that $(\frac{1}{2}, \frac{1}{2})$ is the unique equilibrium.

□

A.6 Proof of Proposition 2

Proof. We first show that if the pair (p_1, p_2) is an equilibrium of the vote share maximization game, then it is an equilibrium of the plurality maximization game.

Let (p_1, p_2) be an equilibrium of the vote share maximization game. It follows from Claim 2 that $VS_1(p_1, p_2) = VS_2(p_1, p_2) = \frac{1}{2}$. Consequently, there is no deviation p_1' such that $VS_1(p_1', p_2) > \frac{1}{2}$ and no deviation p_2' such that $VS_2(p_1, p_2') > \frac{1}{2}$. Equivalently, there is no deviation p_1' such that $V_1(p_1', p_2) - V_1(p_1, p_2) > V_2(p_1', p_2) - V_2(p_1, p_2)$ and no deviation p_2' such that $V_2(p_1, p_2') - V_2(p_1, p_2) > V_1(p_1, p_2') - V_1(p_1, p_2)$. Rearranging the terms of the two last equations, we obtain $\mathcal{P}_1(p_1', p_2) > \mathcal{P}_1(p_1, p_2)$ and $\mathcal{P}_2(p_1, p_2') > \mathcal{P}_2(p_1, p_2)$. We can conclude that if (p_1, p_2) is an equilibrium of the vote share maximization game, then it is an equilibrium of the plurality maximization game.

We can also prove that if (p_1, p_2) is an equilibrium of the plurality maximization game, then it is an equilibrium of the vote share maximization game.

Let (p_1, p_2) be an equilibrium of the plurality maximization game. Using the same reasoning as in the proof of Claim 2, we obtain $\mathcal{P}_1(p_1, p_2) = \mathcal{P}_2(p_1, p_2) = 0$. It follows that there is no deviation p_1' such that $V_1(p_1', p_2) - V_2(p_1', p_2) > 0$ and no deviation p_2' such that $V_2(p_1, p_2') - V_1(p_1, p_2') > 0$. With some algebra, the two last equations can be rewritten as: $VS_1(p_1', p_2) > \frac{1}{2}$ and $VS_2(p_1, p_2') > \frac{1}{2}$. We can conclude that if (p_1, p_2) is an equilibrium of the plurality maximization game, then it is an equilibrium of the vote share maximization game. □

A.7 Proof of Proposition 3

Proof. Let (p_1, p_2) be an equilibrium of the winner-takes-all game. Using the same reasoning as in the proof of Claim 2, we obtain $\pi_1(p_1, p_2) = \pi_2(p_1, p_2) = \frac{1}{2}$. Consequently, there is no deviation p_1' such that $\pi_1(p_1', p_2) > \frac{1}{2}$ and no deviation p_2' such that $\pi_2(p_1, p_2') > \frac{1}{2}$. The two last equations can be rewritten as $VS_1(p_1', p_2) > \frac{1}{2}$ and $VS_2(p_1, p_2') > \frac{1}{2}$. This is exactly the definition of an equilibrium in the vote share maximization game. We can conclude that if (p_1, p_2) is an equilibrium of the vote share maximization game, then it is an equilibrium of the winner-takes-all game, and vice versa.

□

A.8 Proof of Theorem 2 and Theorem 3

Claim 9. *If* (p_1^*, p_2^*) *is an equilibrium, then* $p_1^* \leq \frac{1}{2} \leq p_2^*$.

Proof. Let (p_1, p_2) with $p_1 > \frac{1}{2}$ be an equilibrium. First, we can show that we cannot have $I_1 \cap I_2 = \emptyset$. Candidate 1 can indeed increase her attraction interval (and thus her number of votes) by deviating from p_1 toward her initial position.

Now, let $I_1 \cap I_2 \neq \emptyset$, we show that it contradicts that (p_1, p_2) is an equilibrium. First, consider $p_1 = p_2$. It follows from Lemma 2 that $V_1(p_2, p_2) = 0$, candidate 1 thus has a profitable deviation. We now consider two cases: (i) $p_1 < p_2$ and (ii) $p_1 > p_2$.

Case (i). We first assume that $p_1 < p_2$. It follows from $I_1 \cap I_2 \neq \emptyset$ that $\theta_1^{sup}(p_1) > \theta_2^{inf}(p_2)$. If $\theta_1^{inf}(p_1) \leq 0$, then $\theta_1^{sup}(p_1) > 1$, and it follows from Lemma 2 that candidate 2 has a profitable deviation $p_2' = p_1$ as $V_2(p_1, p_1) = 1$. If $\theta_1^{inf}(p_1) > 0$, then $V_1(p_1, p_2) = \overline{\theta}(p_1, p_2) - \theta_1^{inf}(p_1)$. As the function $\overline{\theta}(p_1, p_2) - \theta_1^{inf}(p_1)$ is concave in p_1 and maximized in $q_1 - \frac{1}{3\gamma} < \frac{1}{2}$, we can deduce that $p_1' = \frac{1}{2}$ is a profitable deviation for candidate 1. It contradicts that $p_1 > \frac{1}{2}$ and $p_1 < p_2$ hold together.

Case (ii). Suppose now that $p_1 > p_2$. It follows from $I_1 \cap I_2 \neq \emptyset$ that $\theta_1^{inf}(p_1) > \theta_2^{sup}(p_2)$. We first show that we must have $p_2 \geq \frac{1}{2}$.

Let $p_2 < \frac{1}{2}$, we show that we will obtain a contradiction. Candidate 1's number of votes can be written either as $V_1(p_1, p_2) = 1 - \overline{\theta}_0(p_1, p_2)$ or as $V_1(p_1, p_2) = \theta_1^{sup}(p_1) - \overline{\theta}_0(p_1, p_2)$. As $\arg\max_{p_1}\{1 - \overline{\theta}_0(p_1, p_2)\} = \{q_1 - \frac{1}{\gamma}\}$ and $q_1 - \frac{1}{\gamma} < \frac{1}{2}$, it follows that $I_1 \subseteq X$ at equilibrium. Also, as $\arg\max_{p_2}\{\overline{\theta}_0(p_1, p_2)\} = \{q_2 + \frac{1}{\gamma}\}$ and $q_2 + \frac{1}{\gamma} > \frac{1}{2}$, it follows that $I_2 \subseteq X$ at equilibrium.

