école d'économie d'aix-marseille
aix-marseille school of economics

Working Papers / Documents de travail

# Non-Balanced Endogenous Growth and Structural Change: When Romer Meets Kaldor and Kuznets

Christian Ghiglino
Kazuo Nishimura
Alain Venditti

WP 2018 - Nr 31

# Non-balanced endogenous growth and structural change: when Romer meets Kaldor and Kuznets*

Christian GHIGLINO

*Department of Economics, University of Essex, UK*

Kazuo NISHIMURA

*RIEB, Kobe University, Japan*

and

Alain VENDITTI

*Aix-Marseille Univ., CNRS, EHESS, Centrale Marseille, AMSE*
*& EDHEC Business School, France*

**Abstract:** *We propose a model of non-balanced endogenous growth in which the final good, which can be either consumed or used as capital, is produced using two intermediate inputs, one being "knowledge-intensive". Agents working in the knowledge-intensive sector need to accumulate technological knowledge and thus have to decide how to split their individual unit of time between accumulation of technological knowledge (research) and work. Agents working in the second sector do not need to accumulate knowledge and thus devote all their individual unit of time to work. Individual knowledge therefore becomes a labor-augmenting factor, and knowledge accumulation leads to an unbounded increase in TFP in the knowledge-intensive sector, and thus to endogenous capital deepening. The asymmetry in the growth rates of TFP leads to non-balanced growth. Labor (number of workers) reallocations across sectors occur, leading to a greater increase in output for the knowledge-intensive sector. We show that non-balanced growth is consistent with Kaldor facts, as the asymptotic equilibrium is above all characterized by a constant interest rate and capital share in national income. However, the economy follows a growth path converging to a particular level of wealth that depends on the initial price of capital and knowledge. As a consequence, countries with the same fundamentals but lower initial wealth will be characterized by lower asymptotic wealth. We therefore extend the Lucas [19] finding and prove the existence of non-convergence across countries in a framework with structural change.*

**Keywords:** *Two-sector model, technological knowledge, non-balanced endogenous growth, structural change, Kaldor and Kuznets facts*

*Journal of Economic Literature* Classification Numbers: C62, E32, O41.

*This work was supported by French National Research Agency Grants ANR-08-BLAN-0245-01 and ANR-17-EURE-0020. We thank R. Boucekkine, E. Cruz, J.P. Drugeon, C. Nourry, X. Raurich, T. Seegmuller, G. Sorger and H. Takahashi for useful comments and suggestions.

## 1 Introduction

Building on multisector growth models derived from the benchmark formulation of Uzawa [24], a recent literature has been trying to reconcile the standard concept of balanced growth with both Kaldor [16] and Kuznets [18] facts. Kaldor facts state basically that growth rates, the capital-output ratio, the share of capital income in GDP and the real interest rate are constant. Kuznets facts state basically that there are both structural change due to strong reallocations of labor across sectors, and variations in the relative importance of these sectors.

The literature essentially gives two main explanations for the existence of structural transformations:

i) income effects derived from a specific form of utility functions. This demand-side mechanism is based on the fact that the income effect differs among consumption goods, while the relative prices of all goods are constant (see for instance Alonso-Carrera and Raurich [3], Boppart [8], Buera and Kaboski [9, 10], Echevarria [13], Foellmi and Zweimuleler [14] or Kongsamut *et al.* [17]);

ii) relative price effects of consumption or intermediary goods. This is a supply-side mechanism which relies on relative price variations coming from differences in sectoral growth rates of productivity and/or sectoral differences in the degree of capital-labor substitutability (see for instance Acemoglu and Guerrieri [1], Alvarez-Cuadrado *et al.* [4], Caselli and Coleman [12] or Ngai and Pissarides [22]).[1]

Two types of models are actually found in this literature. There are models that accept structural change along the long-run equilibrium although this still keep its Kaldor characteristics. Others are characterized by equilibria that asymptotically possess Kaldor properties with no structural change but require that structural change occurs along the transition path to the long-run equilibium, and that this transition does not occur in finite time. However, a common property of all these papers is that they rely on exogenous growth arguments with exogenous formulations of technologi-

[1] For additional references, see the excellent survey of Herrendorf *et al.* [15].

cal progress.

Our strategy in this paper is to provide a simple and standard endogenous growth model compatible with structural change that is based on Lucas' [19] human capital accumulation.[2] As in Acemoglu and Guerrieri [1], we focus on a mechanism relying on relative price effects in which the potential effect of growth on structural transformations results from differential productivity growth across sectors.[3] Acemoglu and Guerreri [1] revisited Baumol's cost disease argument. There are two intermediate sectors and these are complements.[4] The capital-intensive sector grows faster than the less capital-intensive sector when capital deepens, and there is high sector-specific growth. The relative price of the less capital-intensive sector good will rise because the "productivity" growth is lower. This will be the dominating force for the increase in the value-added share. More and more resources will be shifted into the less capital-intensive sector because the two goods are complements, which makes up for the productivity growth differential. However, since the two goods are not perfect complements, the economy will want to consume more of the high productivity sector good as it becomes relatively cheap. Thus, the relative real quantities will move in the opposite direction from the value-added share.

In this paper, we use the same structure but with three important distinctions:

- Instead of considering two intermediary sectors with different exogenous rates of technological progress, we assume that the first sector, called "knowledge-intensive", needs technological knowledge which is endogenously accumulated, while the second sector does not.

- The final good is produced through a Cobb-Douglas technology with a unitary elasticity of substitution between the two sectors. This eliminates

[2]We follow Lucas' [19] interpretation of Romer's [23] technological knowledge. Using a Lucas-type two-sector endogenous growth model, Alonso-Carrera and Raurich [2] also analyze the sectoral composition of GDP but with a balanced growth rate common to both sectors.

[3]See also the seminal contribution of Baumol [8].

[4]Alvarez-Cuadrado *et al.* [4] extend the oAcemoglu and Guerrieri [1] formulation, considering that each intermediate good is produced through a CES technology.

the price effect, i.e., the change in the price of one factor does not affect the demand for the other factor, and consequently does not affect Baumol's cost disease either.

- We consider many agents who are homogeneous with respect to their preferences but heterogeneous with respect to their behavior depending on the sector in which they work. While the share of capital and the share of working hours is constant on all equilibrium paths due to the Cobb-Douglas formulation, the number of workers in each sector might vary along the transition path.

Structural change is still generated by differential growth, which is the outcome of knowledge accumulation in one of the intermediate sectors. Agents working in the "knowledge-intensive" sector need to accumulate technological knowledge and thus have to decide how to allocate their individual unit of time between accumulation of technological knowledge (research) and work. We assume, as in Lucas [19] and Romer [23], that the equation governing knowledge accumulation is linear. By contrast, agents working in the second sector do not need to accumulate knowledge and therefore devote all their individual unit of time to work. While the total population is growing at an exogenously given rate, the number of agents working in each sector is endogenously determined over time.

Aggregate knowledge leads to an unbounded increase in TFP in the knowledge-intensive sector, and thus to endogenous capital deepening. It is an externality to the firm, and we consider the planner's solution in which this externality is internalized. The model sheds light on the forces of structural change among sectors characterized by heterogeneous knowledge accumulation and different degrees of knowledge-intensity. It also points to the socially optimal government intervention. The asymmetry in the growth rates of TFP leads to non-balanced growth, and this mechanism does not rely on capital intensities. Moreover, even though the production function of the final good is Cobb-Douglas, labor (number of workers) reallocations across sectors occur, leading to a greater increase in the output of the knowledge-intensive sector.

Our main results are the following. There exists a unique set of constant non-trivial endogenous non-balanced growth paths. The presence of technological knowledge ensures that output growth is larger within the "knowledge-intensive" sector, and thus that its real share is increasing while the share of the second sector is decreasing. However, both nominal shares are constant. These are BGP properties that do not need to hold on the transition. Structural change is therefore consistent with Kaldor [16] stylized facts, in particular the constant interest rate and share of capital income in national income.

After detrending all the variables by their respective endogenous non-balanced growth rates, we prove that there exists a manifold of steady states parameterized by the initial value of the price of knowledge. Although each of these steady states is saddle-point stable and associated with a set of unique non-balanced growth rates, depending on the initial value of capital and knowledge, the economy will follow a particular growth path converging to a particular level of wealth. As a consequence, countries with the same fundamentals but lower initial wealth will be characterized by lower asymptotic wealth. Contrary to Acemoglu and Guerrieri [1] but like Lucas [19], we therefore prove the existence of non-convergence across countries in a framework with structural change.

We also provide a numerical illustration to characterize the transitional dynamics of the main variables. We show that the economy takes approximately 50 years to reach the asymptotic equilibrium, during which there are significant reallocations of capital and labor into the knowledge-intensive sector. Knowledge adjustments thus generate quite fast structural change, actually faster than under the mechanism explored by Acemoglu and Guerrieri [1]. While there is non-balanced growth at the sectoral level, even along the transition path the share of capital income in GDP is constant. Finally, along the transition path the speed of convergence is decreasing for all economic variables and their speeds are comparable, even that of the interest rate.

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 characterizes the intertemporal equilibrium. Section 4 shows that our model generates non-balanced growth and structural change consistent with Kaldor facts. Section 5 establishes the existence of a manifold of steady states of the stationarized dynamical system and provides a local stability analysis. Section 6 contains our numerical illustration. Section 7 presents conclusions and the Appendix contains all the proofs.

## 2 The model

We consider an economy in which at each time $t$ there is a continuum $[0, N(t)]$ of infinitely-lived agents characterized by homogeneous preferences. We assume a standard formulation for the utility function which is compatible with endogenous growth such that

$$u(c_i(t)) = \int_0^{+\infty} \frac{c_i(t)^{1-\theta}}{1-\theta} e^{-\rho t} dt \tag{1}$$

where $c_i(t)$ is consumption of agents of type $i \in [0, N(t)]$ at time $t$, $\theta > 0$ is the inverse of the elasticity of intertemporal substitution in consumption and $\rho > 0$ is the discount factor. We assume that the total population grows at the constant exponential rate $n \in [0, \rho)$, so that $N(t) = e^{nt}N(0)$. Agents are heterogeneous only with respect to the sector in which they work. As explained below, this difference arises from different decisions on the allocation of labor time.

Our economy consists of two sectors, with three factors of production, capital, $K(t)$, labor, $L(t)$, and individual technological knowledge, $a(t)$. In line with Lucas (1988), we interpret $a(t)$ as the outcome of an individual's choice. Final output, $Y(t)$, is an aggregate of the output of two intermediate sectors, $Y_1(t)$ a "knowledge-intensive sector", and $Y_2(t)$. The numbers of workers $N_1(t)$ and $N_2(t)$ in these two sectors, together with their respective growth rates $\dot{N}_1(t)/N_1(t) = g_{N_1}(t)$ and $\dot{N}_2(t)/N_2(t) = g_{N_2}(t)$, are endogenously determined at the equilibrium.

Agents working in the "knowledge-intensive sector" devote a fraction $u(t) \in (0, 1)$ of their unit of time to production. Total labor in this sector is $L_1(t) = u(t)N_1(t)$. The rest of time $1 - u(t)$ is devoted to the accumulation of individual technological knowledge $a(t)$. Here, $L_1(t)$ denotes the total number of hours worked in the "knowledge-intensive sector" and differs from the number $N_1(t)$ of workers. Similarly to how Lucas (1988) treats newborns, in our model each agent entering the knowledge-intensive sector at any time $t_0$ acquires the available knowledge $a(t_0)$. We assume that

$$\dot{a}(t) = z[1 - u(t)]a(t) - \eta a(t) \tag{2}$$

with $z > 0$ and $\eta > 0$ the depreciation rate of knowledge. By contrast, agents working in the second sector will spend all their unit of time working so that total labor in this sector is $L_2(t) = N_2(t)$. Contrary to the "knowledge-intensive sector", in this sector the number of hours worked is identical to the number of workers, since the total individual available working time is normalized to one.

