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Equality among Unequals

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Abstract

This paper establishes an equivalence between three incomplete rankings of distributions of income among agents that are vertically differentiated with respect to some other non-income characteristic (health, household size, etc.). The first ranking is that associated with the possibility of going from one distribution to the other by a finite sequence of income transfers from richer and more highly ranked agents to poorer and less highly ranked ones. The second ranking is the unanimity of all comparisons of two distributions made by a utilitarian planer who assumes that agents convert income into utility by the same function exhibiting a marginal utility of income that is decreasing with respect to both income and the source of vertical differentiation. The third ranking is the Bourguignon (1989) ordered poverty gap dominance criterion.

Keywords: Equalization, transfers, heterogenous agents, poverty gap, dominance, utilitarianism

JEL Classification: D30, D63, I32.

1 Introduction

The foundations to the comparisons of alternative distributions of a *single* cardinally meaningful attribute between a given number of agents from the view point of *equality* are by now well-established. These foundations ride on an *equivalence* between *three* answers that can be provided to the basic question of when a distribution A can be considered "more equal" than a distribution B. These three answers, whose equivalence was apparently first established by Hardy, Littlewood, and Polya (1952) and popularized among economists by Dasgupta, Sen, and Starrett (1973) (see also Kolm (1969), Sen (1973) and Fields and Fei (1978)), are:

(1) A is more equal than B if it can be obtained from B by a means of a finite sequence of bilateral Pigou-Dalton transfers.

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(2) A is more equal than B if all utilitarian (or for that matter - see e.g. Gravel and Moyes (2013)- all utility-inequality averse welfarist) philosophers who assume that individuals convert the attribute into well-being by the same concave utility function would agree so.

(3) A is more equal than B if poverty, as measured by the poverty gap, is lower in A than in B for every definition of the poverty line or, equivalently, if the Lorenz curve associated to A lies everywhere above that associated to B.

The remarkable result of the equivalence between these three answers is of fundamental importance for (in)equality measurement. It shows, in effect, the congruence of three *a priori* distinct approaches to the matter. The first approach focuses on the *elementary operations* that intuitively captures the very notion of inequality reduction. There is indeed a strong presumption that inequalities are unambiguously reduced when a Pigou-Dalton transfer is performed between two individuals. The second approach links inequality measurement to a set of explicit normative principles and seeks for the *unanimity* of all principles in this set. While the original statement of the equivalence was focusing on a welfarist normative theory, it is possible to generalize this approach and to interpret the individual "utility" functions in a non-welfarist way. Finally, the third approach provides *empirically implementable tests* for checking whether or not one distribution dominates the other. These tests have been shown in numerous studies to be quite useful in comparing conclusively several distributions. Furthermore, when more ethically demanding "inequality indices" are used to compare distributions, the compatibility of these indices with any of these answers is seen as a very natural requirement.

Yet, remarkable as they are, these foundations only concern distributions of a single attribute, often identified with income, between otherwise perfectly homogenous agents. Yet, it seems quite clear that income is not the only ethically relevant source of differentiation of economic agents. If these agents are collectivities such as households or jurisdictions, they differ not only by their total income but, also, by the number of members between which the income must be shared. If these agents are individuals, they may also differ by non-income characteristics such as age, health, education or effort. What can be the meaning of "being more equal" when applied to distributions of an attribute between differentiated agents ? In short, how can one define equality among unequals ? This is the basic question addressed in this paper. Precisely, we propose what we view as a plausible "elementary" definition of an "increase in equality" - analogous to the transfer principle of answer (1) - that is applicable to income distributions between agents that are vertically differentiated with respect to a non-income characteristic. We also provide an implementable criterion that coincides with the fact of going from a distribution to another by a finite sequence of such transfers and that is also equivalent to a wide class of normative principles.

For sure, this paper can be seen as a contribution to the multidimensional - in fact the two-dimensional - inequality measurement literature which has emerged, somewhat slowly, in the last forty years or so. Yet, to the very best of our knowledge, no contribution to this literature has succeeded in establishing an *equivalence* between an empirically implementable criterion (such as Lorenz or poverty gap dominance), a welfarist (or otherwise) unanimity over a class of functions that transform the attributes into achievement and an elementary operation that captures in an intuitive way the nature of equalization that is looked for.

For instance Atkinson and Bourguignon (1982) (and before them Hadar and Russell (1974)) have shown that first and second-order multidimensional stochastic dominance imply utilitarian dominance over a class of individual utility functions that is specific to the order of dominance. They also suggest (without providing any proof) that there could be an equivalence between their multidimensional stochastic dominance criteria and utilitarian unanimity over their class of individual utility functions. But they have not provided any elementary transformation that could be implied by their criteria or that could imply them. Atkinson and Bourguignon (1987) have proposed a nice interpretation of one of the Atkinson and Bourguignon (1982) stochastic dominance criteria in the specific case of two attributes, one of which being interpreted as an ordinal index of needs (such as household size). Yet, they have not identified the elementary transformation which, when performed a finite number of times, would coincide with the criterion. Their criterion and equivalence results, developed originally for distributions of attributes with an identical (marginal) distribution of needs, have been extended to more general situations by Jenkins and Lambert (1993) and Bazen and Moyes (2003). Yet, in performing these extensions, the authors have imposed extra-assumptions on the individual utility function and have not identified the underlying elementary transformations.

An interesting empirically implementable criterion, lying between the first and the second order Atkinson and Bourguignon (1982) criteria, has been proposed by Bourguignon (1989) in the same two dimensional context as that considered in Atkinson and Bourguignon (1987). Bourguignon also identified the class of utility functions over which utilitarian unanimity was equivalent to his criterion. However, he did not identify the elementary transformations that would be equivalent to it. The Bourguignon criterion has been extended to distributions with varying marginal distribution of the ordinal index of needs by Fleurbaey, Hagneré, and Trannoy (2003), albeit at the cost of imposing, here again, extra assumptions on individual utility functions.

Elementary transformations believed to lie behind the criteria proposed by Atkinson and Bourguignon (1982), Atkinson and Bourguignon (1987) and Bourguignon (1989) have been discussed by various authors, including Atkinson and Bourguignon (1982) themselves, Ebert (1997), Fleurbaey, Hagneré, and Trannoy (2003) and Moyes (2012) (among many others). Yet none of these papers has shown that performing these elementary transformations a finite number of times was equivalent to the implementable criteria. In a related vein Muller and Scarsini (2012) have established an equivalence between a class of elementary transformations (multidimensional transfers and correlation reducing permutations, to be discussed below) and a utilitarian dominance over the class of increasing and submodular utility functions.[1] However, they have not succeeded in identifying an implementable test - such as Lorenz or poverty gap dominance - that coincides with either their elementary transformations or the utilitarian unanimity over their class of utility functions.

Progresses in establishing equivalence between an empirically implementable criterion, a utilitarian unanimity over a suitable class of individual utility functions and a finite sequence of elementary transformations have been made recently in two streams of the literature. One of them, initiated by Epstein and

[1]See e.g. Marinacci and Montrucchio (2005) for a definition of these properties.

Tanny (1980) (see also Tchen (1980)), and significantly generalized by Decancq (2012), has considered first order stochastic dominance rankings of multivariate distributions in the context of decision making under uncertainty. In this setting, Decancq (2012) has established an equivalence between first order dominance among two multivariate distributions with the same marginals and the possibility of going from the dominated distribution to the dominating one by a finite sequence of Frechet rearrangements (an elementary operation that reduces to correlation reducing permutations when applied to bivariate distributions). By significantly generalizing results from Lehmann (1955) and Levhari, Paroush, and Peleg (1975), Osterdal (2010) also succeeded in establishing an equivalence between a utilitarian unanimity over the class of all increasing utility functions, the possibility of going from one distribution to another by a finite sequence of improving mass transfers, and a specific first order stochastic dominance test that is less discriminant than the usual multivariate one considered in Hadar and Russell (1974) and Atkinson and Bourguignon (1982). None of these results, however, sheds light on the meaning of equalizing income (or other attribute) in a bi or multivariate context.