We just proved that we must have $I_1 \subseteq X$ and $I_2 \subseteq X$, we are now ready to contradict that $p_2 < \frac{1}{2}$. First, if $p_1 \leq q_2$, then $p_2' \in (p_2, p_1]$ such that $I_1 \subset I_2$ and $I_2 \subseteq X$ is a profitable deviation for candidate 2. She would indeed increase the length of her attraction interval, while ensuring that all citizens included in this interval vote for her. It follows that we have $p_1 > q_2$. A symmetric argument also provides that $p_2 < q_1$. Finally, we have $|p_2 - q_1| > |p_1 - q_1|$ and $|p_1 - q_2| > |p_2 - q_2|$. Otherwise, similar deviations are still profitable. However, the first inequality is equivalent to $p_1 + p_2 < 2q_1 \leq 1$ while the second one is equivalent to $p_1 + p_2 > 2 - 2q_1 \geq 1$. We therefore contradicted that $p_2 < \frac{1}{2}$.

To contradict that $p_1 > p_2 \geq \frac{1}{2}$, we are going to show that $p_1' = 1 - p_1$ is a better response against p_2 than $p_1 > \frac{1}{2}$, i.e $V_1(1 - p_1, p_2) > V_1(p_1, p_2)$.

We proved that we have $I_1 \subseteq X$. It follows that $V_1(p_1,p_2) = \theta_1^{sup}(p_1) - \overline{\theta}_0(p_1,p_2)$ with $\theta_1^{sup}(p_1) \leq 1$. We divide the interval $V_1(p_1,p_2)$ into two parts: $l(p_1,p_2)$ on the left of p_1 and $r(p_1,p_2)$ on the right. We have $l(p_1,p_2) = p_1 - \overline{\theta}_0(p_1,p_2)$ and $r(p_1,p_2) = \theta_1^{sup}(p_1) - p_1$. We consider now $p_1' = 1-p_1$ $(1-p_1 \leq p_2)$, we have $V_1(1-p_1,p_2) = l(1-p_1,p_2) + r(1-p_1,p_2)$ with:

$$l(1-p_1,p_2) = \begin{cases} 1-p_1 & \text{if } \theta_1^{inf}(1-p_1) < 0 \\ 1-p_1-\theta_1^{inf}(1-p_1) & \text{otherwise,} \end{cases}$$

and:

$$r(1-p_1,p_2) = \begin{cases} \theta_1^{sup}(1-p_1)-(1-p_1) & \text{if } \theta_1^{sup}(1-p_1) \leq \theta_2^{inf}(p_2) \\ \overline{\theta}(1-p_1,p_2)-(1-p_1) & \text{otherwise.} \end{cases}$$

We will show that $l(1-p_1,p_2) \geq r(p_1,p_2)$ and that $r(1-p_1,p_2) > l(p_1,p_2)$. We first prove that $l(1-p_1,p_2) \geq r(p_1,p_2)$.

We know that I_1 is symmetric around candidate 1's platform. Also, it follows from $q_1 < \frac{1}{2}$ that $|1-p_1-q_1| < |p_1-q_1|$. Consequently, $1-p_1-\theta_1^{inf}(1-p_1) > \theta_1^{sup}(p_1) - p_1$. We can deduce that $l(1-p_1,p_2) \geq r(p_1,p_2)$.

We now prove that $r(1-p_1,p_2) > l(p_1,p_2)$. It follows from $|1-p_1-q_1| < |p_1-q_1|$ that $\theta_1^{sup}(1-p_1) - (1-p_1) > p_1 - \theta_1^{inf}(p_1)$. As $p_1 - \theta_1^{inf}(p_1) > p_1 - \overline{\theta}_0(p_1,p_2)$, then we can conclude that $\theta_1^{sup}(1-p_1) - (1-p_1) > l(p_1,p_2)$. We are left to prove that $\overline{\theta}(1-p_1,p_2) - (1-p_1) > p_1 - \overline{\theta}_0(p_1,p_2)$. This inequality can be simplified as: $p_2 - \frac{1}{2} > \frac{\gamma}{4}(1-p_1-q_1)^2 - \frac{\gamma}{4}(p_1-q_1)^2$. It follows from $p_2 \geq \frac{1}{2}$ and from $|1-p_1-q_1| < |p_1-q_1|$ that this inequality holds.

We can conclude that $V_1(1-p_1,p_2) > V_1(p_1,p_2)$. We therefore contradicted that (p_1,p_2) with $p_1 > \frac{1}{2}$ is an equilibrium. By a symmetric argument, it can be deduced that $p_2 \geq \frac{1}{2}$ at equilibrium.

In the subsequent sections of this proof, we will first prove Theorem 2. Then, we will study the symmetric equilibrium stated in Theorem 3 as well as all additional asymmetric equilibria. The case where $\frac{1}{4} < q_1 - \frac{1}{3\gamma}$ will be presented through Claim 13, Claim 14 and Claim 15. Note that in this proof, $br_j(p_{-j}) = \arg\max_{p_j} V_j(p_1,p_2)$, and that Claim 4 holds (the proof of this claim holds true in both the vote share maximization and the number of votes maximization games).

□

A.8.1 Theorem 2

Claim 10. *If $\overline{u} \geq -q_1$, then $(p_1^*, 1-p_1^*)$ with $p_1^* = q_1$ is the unique pure strategy equilibrium.*

Proof. Let $\overline{u} \geq -q_1$. If $(p_1, p_2) = (q_1, 1-q_1) = (q_1, q_2)$, it follows from Lemma 1 that $V_1(p_1, p_2) = \ell(I_1)$. As $\arg\max_{p_1}\{\ell(I_1)\} = \{q_1\}$, it follows that $br_1(q_2) = \{q_1\}$. By a symmetrical argument, it can be deduced that $br_2(q_1) = \{q_2\}$.