The final good is produced through a Cobb-Douglas technology such that

$$Y(t) = Y_1(t)^{\gamma} Y_2(t)^{1-\gamma} \tag{3}$$

with $\gamma \in (0,1)$. Sector 1 produces the knowledge-intensive intermediate good using capital, labor and knowledge through the following Cobb-Douglas technology:

$$Y_1(t) = [L_1(t)a(t)]^{\alpha} K_1(t)^{1-\alpha} \tag{4}$$

with $\alpha \in (0,1)$. The product $L_1(t)a(t) = u(t)N_1(t)a(t)$ then represents total efficient labor[5]. Note that, considering capital and labor (hours worked) as inputs, the total factor productivity (TFP) is given by $a(t)^{\alpha}$. Sector 2 produces the second intermediate good using only capital and labor through the following Cobb-Douglas technology:

$$Y_2(t) = L_2(t)^{\beta} K_2(t)^{1-\beta} \tag{5}$$

with $L_2(t) = N_2(t)$ and $\beta \in (0,1)$. Contrary to the "knowledge-intensive" sector, the second sector has a constant TFP.

Denoting total capital by $K(t)$ and total labor by $L(t)$, capital and labor market clearing require at each date $K(t) \geq K_1(t) + K_2(t)$ and $L(t) \geq L_1(t) + L_2(t) = N_1(t)u(t) + N_2(t)$. The capital accumulation equation is standard

[5]As in Lucas (1988), we define each worker's output according to $(u(t)a(t))^{\alpha} k(t)^{1-\alpha}$ where $k(t) = K_1(t)/N_1(t)$.

$$\dot{K}(t) = Y(t) - \delta K(t) - N_1(t)c_1(t) - N_2(t)c_2(t) \tag{6}$$

where $\delta > 0$ is the depreciation rate of capital.

# 3 Planner solution and intertemporal equilibrium

In the model, as in Lucas (1988), the welfare theorems hold and the intertemporal competitive equilibrium can be found via the planner's problem[6]. Assume that the planner has Benthamite objective function and consider the following intertemporal optimization problem

$$\begin{aligned} \max_{\{c_i(t),K_i(t),L_i(t)\}_{i=1,2},u(t),A(t)} \quad & \int_0^{+\infty} \left( N_1(t)\frac{c_1(t)^{1-\theta}}{1-\theta} + N_2(t)\frac{c_2(t)^{1-\theta}}{1-\theta} \right) e^{-\rho t} dt \\ s.t. \quad & (2),(3),(4),(5),(6) \text{ and} \\ & K(t) \geq K_1(t) + K_2(t) \\ & L(t) \geq L_1(t) + L_2(t) = N_1(t)u(t) + N_2(t) \\ & K(0), a(0), N(0) \text{ given} \end{aligned} \tag{7}$$

The Hamiltonian in current value is (we omit subscript for $t$ to simplify notations):

$$\begin{aligned} \mathbb{H} = \; & N_1 \frac{c_1^{1-\theta}}{1-\theta} + N_2 \frac{c_2^{1-\theta}}{1-\theta} + P_1 \Big[ (L_1 a)^{\alpha} K_1^{1-\alpha} - Y_1 \Big] + P_2 \Big[ L_2^{\beta} K_2^{1-\beta} - Y_2 \Big] \\ + \; & P \Big[ Y_1^{\gamma} Y_2^{1-\gamma} - \delta K - N_1 c_1 - N_2 c_2 \Big] + Q \Big[ z(1-u) - \eta \Big] a \\ + \; & \lambda \left[ K - K_1 - K_2 \right] + \mu [L - L_1 - L_2] \end{aligned}$$

with $L_1 = uN_1$ and where the value of increments in aggregate capital is given by the "price" $P$. Similarly, the price of the knowledge-intensive good is $P_1$, the price of the second good is $P_2$, the price of knowledge is $Q$ and $\lambda$ and $\mu$ are the Lagrange multipliers associated with the capital and labor market clearing conditions. The first-order conditions with respect to the control variables $c_1, c_2, u, L_1, L_2, K_1, K_2, Y_1$ and $Y_2$ give:

[6]The economic interpretation of competitive equilibrium is as in Lucas (1988)

$$c_i^{-\theta} = P \text{ for any } i = 1,2 \tag{8}$$

$$P_1 \alpha \frac{Y_1}{L_1 a} N_1 = Qz \tag{9}$$

$$P_1 = P\gamma \frac{Y}{Y_1} \tag{10}$$

$$P_2 = P(1-\gamma)\frac{Y}{Y_2} \tag{11}$$

$$\lambda = P_1(1-\alpha)\frac{Y_1}{K_1} = P_2(1-\beta)\frac{Y_2}{K_2} \tag{12}$$

$$\mu = P_1 \alpha \frac{Y_1}{L_1} = P_2 \beta \frac{Y_2}{L_2} \tag{13}$$

Substituting (10) and (11) into (12) and (13) gives

$$\gamma(1-\alpha)\frac{Y}{K_1} = (1-\gamma)(1-\beta)\frac{Y}{K_2} \tag{14}$$

$$\gamma\alpha\frac{Y}{L_1} = (1-\gamma)\beta\frac{Y}{L_2} \tag{15}$$

Despite the fact that agents work in different sectors, they all consume the same amount since $c_1(t) = c_2(t) = P^{-1/\theta}$ for any $t \geq 0$. This result can easily be explained. In the "knowledge-intensive sector", the rental rate of capital $r_1$ and the individual wage rate $w_1$ are given by

$$r_1 = (1-\alpha)\tfrac{Y_1}{K_1}, \quad w_1 = \alpha\tfrac{Y_1}{N_1 ua} \tag{16}$$

while in the second sector, $r_2$ and $w_2$ are given by

$$r_2 = (1-\beta)\tfrac{Y_2}{K_2}, \quad w_2 = \beta\tfrac{Y_2}{N_2} \tag{17}$$

Note first that substituting (10) and (11) into (16) and (17) allows (15) to be written as follows

$$w_1(t)a(t)P_1(t) = w_2(t)P_2(t)$$

which gives the equality between nominal wages per hour devoted to production in the relevant sector. Similarly (14) is equivalent to

$$r_1(t)P_1(t) = r_2(t)P_2(t) \tag{18}$$

which gives the equality between the capital return in the two sectors. These two properties imply, as expected, that all agents, no matter which sector they work in, will have the same labor and capital income at the equilibrium and will thus be able to have the same intertemporal profile of consumption.

We also derive from the first-order conditions:

**Proposition 1.** *Solving the first-order conditions (10)-(11) and (14)-(15) gives the optimal demand functions for capital and labor*

$$\begin{aligned} K_1(t) &= \tfrac{\gamma(1-\alpha)K(t)}{\gamma(1-\alpha)+(1-\gamma)(1-\beta)} \equiv A_1K(t),\ L_1(t) = \tfrac{\gamma\alpha L(t)}{\gamma\alpha+(1-\gamma)\beta} \equiv B_1L(t) \\ K_2(t) &= \tfrac{(1-\gamma)(1-\beta)K(t)}{\gamma(1-\alpha)+(1-\gamma)(1-\beta)} \equiv A_2K(t),\ L_2(t) = \tfrac{(1-\gamma)\beta L(t)}{\gamma\alpha+(1-\gamma)\beta} \equiv B_2L(t) \end{aligned} \tag{19}$$

*Moreover, there exists a function $v : \mathbb{R}^5 \rightarrow$ such that*

$$u(t) = v\left(K(t), a(t), P(t), Q(t), N(t)\right) \tag{20}$$

*with*

$$\tfrac{\partial v}{\partial K(t)} > 0, \quad \tfrac{\partial v}{\partial a(t)} < 0 \tag{21}$$

*and we derive*

$$\begin{aligned} N_1(t) &= \tfrac{B_1N(t)}{u(t)+B_1(1-u(t))},\ N_2(t) = \tfrac{u(t)(1-B_1)N(t)}{u(t)+B_1(1-u(t))} \\ L(t) &= \tfrac{u(t)N(t)}{u(t)+B_1(1-u(t))} \equiv \ell\left(K(t), a(t), P(t), Q(t), N(t)\right) N(t) \end{aligned} \tag{22}$$

*together with the optimal production levels*

$$\begin{aligned} Y_1(t) &= D_1K(t)^{1-\alpha}a(t)^{\alpha}\left[\ell\left(K(t), a(t), P(t), Q(t), N(t)\right) N(t)\right]^{\alpha} \\ Y_2(t) &= D_2K(t)^{1-\beta}\left[\ell\left(K(t), a(t), P(t), Q(t), N(t)\right) N(t)\right]^{\beta} \\ Y(t) &= DK(t)^{1-\gamma\alpha-(1-\gamma)\beta}a(t)^{\gamma\alpha}\left[\ell\left(K(t), a(t), P(t), Q(t), N(t)\right) N(t)\right]^{\gamma\alpha+(1-\gamma)\beta} \end{aligned}$$

*with*

$$D_1 = A_1^{1-\alpha}B_1^{\alpha},\ D_2 = A_2^{1-\beta}B_2^{\beta},\ D = D_1^{\gamma}D_2^{1-\gamma}$$

*Proof.* See Appendix 8.1.

Contrary to Acemoglu and Guerrieri [1], our use of a Cobb-Douglas technology for the final good sector implies from equations (19) that there are no capital or labor (hours worked) reallocations across sectors along the equilibrium path. However, as shown by equations (22), there are reallocations of workers across sectors. In fact, equations (20)-(21) show that, for a given level of knowledge, an increase in aggregate capital entails an increase in the time $u(t)$ allocated to production in the knowledge-intensive sector. Similarly, for a given level of aggregate capital, an increase in knowledge entails a decrease in the time $u(t)$ allocated to production in the knowledge-intensive sector. As a result, along an equilibrium path, the ratio $N_2(t)/N_1(t)$ evolves in the same direction as $u(t)$.

In particular, in the event that $N_2(t)$ decreases, as previously noted, the agents that migrate from the second sector to the knowledge-intensive sector inherit the available individual stock of knowledge. Moreover, as in standard multisector models with sectors that differ in their capital and labor shares, the sector with the larger capital share has the larger capital-labor ratio. For instance, $K_1/L_1 > K_2/L_2$ if and only if $\beta > \alpha$.

It is worth noting here that the equilibrium rental rate of capital $R(t)$ can be derived from the expression of $Y(t)$ as given in Proposition 1:

$$R(t) = \frac{\partial Y(t)}{\partial K(t)} = [1-\gamma\alpha-(1-\gamma)\beta]\frac{Y(t)}{K(t)} = \frac{P_1(t)r_1(t)}{P(t)} = \frac{P_2(t)r_2(t)}{P(t)} \tag{23}$$

From the Hamiltonian, we also derive the optimality conditions that provide differential equations for the prices $P$ and $Q$ of aggregate capital and knowledge:

$$\dot{P} = \rho P - \frac{\partial \mathbb{H}}{\partial K} = -P_1(1-\alpha)\frac{Y_1}{K_1} + P(\delta+\rho) \tag{24}$$

$$\dot{Q} = \rho Q - \frac{\partial \mathbb{H}}{\partial a} = -Q(z-\eta-\rho) + Qzu - P_1\alpha\frac{Y_1}{a} \tag{25}$$

Substituting equation (10) into (24) and using (23) then gives

$$\dot{P} = -P\left[[1-\gamma\alpha-(1-\gamma)\beta]\frac{Y}{K} - \delta - \rho\right] = -P\left[R-\delta-\rho\right] \tag{26}$$

Note now that using $L_1 = uN_1$, equation (9) becomes $P_1\alpha Y_1/(ua) = Qz$. Substituting this expression into (25) gives

$$\dot{Q} = -Q(z-\eta-\rho) \tag{27}$$

Moreover, since $c_1(t) = c_2(t) = P^{-\frac{1}{\theta}}$, we get aggregate consumption as

$$C_1 + C_2 = N_1c_1 + N_2c_2 = NP^{-\frac{1}{\theta}}$$

Finally, using all these results together with Proposition 1, we obtain the following differential equations for prices and stocks:

$$\frac{\dot{P}}{P} = -\left[R(K, a, P, Q, N) - \delta - \rho\right] \tag{28}$$

$$\frac{\dot{Q}}{Q} = -(z - \eta - \rho) \tag{29}$$

$$\frac{\dot{K}}{K} = \frac{Y(K, a, P, Q, N)}{K} - \delta - \frac{NP^{-\frac{1}{\theta}}}{K} \tag{30}$$

$$\frac{\dot{a}}{a} = z\left[1 - v(K, a, P, Q, N)\right] - \eta \tag{31}$$

It follows that any path $\{K(t), a(t), P(t), Q(t)\}_{t\geq 0}$ that satisfies the conditions (19)-(31) together with the transversality conditions

$$\lim_{t\to+\infty} P(t)K(t)e^{-\rho t} = 0 \text{ and } \lim_{t\to+\infty} Q(t)a(t)e^{-\rho t} = 0 \tag{32}$$

for any given initial conditions $(K(0), a(0))$ is an optimal solution of problem (7) and therefore an intertemporal equilibrium.