Progress in this direction was made by Gravel and Moyes (2012) in a setting with only two attributes, one of which being cardinally measurable. In such a setting, Gravel and Moyes (2012) have established a *form of equivalence* between the *three* following answers to the basic question of when a distribution A of two attributes is normatively better than a distribution B:

(a) When A could be obtained from B by performing a finite sequence of *either* Pigou-Dalton transfers of the cardinally measurable attribute between agents endowed with the same quantity of the other attributes *or* correlation reducing permutations.

(b) When A is considered better than B by all utilitarian planners who assume that households transform attributes into well-being by the same utility function that is *increasing in both attributes* and have a *marginal utility* of the cardinally measurable attribute that is *decreasing* with respect to the two attributes.

(c) When poverty gap in the cardinally measurable attribute is lower in A than in B, with poverty gap calculated by assigning to each individual a poverty line that is *negatively* related to the individual's endowment of the other attribute, no matter what is the rule used for assigning poverty line to individuals as a (decreasing) function of their endowment in the other attribute.

Answer (a) combines two intuitive elementary transformations. The first one is the standard Pigou-Dalton transfer performed between agents who are homogenous with respect to the other characteristic. The second one reflects an aversion toward *correlation between the two attributes*. For instance, in a two-individual world where one individual is endowed with more of the two attributes than the other, a transfer of the *difference* in the quantities of the cardinally measurable attribute between the rich and the poor will be considered to be a normative improvement by answer (c). Yet such a "transfer", which happens to be a switch of the quantities of the cardinally measurable attribute between the two individuals, does not affect the dispersion of the cardinally measurable attribute in the population. It only affects the correlation between the two attributes. The favorable permutation is closely related to the notion of Frechet rearrangement considered by Decancq (2012) (see also Tsui (1999), Atkinson and Bourguignon (1982) and Epstein and Tanny (1980)) in the same way as the Pigou-Dalton transfer is related to the mean-preserving spread in the one-dimensional risk literature.

Answer (b) corresponds to the incomplete ranking of distributions of two attributes that commands unanimity over *all* utility-averse welfarist ethics who assume that individual welfare is increasing in both attributes, and whose rate of increase in the cardinally measurable attribute is decreasing with respect to the two attributes. This class of individual utility functions has been considered by Bourguignon (1989) in the specific case where the marginal distribution of the non-cardinally measurable attribute (interpreted in Bourguignon (1989) as the household's size) is fixed.

Answer (c) is nothing else than a generalization of the empirically implementable criterion examined by Bourguignon (1989) and shown by him (under the assumption of a fixed distribution of the non-cardinally measurable attribute) to be equivalent to answer (b). It is a nice poverty dominance criterion which, in our view, has not been sufficiently used in applied works (see however Gravel, Moyes, and Tarroux (2009)). It emphasizes the requirement that, in order to measure income poverty among vertically differentiated agents, the poverty line must depend negatively upon the source of vertical differentiation. After all, a person with a given amount of income should be less likely to be considered poor when she has a good health than when she has a bad one.

However, Gravel and Moyes (2012) did not quite prove that answer (c) (or answer (b)) implies answer (a) (it is easy to see that answer (a) implies answer (b) which implies in turn answer (b)). What they prove is that *if* distribution A dominates distribution B for the Bourguignon criterion, *then* it is possible to add dummy individuals - or *phantoms* - to both distributions A and B in such a way as to be able to go from the phantoms-augmented distribution B to the phantoms-augmented distribution A by first performing a finite sequence of Pigou-Dalton transfers among agents of the same type and, *second*, by performing a finite sequence of favorable permutations. It is not hard to see that the pairwise ranking of the phantoms-augmented distributions A and B by utilitarian or Bourguignon dominance is the same as the pairwise ranking of the real distributions A and B. Indeed the "unconcerned" phantoms do not affect the (additively separable) ranking of A and B by either criteria. Yet the inability of Gravel and Moyes (2012) to prove the equivalence of statements (a) (b) and (c) without resorting to phantoms is somewhat disappointing. What matters, after all, are distributions of attributes between *actua*l individuals. The fact that these actual individuals could make (receive) transfers to (from) non-existing phantoms may seem to be of second order of importance.

In this paper, we provide an equivalence between answers (b) and (c) above on the one hand and the fact of going from the dominated distributions to the dominating one by a finite sequence of elementary transfers of income on the other. The elementary transfer that we consider is quite intuitive. It says that inequalities among vertically differentiated agents are "unquestionably reduced" every time an income transfer is performed from a relatively rich and highly ranked agent to a relatively poor and less highly ranked one provided that the amount transferred does not exceeds the income difference between the two agents. This kind of transfer has been discussed by many authors, including for our purpose Ebert (1997). It strikes us as a very natural notion of equalization among unequals.

We establish the equivalence between either answer (b) or (c) on the one hand and the fact of going from the dominated distribution to the dominating one by a finite sequence of this kind of transfers without any resort whatsoever to phantoms or dummies. We therefore view this paper as providing the first dominance foundation that we are aware of about income equalization between heterogenous agents.

The organization of the remaining of the paper is as follows. In the next section, we introduce notations and give the definitions of the main criteria and elementary transformations considered. The main results are stated and discussed in Section 3 and Section 4 concludes.

2 The formal setting

2.1 Notations

The sets of integers, non-negative integers, real numbers and non-negative real numbers are denoted respectively by $\mathbb{N}$, $\mathbb{N}_+$, $\mathbb{R}$ and $\mathbb{R}_+$. The cardinality of any set A is denoted by $\#A$ and the k-fold Cartesian product of a set A with itself is denoted by A^k.

We consider a finite *population* of n agents ($n \geqq 2$) that are vertically differentiated into k *categories* or *types*, indexed by h. Agents in lower categories are assumed to be more needy (or worse off) ceteris paribus than agents in higher categories. These categories may refer to any non-pecuniary source of agent's differentiation such as health, number of members, education level, labor effort, etc. For any category h, we denote by $\mathcal{N}(h)$ the set of agents in category h and by $n(h) = \#\mathcal{N}(h)$ the number of those agents. Our objective is to provide a ranking of alternative distributions of income (or any other cardinally meaningful variable) between these differentiated agents on the basis of *equality*. Any such income distribution, $\mathbf{x}$ say, is depicted as a collection of k vectors $(x_1^h, ..., x_{n(h)}^h) \in [\underline{v}, \overline{v}\,]^{n(h)}$ (for $h = 1, ..., k$) where $[\underline{v}, \overline{v}\,] \subset \mathbb{R}$ is the interval of possible income levels. The criteria used in this paper for comparing alternative distributions are all *anonymous* conditional on the agent's type. Because of this, we find convenient to index the agents in category h (for $h = 1, ..., k$) according to their income so that one has, for any distribution $\mathbf{x}$ and type h, $x_i^h \leq x_{i+1}^h$ for $i = 1, ..., n(h) - 1$. More compactly, we write $\mathbf{x} = \{(x_1^h, ..., x_{n(h)}^h)\}_{h=1}^{n(h)}$. Since we focus on pure equality considerations, we restrict attention to income distributions $\mathbf{x}$ such that $\sum_{h=1}^{k} \sum_{i \in \mathcal{N}(h)} x_i^h = I$ for some real number $I \in [n\underline{v}, n\overline{v}\,]$. We let $\mathcal{D}(I)$ denote the set of all such income distributions.

For any income poverty *threshold* $t \in [\underline{v}, \overline{v}\,]$ and any distribution $\mathbf{x}$, we also denote by $\overline{\mathcal{P}}^{\mathbf{x}}(h, t)$ and $\mathcal{P}^{\mathbf{x}}(h, t)$ the (possibly empty) sets of agents of type h who are, respectively, weakly and strictly poor for the threshold t in the distribution $\mathbf{x}$. These sets are defined by:

$$
\begin{aligned}
\overline{\mathcal{P}}^{\mathbf{x}}(h, t) &= \{i \in \mathcal{N}(h) : x_i \leq t\} \text{ and} \\
\mathcal{P}^{\mathbf{x}}(h, t) &= \{i \in \mathcal{N}(h) : x_i < t\}
\end{aligned}
$$

while the number of poor that these sets contain are denoted respectively by $\overline{p}^{\mathbf{x}}(h, t) = \#\overline{\mathcal{P}}^{\mathbf{x}}(h, t)$ and $p^{\mathbf{x}}(h, t) = \#\mathcal{P}^{\mathbf{x}}(h, t)$.