We proved that $(q_1, 1-q_1)$ is an equilibrium, we now prove that it is unique. Let (p_1, p_2) with $p_1 \neq q_1$ and $p_2 \neq q_2$ be an equilibrium. We will obtain a contradiction. If $p_1 > q_1$, as $\theta_1^{inf}(p_1) > 0$ and as $\arg\max_{p_1}\{\overline{\theta}(p_1, p_2) - \theta_1^{inf}(p_1)\} = \{q_1 - \frac{1}{3\gamma}\}$, then $p_1' = q_1$ is a profitable deviation for candidate 1. If $p_1 < q_1 - \frac{1}{3\gamma}$, as $\arg\max_{p_1}\{\overline{\theta}(p_1, p_2)\} = \arg\max_{p_1}\{\theta_1^{sup}(p_1)\} = \{q_1 + \frac{1}{\gamma}\}$, then candidate 1 has a profitable marginal deviation toward the right.

We thus have $p_1 \in [q_1 - \frac{1}{3\gamma}, q_1)$. Let $p_1 \in (q_1 - \frac{1}{3\gamma}, q_1)$. If $\theta_1^{sup}(p_1) < \theta_2^{inf}(p_2)$, then candidate 1 has a profitable marginal deviation toward the right. However, if $\theta_1^{sup}(p_1) > \theta_2^{inf}(p_2)$, then she has a profitable deviation toward the left.

Consequently, we either have $(p_1, p_2) = (q_1 - \frac{1}{3\gamma}, p_2)$ with $\theta_1^{sup}(q_1 - \frac{1}{3\gamma}) > \theta_2^{inf}(p_2)$, or (p_1, p_2) with $p_1 \in (q_1 - \frac{1}{3\gamma}, q_1)$ and $\theta_1^{sup}(p_1) = \theta_2^{inf}(p_2)$. The first pair is not an equilibrium. Indeed, it follows from $\overline{u} \geq -q_1$ that $p_2 < \arg\max_{p_2}\{\theta_2^{sup}(p_2) - \overline{\theta}(p_1, p_2)\}$. Candidate 2 thus has a marginal profitable deviation toward the right. The second one also contradicts that (p_1, p_2) is an equilibrium. As $\overline{u} \geq -q_1$, then $p_2 < q_2$, $p_2' = q_2$ is thus a profitable deviation for candidate 2.

We contradicted that (p_1, p_2) with $p_1 \neq q_1$ and $p_2 \neq q_2$ is an equilibrium. We can conclude that $(q_1, 1-q_1)$ is the unique equilibrium.

□

Claim 11. *If $\overline{u} \in [\overline{u}_0, -q_1)$, then $(p_1^*, 1-p_1^*)$ with $p_1^* = \hat{p}_1$ is the unique pure strategy equilibrium.*

Let $\overline{u} \in [\overline{u}_0, -q_1)$, and $p_2 = \hat{p}_2 = 1 - \hat{p}_1$, we first prove that $br_1(\hat{p}_2) = \{\hat{p}_1\}$.

As stated in Claim 7's proof, we have $\hat{p}_1 \leq \overline{p}_1$. Consequently, if $p_1 < \hat{p}_1$, then[10] $\theta_1^{inf}(p_1) < 0$ and $p_1' = \hat{p}_1$ is a profitable deviation for candidate 1. More generally, $\hat{p}_1$

[10]If $p_1 < q_1 - \frac{1}{\gamma}$, it is possible that $\theta_1^{inf}(p_1) > 0$. In this case, candidate 1 has a marginal profitable deviation toward the right, as $\theta_1^{inf}(p_1)$ would decrease, while $\theta_1^{sup}(p_1)$ and $\overline{\theta}(p_1, p_2)$ would increase.

is a better response for candidate 1 than any $p_1 < \hat{p}_1$ against a platform $p_2 \geq \frac{1}{2}$. We proved that there is no $p_1 < \hat{p}_1$ that belongs to $br_1(\hat{p}_2)$.

Now we consider $p_1 > \hat{p}_1$. Note that we can show that $\hat{p}_1 > q_1$. It follows that if $\theta_1^{sup}(p_1) \leq \theta_2^{inf}(\hat{p}_2)$, then $p_1' = \hat{p}_1$ is a profitable deviation for candidate 1. If $\theta_1^{sup}(p_1) \in (\theta_2^{inf}(\hat{p}_2), \theta_2^{sup}(\hat{p}_2)]$, it follows from $\arg\max_{p_1}\{\overline{\theta}(p_1, p_2) - \theta_1^{inf}(p_1)\} = \{q_1 - \frac{1}{3\gamma}\}$ that p_1' such that $\theta_1^{sup}(p_1') = \theta_2^{inf}(\hat{p}_2)$ is a profitable deviation for candidate 1. As $q_1 \leq \frac{1}{4}$, then $p_1' \geq \hat{p}_1$. Consequently, $p_1'' = \hat{p}_1$ is a profitable deviation for candidate 1. We proved that there is no $p_1 > \hat{p}_1$ such that $\theta_1^{sup}(p_1) \leq \theta_2^{sup}(\hat{p}_2)$ that belongs to $br_1(\hat{p}_2)$.

Now let $p_1 > \hat{p}_1$ such that $\theta_1^{sup}(p_1) > \theta_2^{sup}(\hat{p}_2)$. We proved in Claim 9 that if $p_1 > p_2 \geq \frac{1}{2}$, then $p_1''' = 1 - p_1$ is a better response against p_2 than p_1. We can deduce that there is no $p_1 > \hat{p}_1$ such that $\theta_1^{sup}(p_1) > \theta_2^{sup}(\hat{p}_2)$ that belongs to $br_1(\hat{p}_2)$.

We can finally conclude that $br_1(\hat{p}_2) = \{\hat{p}_1\}$. By a symmetric argument, we can deduce that $br_2(\hat{p}_1) = \{\hat{p}_2\}$.