So far we have interpreted knowledge accumulation along the lines of Lucas (1988), see footnote 4. Another interpretation of the model is as follows. Knowledge now becomes an aggregated variable, a public good used by the firms in the high-tech sector. Since there is no private incentive to invest in $a(t)$, the regulator has to step in. The government manages the production of $a(t)$ by imposing lump-sum taxes on consumers and use the revenue to hire workers from the high-tech sector). As the two other sectors (low- and high-tech) are perfectly competitive, the wage per hour needs to be identical across sectors. Note that the tax revenue used to buy efforts from consumers working in the high-tech industry might involve a time dependent tax system.

## 4 Non-balanced growth paths

We now provide the existence result and characterize the non-balanced growth paths along which the variables $P, Q, K, a$ grow at constant rates. Note that this implies that $C, Y, C_i, K_i$ and $Y_i$ also grow at constant rates. We introduce the following notation valid along the non-balanced growth path, denoted NBGP, which highlights the fact that the growth rates of the various variables can differ:

$$g_C = \frac{\dot{C}}{C}, g_Y = \frac{\dot{Y}}{Y}, g_K = \frac{\dot{K}}{K}, \; g_{Y_i} = \frac{\dot{Y}_i}{Y_i},$$

$$g_{K_i} = \frac{\dot{K}_i}{K_i}, \; g_{c_i} = \frac{\dot{c}_i}{c_i}, \; g_a = \frac{\dot{a}}{a}, g_P = \frac{\dot{P}}{P}, \; g_Q = \frac{\dot{Q}}{Q}$$

for $i = 1,2$. Along an NBGP we have $K(t) = ke^{g_K t}$, $a(t) = xe^{g_a t}$, $Q(t) = qe^{g_Q t}$ and $P(t) = pe^{g_P t}$. Therefore an NBGP is an 8-tuple in $\mathbb{R}^8$

We also introduce the following restrictions which guarantee positiveness of growth rates and interiority of the share $u$ of time devoted to work by agents in the knowledge-intensive sector:

**Assumption 1.** *$z > \eta + \rho - n$ and $\theta > 1 - (\rho - n)\frac{\gamma\alpha + (1-\gamma)\beta}{\gamma\alpha(z-\eta)}$.*

From the first-order conditions (8)-(15), Proposition 1 and the differential equations (28)-(31) we derive:

**Theorem 1.** *Let Assumption 1 hold. The set of non-balanced growth paths NBGR is non-empty. The growth rates are constant across this set and given by:*

$$g_Y = g_K = g_{K_2} = g_{K_1} = g_C = n + g_{c_1} = n + g_{c_2} = n - \frac{g_P}{\theta} = \frac{\gamma\alpha(z-\eta+n-\rho)}{\theta\gamma\alpha + (1-\gamma)\beta} + n$$

$$g_a = \frac{[\gamma\alpha + (1-\gamma)\beta](z-\eta+n-\rho)}{\theta\gamma\alpha + (1-\gamma)\beta}$$

$$g_{N_1} = g_{N_2} = n$$

$$g_{Y_1} = \frac{\alpha[\gamma + (1-\gamma)\beta](z-\eta+n-\rho)}{\theta\gamma\alpha + (1-\gamma)\beta} + n$$

$$g_{Y_2} = \frac{\gamma\alpha(1-\beta)(z-\eta+n-\rho)}{\theta\gamma\alpha + (1-\gamma)\beta} + n$$

$$g_Q = -(z - \eta - \rho)$$

*Moreover, the time devoted to production in the knowledge-intensive sector and the rental rate of capital are both constant:*

$$u^* = \frac{z-\eta-g_a}{z} = \frac{1}{z}\left[\frac{(z-\eta)\gamma\alpha(\theta-1) + (\rho-n)[\gamma\alpha + (1-\gamma)\beta]}{\theta\gamma\alpha + (1-\gamma)\beta}\right] \in (0,1) \tag{33}$$

*and*

$$R^* = \delta + \rho - g_P = \delta + \rho - \theta(n - g_K) \tag{34}$$

*Proof.* See Appendix 8.2.

As we will see later each, element of the set of NBGP corresponds to a different initial condition. Although all of these NBGP differ in the level of their economic variables, their growth rates are the same. We can also derive a number of important implications from Proposition 1 and Theorem 1. First, along the NBGP, workforces $N_1$ and $N_2$ in the two sectors are growing at the same rate $n$ as total population $N$, and the ratio $L_1/L_2$ is constant. Second, $g_{Y_1} > g_Y > g_{Y_2}$ and the growth rate of output $g_Y$ is a convex combination of the growth rates in the dominant and dominated intermediate sectors

$$g_Y = \gamma g_{Y_1} + (1-\gamma) g_{Y_2}$$

The latter is a consequence of the fact that the ratios $K_1/K_2$, $L_1/L_2$ and $N_1/N_2$ are constant, which is a consequence of the Cobb-Douglas technology in the final good sector. Although the knowledge-intensive sector is asymptotically dominant in output, the amount of inputs used are not vanishing and workforce does not shrink.

To conclude, we show that the properties of the NBGP agree with Kaldor facts, obtaiing the following characterization:

**Corollary 1.** *The non-balanced growth path has the following properties:*

1. *There is capital deepening, i.e. the ratio $K/L$ is increasing.*
2. *The growth rate of real GDP, or $Y$, is constant.*
3. *The capital-output ratio $K/Y$ is constant.*
4. *The nominal share of capital income in GDP is constant and equal to $s_K = 1 - \gamma\alpha - (1-\gamma)\beta$.*
5. *The real interest rate $R$ is constant.*
6. *The relative prices $P_1/P$ and $P_2/P$ are respectively decreasing and increasing. As a result, $P_2/P_1$ is increasing.*
7. *The real shares in GDP of the knowledge-intensive sector and of the dominated sector, $Y_1/Y$ and $Y_2/Y$, are respectively increasing and decreasing while the nominal shares, $P_1Y_1/PY$ and $P_2Y_2/PY$, are constant.*

*Proof.* See Appendix 8.3.

The mechanism at work in the economy can be described as follows. The accumulation of individual knowledge in the knowledge-intensive sector is used as labor-augmenting technological progress. Knowledge accumulation thus leads to an unbounded increase in TFP in the knowledge-intensive sector and to capital deepening. The relative price of the knowledge-intensive good decreases because of knowledge accumulation and TFP growth, while the relative price of the less knowledge-intensive good increases. The change in relative prices is associated with changes in the demand for both intermediate goods by the final sector. This mechanism endogenously determines the growth rates of the two sectors such that the growth rate of real GDP remains constant and is a convex combination of the growth rates in the two intermediate sectors. Note that this growth rate heterogeneity does not rely on differences in capital intensity across sectors, on a non-unitary elasticity of substitution between the two sectors, or on the existence of exogenous technological progress.

It is interesting to compare the results obtained so far with those of Acemoglu and Guerrieri [1]. In their case, non-balanced growth is due to capital deepening driven by the growth of exogenous technological progress in the intermediate sectors, together with differences in capital intensities. When the elasticity of substitution between the two sectors is below one, an increase in the capital-labor ratio (i.e., capital deepening) raises output more in the sector with greater capital intensity. Along the NBGP, the growth rate of GDP is equal to the growth rate of the dominant intermediate sector. In their model, exogenous technological progress is a necessary condition for the existence of non-balanced growth. Note that with an elasticity of substitution between the two intermediate sectors of less than one, capital and labor are allocated away from the sector that is less capital-intensive. It follows that the capital-intensive sector becomes dominant and the labor intensive sector becomes asymptotically vanishing.

Our model, like that of Acemoglu and Guerrieri, describes an economy featuring non-balanced growth at the sectoral level, i.e., unequal growth between the two sectors, and aggregate growth consistent with Kaldor facts. In Acemoglu and Guerrieri, along the NBGP one of the sectors has already become (vanishingly) small relative to the other. Contrastingly, in the model considered here employment levels are constant along the NBGP even though the real shares of the two sectors are not constant and the growth rates of the two sectors differ. The challenge we face could be considered the reverse of that faced by Acemoglu and Guerrieri. The employment level, rather than being trivial, is here strictly positive but constant.

## 5 Local dynamics

As argued by Acemoglu and Guerrieri, the asymptotic properties of the NBGP described in the previous section only provide a long-run necessary condition. Yet interesting and empirically relevant properties also occur along the transition path. We now aim to generate structural change along the transition path in which both sectors retain nontrivial employment levels. We can reformulate the dynamical system given by equations (28)-(31) using the normalization of variables introduced by Caballe and Santos [11]. The stationarized NBGP is obtained by "removing" the NBGR trend from the variables, namely: $k(t) = K(t)e^{-g_K t}$, $x(t) = a(t)e^{-g_a t}$, $q(t) = Q(t)e^{-g_Q t}$ and $p(t) = P(t)e^{-g_P t}$, for all $t \geq 0$, with $k(t)$, $x(t)$, $q(t)$ and $p(t)$ the stationarized values for $K(t)$, $a(t)$, $Q(t)$ and $P(t)$. As the price of knowledge $Q$ is characterized by a constant growth rate $g_Q$, the solution of (29) is given by $Q(t) = Q(0)e^{-g_Q t}$ and its stationarized value is constant with $\dot{q}(t) = 0$. We then get $q(t) = q(0) = q_0$ for all $t \geq 0$. Recall that as the population is growing at the exponential rate $n$, we have $N(t) = e^{nt}N(0)$ with $N(0) = N_0$ given.

Substituting these stationarized variables into (28)-(31), we obtain an equivalent stationarized system of differential equations that characterizes the equilibrium path.

**Lemma 1.** *Let Assumption 1 hold and $N_0$ be given. Along a stationarized equilibrium path and for any given $q_0$, knowledge $a$, capital $k$ and its price $p$ are solutions of the following dynamical system*

$$\begin{aligned}\frac{\dot{p}}{p} &= -\left[Fk^{-\gamma\alpha-(1-\gamma)\beta}x^{\gamma\alpha}\ell\left(k,x,p,q_0,N_0\right)^{\gamma\alpha+(1-\gamma)\beta}+g_P-\delta-\rho\right]\\ \frac{\dot{k}}{k} &= Ek^{-\gamma\alpha-(1-\gamma)\beta}x^{\gamma\alpha}\ell\left(k,x,p,q_0,N_0\right)^{\gamma\alpha+(1-\gamma)\beta}-g_K-\delta-\frac{N_0p^{-\frac{1}{\theta}}}{k} \\ \frac{\dot{x}}{x} &= z\left[1-v(k,x,p,q_0,N_0)\right]-g_a-\eta\end{aligned} \tag{35}$$

*with* $E = DN_0^{\gamma\alpha+(1-\gamma)\beta}$ *and* $F = [1-\gamma\alpha-(1-\gamma)\beta]E$.

*Proof.* See Appendix 8.4.

The transversality conditions (32) become

$$\lim_{t\to+\infty} p(t)k(t)e^{-[\rho-g_K-g_P]t} = 0 \text{ and } \lim_{t\to+\infty} x(t)e^{-(\rho-g_a-g_Q)t} = 0$$

Using the expressions of the growth rates $g_K$, $g_a$, $g_Q$ and $g_P$ given in Theorem 1, we easily derive under Assumption 1 that $zu^* = z - g_a - \eta = \rho - g_K - g_P = \rho - g_a - g_Q > 0$.