Finally, for any two distributions $\mathbf{x}$ and $\mathbf{y} \in \mathcal{D}(I)$, we denote by $\mathcal{I}(\mathbf{x}, \mathbf{y})$ the income support of these two distributions. This set, which consists of all income levels observed in the two distributions, is defined by:

$$\mathcal{I}(\mathbf{x}, \mathbf{y}) = \{a : \exists h \in \{1, ..., k\},\ i \in \{1, ..., n(h)\} \text{ such that } x_i^h = a \text{ or } y_i^h = a\}$$

We now introduce the elementary transformations, the notion of utilitarian dominance, and the implementable ordered poverty gap criterion between which an equivalence shall be established.

2.2 Elementary transformation

The main elementary transformation considered in this paper is the following notion of Between-Type transfers, discussed in many papers, including Ebert (1997), Atkinson and Bourguignon (1982), Fleurbaey, Hagneré, and Trannoy (2003), Muller and Scarsini (2012) and Gravel and Moyes (2012).

Definition 1 *(Between-Type Progressive Income Transfer). Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be two distributions in* $\mathcal{D}(I)$*. We say that* $\mathbf{x}$ *is obtained from* $\mathbf{y}$ *by means of a Between-Type Progressive Income Transfer (BTPIT) if there are categories* g *and* h *for which* $g \leq h$*, two agents* $i^g \in \mathcal{N}(g)$ *and* $i^h \in \mathcal{N}(h)$ *for which* $y_{i^h}^h > y_{i^g}^g$ *and a number* $\alpha \in [0, \frac{y_{i^h}^h - y_{i^g}^g}{2}]$ *such that:*

(i) $x_i^g = y_{i+1}^g$ *for all* $i \in \mathcal{N}(g)$ *such that* $i^g \leq i < r_+^g$ *(if any).*

(ii) $x_{r_+^g}^g = y_{i^g}^g + \alpha$.

(iii) $x_i^h = y_{i-1}^h$ *for all* $i \in \mathcal{N}(h)$ *such that* $r_-^h < i \leq i^h$ *(if any)*

(iv) $x_{r_-^h}^h = y_{i^h}^h - \alpha$.

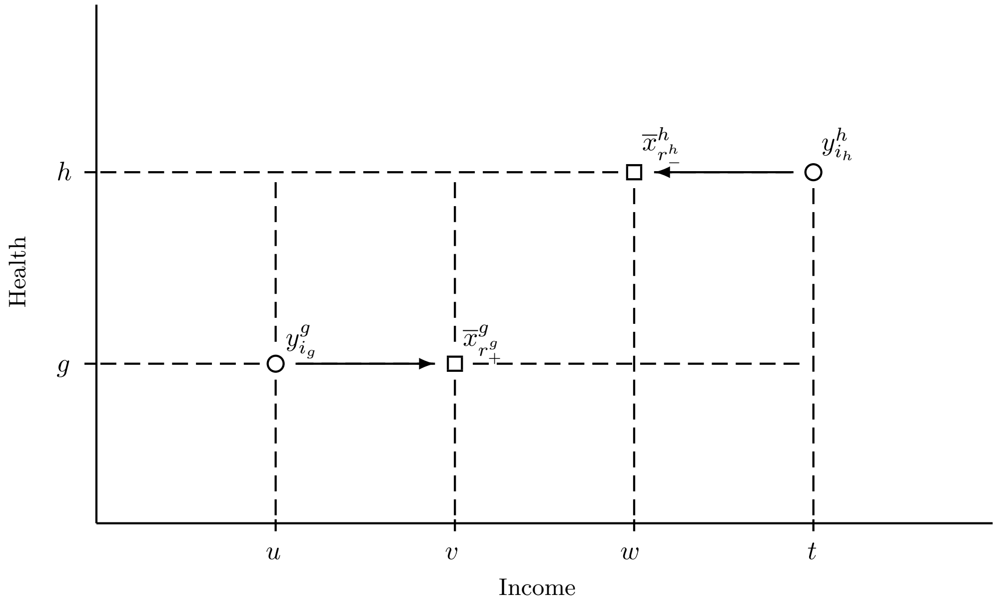
(v) $x_i^l = y_i^l$ *for any other pair* (i, l) *where* $l \in \{1, ..., k\}$ *and* $i \in \mathcal{N}(l)$.

where $r_+^g := \max\{i \in \mathcal{N}(g) : y_i < y_i^g + \alpha\}$, $r_-^h := \min\{i \in \mathcal{N}(h) : y_i > y_i^h - \alpha\}$.

A BTPIT resembles a standard one-dimensional Pigou-Dalton transfer. There is however a major difference: the beneficiary of the transfer must be both poorer than the donor and must also have a (weakly) lower status. Put differently, the transfer recipient must be deprived in both dimensions – income and status – compared to the donor. This kind of transfer is a particular case of the *equalizing transformation* considered by Muller and Scarsini (2012) where the transfers occur in *possibly* all dimensions. In the current setting, it would not make much sense to transfer the (ordinal) non-pecuniary variable by which agents differentiate themselves. We have represented in the Figure above a BTPIT where agent i_g of type g with income u receives an additional income of $\alpha = v - u$ that is taken from individual i_h of a better type h. Note that our definition of a BTPIT allows the donor to be of the same type than the receiver. Hence, the standard one-dimensional Pigou-Dalton transfer (conditional on the type) is a particular case of BTPIT. Notice also that our definition of BTPIT rules out the possibility that the amount transferred be more than half the income difference between the giver and the receiver.

It is easy to eliminate this restriction by considering the following elementary transformation, called *Favourable Income Permutation* in Gravel and Moyes (2012).

Figure 1: A between-type progressive income transfer



Definition 2 *(Favourable Income Permutation).Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be two distributions in* $\mathcal{D}(I)$*. We say that* $\mathbf{x}$ *is obtained from* $\mathbf{y}$ *by means of a Favourable Income Permutation (FIP) if there are categories* g *and* h *for which* $g < h$ *and two agents* $i^g \in \mathcal{N}(g)$ *and* $i^h \in \mathcal{N}(h)$ *for which* $y^h_{i^h} > y^g_{i^g}$ *such that:*

(i) $x^g_i = y^g_{i+1}$ *for all* $i \in \mathcal{N}(g)$ *such that* $i^g \leq i < r^g(i^h)$ *(if any).*

(ii) $x^g_{r^g(i^h)} = y^h_{i^h}$.

(iii) $x^h_i = y^h_{i-1}$ *for all* $i \in \mathcal{N}(h)$ *such that* $r^h(i^g) < i \leq i^h$ *(if any).*

(iv) $x^h_{r^h(i^g)} = y^g_{i^g}$.