We proved that $(\hat{p}_1, \hat{p}_2) = (\hat{p}_1, 1-\hat{p}_1)$ is an equilibrium, we now show that it is unique. Let (p_1, p_2) with $p_1 \neq \hat{p}_1$ and $p_2 \neq \hat{p}_2$ be an equilibrium. We will obtain a contradiction. We have seen in the first part of the proof that $\hat{p}_1$ is a better response for candidate 1 than any $p_1 < \hat{p}_1$ against a platform $p_2 \geq \frac{1}{2}$. We can deduce that $p_1 > \hat{p}_1$ and symmetrically that $p_2 < \hat{p}_2$.

If $\theta_1^{sup}(p_1) \leq \theta_2^{inf}(p_2)$, then it follows from $\hat{p}_1 > q_1$ that $p_1' = \hat{p}_1$ is a profitable deviation for candidate 1, which contradicts that (p_1, p_2) is an equilibrium.

If $\theta_1^{sup}(p_1) > \theta_2^{inf}(p_2)$, then it follows from $\arg\max_{p_1}\{\overline{\theta}(p_1, p_2)\} = \{q_1 - \frac{1}{3\gamma}\}$ that candidate 1 has profitable deviations which contradicts that (p_1, p_2) is an equilibrium.

We contradicted that (p_1, p_2) with $p_1 \neq \hat{p}_1$ and $p_2 \neq \hat{p}_2$ is an equilibrium. We can conclude that $(\hat{p}_1, 1 - \hat{p}_1)$ is the unique equilibrium.

Claim 12. *If $\overline{u} < \overline{u}_0$, then $(p_1^*, 1 - p_1^*)$ with $p_1^* = \overline{p}_1$ is the unique pure strategy equilibrium.*

Proof. Let $\overline{u} < \overline{u}_0$. I will use the same method as in Claim 8, considering the two following cases: (i) $\overline{u}_0 = -q_1 - \frac{3}{2\gamma}$ and (ii) $\overline{u}_0 = -\frac{\gamma}{2}(\frac{1}{2} - q_1)^2 - \frac{1}{2}$. We will show in case (i) that $(q_1 + \frac{1}{\gamma}, 1 - q_1 - \frac{1}{\gamma})$ is the unique equilibrium. In case (ii), we will prove that $(\frac{1}{2}, \frac{1}{2})$ is the unique equilibrium.

Case (i). We first show that $br_1(q_2 - \frac{1}{\gamma}) = \{q_1 + \frac{1}{\gamma}\}$. We proved in Claim 8 that if $p_1 \leq q_2 - \frac{1}{\gamma}$ and $p_1 \neq q_1 + \frac{1}{\gamma}$, then $V_1(q_1 + \frac{1}{\gamma}, q_2 - \frac{1}{\gamma}) > V_1(p_1, q_2 - \frac{1}{\gamma})$. Consider now

$p_1 > q_2 - \frac{1}{\gamma}$. We proved in Claim 9 that if $p_1 > p_2 \geq \frac{1}{2}$, then $p_1' = 1 - p_1$ is a better response against p_2 than p_1. Consequently, there is no $p_1 > q_2 - \frac{1}{\gamma}$ that belongs to $br_1(q_2 - \frac{1}{\gamma})$.

We can conclude that $br_1(q_2 - \frac{1}{\gamma}) = \{q_1 + \frac{1}{\gamma}\}$. By a symmetric argument, it can be deduced that $br_2(q_1 + \frac{1}{\gamma}) = \{q_2 - \frac{1}{\gamma}\}$

We proved that $(q_1 + \frac{1}{\gamma}, q_2 - \frac{1}{\gamma}) = (q_1 + \frac{1}{\gamma}, 1 - q_1 - \frac{1}{\gamma})$ is an equilibrium. We now show that it is unique. It follows from $\arg\max_{p_1}\{\overline{\theta}(p_1, p_2)\} = \arg\max_{p_1}\{\theta_1^{sup}(p_1)\} = \{q_1 + \frac{1}{\gamma}\}$ and from $q_1 + \frac{1}{\gamma} < \hat{p}_1$ that $p_1 = q_1 + \frac{1}{\gamma}$ is the best response of candidate 1 against any $p_2 \geq \frac{1}{2}$. Symmetrically, $p_2 = q_2 - \frac{1}{\gamma}$ is the best response of candidate 2 against any $p_1 \leq \frac{1}{2}$. We can conclude that $(q_1 + \frac{1}{\gamma}, 1 - q_1 - \frac{1}{\gamma})$ is the unique equilibrium.

Case (ii). We first show that $br_1(\frac{1}{2}) = \{\frac{1}{2}\}$. Consider first $p_1 < \frac{1}{2}$. It follows from $\frac{1}{2} \leq q_1 + \frac{1}{\gamma} < \hat{p}_1$ that $p_1' = \frac{1}{2}$ is a profitable deviation for candidate 1 against any $p_2 \geq \frac{1}{2}$. It increases $\theta_1^{sup}(p_1)$ and $\overline{\theta}(p_1, p_2)$ while $\theta_1^{inf}(p_1)$ decreases or remains negative. We can deduce that there is no $p_1 < \frac{1}{2}$ that belongs to $br_1(\frac{1}{2})$. Consider now $p_1 > \frac{1}{2}$ and let $p_2 = \frac{1}{2}$. We proved in Claim 9 that if $p_1 > p_2 \geq \frac{1}{2}$, then $p_1' = 1 - p_1$ is a better response for candidate 1 against p_2 than p_1. Consequently, there is no $p_1 > \frac{1}{2}$ that belongs to $br_1(\frac{1}{2})$. We can conclude that $br_1(\frac{1}{2}) = \{\frac{1}{2}\}$. Using a symmetric argument, it can be deduced that $br_2(\frac{1}{2}) = \{\frac{1}{2}\}$.

We proved that $(\frac{1}{2}, \frac{1}{2})$ is an equilibrium. Now, we show that it is unique.