Considering the stationarized dynamical system given in Proposition 1, we can now focus on proving the existence of a steady state solution, i.g., $\dot{p}/p = \dot{k}/k = \dot{x}/x = 0$, and $\dot{q}(t) = 0$ which obviously corresponds to the NBGR exhibited in Theorem 1.

**Theorem 2.** *Let Assumption 1 hold and* $N_0$ *be given. The projection of the set of NBGP on the subspace* $(k,x,p,q)$ *is a one-dimensional manifold, noted* $\mathcal{M} \subset \mathcal{R}^4$ *paramerterized by* $q_0$*. Then for any given* $q_0 > 0$*, there exists a unique steady state* $(k^*(q_0), x^*(q_0), p^*(q_0))$*, solution of the dynamical system (35). Moreover,* $k^{*'}(q_0) < 0$*,* $x^{*'}(q_0) < 0$ *and* $p^{*'}(q_0) > 0$*.*

*Proof.* See Appendix 8.5.

Theorem 2 proves that there exists a one-dimensional manifold of steady states for the capital stock $k$, technological knowledge $x$ and the price of capital $p$ parameterized by the constant price of knowledge $q_0$. It is worth noting, however, that the asymptotic amount of time devoted to production in the knowledge-intensive sector $u^*$ and the asymptotic rental rate of capital $R^*$, as given respectively by (33) and (34), do not depend on $q_0$.

The existence of a manifold of steady states in levels is fairly standard in endogenous growth models (see for instance Lucas [19]) where the asymptotic equilibrium of the economy depends on some initial conditions. We will show that there exists a set $\mathcal{K}$ containing the set $\mathcal{M}$ such that for initial values of physical capital $k(0)$ and technological knowledge $x(0)$ in $\mathcal{K}$, a value of $q_0$ is "automatically" selected in order for the economy to leap onto the optimal path (i.e., the stable manifold) and then converge to the particular steady state $(k^*(q_0), x^*(q_0), p^*(q_0))$ situated on the manifold $\mathcal{M}$. Note that for any given pair $k(0), x(0)) \in \mathcal{K}$ there exists a unique value of the price of knowledge $q_0$ compatible with the equilibrium conditions.

To prove such a result, we need to study the local stability properties of the steady state. Linearizing the dynamical systems (35) around the steady state $(k^*(q_0), x^*(q_0), p^*(q_0))$ for a given $q_0 > 0$, the local stability property of $(k^*(q_0), x^*(q_0), p^*(q_0))$ is appraised through the characteristic roots of the associated Jacobian matrix. As shown by Martínez-García [20] (see also Bond, Wang and Yip [7] and Xie [25]), since we have two state variables, $k$ and $a$, and two forward variables, $p$ and $q$, with $q$ being constant, the standard saddle-point stability occurs if there exists a one-dimensional stable manifold, i.e. if only one characteristic root is negative.

**Lemma 2.** *Let Assumption 1 hold. Then for any given $q_0 > 0$, the steady state $(k^*(q_0), x^*(q_0), p^*(q_0))$ is saddle-point stable.*

*Proof.* See Appendix 8.6.

In the dynamical system (35), the predetermined variables are the capital stock and the level of individual knowledge. For given initial conditions $K(0) = k(0) = k_0$ and $a(0) = x(0) = x_0$, we generically cannot find a value of $q_0$ such that $(k_0, x_0) = (k^*(q_0), x^*(q_0))$ and the economy is not in the set of NBGP from the initial date. In other words, non-trivial transitional dynamics generically occurs starting from $(k_0, x_0) \neq (k^*, x^*)$. The initial values of the forward variables $p(0) = p_0$ and $q(0) = q_0$ are chosen such that the one-dimensional stable optimal path converging toward an NBGP is selected. As the stable manifold is one-dimensional, for any given $K(0) = k(0) = k_0$ and $a(0) = x(0) = x_0$ there exists a unique pair $(q_0, p_0)$

compatible with an equilibrium path.

The arguments supporting the truth of the previous statement are as follows. The dynamical system has two state and two forward looking variables. The steady state is then a $(k, x, p, q)$. There is a one-dimensional manifold of steady states. Each of these is saddle-path stable and $q$ is constant on any equilibrium path. For each of these $q$ there is a one-dimensional stable manifold leading to the NBGP. When $q$ spans the feasible set, this describes a stable planar manifold. Generically there is a two-dimensional plane in space of dimension four with a given $(x(0), k(0))$. The intersection of the two planes in dimension four is a set of dimension zero, a point. So for a given $(x(0), k(0))$ there is a unique $(p(0), q(0))$.

A striking property is that, although the steady state values $(k^*(q_0), x^*(q_0), p^*(q_0))$ depend on the selected $q_0$, the eigenvalues do not. It follows that the rate of convergence along any transitional path is the same, regardless of the initial conditions and thus of the asymptotic value of the steady state.

This property can be illustrated by the following Figure. From the expressions of $x^*$ and $k^*$ as given respectively by (55) and (56) in the proof of Theorem 2 (Appendix 8.5), we get the following relationship as $q_0$ varies:

$$k^* = \mathcal{K}\ell^* x^{*\frac{\gamma\alpha}{\gamma\alpha+(1-\gamma)\beta}} \tag{36}$$

with $\mathcal{K}$ as defined in (51).

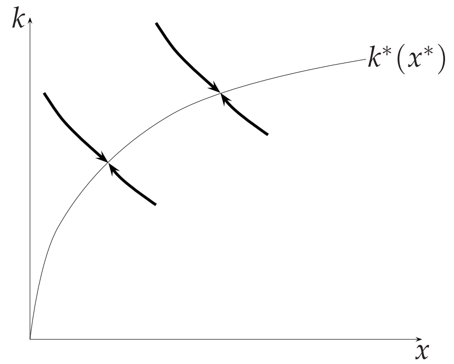


Figure 1: Manifold of steady states.

All pairs $(k^*, x^*)$ satisfying (36) correspond to a common asymptotic

NBGP but different optimal paths along the transition according to the initial condition $(k_0, x_0)$. For a given pair $(k_0, x_0)$, the optimal path will converge toward an asymptotic position located on the curve $k^*(x^*)$ as given by (36) which depends on the initial position $(k_0, x_0)$ that defines the admissible initial value $q_0$. The arrows in Figure 1 illustrate some possible trajectories.

The following theorem sumarizes the results.

**Theorem 3.** *Let Assumption 1 hold. There exists a set $\mathcal{K}$ containing the set $\mathcal{M} \subset \mathbb{R}^4$ such that for initial values of physical capital $k(0)$ and technological knowledge $x(0)$ in $\mathcal{K}$, the economy converges to the steady state $(k^*(q_0), x^*(q_0), p^*(q_0))$ situated on the manifold $\mathcal{M}$, where the price of knowledge $q_0$ and $p_0$ are uniquely determined and the optimal path converging to the NBGP is unique.*

*Proof.* See Appendix 8.6.

The existence of a manifold of steady states, while standard in the endogenous growth literature, is a fundamental difference with respect to the results of Acemoglu and Guerrieri [1]. In the latter, as technological progress is exogenous, the steady state is unique and countries with the same fundamentals but different initial endowments of physical capital and technological knowledge will all converge toward the same asymptotic level of wealth and the same asymptotic growth rate. By contrast, in the present model, while all countries with the same fundamentals are characterized by the same growth rate, they will follow different optimal paths along the transition and be asymptotically characterized by different wealth levels.

## 6 Transitional structural change

Having shown the existence of paths converging to the NBGP, we turn to the properties of these paths. For a given pair $(k_0, x_0)$ the optimal path will converge toward an asymptotic position located on the manifold $\mathcal{M}$. From the property that the manifold is parameterized by $q_0$ and that $k^{*'}(q_0) < 0$, $x^{*'}(q_0) < 0$ in Theorem 2, we know that the set of NBGP is such that greater value of capital along the NBGP is associated with greater value of knowledge (also seen in Fig. 1). However, analytical considerations prevent us from determining the dynamics on the transition path. It seems likely that an economy with initially low levels of physical capital and technological knowledge will remain permanently behind an initially better-endowed economy, as suggested by Fig. 1. In other words, $q_0$ is decreasing when $k(0)$ and $a(0)$ increase. Fig. 1 also indicates that if an economy starts out with a combination of capital and knowledge above the NBGP knowledge-intensiveness level, the knowledge will increase less than at the NBGP and capital will increase more than along the NBGP. An analogous result is expected to hold for capital. The numerical simulations of Section 6.1 confirm these conjectures.

Along the transition path, the proportion of capital and hours worked allocated to the two sectors is constant. However, workers in the knowledge-intensive sector devote time to acquiring skills, so that although the share of hours worked $L_2/L_1$ remains constant, the share of the workforce $N_2/N_1$ is not constant along the transition path (but it is asymptotically constant). Although the initial allocation of workers between the two sectors is endogenously determined (Proposition 1), the numbers of individuals in each sector can in principle still increase or decrease along the transition path while converging toward their stationary values. In the event that the labor force in the less knowledge-intensive sector decreases, the migrating workers will receive the knowledge available in the knowledge-intensive sector when they join it. Note that, as the population grows, this situation is not necessarilly implied by a rise of the skilled-population. Rather than being reallocation of workers, it might simply be variable distribution of newborns. In the next section we investigate a plausible numerical example and show that variables move monotonically along the transition path

## 6.1 Transitional dynamics: the numerical strategy and an illustrative calibration

The economy is characterized by two initial conditions, $k(0)$ and $x(0)$, while $p(0)$ and $q(0)$ are pinned down by the equilibrium conditions. To study the transition dynamics, one way to proceed is to calibrate the model using a "plausible"' asymptotic value of the NBGP. As this depends on $q_0$ (which of course depends on $k(0), x(0)$); the calibration delivers a value of $q_0$ in line with the realistic NBGP. As $q_0$ is now fixed, the set of initial conditions $(k(0), x(0))$ leading to the given NBGP, or equivalently the given $q_0$, is a one-dimensional manifold. The transition is then numerically evaluated by letting values of $k(0)$ (or $x(0)$) be in a neighborhood of $k^*$, the corresponding value of $x(0)$ and $p(0)$ being pinned down by the equilibrium conditions. Having fixed the asymptotic equilibrium at a plausible value, an interesting exercise is to consider an economy initially poorer in capital than the NBGP level and to study how the relevant variables evolve along the transition path. A similar approach is to look at an economy initially poorer in knowledge than the NBGP level.

Following this strategy, the first step is to obtain a plausible NBGP. In this numerical analysis we use the same data as in Acemoglu and Guerrieri [1]. The data are taken from the National Income and Product Accounts (NIPA) between 1948 and 2005 and industries are classified according to the North American Industrial Classification System at the 22-industry level of detail. Unlike Acemoglu and Guerrieri [1], we do not classify industries according to their capital intensity but according to whether or not they require technological knowledge, broadly defined. We consider an industry to be knowledge-intensive if working in this industry requires extensive initial training. We compare conclusions from two groups of 9 industries, as in Acemoglu and Guerrieri [1]. The following Table shows the average capital share of each industry together with the sector classification.