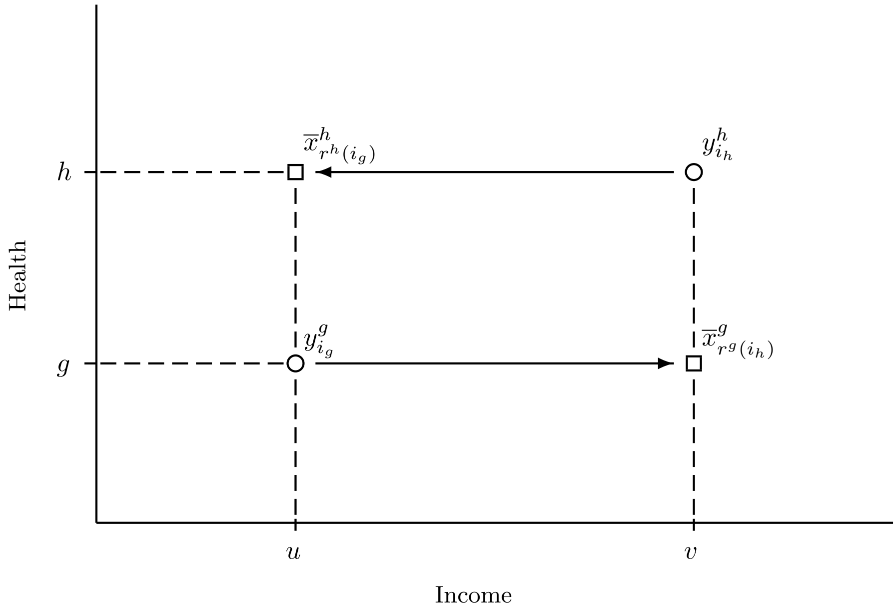
(v) $x^l_i = y^l_i$ *for any other pair* (i, l) *where* $l \in \{1, ..., k\}$ *and* $i \in \mathcal{N}(l)$.

where $r^g(i^h) := \max\{i \in \mathcal{N}(g) : y^g_i < y^h_{i^h}\}$ *and* $r^h(i^g) := \min\{i \in \mathcal{N}(h) : y^h_i > y^g_{i^g}\}$.

A FIP consists in exchanging the income endowment of relatively rich agent belonging to a relatively high category with that of a poorer agent from a lower category. It can be thus be viewed as an extreme form of progressive between-type transfers in which the integrality of the income difference between the two individuals is transferred.

In Gravel and Moyes (2012), it was shown that a BTPIT can always be decomposed into a (within-type) conventional Pigou-Dalton transfer followed by a FIP provided that one adds a phantom individual endowed with the income of the beneficiary and the health status of the donor prior to the transfer. In this

Figure 2: A favourable income permutation



paper, we show that the possibility of going from a distribution $\mathbf{y}$ to a distribution $\mathbf{x}$ by a finite sequence of BTPIT that include FIP as a special (extreme) case is equivalent to the utilitarian dominance of $\mathbf{y}$ by $\mathbf{x}$ for a somewhat large class of utility functions, to the definition of which we now turn.

2.3 Utilitarian dominance.

This notion of dominance rides on the assumption that all agents of a given type transform their income into some type-dependant ethically meaningful achievement (well-being, happiness, freedom, etc.) by means of the same (utility) function satisfying some minimal property. Specifically, the utility achieved by agent i of type h in distribution $\mathbf{x}$ is indicated by $U^h(x_i^h)$, where $U^h : [\underline{v}, \overline{v}] \to \mathbb{R}$. The *utilitarian rule* ranks the distributions on the basis of the *sum of the utilities* they generate. More precisely, the utilitarian rule considers distribution $\mathbf{x}$ to be no worse than distribution $\mathbf{y}$ if and only if

$$\sum_{h=1}^{k} \sum_{i=1}^{n(h)} U^h\left(x_i^h\right) \geqq \sum_{h=1}^{k} \sum_{i=1}^{n(h)} U^h\left(y_i^h\right). \tag{1}$$

The list of type-dependant utility functions $U^1, ..., U^k$ used by the utilitarian rule reflects one's evaluation of the contribution of income to every agent's achievement, conditional on the agent's type. In order to obtain some robustness in the normative evaluation, it is common in the dominance approach to require a consensus among a somewhat large class, $\mathcal{U}^*$ say, of such lists of utility functions. This gives rise to the following general notion of utilitarian dominance.

Definition 3 *(Utilitarian Dominance). We say that distribution* $\mathbf{x}$ *is no worse than distribution* $\mathbf{y}$ *for the utilitarian rule over a class* $\mathcal{U}^*$ *of collections of* k *utility functions if and only if*

$$\sum_{h=1}^{k}\sum_{i=1}^{n(h)} U^h(x_i^h) \geqq \sum_{h=1}^{k}\sum_{i=1}^{n(h)} U^h(y_i^h),\ \forall\ (U^1,...,U^k) \in \mathcal{U}^*. \tag{2}$$

In this paper, we specifically consider the class $\mathcal{U}^*$ of type-dependent $U^1,...,U^k$ that satisfy:

$$U^h(w+a) - U^h(w) \geq U^{h+1}(w'+a) - U^{h+1}(w') \tag{3}$$

for any non-negative real number a, any category $h \in \{1,...,k-1\}$, and any pair of income $(w,w^{'}) \in [\underline{v},\overline{v}]^2$ such that $w \leq w'$. In words, $\mathcal{U}^*$ is the class of collections of utility functions U^h (for $h = 1,...,k$) with the property that the contribution of an additional unit of income to the individual's advantage (as measured by the function U^h) is decreasing with respect to *both* income and the type.

2.4 Ordered poverty gap dominance

The ordered poverty gap criterion has been proposed by Bourguignon (1989) for comparing distributions of incomes between households of differing sizes. In order to define this criterion in the current context, we first define the set $\mathcal{V} \subset \mathbb{R}_+^k$ by:

$$\mathcal{V} = \{(v_1,...,v_k) \in \mathbb{R}^k : \overline{v} \geq v_1 \geq v_2 \geq ... \geq v_k \geq \underline{v}\} \tag{4}$$

The set $\mathcal{V}$ comprises all combinations of poverty lines (one such line for every type) that are weakly decreasing with respect to type. Given this set, we define the Ordered Poverty Gap (OPG) dominance criterion as follows.

Definition 4 *(Ordered Poverty Gap Dominance). Given two distributions* $\mathbf{x}$ *and* $\mathbf{y} \in \mathcal{D}(I)$*, we say that* $\mathbf{x}$ *dominates* $\mathbf{y}$ *for the Ordered Poverty Gap criterion, that we denoted by* $\mathbf{x} \succsim^{OPG} \mathbf{y}$*, if the following inequality:*

$$\sum_{h=1}^{k}\sum_{i\in\mathcal{N}^{\mathbf{x}}(h)} \max(v_h - x_i^h, 0) \leq \sum_{h=1}^{k}\sum_{i\in\mathcal{N}^{y}(h)} \max(v_h - y_i^h, 0) \tag{5}$$

holds for all $(v_1,...,v_k) \in \mathcal{V}$

In words $\mathbf{x}$ dominates $\mathbf{y}$ for the OPG criterion if, for all possible poverty lines that are (weakly) decreasing with respect to the agent's type, the minimal amount of income that is required to eliminate poverty defined by these lines is lower in $\mathbf{x}$ than in $\mathbf{y}$. This ordered poverty dominance criterion is easily implementable (see e.g. Decoster and Ooghe (2006) or Gravel, Moyes, and Tarroux (2009)). In the next section, we show that this criterion, is equivalent to both utilitarian dominance and the possibility of going from the dominated distribution to the dominating one by a finite sequence of BTPIT and/or FIP. For later use, and for any distribution $\mathbf{x} \in D(I)$ and (ordered) poverty lines

$(v_1, ..., v_k) \in \mathcal{V}$, we denote by $P^{\mathbf{x}}(v_1, ..., v_k)$ the Ordered Poverty Gap of this distribution for those poverty lines defined by:

$$P^{\mathbf{x}}(v_1, ..., v_k) = \sum_{h=1}^{k} \sum_{i \in \mathcal{N}^{\mathbf{x}}(h)} \max(v_t - x_i^h, 0) \tag{6}$$

We carefully notice that $P^{\mathbf{x}}(v_1, ..., v_k)$ can also equivalently be written as:

$$P^{\mathbf{x}}(v_1, ..., v_k) = \sum_{h=1}^{k} [\overline{p}^{\mathbf{x}}(h, v_h) v_h - \sum_{i \in \overline{\mathcal{P}}^{\mathbf{x}}(h, v_h)} x_i^h] \tag{7}$$

or as:

$$P^{\mathbf{x}}(v_1, ..., v_k) = \sum_{h=1}^{k} [p^{\mathbf{x}}(h, v_h) v_h - \sum_{i \in \mathcal{P}^{\mathbf{x}}(h, v_h)} x_i^h] \tag{8}$$

3 Main result

The main theorem proved in this paper is the following.