We have already proved that $p_1 = \frac{1}{2}$ is a better response for candidate 1 than any $p_1 < \frac{1}{2}$ against any $p_2 \geq \frac{1}{2}$. Symmetrically, $p_2 = \frac{1}{2}$ is a better response for candidate 2 than any $p_2 > \frac{1}{2}$ against any $p_1 \geq \frac{1}{2}$. It follows from Claim 9 that $(\frac{1}{2}, \frac{1}{2})$ is the unique equilibrium. □

A.8.2 Theorem 3

Let $q_1 > \frac{1}{4}$. A couple of results presented in Theorem 2 and proofs associated still hold. If $\overline{u} \geq q_1 - \frac{1}{2}$ (note that $q_1 - \frac{1}{2} > -q_1$), then $(p_1, 1 - p_1)$ with $p_1 = q_1$ is the unique pure strategy equilibrium. If $\overline{u} < \overline{u}_0$, then $(p_1, 1 - p_1)$ with $p_1 = \min\{q_1 + \frac{1}{\gamma}, \frac{1}{2}\}$ is the unique pure strategy equilibrium. The region of the alienation threshold $\overline{u}$ for which $(\hat{p}_1, 1 - \hat{p}_1)$ is the unique pure strategy equilibrium changes a bit. When $\frac{1}{4} \geq q_1 - \frac{1}{3\gamma}$, $(\hat{p}_1, 1 - \hat{p}_1)$ is the unique pure strategy equilibrium if $\overline{u}$ belongs to the interval $[\overline{u}_0, -\frac{\gamma}{2}(q_1 - \frac{1}{4})^2 - \frac{1}{4})$. However, when $\frac{1}{4} < q_1 - \frac{1}{3\gamma}$, $(\hat{p}_1, 1 - \hat{p}_1)$ is the unique pure strategy equilibrium if

$\overline{u} \in [\overline{u}_0, -q_1 + \frac{5}{18\gamma})$. We will prove it in Claim 13. In Claim 14, we will treat the case where $\overline{u} \in [\max\{q_1 - \frac{9\gamma+7}{18\gamma}, -\frac{\gamma}{2}(q_1 - \frac{1}{4})^2 - \frac{1}{4}\}, q_1 - \frac{1}{2})$. Finally, in Claim 15, we will study the case where $\frac{1}{4} < q_1 - \frac{1}{3\gamma}$ and $\overline{u} \in [-q_1 + \frac{5}{18\gamma}, q_1 - \frac{9\gamma+7}{18\gamma})$.

Claim 13. *If $\frac{1}{4} \geq q_1 - \frac{1}{3\gamma}$ and $\overline{u} \in [\overline{u}_0, -\frac{\gamma}{2}(q_1 - \frac{1}{4})^2 - \frac{1}{4})$, or if $\frac{1}{4} < q_1 - \frac{1}{3\gamma}$ and $\overline{u} \in [\overline{u}_0, -q_1 + \frac{5}{18\gamma})$, then $(p_1^*, 1 - p_1^*)$ with $p_1^* = \hat{p}_1$ is the unique pure strategy equilibrium.*

Proof. Let $\frac{1}{4} \geq q_1 - \frac{1}{3\gamma}$ and $\overline{u} \in [\overline{u}_0, -\frac{\gamma}{2}(q_1 - \frac{1}{4})^2 - \frac{1}{4})$. Similar and thus omitted arguments can be used to treat the case where $\frac{1}{4} < q_1 - \frac{1}{3\gamma}$ and $\overline{u} \in [\overline{u}_0, -q_1 + \frac{5}{18\gamma})$.

We first show that $br_1(\hat{p}_2) = \{\hat{p}_1\}$. We proved in Claim 11 that $p_1 = \hat{p}_1$ is a better response for candidate 1 than any $p_1 < \hat{p}_1$ against a platform $p_2 \geq \frac{1}{2}$. We consider now $p_1 > \hat{p}_1$. We saw in Claim 11 that if $\theta_1^{sup}(p_1) > \theta_2^{inf}(\hat{p}_2)$, then $\hat{p}_1$ is a profitable deviation[11] for candidate 1. Now let's focus on $\theta_1^{sup}(p_1) \leq \theta_2^{inf}(\hat{p}_2)$. It follows from $\overline{u} < -\frac{\gamma}{2}(q_1 - \frac{1}{4})^2 - \frac{1}{4}$ that[12] $\theta_1^{sup}(\hat{p}_1) > \theta_2^{inf}(\hat{p}_2)$. Since $\theta_1^{sup}(p_1)$ is a decreasing function of p_1 for $p_1 > \overline{p}_1$, it follows that $p_1 > \overline{p}_1$. Consequently, $\overline{p}_1$ is a profitable deviation for candidate 1, we can deduce that there is no $p_1 > \hat{p}_1$ such that $\theta_1^{sup}(p_1) \leq \theta_2^{inf}(\hat{p}_2)$ that belongs to $br_1(\hat{p}_2)$.

We can conclude that $br_1(\hat{p}_2) = \{\hat{p}_1\}$. It can be deduced by a symmetric argument that $br_2(\hat{p}_1) = \{\hat{p}_2\}$. We thus proved that $(\hat{p}_1, \hat{p}_2) = (\hat{p}_1, 1 - \hat{p}_1)$ is an equilibrium. We now show that it is unique.

Let (p_1, p_2) with $p_1 \neq \hat{p}_1$ and $p_2 \neq \hat{p}_2$ be an equilibrium. As proved in Claim 11, we have $p_1 > \hat{p}_1$ and $p_2 < \hat{p}_2$.

If $\theta_1^{sup}(p_1) > \theta_2^{inf}(p_2)$, then it follows from $\arg\max_{p_1}\{\overline{\theta}(p_1, p_2) - \theta_1^{inf}(p_1)\} = \{q_1 - \frac{1}{3\gamma}\}$ that candidate 1 has profitable deviations which contradicts that (p_1, p_2) is an equilibrium.