Our model is characterized by 9 parameters, namely $\delta, \rho, \theta, \gamma, \alpha, \beta, n, z$ and $\eta$. We adopt the standard values for the annual depreciation rate $\delta = 0.05$ and the annual discount rate $\rho = 0.02$. We use the annual popu-

| Industry | Sector | Capital share |
|---|---|---|
| Educational servives | 1 | 0.10 |
| Management of companies and enterprises | 1 | 0.20 |
| Health care and social asssitance | 1 | 0.22 |
| Durable goods | 1 | 0.27 |
| Administrative and waste management services | 1 | 0.28 |
| Professional, scientific and technical services | 1 | 0.34 |
| Arts, entertainment and recreation | 1 | 0.42 |
| Finance and insurance | 1 | 0.45 |
| Information | 1 | 0.53 |
| Construction | 2 | 0.32 |
| Other services, except government | 2 | 0.33 |
| Transportation and warehousing | 2 | 0.35 |
| Accommodation and food services | 2 | 0.36 |
| Retail trade | 2 | 0.42 |
| Wholesale trade | 2 | 0.46 |
| Nondurable goods | 2 | 0.47 |
| Mining | 2 | 0.66 |
| Utilities | 2 | 0.77 |

Table 1: Industry capital shares

lation growth rate $n = 0.018$ from the NIPA data on employment growth for 1948-2005. Our classification of industries allows us to compute average shares of capital for two "aggregate sectors" in which $\alpha = 0.68$ and $\beta = 0.54$. In our model, the capital share in national income is constant and given by $s_K = 1 - \alpha\gamma - (1-\gamma)\beta$. Since the US data give $s_K = 0.4$, we choose the value of $\gamma$ that leads to this share given the values of $\alpha$ and $\beta$. We then get $\gamma = (0.6 - \beta)/(\alpha - \beta) \approx 0.4286$. As in Acemoglu and Guerrieri [1], we choose $\theta = 4$, leading to reasonable elasticity of intertemporal substitution in consumption. As we do not have any empirical evidence to calibrate the values of the parameters $z$ and $\eta$ characterizing the differential equation of individual knowledge, we will adjust these values to match the endogenous annual output growth rate $g_Y$ and the endogenous annual (asymptotic) net interest rate $r^* = R^* - \delta = \rho + \theta(n + g_Y)$. The total output growth between 1948 and 2005 in the NIPA is 3.4%, leading to $g_Y = 0.33$. Following Acemoglu and Guerrieri [1], we choose the value of the asymptotic net interest rate used by Barro and Sala-i-Martin [5], namely $r^* = 0.08$. The corresponding values of $z$ and $\eta$ are then $z = 0.09$ and $\eta = 0.012$.

As shown in Proposition 1, the ratio of capital used in each sector $K_2/K_1$ and the ratio of hours worked in each sector $L_2/L_1$ are constant over time and, considering our calibration, equal to

$$\frac{K_2}{K_1} = \frac{(1-\gamma)(1-\beta)}{\gamma(1-\alpha)} \approx 1.916 \quad \text{and} \quad \frac{L_2}{L_1} = \frac{(1-\gamma)\beta}{\gamma\alpha} \approx 1.059$$

However, there are some reallocations of labor from one sector to the other as the ratio of workers $N_1/N_2$ is not constant. The asymptotic value (along the NBGP) of this ratio is

$$\frac{N_2}{N_1} = u^* \frac{(1-\gamma)\beta}{\gamma\alpha} \approx 0.554$$

with the asymptotic fraction of time devoted to production in the knowledge-intensive sector $u^* \approx 0.523$.

We are now ready to complete our analysis by looking at the transitional dynamics to the NBGP. We are interested in the properties of the transition path of the following variables:

$$\begin{array}{rcl} \frac{Y_2(t)}{Y_1(t)} & = & \frac{A_2^{1-\beta} B_2^{\beta}}{A_1^{1-\alpha} B_1^{\alpha}} k(t)^{\alpha-\beta} x(t)^{-\alpha} \ell(t)^{\beta-\alpha} N_0^{\beta-\alpha} e^{-\frac{\alpha\beta(z-\eta+n-\rho)}{\theta\gamma\alpha+(1-\gamma)\beta}t} \\ \frac{N_2(t)}{N_1(t)} & = & u(t) \frac{(1-\gamma)\beta}{\gamma\alpha} \\ r(t) & = & Mk(t)^{-\gamma\alpha-(1-\gamma)\beta} x(t)^{\gamma\alpha} \ell(t)^{\gamma\alpha+(1-\gamma)\beta} N_0^{\gamma\alpha+(1-\gamma)\beta} - \delta \end{array} \tag{37}$$

with $A_1, A_2, B_1, B_2$, $M = [1 - \gamma\alpha - (1-\gamma)\beta]D$ and $D$ as given in Proposition 1.

For our simulations, we consider that the initial date $t = 0$ corresponds to 1948 and thus $t = 57$ corresponds to 2005. Using the NIPA data for 1948 gives $N(0) = 40,336$ (in thousands) and $K(0) = k(0) = 244,900$ (in millions of dollars).[7] The initial value of knowledge $a(0) = x(0)$ is adjusted

[7] To avoid any convergence problem for the simulations, we normalize the initial population to 1, i.e. $N(0) = 1$, and we proceed with the per-capita initial capital stock $K(0) = k(0) = 6.07$.

to obtain the same initial net interest rate as in Acemoglu and Guerrieri [1], namely $r(0) = 0.095$. We immediately conclude that along the transition, the interest rate must decline as its asymptotic value $r^*$ is lower. Considering the expression of $Y_2(t)/Y_1(t)$ and the fact that the knowledge-intensive sector is dominant in the long-run, we also know that this ratio has to decline.

### 6.2 Transitional patterns

The numerical strategy highlighted a few interesting properties. Two types of results were sought. First, we wanted to know what the converging paths in Fig 1 looked like. That is, how different choices of $k(0)$ and $x(0)$ would affect $q(0)$ and thus the NBGP $(k^*, x^*, p^*, q^*)$.

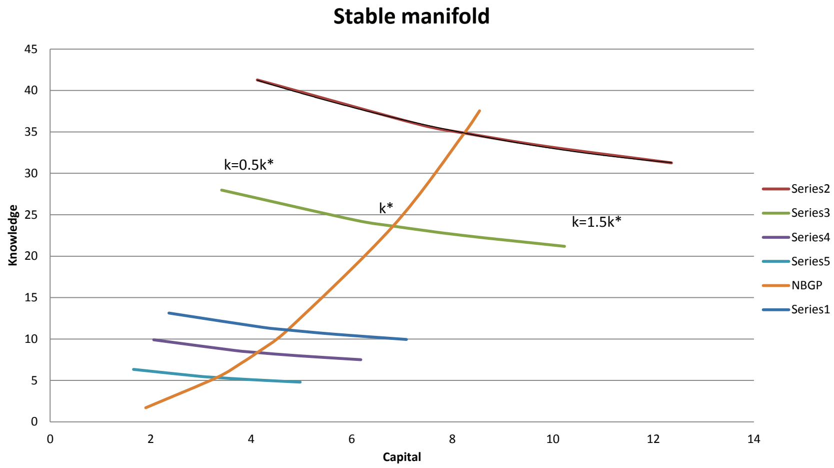


Figure 2: Manifold of steady states and convergence process

As suggested, Figure 2 shows that lower initial endowment in capital or knowledge are associated with higher values of $q_0$ and, according to Theorem 2, lead to lower levels of capital, knowledge and output.

Second, we wanted to reveal the dynamics along the transition path of the various economic variables, and their speed of convergence.

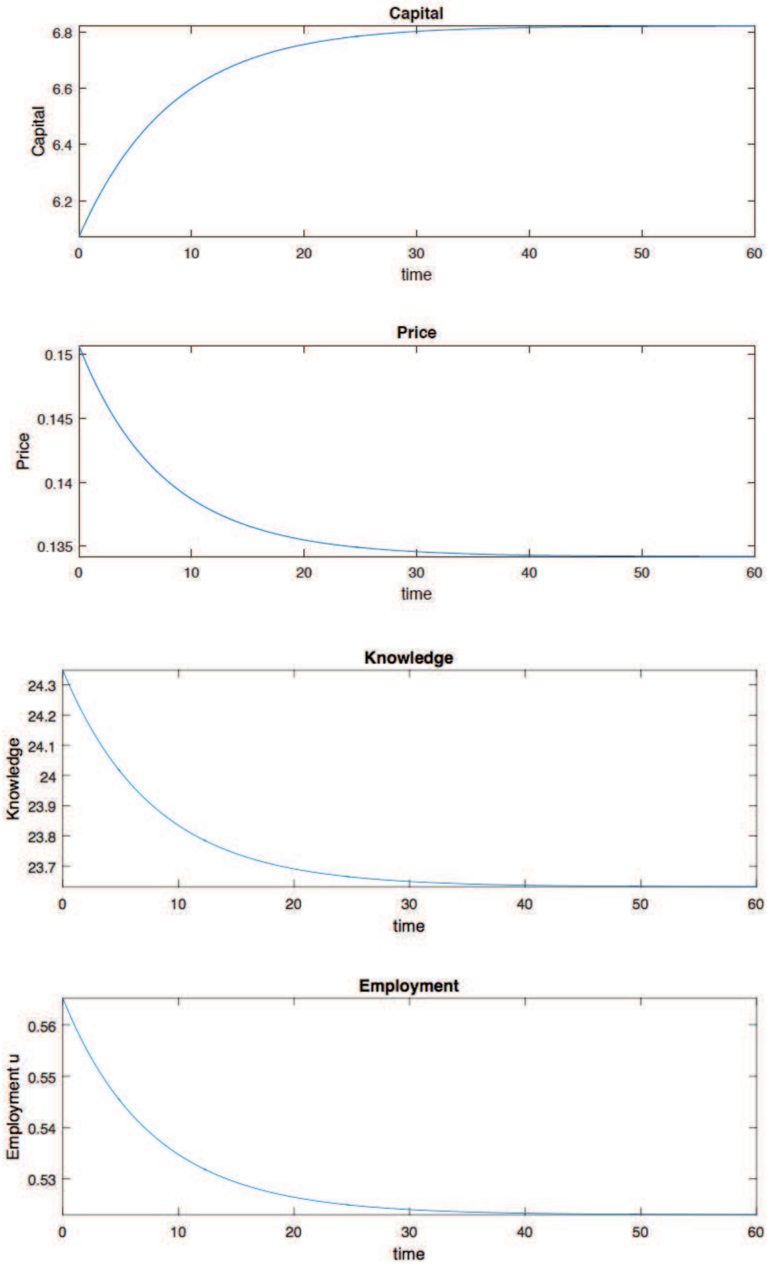


Figure 3: Transitional dynamics of the main variables

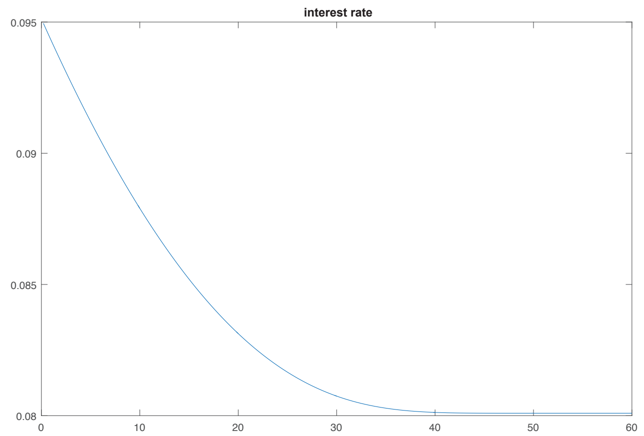


Figure 4: Transitional dynamics of the interest rate

A number of features are worth noting. First, as shown in Figure 3, the economy takes approximately 50 years to reach the asymptotic equilibrium, during which there are significant reallocations of capital and labor to the knowledge-intensive sector. This shows that knowledge adjustments generate fairly rapid structural change, actually faster than under the mechanism explored by Acemoglu and Guerrieri [1]. Second, while there is non-balanced growth at the sectoral level, and as we have shown previously, even along the transition path, the nominal share of capital income in GDP is constant and equal to $s_K = 1 - \gamma\alpha - (1-\gamma)\beta$. Third, along the transition path, the speed of convergence is decreasing and comparable for all economic variables, even for the interest rate, as shown in Figure 4. This means that we fail to reproduce the slow convergence of the interest rate achieved by Acemoglu and Guerrieri. We believe that the difference is the result of the endogenous growth mechanism, which casts doubt on the generality of the result in Acemoglu and Guerrieri. In our model, along the planner's solution, the relative quantities and the relative prices move in opposite directions in such a way that the value-added shares stay constant. On the transition path, although hours worked do not change, Figure 3 clearly shows that employment does. However, numerical analysis seems to suggest that along the transition path, employment follows the other variables, which would be consistent with the data.