Theorem 1 *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be two distributions in* $\mathcal{D}(I)$*. Then the following three statements are equivalent.*

(i) *It is possible to go from* $\mathbf{y}$ *to* $\mathbf{x}$ *by a finite sequence of BTPIT and/or FIP.*

(ii) $\mathbf{x}$ *utilitarian dominates* $\mathbf{y}$ *for all lists of* k *utility functions* U^h *(for* $h = 1, ..., k$*) in the class* $\mathcal{U}^*$*.*

(iii) $\mathbf{x} \succsim^{OPG} \mathbf{y}$.

The proof of this theorem proceeds in several steps. The first of them, consisting in proving that (i) implies (ii) and that (ii) implies (iii), is easy and somewhat known (see for example Ebert (1997) or Gravel and Moyes (2012)). It is described in Proposition 3.1 that is proved for the sake of completeness. We can then turn, in Theorem 1, to the proof of the crucial implication that statement (iii) implies statement (i). Proving this implication amounts to constructing an algorithm for going from a distribution $\mathbf{y}$ to a distribution $\mathbf{x}$ by a finite sequence of either BTPIT and/or FIP from the mere information that $\mathbf{x} \succsim^{OPG} \mathbf{y}$. In every step of the algorithm, either a BTPIT or a FIP must be performed in such a way that the result of this elementary operation remains dominated by the distribution $\mathbf{x}$. Section 3.2 is devoted to technical lemmas while Section 3.3 establishes a very important first step in the construction of the algorithm. Specifically, we prove in Section 3.3 that if $\mathbf{x}$ strictly dominates $\mathbf{y}$ as per the OPG criterion, it is always possible to perform either a FIP or a BTPIT in a way that preserves weakly the OPG dominance of the newly created distribution by $\mathbf{x}$. We actually propose a diagnostic tool that allows one to identify wether the possible elementary operation is a FIP or BTPIT. Finally, in section 3.4, we define our algorithm and prove its finiteness, which concludes the proof of $(iii) \Rightarrow (i)$.

3.1 A known result $(i) \Rightarrow (ii) \Rightarrow (iii)$

Proposition 1 *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be two distributions in* $\mathcal{D}(I)$*. Then, in Theorem 1, Statement (i) implies Statement (ii) and Statement (ii) implies statement (iii).*

Proof. $(i) \Rightarrow (ii)$ We must prove that both BTPIT and FIP increase the sum of utilities utility for any collection of utility functions $\{U^h\}_{h=1}^k \in \mathcal{U}^*$.

BTPIT: Assume that $\mathbf{x}$ has been obtained from $\mathbf{y}$ by a BTPIT. Then, using Definition 1, there are categories g and h satisfying $g \leq h$, agents $i^g \in \mathcal{N}(g)$ and $i^h \in \mathcal{N}(h)$ satisfying $y^g_{i^g} < y^h_{i^h}$ and a number $\alpha \in [0, (y^h_{i^h} - y^g_{i^g})/2]$ for which one has:

$$\begin{aligned}
& \sum_{j=1}^{k}\sum_{i=1}^{n(j)}(U^j(x_i^j) - U^j(y_i^j)) \\
= \;& U^g(x^g_{r^g_+}) - U^g(y^g_{i^g}) + U^h(x^h_{r^h_-}) - U^h(y^h_{i^h}) \\
= \;& U^g(y^g_{i^g} + \alpha) - U^g(y^g_{i^g}) - [U^h(y^h_{i^h}) - U^h(y^h_{i^h} - \alpha)] \\
\geq \;& 0 \text{ (if the functions } U^1, ..., U^k \text{ belong to } \mathcal{U}^*)
\end{aligned}$$

FIP: Assume that $\mathbf{x}$ has been obtained from $\mathbf{y}$ by a FIP. Then, using Definition 2, there are categories g and h satisfying $g < h$, agents $i^g \in \mathcal{N}(g)$ and $i^h \in \mathcal{N}(h)$ satisfying $y^g_{i^g} < y^h_{i^h}$ for which one has:

$$\begin{aligned}
& \sum_{j=1}^{k}\sum_{i=1}^{n(j)}(U^j(x_i^j) - U^j(y_i^j)) \\
= \;& U^g(x^g_{r^g(i^h)}) - U^g(y^g_{i^g}) + U^h(x^h_{r^h(i^g)}) - U^h(y^h_{i^h}) \\
= \;& U^g(y^h_{i^h}) - U^g(y^g_{i^g}) + U^h(y^g_{i^g}) - U^h(y^h_{i^h}) \\
= \;& U^g(y^h_{i^h}) - U^g(y^g_{i^g}) - [U^h(y^h_{i^h}) - U^h(y^g_{i^g})] \\
\geq \;& 0 \text{ (if the functions } U^1, ..., U^k \text{ belong to } \mathcal{U}^*)
\end{aligned}$$

Repeating the arguments (for the FIP and/or the BTPIT) for any finite sequence of distributions of income completes the proof of the first implication for the theorem.

$(ii) \Rightarrow (iii)$. Let $\mathbf{x}$ and $\mathbf{y}$ be two distributions in $\mathcal{D}(I)$ for which the inequality:

$$\sum_{h=1}^{k}\sum_{i=1}^{n(t)} U^h(x_i^h) - \sum_{h=1}^{k}\sum_{i=1}^{n(t)} U^h(y_i^h) \geq 0 \tag{9}$$

holds for all list of utility functions $\{U^h\}_{h=1}^k$ in $\mathcal{U}^*$. Choose any vector $\mathbf{v} = (v_1, ..., v_k)$ in the set $\mathcal{V}$ and define the k functions $U^{v_h} : [v, \overline{v}] \longrightarrow \mathbb{R}$ (for $h = 1, ..., k$) by:

$$U^{v_h}(w) = \min[w - v_h, 0]$$

It is not difficult to see that the k functions $U^{\mathbf{v}_h}$ (for $h = 1, ..., k$) satisfy inequality (3) whatever is the vector $\mathbf{v} = (v_1, ..., v_k)$ in $\mathcal{V}$: consider any $u \geq 0$, $w \leq w'$ and $h \leq h'$. First note that the quantities $U^{v_h}(w + u) - U^{v_h}(w)$ and $U^{v_{h'}}(w' + u) - U^{v_{h'}}(w')$ belong to $[0, u]$.

If $w \geq v_h$ then $w + u \geq v_h$ and $w' + u \geq w' \geq v_{h'}$. Thus (3) holds with both sides equal to zero.

If $w \leq v_h$ then $U^{v_h}(w) = w - v_h$ and $U^{v_h}(w+u) - U^{v_h}(w) = \min(u, v_h - w)$. Notice also that:

$$U^{v_{h'}}(w' + u) - U^{v_{h'}}(w') \leq -U^{v_{h'}}(w') \leq v_{h'} - w' \leq v_h - w.$$

Hence $U^{v_{h'}}(w' + u) - U^{v_{h'}}(w') \leq \min(u, v_h - w)$ and inequality (3) holds. We have therefore proved that the list of functions U^{v_h} (for $h = 1, ..., k$) belongs to the class $\mathcal{U}^*$ for all $\mathbf{v} = (v_1, ..., v_k) \in \mathcal{V}$. As a result, inequality (9) holds for all such functions so that one has:

$$\begin{array}{ccc} \sum_{h=1}^{k} \sum_{i=1}^{n(h)} \min[x_i^h - v_h, 0] & \geq & \sum_{h=1}^{k} \sum_{i=1}^{n(h)} \min[y_i^h - v_h, 0] \\ & \Longleftrightarrow & \\ \sum_{h=1}^{k} \sum_{i=1}^{n(h)} \max[v_h - x_i^h, 0] & \leq & \sum_{h=1}^{k} \sum_{i=1}^{n(h)} \max[v_h - y_i^h, 0] \end{array}$$

for all $\mathbf{v} = (v_1, ..., v_k) \in \mathcal{V}$, as required by the OPG criterion. ■

3.2 Some technical lemmas

We now provide the proof of the most difficult implication of Theorem 1 (Statement (iii) implies Statement (i)). We establish this implication by means of several auxiliary results. In proceeding, we assume without loss of generality that, in the two distributions $\mathbf{x}$ and $\mathbf{y}$ under consideration, one has $x_i^h \neq y_j^h$ for every type $h = 1, ..., k$ and every $i, j \in \mathcal{N}(h)$. In effect, if this condition was not satisfied, that is to say if there were a type h for which $x_i^h = y_j^h$ for some $i, j \in \mathcal{N}(h)$, one could remove these two agents and proceed with the remaining population. Since the OPG criterion is additively separable, such a removal of agents with the same type and income from the two distributions $\mathbf{x}$ and $\mathbf{y}$ would not affect their ranking as per the OPG criterion.