If $\theta_1^{sup}(p_1) \leq \theta_2^{inf}(p_2)$, then we either have $p_1 > \overline{p}_1$, $p_2 < \overline{p}_2$ or both. It contradicts that (p_1, p_2) is an equilibrium.

We contradicted that (p_1, p_2) with $p_1 \neq \hat{p}_1$ and $p_2 \neq \hat{p}_2$ is an equilibrium. We can conclude that $(\hat{p}_1, 1 - \hat{p}_1)$ is the unique equilibrium. □

Claim 14. *If $\overline{u} \in [\max\{q_1 - \frac{9\gamma+7}{18\gamma}, -\frac{\gamma}{2}(q_1 - \frac{1}{4})^2 - \frac{1}{4}\}, q_1 - \frac{1}{2})$, then the set E of equilibria is $E \equiv \{(p_1, p_2) | \theta_1^{sup}(p_1) = \theta_2^{inf}(p_2), p_1 \in [q_1 - \frac{1}{3\gamma}, q_1], p_2 \in [q_2, q_2 + \frac{1}{3\gamma}], \theta_1^{inf}(p_1) \geq 0, \theta_2^{sup}(p_2) \leq 1\}$.*

[11] In this case, $\hat{p}_1$ is a better deviation than p_1' such that $\theta_1^{sup}(p_1') = \theta_2^{inf}(p_2)$ because we can show that $p_1' < \hat{p}_1$.

[12] It is also the case for $\frac{1}{4} < q_1 - \frac{1}{3\gamma}$ and $\overline{u} \in [\overline{u}_0, -q_1 + \frac{5}{18\gamma})$.

Proof. Let $\overline{u} \in [\max\{q_1 - \frac{9\gamma+7}{18\gamma}, -\frac{\gamma}{2}(q_1-\frac{1}{4})^2 - \frac{1}{4}\}, q_1 - \frac{1}{2})$. We first prove that if $(p_1,p_2) \in E$, then there is no profitable unilateral deviation. First, note that if $(p_1,p_2) \in E$, then for all j, $V_j(p_1,p_2) = \ell(I_j)$. As $p_1 \leq q_1$, then a deviation $p_1' < p_1$ is not profitable for candidate 1. Consider now a deviation $p_1'' \in (p_1,p_2)$. It follows from $\arg\max_{p_1}\{\overline{\theta}(p_1,p_2) - \theta_1^{inf}(p_1)\} = \{q_1 - \frac{1}{3\gamma}\}$ and from $p_1 \geq q_1 - \frac{1}{3\gamma}$ that $p_1'' > p_1$ is not a profitable deviation for candidate 1. Finally, as proved in Claim 9, a platform p_1 such that $p_1 > p_2 \geq \frac{1}{2}$ is never a best reponse against p_2.

The argument for candidate 2 is similar and thus omitted. We can deduce that if $\overline{u} \in [\max\{q_1 - \frac{9\gamma+7}{18\gamma}, -\frac{\gamma}{2}(q_1-\frac{1}{4})^2 - \frac{1}{4}\}, q_1 - \frac{1}{2})$ and if $(p_1,p_2) \in E$, then it is an equilibrium.

Before proving that if (p_1,p_2) is an equilibrium, then $(p_1,p_2) \in E$, let's show that E is not empty for $\overline{u} \in [\max\{q_1 - \frac{9\gamma+7}{18\gamma}, -\frac{\gamma}{2}(q_1-\frac{1}{4})^2 - \frac{1}{4}\}, q_1 - \frac{1}{2})$. Denote $p_1(p_2)$ the platform p_1 as a function of the policy p_2 such that $\theta_1^{sup}(p_1) = \theta_2^{inf}(p_2)$ and $p_1 \leq q_1 + \frac{1}{\gamma}$:

$$p_1(p_2) = q_1 + \frac{1 - \sqrt{1 + 2\gamma(q_1 - p_2) - 4\gamma\overline{u} - \gamma^2(p_2 - q_2)^2}}{\gamma}$$

It is a continuous and increasing function of p_2 for $p_2 \in [1 - q_2, q_2 + \frac{1}{3\gamma}]$. Moreover, $p_1(q_2) \in [q_1 - \frac{1}{3\gamma}, q_1]$ is equivalent to $\overline{u} \in [q_1 - \frac{18\gamma+7}{36\gamma}, q_1 - \frac{1}{2}]$ and $p_1(q_2 + \frac{1}{3\gamma}) \in [q_1 - \frac{1}{3\gamma}, q_1]$ is equivalent to $\overline{u} \in [q_1 - \frac{9\gamma+7}{18\gamma}, q_1 - \frac{18\gamma+7}{36\gamma}]$. Consequently, as $p_1(p_2)$ is an increasing function with respect to $\overline{u}$, then there exists a platform $p_1 \in [q_1 - \frac{1}{3\gamma}, q_1]$ against a policy $p_2 \in [q_2, q_2 + \frac{1}{3\gamma}]$ such that $(p_1,p_2) \in E$ for $\overline{u} \in [q_1 - \frac{9\gamma+7}{18\gamma}, q_1 - \frac{1}{2})$. However, a pair $(p_1(p_2), p_2)$ does not belong to the set E if $\theta_1^{inf}(p_1(p_2)) < 0$ and/or $\theta_2^{sup}(p_2) > 1$. We distinguish two cases: $\frac{1}{4} \leq q_1 - \frac{1}{3\gamma}$ and $\frac{1}{4} > q_1 - \frac{1}{3\gamma}$.

If $\frac{1}{4} < q_1 - \frac{1}{3\gamma}$, then we can show that for $p_1 \in [q_1 - \frac{1}{3\gamma}, q_1]$, $\theta_1^{inf}(p_1) \geq 0$. We can deduce by a symmetric argument that for $p_2 \in [q_2, q_2 + \frac{1}{3\gamma}]$, $\theta_2^{sup}(p_2) \leq 1$. We can conclude that if $\overline{u} \in [q_1 - \frac{9\gamma+7}{18\gamma}, q_1 - \frac{1}{2})$, then E is non empty.