## 7 Conclusion

We proposed a two-sector model of non-balanced endogenous growth in which the final good is produced using two intermediary sectors, one being knowledge-intensive *a la* Romer [23]. The endogenous accumulation of knowledge leads to an unbounded increase in TFP in the knowledge-intensive sector and thus to capital deepening. As a result, there are labor reallocations toward the knowledge-intensive sector leading to a greater increase in its output, i.e. structural transformations. We also showed that non-balanced growth is consistent with Kaldor facts, in particular the constant interest rate and capital share in national income. Finally, we characterized the non-balanced growth path together with the equilibrium (Pareto-optimal) dynamics in its neighborhood, showing that saddle-point stability always holds.

We proved that, associated with the non-balanced growth path, there exists a manifold of steady states parameterized by the initial value of the price of knowledge. Although each of these steady states is saddle-point stable and associated with a set of unique non-balanced growth rates, depending on the initial value of capital and knowledge, the economy will follow a particular growth path converging to a particular level of wealth. As a consequence, countries with the same fundamentals but lower initial wealth will be characterized by lower asymptotic wealth. Contrary to Acemoglu and Guerrieri [1] but like Lucas [19], we thus prove the existence of non-convergence across countries in a framework with structural change.

The main contribution of this paper is therefore theoretical. We demonstrated that the interaction between the endogenous accumulation of knowledge and the resulting capital deepening generates non-balanced growth, which is consistent with the main aggregate Kaldor facts. We showed that the transition variables monotonously approach their NBGP values. Contrary to the findings of Acemoglu and Guerrieri, even the interest rate rapidly converges to the NBGP.

# 8 Appendix

## 8.1 Proof of Proposition 1

From equation (14) we get $K_1 = \frac{\gamma(1-\alpha)}{(1-\gamma)(1-\beta)} K_2$. Substituting this expression into $K_1 + K_2 = K$ gives the expressions of $K_1$ and $K_2$. From equation (15) we derive $L_1 = \frac{\gamma\alpha}{(1-\gamma)\beta} L_2$ and using $L_1 + L_2 = L$ we get the expressions of $L_1$ and $L_2$. Recall now that $N = N_1 + N_2$ and $L = uN_1 + N_2$. We derive $uN_1 = B_1 L = B_1(uN_1 + N - N_1) = B_1[N - N_1(1-u)]$. Solving this equation with respect to $N_1$ gives

$$N_1 = \frac{B_1 N}{u + B_1(1-u)} \text{ and thus } N_2 = \frac{u(1-B_1)N}{u+B_1(1-u)} \tag{38}$$

We finally derive

$$L = \frac{uN}{u+B_1(1-u)} \tag{39}$$

Using all these results allows us to write aggregate output as follows

$$Y = DK^{1-\gamma\alpha-(1-\gamma)\beta} a^{\gamma\alpha} \left(\frac{uN}{u+B_1(1-u)}\right)^{\gamma\alpha+(1-\gamma)\beta} \tag{40}$$

with $D = A_1^{\gamma(1-\alpha)} A_2^{(1-\gamma)(1-\beta)} B_1^{\gamma\alpha} B_2^{(1-\gamma)\beta}$. Substituting equation (10) into equation (9) gives

$$P\gamma\alpha\frac{Y}{ua} = Qz \tag{41}$$

Using (38), (39) and (40) into (41) gives the expression

$$G(K, a, P, Q, N, u) = \frac{\gamma\alpha D}{z}\frac{P}{Q} K^{1-\gamma\alpha-(1-\gamma)\beta} a^{\gamma\alpha-1} N^{\gamma\alpha+(1-\gamma)\beta} - g(u) = 0 \tag{42}$$

with $g(u) = u^{1-\gamma\alpha-(1-\gamma)\beta}[u + B_1(1-u)]^{\gamma\alpha+(1-\gamma)\beta}$. Since $g(0) = 0$ and $g'(u) > 0$ for $u \in (0,1)$, we conclude that there exists a unique function $v : \mathbb{R}^5 \to$ such that $u = v(K, a, P, Q, N)$. Obvious computations show that

$$\frac{\partial v}{\partial K} = -\frac{\partial G/\partial K}{g'(u)} > 0, \quad \frac{\partial v}{\partial a} = -\frac{\partial G/\partial a}{g'(u)} < 0$$

Finally, substituting the function $u = v(K, a, P, Q, N)$ into $N_1, N_2, L, Y_1, Y_2$ and $Y$ gives the expressions in the Proposition. □

## 8.2 Proof of Theorem 1

From Proposition 1 we immediately derive that $g_K = g_{K_1} = g_{K_2}$. Differentiating equation (8) gives using (28)

$$g_{c_1} = g_{c_2} = -\frac{1}{\theta} g_P = \frac{1}{\theta}\left[\gamma(1-\alpha)\frac{Y}{K_1} - \delta - \rho\right] \tag{43}$$

It follows that $g_P = -\theta g_{c_1} = -\theta g_{c_2}$. Since $g_P$ is constant along a NBGP, we get $g_K = g_Y$. The capital accumulation equation (6) can be written as

$$g_K = \frac{Y}{K} - \delta - \frac{NP^{-1/\theta}}{K}$$

Differentiating this expression using the fact that along a NBGP $\dot{g}_K = 0$ yields $g_K = n - g_P/\theta$. Since aggregate consumption is given by $C = NP^{-1/\theta}$ we conclude that $g_C = g_K$.

From the equalities $L_1 = uN_1$ and $L_2 = N_2$, and considering that along a NBGP the share $u$ must be constant, we derive that $\dot{L}_1/L_1 = \dot{N}_1/N_1 = g_{N_1}$ and $\dot{L}_2/L_2 = \dot{N}_2/N_2 = g_{N_2}$. Moreover, since $L_1/L = B_1$ and $L_2/L = B_2$, we conclude that $\dot{L}_1/L_1 = \dot{L}_2/L_2 = \dot{L}/L$ and thus $\dot{L}/L = g_{N_1} = g_{N_2} = n$. From (3), (4), (5) and (19) in Proposition 1 we get

$$\frac{Y}{K} = DK^{-\gamma\alpha-(1-\gamma)\beta} a^{\gamma\alpha} L^{\gamma\alpha+(1-\gamma)\beta}$$

Differentiating this expression considering that $g_K = g_Y$ then yields

$$g_K = \frac{\gamma\alpha}{\gamma\alpha+(1-\gamma)\beta} g_a + n \tag{44}$$

Differentiating (41) using (29) yields

$$g_P + g_Y - g_a = g_Q = -(z - \eta - \rho) \tag{45}$$

Substituting $g_P = \theta(n - g_K)$ and $g_Y = g_K$ into (45) and considering (44) we conclude that

$$\begin{aligned} g_a &= (z - \eta + n - \rho)\frac{\gamma\alpha+(1-\gamma)\beta}{\theta\gamma\alpha+(1-\gamma)\beta} \\ g_K &= (z - \eta + n - \rho)\frac{\gamma\alpha}{\theta\gamma\alpha+(1-\gamma)\beta} + n \end{aligned} \tag{46}$$

Recall from (31) that $g_a = z(1-u) - \eta$. Considering the expression of $g_a$ as given in (46), solving this equality with respect to $u$ gives the expression (33). A sufficient condition to get $u \in (0,1)$ is given by Assumption 1. Finally, the differentiation of (4) and (5) gives

$$\begin{aligned} g_{Y_1} &= \alpha(n + g_a) + (1-\alpha) g_K \\ g_{Y_2} &= \beta n + (1-\beta) g_K \end{aligned}$$

We derive from this:

$$\begin{aligned} g_{Y_1} &= (z - \eta + n - \rho)\frac{\alpha[\gamma+(1-\gamma)\beta]}{\theta\gamma\alpha+(1-\gamma)\beta} + n \\ g_{Y_2} &= (z - \eta + n - \rho)\frac{\gamma\alpha(1-\beta)}{\theta\gamma\alpha+(1-\gamma)\beta} + n \end{aligned}$$

and straightforward computations give $g_{Y_1} > g_Y > g_{Y_2}$. □

### 8.3 Proof of Corollary 1

Let us consider the first-order conditions (9)-(15), Proposition 1 and Theorem 1.

i) We derive from Theorem 1 that the growth rate of $K/L$ is equal to

$$g_{K/L} = g_K - n = \frac{(z-\eta+n-\rho)\gamma\alpha}{\theta\gamma\alpha+(1-\gamma)\beta} > 0$$

ii) This result is obvious from Theorem 1.

iii) Along the NBGP the capital-output ratio is given by

$$g_{K/Y} = g_K - g_Y = 0$$

iv) Using (10) and (11), we derive the following expression of GDP

$$PY = P_1Y_1 + P_2Y_2$$

As both intermediary goods are produced with constant returns technologies, we have

$$PY = P_1\left(r_1K_1 + w_1auN_1\right) + P_2\left(r_2K_2 + w_2N_2\right)$$

We then derive the share of capital income in GDP as follows

$$s_K = \frac{P_1r_1K_1+P_2r_2K_2}{PY}$$

and using the expressions of $r_1$ and $r_2$ respectively given by (16) and (17) together with (10) and (11), we get

$$s_K = \gamma(1-\alpha) + (1-\gamma)(1-\beta) = 1 - \gamma\alpha - (1-\gamma)\beta$$

Note that this equality holds for any $t \geq 0$, i.e. during the transition and along the NBGP.

v) As shown by Proposition 1 and equation (23), the real interest rate is given by $R = [1-\gamma\alpha-(1-\gamma)\beta]Y/K$. Along the NBGP we then get $\dot{R}/R = g_Y - g_K = 0$.

vi) From (10) and (11) we derive that

$$\frac{P_1}{P} = \gamma\frac{Y}{Y_1}, \text{ and } \frac{P_2}{P} = (1-\gamma)\frac{Y}{Y_2}$$

Since $g_{Y_1} > g_Y > g_{Y_2}$, we conclude that $g_{P_1/P} < 0$, $g_{P_2/P} > 0$ and thus $g_{P_2/P_1} > 0$.

vii) From the results of point vii), we immediately derive that the real shares of the knowledge-intensive and the second sectors in GDP, $Y_1/Y$ and $Y_2/Y$, are respectively increasing and decreasing while the nominal shares, $P_1Y_1/PY$ and $P_2Y_2/PY$, are constant. □

### 8.4 Proof of Lemma 1

Let us consider the stationarized values for $K(t)$, $A(t)$ and $P(t)$ as defined by $k(t) = K(t)e^{-g_K t}$, $x(t) = a(t)e^{-g_a t}$ and $p(t) = P(t)e^{-g_P t}$, for all $t \geq 0$. Recall also that as population is growing at the exponential rate $n$, we have $N(t) = e^{nt}N(0)$ with $N(0) = N_0$ given. From equation (42), we first derive that

$$G(K,a,P,Q,N,u) = \tfrac{\gamma\alpha D}{z}\tfrac{p}{q}k^{1-\gamma\alpha-(1-\gamma)\beta}x^{\gamma\alpha-1}N_0^{\gamma\alpha+(1-\gamma)\beta}e^{\Phi t} - g(u) = 0$$

with $\Phi = g_P - g_Q + [1-\gamma\alpha-(1-\gamma)\beta]g_K + (\gamma\alpha-1)g_a + [\gamma\alpha+(1-\gamma)\beta]n$. Considering the growth rates as given by Theorem 1, we conclude that $\Phi = 0$. It follows that $v(K,a,P,Q,N) = v(k,x,p,q,N_0)$ and thus $L = \ell(k,x,p,q,N_0)N$. We also get

$$\tfrac{Y}{K} = Dk^{-\gamma\alpha-(1-\gamma)\beta}x^{\gamma\alpha}N_0^{\gamma\alpha+(1-\gamma)\beta}\ell(k,x,p,q,N_0)^{\gamma\alpha+(1-\gamma)\beta}e^{\Psi t}$$

with $\Psi = \gamma\alpha g_a + [\gamma\alpha+(1-\gamma)\beta](n-g_K) = 0$. We finally derive

$$\dot{k} = k\left[\tfrac{\dot{K}}{K} - g_K\right],\ \dot{x} = x\left[\tfrac{\dot{a}}{a} - g_a\right],\ \dot{p} = p\left[\tfrac{\dot{P}}{P} - g_P\right]$$