The first auxiliary result of this section is the following lemma (proved, like all lemmas and formal claims in the Appendix) which says that if $\mathbf{x}$ is a distribution that dominates $\mathbf{y}$ for the OPG criterion, the poorest person in the worst category is weakly richest in $\mathbf{x}$ than in $\mathbf{y}$ and, somewhat conversely, the richest person in the best category is poorer in $\mathbf{x}$ then in $\mathbf{y}$.

Lemma 1 *Let $\mathbf{x}$ and $\mathbf{y}$ be distributions in $\mathcal{D}(I)$, for which $\mathbf{x} \succsim^{OPG} \mathbf{y}$. Then $y_1^1 < x_1^1$ and $y_{n(k)}^k > x_{n(k)}^k$.*

The next lemma states that, if a distribution $\mathbf{x}$ dominates a distribution $\mathbf{y}$ by the OPG criterion, then the sum of incomes held by agents in low categories must be weakly larger in $\mathbf{x}$ than in $\mathbf{y}$ no matter what the the threshold of "lowness" is.

Lemma 2 *Let $\mathbf{x}$ and $\mathbf{y}$ be two distributions in $\mathcal{D}(I)$ for which $\mathbf{x} \succsim^{OPG} \mathbf{y}$. Then $\sum_{h=1}^{\overline{h}} \sum_{i \in \mathcal{N}(h)} x_i^h \geq \sum_{h=1}^{\overline{h}} \sum_{i \in \mathcal{N}(h)} y_i^h$ for all $\overline{h} = 1, ..., k$.*

We next state an important lemma that provides a sufficient condition for the possibility of performing a FIP from a distribution $\mathbf{y}$ in such a way that the distribution obtained after making such a FIP remains dominated by $\mathbf{x}$ as per the OPG criterion.

Lemma 3 *Let $\mathbf{x}$ and $\mathbf{y}$ be two distributions in $\mathcal{D}(I)$ such that $\mathbf{x} \succsim^{OPG} \mathbf{y}$, and assume that $\mathbf{w} \in \mathcal{V}$ is such that $P^{\mathbf{y}}(\mathbf{w}) = P^{\mathbf{x}}(\mathbf{w})$ and $y_{i_1}^1 = w_1 = ... = w_{h_0} > w_{h_0+1}$ for some $h_0 \in \{2, ..., k\}$ and some agent $i_1 \in \mathcal{N}(1)$ (with the convention that $w_{k+1} = \underline{v}$). Assume also that there exists a category g_0 such that $2 \leq g_0 \leq h_0$ and:*

$$\sum_{h=l+1}^{g_0} \overline{p}^{\mathbf{y}}(h, y_{i_1}^1) < \sum_{h=l+1}^{g_0} \overline{p}^{\mathbf{x}}(h, y_{i_1}^1) \tag{10}$$

for all $l = 1, ..., g_0 - 1$. Then, one can define a category $\gamma \in \{2, ..., g_0\}$ by:

$$\gamma = \min\{g : g \geq 2 \text{ and } \exists i \ \in \mathcal{N}(g) \text{ such that } y_i^g > y_{i_1}^1\} \tag{11}$$

and the index i_γ by:

$$i_\gamma = \min\{i \ \in \mathcal{N}(\gamma) : \ y_i^\gamma > y_{i_1}^1\} \tag{12}$$

Then, for any $\mathbf{v} \in \mathcal{V}$, one has:

$$P^{\mathbf{y}}(\mathbf{v}) - P^{\mathbf{x}}(\mathbf{v}) \geq \min\{y_{i_\gamma}^\gamma, v_1\} - \max\{y_{i_1}^1, v_\gamma\}. \tag{13}$$

and there exists a distribution $\overline{\mathbf{x}} \in \mathcal{D}(I)$ such that $\overline{\mathbf{x}}$ has been obtained from $\mathbf{y}$ by a FIP and $\mathbf{x} \succsim^{OPG} \overline{\mathbf{x}}$.

Although this result is important, it is of limited immediate usefulness. In effect, there are no obvious ways to identify the vector $\mathbf{w}$ of poverty lines that is required by this Lemma. We will nonetheless use Lemma 3 on two occasions in what follows.

3.3 Identifying which elementary operation is possible: a diagnostic tool

An important prerequisite for performing any step of the algorithm that we want to construct is a "diagnostic tool" for identifying which of the two elementary operations - FIP or BTPIT - can be performed at any given step of the algorithm. Our diagnostic tool is based on the critical value v_1^c that is defined as follows:

$$v_1^c := \inf\left\{v_1 > y_1^1 : \exists v_2, ..., v_k \text{ s.t. } \mathbf{v} = (v_1, ..., v_k) \in \mathcal{V} \text{ and } P^{\mathbf{x}}(\mathbf{v}) = P^{\mathbf{y}}(\mathbf{v})\right\} \tag{14}$$

In words, v_1^c is the smallest poverty threshold *above* the smallest income in the lowest category in the dominated distribution $\mathbf{y}$ that can be part of a collection of (decreasingly) ordered poverty thresholds for which the ordered poverty gap in the two distributions $\mathbf{x}$ and $\mathbf{y}$ is the same. It is clear that v_1^c is well-defined because the set:

$$\left\{v_1 > y_1^1 : \exists v_2, ..., v_k \text{ s.t. } \mathbf{v} = (v_1, ..., v_k) \in \mathcal{V} \text{ and } P^{\mathbf{x}}(\mathbf{v}) = P^{\mathbf{y}}(\mathbf{v})\right\}$$

is not empty (it contains $\overline{v}$) and is also bounded from below (by $\underline{v}$). Two mutually exclusive cases are possible:

(A) $v_1^c > y_1^1$ and:

(B) $v_1^c = y_1^1$.

As will now be shown, if case (A) holds, there is some margin to make a strict BTPIT to the poorest individual in category 1 (endowed with y_1^1) in such a way that the after transfer distribution remains dominated by $\mathbf{x}$ as per the OPG criterion. This however does not preclude the possibility that a FIP involving one individual in category 1 be also possible in that case. If both a FIP and a BTPIT are possible, then our algorithmic procedure will always choose to perform the FIP. [2] As will also be shown, if on the other hand, case (B) holds, then it is possible to involve the poorest individual of type 1 in a FIP while preserving the OPG dominance of $\mathbf{x}$ over the distribution obtained by doing so.

- **Case** $(A) : v_1^c > y_1^1$**.**

In this case, one can recursively define v_h^c (for any $h = 2, ..., k$) by:

$$v_h^c = \inf \{v_h \geq \underline{v} : \ \exists v_{h+1}, ..., v_k \text{ s.t. } \mathbf{v} = (v_1^c, .., v_{h-1}^c, v_h, .., v_k) \in \mathcal{V}, \ P^{\mathbf{x}}(\mathbf{v}) = P^{\mathbf{y}}(\mathbf{v})\} \quad (15)$$

Just for the same reason as for v_1^c, it is clear that v_h^c is well-defined for any $h = 2, ..., k$. By construction, we have $\mathbf{v}^c := (v_1^c, ..., v_k^c) \in \mathcal{V}$ and call $\mathbf{v}^c$ the *critical vector.*

We start by establishing the following important result that if an ordered list $\mathbf{v} \in \mathcal{V}$ of poverty lines is such that $v_1 > y_1^1$ and $v_{h_0} < v_{h_0}^c$ for some $h_0 \in \{2, ..., k\}$, then $P^{\mathbf{x}}(\mathbf{v}) < P^{\mathbf{y}}(\mathbf{v})$. Specifically, we prove the following result.