If $\frac{1}{4} \geq q_1 - \frac{1}{3\gamma}$, then $\theta_1^{inf}(q_1 - \frac{1}{3\gamma}) < 0$ for some $\overline{u} \in [q_1 - \frac{9\gamma+7}{18\gamma}, q_1 - \frac{1}{2})$. The lower value of $\overline{u}$ for which E is not empty is the one that ensures $\theta_1^{inf}(p_1) = 0$, $\theta_2^{sup}(p_2) = 1$ and $\theta_1^{sup}(p_1) = \theta_2^{inf}(p_2)$, for $p_1 \in [q_1 - \frac{1}{3\gamma}]$ and $p_2 \in [q_2, q_2 + \frac{1}{3\gamma}]$. The only pair (p_1,p_2) that can jointly verify these conditions is $(\frac{1}{4}, \frac{3}{4})$. We can deduce that if $\overline{u} \in [-\frac{\gamma}{2}(q_1 - \frac{1}{4})^2 - \frac{1}{4}, q_1 - \frac{1}{2})$, then E is non empty.

Note that $-\frac{\gamma}{2}(q_1 - \frac{1}{4})^2 - \frac{1}{4} > q_1 - \frac{9\gamma+7}{18\gamma}$ is equivalent to $\frac{1}{4} > q_1 - \frac{1}{3\gamma}$ (under the constraint that $q_1 > \frac{1}{4}$), we thus conclude that E is not empty for $\overline{u} \in [\max\{q_1 - \frac{9\gamma+7}{18\gamma}, -\frac{\gamma}{2}(q_1 -$

$\frac{1}{4})^2 - \frac{1}{4}\}, q_1 - \frac{1}{2})$.

Let $\overline{u} \in [\max\{q_1 - \frac{9\gamma+7}{18\gamma}, -\frac{\gamma}{2}(q_1 - \frac{1}{4})^2 - \frac{1}{4}\}, q_1 - \frac{1}{2})$. We now show that if (p_1, p_2) is an equilibrium, then $(p_1, p_2) \in E$. Let a pair $(p_1, p_2) \notin E$ be an equilibrium. First, note that as $\overline{u} \in [\max\{q_1 - \frac{9\gamma+7}{18\gamma}, -\frac{\gamma}{2}(q_1 - \frac{1}{4})^2 - \frac{1}{4}\}, q_1 - \frac{1}{2})$, then $\theta_1^{inf}(p_1) < 0$ implies $p_1 < q_1$. It follows that if $\theta_1^{inf}(p_1) < 0$, candidate 1 has a profitable deviation p_1' such that $\theta_1^{inf}(p_1') = 0$. Consequently, we have $\theta_1^{inf}(p_1) \geq 0$. It can be deduced by a symmetric argument that $\theta_2^{sup}(p_2) \leq 1$.

If $\theta_1^{sup}(p_1) < \theta_2^{inf}(p_2)$, then it follows from $\overline{u} \in [\max\{q_1 - \frac{9\gamma+7}{18\gamma}, -\frac{\gamma}{2}(q_1 - \frac{1}{4})^2 - \frac{1}{4}\}, q_1 - \frac{1}{2})$ that we[13] have $p_1 < q_1$ and/or $p_2 > q_2$. Consequently, $p_1' \in (p_1, q_1]$ such that $\theta_1^{sup}(p_1') \leq \theta_2^{inf}(p_2)$ is a profitable deviation for candidate 2. We can deduce that we have $\theta_1^{sup}(p_1) \geq \theta_2^{inf}(p_2)$.

Consider $\theta_1^{sup}(p_1) > \theta_2^{inf}(p_2)$. It follows from $\overline{u} \in [\max\{q_1 - \frac{9\gamma+7}{18\gamma}, -\frac{\gamma}{2}(q_1 - \frac{1}{4})^2 - \frac{1}{4}\}, q_1 - \frac{1}{2})$ that $p_1 > q_1 - \frac{1}{3\gamma}$ and/or $p_2 < q_2 + \frac{1}{3\gamma}$. As the function $\overline{\theta}(p_1, p_2) - \theta_1^{inf}(p_1)$ (resp. $\theta_2^{sup}(p_2) - \overline{\theta}(p_1, p_2)$) is strictly concave and maximized for $p_1 = q_1 - \frac{1}{3\gamma}$ (resp. $p_2 = q_2 + \frac{1}{3\gamma}$), then at least one candidate has a profitable deviation. It contradicts that (p_1, p_2) is an equilibrium.

We have $\theta_1^{inf}(p_1) \geq 0$, $\theta_2^{sup}(p_2) \leq 1$ and $\theta_1^{sup}(p_1) = \theta_2^{inf}(p_2)$. Consequently, for all j, $V_j(p_1, p_2) = \ell(I_j)$.

Consider $p_1 > q_1$, it follows from $\theta_1^{inf}(q_1) > 0$ that candidate 1 has a profitable deviation toward her initial position. Consequently, we have $p_1 \leq q_1$. It can be deduced by a symmetric argument that $p_2 \geq q_2$.

Consider now $p_1 < q_1 - \frac{1}{3\gamma}$. It follows from $\arg\max_{p_1}\{\overline{\theta}(p_1, p_2) - \theta_1^{inf}(p_1)\} = \{q_1 - \frac{1}{3\gamma}\}$ that candidate 1 has a profitable deviation toward the center. Consequently, we have $q_1 \geq q_1 - \frac{1}{3\gamma}$. It can be deduced by a symmetric argument that $p_2 \leq q_2 + \frac{1}{3\gamma}$.