The result follows from equations (28)-(31) and the fact that $q(t) = q(0) = q_0$ for all $t \geq 0$. □

### 8.5 Proof of Theorem 2

Consider the stationarized dynamical system as given in Lemma 1. A steady-state is a solution of the following system

$$Fk^{-\gamma\alpha-(1-\gamma)\beta}a^{\gamma\alpha}\ell\left(k,a,p,q_0,N_0\right)^{\gamma\alpha+(1-\gamma)\beta} = \delta+\rho-g_P \quad (47)$$

$$Ek^{-\gamma\alpha-(1-\gamma)\beta}a^{\gamma\alpha}\ell\left(k,a,p,q_0,N_0\right)^{\gamma\alpha+(1-\gamma)\beta} = g_K+\delta+\frac{N_0 p^{-\frac{1}{\theta}}}{k} \quad (48)$$

$$z(1-u) = g_a+\eta \quad (49)$$

From (49) we get $u = (z - g_a - \eta)/z \equiv u^*$ as given by (33) and thus $\ell^* = u^*/[u^* + B_1(1-u^*)]$. It follows that $\ell^*$ does not depend on $q_0$ while along transition $\ell(t)$ does. Taking the ratio of (48) on (47) gives after simplification

$$g_K + \delta + \frac{N_0 p^{-\frac{1}{\theta}}}{k} = \frac{\delta+\rho-g_P}{1-\gamma\alpha-(1-\gamma)\beta} \tag{50}$$

Substituting (50) into (48) and solving for $k$ gives

$$k = \left[\frac{E[1-\gamma\alpha-(1-\gamma)\beta]}{\delta+\rho-g_P}\right]^{\frac{1}{\gamma\alpha+(1-\gamma)\beta}} x^{\frac{\gamma\alpha}{\gamma\alpha+(1-\gamma)\beta}} \ell^* \equiv \mathcal{K} a^{\frac{\gamma\alpha}{\gamma\alpha+(1-\gamma)\beta}} \ell^* \tag{51}$$

Solving (50) with respect to $p$ using (51) gives

$$p = \left(\frac{\mathcal{Z}_1 \mathcal{K}}{N_0}\right)^{-\theta} x^{\frac{-\theta\gamma\alpha}{\gamma\alpha+(1-\gamma)\beta}} \ell^{*-\theta} \equiv \mathcal{P} x^{\frac{-\theta\gamma\alpha}{\gamma\alpha+(1-\gamma)\beta}} \ell^{*-\theta} \tag{52}$$

with

$$\mathcal{Z}_1 \quad = \quad \frac{\delta+\rho-g_P}{1-\gamma\alpha-(1-\gamma)\beta} - \delta - g_K = \frac{\rho-g_P-g_K+(\delta+g_K)[\gamma\alpha+(1-\gamma)\beta]}{1-\gamma\alpha-(1-\gamma)\beta} \tag{53}$$

Note that under Assumption 1, $\mathcal{Z}_1$ is positive. Consider finally equation (41) which can be written as

$$\frac{\gamma\alpha}{z} E \frac{p}{q_0} k^{1-\gamma\alpha-(1-\gamma)\beta} x^{\gamma\alpha-1} \ell^{*\gamma\alpha+(1-\gamma)\beta} = u^* \tag{54}$$

Substituting (51) and (52) into (54) and solving for $a$ finally gives

$$X^* = \left[\frac{\gamma\alpha}{zu^* q_0} E\mathcal{P}\mathcal{K}^{1-\gamma\alpha-(1-\gamma)\beta} \ell^{*1-\theta}\right]^{\frac{\gamma\alpha+(1-\gamma)\beta}{\theta\gamma\alpha+(1-\gamma)\beta}} \equiv x^*(q_0) \tag{55}$$

Therefore, substituting (55) into (52) and (51), we find

$$\begin{aligned} k^* &= \mathcal{K}\left[\frac{\gamma\alpha}{zu^* q_0} E\mathcal{P}\mathcal{K}^{1-\gamma\alpha-(1-\gamma)\beta} \ell^{*1-\theta}\right]^{\frac{\gamma\alpha}{\theta\gamma\alpha+(1-\gamma)\beta}} \ell^* \equiv k^*(q_0) \\ p^* &= \mathcal{P}\left[\frac{\gamma\alpha}{zu^* q_0} E\mathcal{P}\mathcal{K}^{1-\gamma\alpha-(1-\gamma)\beta} \ell^{*1-\theta}\right]^{\frac{-\theta\gamma\alpha}{\theta\gamma\alpha+(1-\gamma)\beta}} \ell^{*-\theta} \equiv p^*(q_0) \end{aligned} \tag{56}$$

We conclude that for any given $q_0 > 0$, there exists a unique steady state $(k^*(q_0), x^*(q_0), p^*(q_0))$ with $k^{*'}(q_0) < 0$, $x^{*'}(q_0) < 0$ and $p^{*'}(q_0) > 0$. □

## 8.6 Proof of Theorem 2

Before linearizing the dynamical system of Lemma 1 around the steady state $(k^*(q_0), x^*(q_0), p^*(q_0))$, we need to compute the derivatives of the function $\ell(k, x, p, q, N_0)$ with respect to $k, x$ and $p$ evaluated at $(k^*(q_0), x^*(q_0), p^*(q_0))$. Consider equation (42) and the definition of the function $g(u)$ in the proof of Proposition 1. We first get

$$\epsilon_{gu} = \frac{g'^{*})u^*}{g(u^*)} = \frac{u^*(1-\gamma)\beta+\gamma\alpha[1-\gamma\alpha-(1-\gamma)\beta]}{u^*(1-\gamma)\beta+\gamma\alpha} \tag{57}$$

Applying the implicit function theorem on equation (42) and considering that $v(K,a,P,Q,N) = v(k,x,p,q,N_0)$ together with (57), we derive the elasticities of the function $v(k,a,p,q,N_0)$:

$$\begin{aligned}
\epsilon_{vk} &= \frac{\partial v}{\partial k}\frac{k}{v} = \frac{[1-\gamma\alpha-(1-\gamma)\beta][u^*(1-\gamma)\beta+\gamma\alpha]}{u^*(1-\gamma)\beta+\gamma\alpha[1-\gamma\alpha-(1-\gamma)\beta]} \\
\epsilon_{vx} &= \frac{\partial v}{\partial x}\frac{x}{v} = -\frac{(1-\gamma\alpha)[u^*(1-\gamma)\beta+\gamma\alpha]}{u^*(1-\gamma)\beta+\gamma\alpha[1-\gamma\alpha-(1-\gamma)\beta]} \\
\epsilon_{vp} &= \frac{\partial v}{\partial p}\frac{p}{v} = \frac{u^*(1-\gamma)\beta+\gamma\alpha}{u^*(1-\gamma)\beta+\gamma\alpha[1-\gamma\alpha-(1-\gamma)\beta]}
\end{aligned}$$

Consider now the function $l(u) = u/[u+B_1(1-u)]$. We get

$$\epsilon_{lu} = \frac{l'(u)u}{l(u)} = \frac{\gamma\alpha}{u^*(1-\gamma)\beta+\gamma\alpha}$$

Since $\ell(k,x,p,q,N_0) = l(v(k,x,p,q,N_0))$ we finally derive

$$\begin{aligned}
\epsilon_{\ell k} &= \frac{\partial \ell}{\partial k}\frac{k}{\ell} = \epsilon_{lu}\epsilon_{vk} = \frac{\gamma\alpha[1-\gamma\alpha-(1-\gamma)\beta]}{u^*(1-\gamma)\beta+\gamma\alpha[1-\gamma\alpha-(1-\gamma)\beta]} \\
\epsilon_{\ell ax} &= \frac{\partial \ell}{\partial x}\frac{x}{\ell} = \epsilon_{lx}\epsilon_{vx} = -\frac{\gamma\alpha(1-\gamma\alpha)}{u^*(1-\gamma)\beta+\gamma\alpha[1-\gamma\alpha-(1-\gamma)\beta]} \\
\epsilon_{\ell p} &= \frac{\partial \ell}{\partial p}\frac{p}{\ell} = \epsilon_{lu}\epsilon_{vp} = \frac{\gamma\alpha}{u^*(1-\gamma)\beta+\gamma\alpha[1-\gamma\alpha-(1-\gamma)\beta]}
\end{aligned}$$

Let us now consider the dynamical system

$$\begin{aligned}
\dot{p} &= -p\left\{Fk^{-\gamma\alpha-(1-\gamma)\beta}x^{\gamma\alpha}\ell\left(k,x,p,q_0,N_0\right)^{\gamma\alpha+(1-\gamma)\beta} + g_P - \delta - \rho\right\} \equiv \mathcal{F}(p,k,x) \\
\dot{k} &= k\left\{Ek^{-\gamma\alpha-(1-\gamma)\beta}x^{\gamma\alpha}\ell\left(k,x,p,1q_0,N_0\right)^{\gamma\alpha+(1-\gamma)\beta} - g_K - \delta - \frac{N_0 p^{-\frac{1}{\theta}}}{k}\right\} \equiv \mathcal{G}(p,k,x) \\
\dot{x} &= a\left\{z\left[1-v(k,x,p,q_0,N_0)\right] - g_a - \eta\right\} \equiv \mathcal{H}(p,k,x)
\end{aligned}$$

The linearization around the steady state yields after tedious but straightforward computations the following Jacobian matrix:

$$\mathcal{J} = \begin{pmatrix} \mathcal{F}_1(p^*(q_0),k^*(q_0),x^*(q_0)) & \mathcal{F}_2(p^*(q_0),k^*(q_0),x^*(q_0)) & \mathcal{F}_3(p^*(q_0),k^*(q_0),x^*(q_0)) \\ \mathcal{G}_1(p^*(q_0),k^*(q_0),x^*(q_0)) & \mathcal{G}_2(p^*(q_0),k^*(q_0),x^*(q_0)) & \mathcal{G}_3(p^*(q_0),k^*(q_0),x^*(q_0)) \\ \mathcal{H}_1(p^*(q_0),k^*(q_0),x^*(q_0)) & \mathcal{H}_2(p^*(q_0),k^*(q_0),x^*(q_0)) & \mathcal{H}_3(p^*(q_0),k^*(q_0),x^*(q_0)) \end{pmatrix}$$

with

$$\mathcal{F}_1(p^*(q_0),k^*(q_0),x^*(q_0)) = -\frac{(\delta+\rho-g_P)\gamma\alpha[\gamma\alpha+(1-\gamma)\beta]}{u^*(1-\gamma)\beta+\gamma\alpha[1-\gamma\alpha-(1-\gamma)\beta]} < 0$$

$$\mathcal{F}_2(p^*(q_0),k^*(q_0),x^*(q_0)) = \frac{p^*(q_0)}{k^*(q_0)}\frac{(\delta+\rho-g_P)u^*(1-\gamma)\beta[\gamma\alpha+(1-\gamma)\beta]}{u^*(1-\gamma)\beta+\gamma\alpha[1-\gamma\alpha-(1-\gamma)\beta]} > 0$$

$$\mathcal{F}_3(p^*(q_0),k^*(q_0),x^*(q_0)) = \frac{p^*(q_0)}{x^*(q_0)}\frac{(\delta+\rho-g_P)\gamma\alpha(1-\gamma)\beta(1-u^*)}{u^*(1-\gamma)\beta+\gamma\alpha[1-\gamma\alpha-(1-\gamma)\beta]} > 0$$

$$\mathcal{G}_1(p^*(q_0),k^*(q_0),x^*(q_0)) = \frac{k^*(q_0)}{p^*(q_0)}\left\{\frac{(\delta+\rho-g_P)\gamma\alpha[\gamma\alpha+(1-\gamma)\beta]}{[1-\gamma\alpha-(1-\gamma)\beta][u^*(1-\gamma)\beta+\gamma\alpha[1-\gamma\alpha-(1-\gamma)\beta]]}\right.$$
$$\left. + \frac{1}{\theta}\left[\frac{\delta+\rho-g_P}{1-\gamma\alpha-(1-\gamma)\beta} - \delta - g_K\right]\right\} > 0$$