We proved that $\theta_1^{inf}(p_1) \geq 0$, $\theta_2^{sup}(p_2) \leq 1$, $\theta_1^{sup}(p_1) = \theta_2^{inf}(p_2)$, $p_1 \in [q_1 - \frac{1}{3\gamma}]$ and $p_2 \in [q_2, q_2 + \frac{1}{3\gamma}]$. It contradicts that $(p_1, p_2) \notin E$. We can conclude that if $\overline{u} \in [\max\{q_1 - \frac{9\gamma+7}{18\gamma}, -\frac{\gamma}{2}(q_1 - \frac{1}{4})^2 - \frac{1}{4}\}, q_1 - \frac{1}{2})$ and if (p_1, p_2) is an equilibrium, then $(p_1, p_2) \in E$. □

Claim 15. *If $\frac{1}{4} < q_1 - \frac{1}{3\gamma}$ and if $\overline{u} \in [-q_1 + \frac{5}{18\gamma}, q_1 - \frac{9\gamma+7}{18\gamma})$, then $(p_1^*, 1 - p_1^*)$ with $p_1^* = q_1 - \frac{1}{3\gamma}$ is the unique pure strategy equilibrium.*

Proof. Let $\overline{u} \in [-q_1 + \frac{5}{18\gamma}, q_1 - \frac{9\gamma+7}{18\gamma})$. We first prove that $br_1(q_2 + \frac{1}{3\gamma}) = \{q_1 - \frac{1}{3\gamma}\}$.

[13] We can also have $p_1 > q_1 + \frac{1}{\gamma}$ and/or $p_2 < q_2 - \frac{1}{\gamma}$. In this case, candidate 1 has a marginal profitable deviation toward the left.

First note that it follows from $\frac{1}{4} < q_1 - \frac{1}{3\gamma}$ and $\overline{u} \in [-q_1 + \frac{5}{18\gamma}, q_1 - \frac{9\gamma+7}{18\gamma})$ that $\theta_1^{inf}(q_1 - \frac{1}{3\gamma}) \geq 0$ and that $\theta_1^{sup}(q_1 - \frac{1}{3\gamma}) > \theta_2^{inf}(q_2 + \frac{1}{3\gamma})$. Consider $p_1 < q_1 - \frac{1}{3\gamma}$. It follows from $\theta_1^{inf}(q_1 - \frac{1}{3\gamma}) \geq 0$ that $\theta_1^{inf}(p_1) < 0$ implies $p_1 < q_1 - \frac{1}{3\gamma} < \overline{p}_1$. Then, if $\theta_1^{inf}(p_1) < 0$, candidate 1 has a profitable deviation toward the right. If $\theta_1^{inf}(p_1) \geq 0$, it follows from $\arg\max_{p_1}\{\overline{\theta}(p_1, p_2) - \theta_1^{inf}(p_1)\} = \{q_1 - \frac{1}{3\gamma}\}$ and from $\arg\max_{p_1}\{\ell(I_1)\} = \{q_1\}$ that candidate 1 has a profitable deviation toward the right. We can deduce that there is no $p_1 < q_1 - \frac{1}{3\gamma}$ that belongs to $br_1(q_2 + \frac{1}{3\gamma})$.

Consider now $p_1 > q_1 - \frac{1}{3\gamma}$. If $\theta_1^{sup}(p_1) \leq \theta_2^{sup}(q_2 + \frac{1}{3\gamma})$, then candidate 1 has a profitable deviation toward the left. Finally, if $p_1 > q_2 + \frac{1}{3\gamma}$ such that $\theta_1^{sup}(p_1) > \theta_2^{sup}(q_2 + \frac{1}{3\gamma})$, then it follows from Claim 9's proof that $p_1' = 1 - p_1$ is a better reponse against $q_2 + \frac{1}{3\gamma}$ than p_1. We thus proved that there is no $p_1 > q_1 - \frac{1}{3\gamma}$ that belongs to $br_1(q_2 + \frac{1}{3\gamma})$.

We can conclude that $br_1(q_2 + \frac{1}{3\gamma}) = \{q_1 - \frac{1}{3\gamma}\}$. It can be deduced by a symmetric argument that $br_2(q_1 - \frac{1}{3\gamma}) = \{q_2 + \frac{1}{3\gamma}\}$.

We proved that $(q_1 - \frac{1}{3\gamma}, q_2 + \frac{1}{3\gamma})$ is an equilibrium, we now show that it is unique. Let a pair $(p_1, p_2) \neq (q_1 - \frac{1}{3\gamma}, 1 - q_1 + \frac{1}{3\gamma})$ be an equilibrium. First, it follows from $\theta_1^{inf}(q_1 - \frac{1}{3\gamma}) \geq 0$ that $\theta_1^{inf}(p_1) < 0$ implies $p_1 < \overline{p}$. Consequently, we have $\theta_1^{inf}(p_1) \geq 0$ and $\theta_2^{sup}(p_2) \leq 1$.

Consider the case where $\theta_1^{sup}(p_1) > \theta_2^{inf}(p_2)$. If $p_1 < q_1 - \frac{1}{3\gamma}$, then candidate 1 has a profitable deviation toward the right. If $p_1 > q_1 - \frac{1}{3\gamma}$, candidate 1 has a profitable deviation toward the left. Consequently, we have $\theta_1^{sup}(p_1) \leq \theta_2^{inf}(p_2)$.

We consider $\theta_1^{sup}(p_1) < \theta_2^{inf}(p_2)$. In this case[14], $p_1 < q_1$ and/or $p_2 > q_2$, it follows that at least one candidate has a profitable deviation toward her initial position.

Finally, let $\theta_1^{sup}(p_1) = \theta_2^{inf}(p_2)$. It follows that $p_1 < q_1 - \frac{1}{3\gamma}$ and/or $p_2 > q_2 + \frac{1}{3\gamma}$. Consequently, at least one candidate has a profitable deviation toward the center. It contradicts that (p_1, p_2) is an equilibrium.

We contradicted that $(p_1, p_2) \neq (q_1 - \frac{1}{3\gamma}, q_2 + \frac{1}{3\gamma})$ is an equilibrium. We can conclude that $(q_1 - \frac{1}{3\gamma}, 1 - q_1 + \frac{1}{3\gamma})$ is the unique equilibrium.

□

[14] We can also have $p_1 > \overline{p}_1$ and/or $p_2 < \overline{p}_2$. It contradicts that (p_1, p_2) is an equilibrium. This argument holds for $\theta_1^{sup}(p_1) = \theta_2^{inf}(p_2)$.