$$\mathcal{G}_2(p^*(q_0),k^*(q_0),x^*(q_0)) = \frac{(\delta+\rho-g_P)u^*(1-\gamma)\beta+\gamma\alpha}{u^*(1-\gamma)\beta+\gamma\alpha[1-\gamma\alpha-(1-\gamma)\beta]} - \delta - g_K > 0$$

$$\mathcal{G}_3(p^*(q_0),k^*(q_0),x^*(q_0)) = -\frac{k^*(q_0)}{x^*(q_0)}\frac{(\delta+\rho-g_P)}{1-\gamma\alpha-(1-\gamma)\beta}\frac{\gamma\alpha(1-\gamma)\beta(1-u^*)}{u^*(1-\gamma)\beta+\gamma\alpha[1-\gamma\alpha-(1-\gamma)\beta]} < 0$$

$$\mathcal{H}_1(p^*(q_0),k^*(q_0),x^*(q_0)) = -zu^*\frac{x^*(q_0)}{p^*(q_0)}\frac{u^*(1-\gamma)\beta+\gamma\alpha}{u^*(1-\gamma)\beta+\gamma\alpha[1-\gamma\alpha-(1-\gamma)\beta]} < 0$$

$$\mathcal{H}_2(p^*(q_0),k^*(q_0),x^*(q_0)) = -zu^*\frac{x^*(q_0)}{k^*(q_0)}\frac{[1-\gamma\alpha-(1-\gamma)\beta][u^*(1-\gamma)\beta+\gamma\alpha]}{u^*(1-\gamma)\beta+\gamma\alpha[1-\gamma\alpha-(1-\gamma)\beta]} < 0$$

$$\mathcal{H}_3(p^*(q_0),k^*(q_0),x^*(q_0)) = zu^*\frac{(1-\gamma\alpha)[u^*(1-\gamma)\beta+\gamma\alpha]}{u^*(1-\gamma)\beta+\gamma\alpha[1-\gamma\alpha-(1-\gamma)\beta]} > 0$$

We then derive the characteristic polynomial

$$\mathcal{Q}(\lambda) = \lambda^3 - \lambda^2\mathcal{T} + \lambda\mathcal{S} - \mathcal{D} \tag{58}$$

with[8]

$$\begin{aligned}
\mathcal{T} &= \mathcal{F}_1 + \mathcal{G}_2 + \mathcal{H}_3 \\
\mathcal{S} &= \mathcal{F}_1\mathcal{G}_2 - \mathcal{G}_1\mathcal{F}_2 + \mathcal{G}_2\mathcal{H}_3 - \mathcal{H}_2\mathcal{G}_3 + \mathcal{F}_1\mathcal{H}_3 - \mathcal{H}_1\mathcal{F}_3 \\
\mathcal{D} &= \mathcal{F}_1\left[\mathcal{G}_2\mathcal{H}_3 - \mathcal{H}_2\mathcal{G}_3\right] - \mathcal{F}_2\left[\mathcal{G}_1\mathcal{H}_3 - \mathcal{H}_1\mathcal{G}_3\right] + \mathcal{F}_3\left[\mathcal{G}_1\mathcal{H}_2 - \mathcal{H}_1\mathcal{G}_2\right]
\end{aligned}$$

Consider first the expression of $\mathcal{T}$. From (53) and Assumption 1 we get

$$\mathcal{T} = \rho - g_K - g_P + zu^*\frac{(1-\gamma\alpha)[u^*(1-\gamma)\beta+\gamma\alpha]}{u^*(1-\gamma)\beta+\gamma\alpha[1-\gamma\alpha-(1-\gamma)\beta]} > 0$$

which does not depend on $q_0$. Consider now the expression of $\mathcal{D}$. We derive

[8] The argument of the derivatives is omitted to save notations.

after tedious computations

$$\begin{aligned}
\mathcal{G}_2\mathcal{H}_3 - \mathcal{H}_2\mathcal{G}_3 &= zu^* \tfrac{[u^*(1-\gamma)\beta+\gamma\alpha][\rho-g_K-g_P+\gamma\alpha(g_K+\delta)]}{u^*(1-\gamma)\beta+\gamma\alpha[1-\gamma\alpha-(1-\gamma)\beta]} \\
\mathcal{G}_1\mathcal{H}_2 - \mathcal{H}_1\mathcal{G}_2 &= -zu^* \tfrac{x^*}{p^*} \tfrac{[u^*(1-\gamma)\beta+\gamma\alpha][(\rho-g_K-g_P)(1-\theta)+[\gamma\alpha+(1-\gamma)\beta](g_K+\delta)]}{\{u^*(1-\gamma)\beta+\gamma\alpha[1-\gamma\alpha-(1-\gamma)\beta]\}\theta} \\
\mathcal{G}_1\mathcal{H}_3 - \mathcal{H}_1\mathcal{G}_3 &= zu^* \tfrac{k^*}{p^*} \tfrac{[u^*(1-\gamma)\beta+\gamma\alpha]\left\{\gamma\alpha(\delta+\rho-g_P)+\frac{1}{\theta}[\rho-g_K-g_P+[\gamma\alpha+(1-\gamma)\beta](g_K+\delta)]\right\}}{\{u^*(1-\gamma)\beta+\gamma\alpha[1-\gamma\alpha-(1-\gamma)\beta]\}[1-\gamma\alpha-(1-\gamma)\beta]}
\end{aligned}$$

We conclude that

$$\begin{aligned}
\mathcal{D} = & -zu^* \tfrac{(\rho-g_K-g_P)[u^*(1-\gamma)\beta+\gamma\alpha]}{\{u^*(1-\gamma)\beta+\gamma\alpha[1-\gamma\alpha-(1-\gamma)\beta]\}^2} \Big\{ \gamma\alpha \Big[ (\rho - g_K - g_P) \Big[ u^*(1-\gamma)\beta \\
& + \gamma\alpha + (1-\gamma)\beta[\gamma\alpha + (1-\gamma)\beta] \Big] + [\gamma\alpha + (1-\gamma)\beta]^2 (g_K + \delta) \Big] \\
& + \tfrac{(1-\gamma)\beta}{\theta} \Big[ \rho - g_K - g_P + [\gamma\alpha + (1-\gamma)\beta](g_K+\delta) \Big[ \gamma\alpha[1-\gamma\alpha-(1-\gamma)\beta](1-u^*) \\
& + u^*[\gamma\alpha + (1-\gamma)\beta] \Big] \Big] \Big\} < 0
\end{aligned}$$

which does not depend on $q_0$. Finally, straightforward computations also show that $\mathcal{S}$ does not depend on $q_0$ either. We conclude that the eigenvalues do not depend on the value of $q_0$ and nor, therefore, on the value of the steady state $(p^*(q_0), k^*(q_0), x^*(q_0))$. Therefore, since $\mathcal{T} > 0$ and $\mathcal{D} < 0$, we conclude that for any given steady state $(p^*(q_0), k^*(q_0), x^*(q_0))$ on the manifold, the local stability properties are the same. For any given $q_0 > 0$, there exists a unique characteristic root with negative real part and the steady state $(p^*(q_0), k^*(q_0), x^*(q_0))$ is saddle-point stable. Therefore, for any given $q_0 > 0$, there exists a unique $p_0 > 0$ such that the unique converging path is on the stable manifold of dimension one. Along this converging path all the variables are bounded and the transversality conditions are satisfied. Therefore this converging path is the unique optimal solution.

□

# References


[1] Acemoglu, D. and V. Guerrieri (2008): "Capital Deepening and Non-balanced Economic Growth," *Journal of Political Economy*, 116, 467-498.

[2] Alonso-Carrera, J. and X. Raurich (2010): "Growth, Sectoral Composition and the Evolution of Income Levels," *Journal of Economic Dynamics and Control*, 34, 2440-2460.

[3] Alonso-Carrera, J. and X. Raurich (2015): "Demand-Based Structural Change and Balanced Economic Growth," *Journal of Macroeconomics*, 46, 359-374.

[4] Alvarez-Cuadrado, F., N. Van Long and M. Poschke (2017): "Capital-Labor, Substitution, Structural Change and Growth," *Theoretical Economics*, 12, 1229-1266.

[5] Barro, R. and X. Sala-i-Martin (2004): *Economic Growth*, Cambridge, MA: MIT Press..

[6] Baumol, W. (1967): "Macroeconomics of Unbalanced Growth: The Anatomy of Urban Crisis", *American Economic Review*, 57, 415-426.

[7] Bond, E., P. Wang and C. Yip (1996): "A General Two-Sector Model of Endogenous Growth with Human and Physical Capital: Balanced Growth and Transitional Dynamics," *Journal of Economic Theory*, 68, 149-173.

[8] Boppart, T. (2014): "Structural Change and the Kaldor Facts in a Growth Model with Relative Price Effects and Non-Gorman Preferences," *Econometrica*, 82, 2167-2196.

[9] Buera, F. and J.P. Kaboski (2012): "The Rise of the Service Economy," *American Economic Review*, 102, 2540-2569.

[10] Buera, F. and J.P. Kaboski (2012): "Scale and Origins of Structural Change," *Journal of Economic Theory*, 147, 684-712.

[11] Caballe, J. and M. Santos (1993): "On Endogenous Growth with Physical and Human Capital," *Journal of Political Economy*, 101, 1042-1068.

[12] Caselli, F. and J. Coleman (2001): "The U.S. Structural Transformation and Regional Convergence: A Reinterpretation," *Journal of Political Economy*, 109, 584- 617.

[13] Echevarria, C. (1997): "Changes in Sectoral Composition Associated with Economic Growth," *International Economic Review*, 38, 431-452.

[14] Foellmi, R. and J. Zweimueller (2008): "Structural Change, Engel's Consumption Cycles and Kaldor's Facts of Economic Growth," *Journal of Monetary Economics*, 55, 1317-1328.

[15] Herrendorf, B., R. Rogerson and A. Valentinyi (2014): "Growth and Structural Transformation," in *Handbook of Economic Growth* Vol. 2, edited by Philippe Aghion and Steven Durlauf, Elsevier, pp. 855-941.

[16] Kaldor, N. (1961): "Capital Accumulation and Economic Growth," in *The Theory of Capital: Proceedings of a Conference of the International Economic Association*, edited by Friedrich A. Lutz and Douglas C. Hague. London: Macmillan.

[17] Kongsamut, P., S. Rebelo and D. Xie (2001): "Beyond Balanced Growth," *Review of Economic Studies*, 68, 869-882.

[18] Kuznets, S. (1957): "Quantitative Aspects of the Economic Growth of Nations II: Industrial Distribution of National Product and Labor Force," *Economic Development and Cultural Change*, 5, 1-111.

[19] Lucas, R. (1988): "On the Mechanics of Economic Development," *Journal of Monetary Economics*, 22, 3-42.

[20] Martínez-García, M.P. (2003): "The General Instability of Balanced Paths in Endogenous Growth Models: the Role of Transversality Conditions," *Journal of Economic Dynamics and Control*, 27, 599-618.

[21] Mulligan, C. and X. Sala-i-Martin (1993): "Transitional Dynamics in Two Sector Models of Endogenous Growth," *Quarterly Journal of Economics*, 103, 739-773.

[22] Ngai, R. and C. Pissarides (2007): "Structural Change in a Multi-Sector Model of Growth," *American Economic Review*, 97, 429-443.

[23] Romer, P. (1990): "Endogenous Technological Change," *Journal of Political Economy*, 98, S71-S102.

[24] Uzawa, H. (1963): "On a Two-Sector Model of Economic Growth II," *Review of Economic Studies*, 30, 105-118.

[25] Xie, D. (1994): "Divergence in Economic Performance: Transitional Dynamics with Multiple Equilibria," *Journal of Economic Theory*, 63, 97-112.