Lemma 4 *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be two distributions in* $\mathcal{D}(I)$ *such that* $\mathbf{x} \succsim^{OPG} \mathbf{y}$*. Suppose that* $v_1^c > y_1^1$*. Then, if* $\mathbf{v} \in \mathcal{V}$ *is such that* $v_1 > y_1^1$ *and* $v_{h_0} < v_{h_0}^c$ *for some* $h_0 \in \{2, ..., k\}$*, it is the case that* $P^{\mathbf{x}}(\mathbf{v}) < P^{\mathbf{y}}(\mathbf{v})$*.*

We now state as a corollary of Lemma 4 the following alternative definition of the critical vector $\mathbf{v}^c$.

Corollary 1 *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be two distributions in* $\mathcal{D}(I)$ *such that* $\mathbf{x} \succsim^{OPG} \mathbf{y}$*. Suppose that* $v_1^c > y_1^1$*. Then, for every* $h = 2, ..., k$*, one has:*

$$v_h^c = \min_{v_h} \left\{ \exists v_{-h} \in [\underline{v}, \overline{v}]^{k-1} : \ v_1 > y_1^1, \ \mathbf{v} = (v_h, v_{-h}) \in \mathcal{V} \text{ and } P^{\mathbf{x}}(\mathbf{v}) = P^{\mathbf{y}}(\mathbf{v}) \right\}.$$

The next lemma establishes an important comparative statement about adjacent sets of strictly and weakly poor agents in $\mathbf{x}$ and $\mathbf{y}$ when these sets are defined with respect to the vector of ordered poverty lines $\mathbf{v}^c$ in the case where the poverty lines assigned to the adjacent categories are the same. Specifically, the next lemma establishes the following.

Lemma 5 *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be two distributions in* $\mathcal{D}(I)$ *such that* $\mathbf{x} \succsim^{OPG} \mathbf{y}$*. Suppose that* $v_1^c > y_1^1$*. Then, for any* $h_0 \in \{1, ..., k\}$ *and* $\overline{h} \in \{0, ..., k - h_0\}$ *such*

[2] We explain in details in Section 3.4 why we make this choice.

that $v^c_{h_0-1} > v^c_{h_0} = v^c_{h_0+\overline{h}} > v^c_{h_0+\overline{h}+1}$[3], *one has, for any* $l = 0, ..., \overline{h}$:

$$\sum_{h=h_0}^{h_0+l} \overline{p}^{\mathbf{x}}(h, v^c_h) \leq \sum_{h=h_0}^{h_0+l} \overline{p}^{\mathbf{y}}(h, v^c_h) \tag{16}$$

and:

$$\sum_{h=h_0+l}^{h_0+\overline{h}} p^{\mathbf{x}}(h, v^c_h) > \sum_{h=h_0+l}^{h_0+\overline{h}} p^{\mathbf{y}}(h, v^c_h). \tag{17}$$

Lemma 5 has the following important corollary, that will be quite useful in establishing the possibility of making a non-zero BTPIT to the poorest individual in the worst health category of distribution $\mathbf{y}$ when the critical value v^c_1 is strictly larger than the income (y^1_1) of this individual. This corollary establishes indeed the existence of (potential donators) individuals in a weakly higher health category who has, in distribution $\mathbf{y}$, a income of v^c_1.

Corollary 2 *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be two distributions in* $\mathcal{D}(I)$ *such that* $\mathbf{x} \succsim^{OPG} \mathbf{y}$. *Suppose that* $v^c_1 > y^1_1$. *Let* $h_0 \in \{0, ..., k-1\}$ *be such that* $v^c_1 = v^c_{h_0+1} > v^c_{h_0+2}$ *Then, there exists some* $j \in \{1, ..., h_0+1\}$ *for which one has* $y^j_i = v^c_1$ *for some* $i \in \mathcal{N}(j)$.

The next important lemma shows that, when the critical value v^c_1 is strictly larger than y^1_1, one has indeed some "margin of maneuver" for performing a BTPIT while preserving OPG dominance. Specifically, the following lemma deals with ordered vectors of poverty lines who assign to the worst category a poverty line that is larger than the lowest income observed in that category by a "tiny" margin. This lemma says, roughly, that for any such ordered vector of poverty lines, the poverty gap in the dominated distribution must exceed that of the dominating one by still a larger margin. The precise statement of this lemma is as follows.

Lemma 6 *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be two distributions in* $\mathcal{D}(I)$ *such that* $\mathbf{x} \succsim^{OPG} \mathbf{y}$. *Suppose that* $v^c_1 > y^1_1$. *Then, for some strictly positive but suitably small real number* ε_1, *one has:*

$$P^{\mathbf{y}}(y^1_1 + \varepsilon_1, v_2, ..., v_k) \geq P^{\mathbf{x}}(y^1_1 + \varepsilon_1, v_2, ..., v_k) + \varepsilon_1,$$

provided $(y^1_1 + \varepsilon_1, v_2, ..., v_k) \in \mathcal{V}$.

We now establish in the following theorem, that if $\mathbf{x} \succsim^{OPG} \mathbf{y}$, then it is possible to find a distribution $\widehat{\mathbf{x}}$ that is OPG dominated by $\mathbf{x}$ and that has been obtained from $\mathbf{y}$ by a BTPIT whenever the critical value v^c_1 is strictly larger than y^1_1.

Theorem 2 *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be two distributions in* $\mathcal{D}(I)$ *such that* $\mathbf{x} \succsim^{OPG} \mathbf{y}$. *Suppose that* $v^c_1 > y^1_1$. *Then, there exists a distribution* $\widehat{\mathbf{x}} \in \mathcal{D}(I)$ *such that:*

- $\mathbf{x} \succsim^{OPG} \widehat{\mathbf{x}}$,

[3] using if necessary convention that $v^c_0 = \overline{v}$ and $v^c_{k+1} = \underline{v}$

- $\widehat{\mathbf{x}}$ *has been obtained from* $\mathbf{y}$ *by a BTPIT involving some agent* $i^j \in \mathcal{N}(j)$ *for some category* $j \in \{1, ..., k\}$ *and agent* $1 \in \mathcal{N}(1)$.

Proof. There clearly exists some $h_0 \in \{1, ..., k\}$ such that $v_1^c = v_2^c = ...v_{h_0}^c > v_{h_0+1}^c$ (using, if necessary, the convention that $v_{k+1}^c = \underline{v}$). Then, using Corollary 2, we conclude in the existence of some category $j \in \{1, ..., h_0\}$ and some individual $i^j \in \mathcal{N}(j)$ such that $y_{i^j}^j = v_1^c$ and $\forall h \in \{j, j+1, ..., h_0\}$, $i \in \mathcal{N}(h)$, $y_i^h \neq v_1^c$ (that is, j is the highest category in the set $\{1, ..., h_0\}$ for which there is an individual in distribution $\mathbf{y}$ whose income is equal to v_1^c.). We notice carefully that we do not preclude the possibility that $j = 1$. Let us show that there exists a distribution $\widehat{\mathbf{x}} \in \mathcal{D}(I)$ such that $\mathbf{x} \succsim^{OPG} \widehat{\mathbf{x}}$ and $\widehat{\mathbf{x}}$ has been obtained from $\mathbf{y}$ by a BTPIT. For any strictly positive integer m, let $\widehat{\mathbf{x}}^m$ be the distribution obtained from distribution $\mathbf{y}$ by performing a BTPIT of an amount of $1/m$ from agent $i^j \in \mathcal{N}(j)$ to agent $1 \in \mathcal{N}(1)$. We claim that there exists some m sufficiently large for which $\mathbf{x} \succsim^{OPG} \widehat{\mathbf{x}}^m$. Assume by contradiction that no such m exists. This implies the existence of a sequence of ordered poverty lines vectors $\mathbf{v}^m \in \mathcal{V}$ such that $P^{\widehat{\mathbf{x}}^m}(\mathbf{v}^m) < P^{\mathbf{x}}(\mathbf{v}^m)$. Notice that, for every strictly positive real integer m, and whatever is the ordered vector of poverty lines $\mathbf{v} \in \mathcal{V}$, one has:

$$
\begin{aligned}
P^{\widehat{\mathbf{x}}^m}(\mathbf{v}) &= P^{\mathbf{y}}(\mathbf{v}) - 1/m \text{ if } v_1 \geq y_1^1 + 1/m \text{ and } v_j \leq y_{i^j}^j - 1/m, && (18)\\
&= P^{\mathbf{y}}(\mathbf{v}) - \max(v_1 - y_1^1, 0) \text{ if } v_1 < y_1^1 + \frac{1}{m},\ v_j \leq y_{i^j}^j - \frac{1}{m} && (19)\\
&= P^{\mathbf{y}}(\mathbf{v}) + \min(v_j - y_{i^j}^j, 0) \text{ if } v_1 \geq y_1^1 + \frac{1}{m},\ v_j > y_{i^j}^j - \frac{1}{m}. && (20)
\end{aligned}
$$

Because of this, one can assume without loss of generality that $v_1^m \geq y_1^1 + 1/m$ and $v_j^m \leq v_1^c - 1/m$. Since the set $\mathcal{V}$ is compact, $\mathbf{v}^m$ admits a subsequence that converges to some vector of ordered poverty lines $\mathbf{v} \in \mathcal{V}$. By continuity, one must have $P^{\mathbf{y}}(\mathbf{v}) = P^{\mathbf{x}}(\mathbf{v})$. Hence, by definition of the critical value v_1^c, either:
(i) $v_1 = y_1^1$ or
(ii) $v_1 \geq v_1^c$

- If case (i) holds, then one has:

$$
\begin{aligned}
P^{\widehat{\mathbf{x}}^m}(\mathbf{v}^m) &\geq P^{\mathbf{y}}(\mathbf{v}^m) - 1/m \text{ by (18)-(20)}\\
&\geq P^{\mathbf{x}}(\mathbf{v}^m) - 1/m + v_1^m - y_1^1 \text{ by Lemma 6, taking } v_1^m - y_1^1 = \varepsilon_1\\
&\geq P^{\mathbf{x}}(\mathbf{v}^m)
\end{aligned}
$$

which is a contradiction.

- If now case (ii) holds and $v_1 \geq v_1^c$ then by Corollary 1, we must have $v_h \geq v_h^c$ for $h = 2, ..., k$. In particular since $v_j^m \leq v_1^c - 1/m$ and $\mathbf{v}^m$ admits a subsequence that converges to $\mathbf{v}$, one must have $v_j = v_j^c = v_1^c$. We may actually assume without loss of generality that that, for every $h = 1, ..., h_0$, $v_h^m \in \{v_1^c - 1/m, v_1^c\}$ (for m large enough, $v_1^c - 1/m$ and v_1^c are the only two incomes observed in distributions $\mathbf{y}$, $\widehat{\mathbf{x}}^m$ and $\mathbf{x}$ for the poverty lines v_h^m relevant for the categories $h = 1, ..., h_0$). Hence, for some $g \in \{1, ..., j\}$, one has: $v_g^m = ... = v_j^m = ...v_{h^0}^m = v_1^c - 1/m$, $v_1^m = v_2^m = ... = v_{g-1}^m = v_1^c$. Since:

$$
\sum_{h=g}^{h_0} p^{\mathbf{x}}(h, v_1^c) > \sum_{h=g}^{h_0} p^{\mathbf{y}}(h, v_1^c)
$$

one has:

$$\begin{aligned}
& \sum_{h=g}^{h_0} \sum_{i \in \mathcal{P}^y(h, v_h^m)} [v_h^m - y_i^h] - \sum_{h=g}^{h_0} \sum_{i \in \mathcal{P}^x(h, v_h^m)} [v_h^m - x_i^h] - 1/m \\
\geq & \sum_{h=g}^{h_0} \sum_{i \in \mathcal{P}^y(h, v_1^c)} [v_1^c - y_i^h] - \sum_{h=g}^{h_0} \sum_{i \in \mathcal{P}^x(h, v_1^c)} [v_1^c - x_i^h]
\end{aligned}$$

and, therefore:

$$\begin{aligned}
& P^{\mathbf{y}}(\mathbf{v}^m) - P^{\mathbf{x}}(\mathbf{v}^m) - 1/m \\
\geq & P^{\mathbf{y}}(v_1^c, ..., v_1^c, v_{h_0+1}^c, ..., v_k^c) - P^{\mathbf{x}}(v_1^c, ..., v_1^c, v_{h_0+1}^c, ..., v_k^c) \\
\geq & 0
\end{aligned}$$

Finally:

$$P^{\widehat{\mathbf{x}}^m}(\mathbf{v}^m) \geq P^{\mathbf{y}}(\mathbf{v}^m) - 1/m \geq P^{\mathbf{x}}(\mathbf{v}^m),$$

a contradiction. ■

This theorem (and its proof) identifies a particular category $j \geq 1$ and a particular agent in that category (labeled i^j), whose income is exactly equal to v_1^c, that can transfer a strictly positive quantity of income to the poorest agent in category 1. Since we proved the theorem with the objective of constructing a finite sequence of such transfers, it is of some importance that the sequence be not unnecessarily long and, therefore, that each transfer be as large as possible. This motivates the following notion of a *maximal transfer.*

Definition 5 *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be two distributions in* $\mathcal{D}(I)$ *such that* $\mathbf{x} \succsim^{OPG} \mathbf{y}$. *Suppose that* $v_1^c > y_1^1$. *Let the category* $j \in \{1, ..., k\}$, *the agent* $i^j \in \mathcal{N}(j)$ *and the distribution* $\widehat{\mathbf{x}}$ *be defined as in Theorem 2. We will say that the transfer between agent* i^j *and agent* 1 *is* maximal *if one of the following occurs:*

(MT1) ***Equalizing transfer:*** *there exist some* $i, i' \in \mathcal{N}(j)$ *such that* $\widehat{x}_{i'}^j = x_i^j$ *or there exists some* $i, i' \in \mathcal{N}(1)$ *such that* $\widehat{x}_{i'}^1 = x_i^1$ *(that is, one of the two agents involved in the transfer obtains the income that they will have in the final distribution* $\mathbf{x}$*).*

(MT2) ***Breaking transfer:*** *no equalizing transfer is possible between these two agents and, by transferring strictly more, one would have* $\mathbf{x} \not\succsim^{OPG} \widehat{\mathbf{x}}$. *(i.e. transferring strictly more between agent* $i^j \in \mathcal{N}(j)$ *and agent* $1 \in \mathcal{N}(1)$ *would break at least one of the ordered poverty gap inequalities that define OPG dominance).*

(MT3) ***Half transfer:*** *no equalizing transfer is possible between these two agents and the two individuals involved have the same income in the resulting distribution (that is he cannot transfer more, by the very definition of a transfer).*

Example 1 *We illustrate this definition in the case where* $k = 2$ *by providing three examples of pairs of distributions* $\mathbf{x}$ *and* $\mathbf{y}$ *for which* $v_1^c > y_1^1$ *and that give rise to the three different possibilities of maximal transfer.*

• *As an example of an "equalizing transfer", consider the distributions where* $\mathcal{N}(1) = \{1, 2\}$ *and* $\mathcal{N}(2) = \{1\}$ *and where:*

$$y_1^1 = 0,\ y_2^1 = 1,\ y_1^2 = 7,$$

$$x_1^1 = 2,\ x_2^1 = 4 \text{ and } x_1^2 = 2.$$

It is not hard to check that $\mathbf{x} \succsim^{OPG} \mathbf{y}$ *and that* $v_1^c = 7 > y_1^1$.

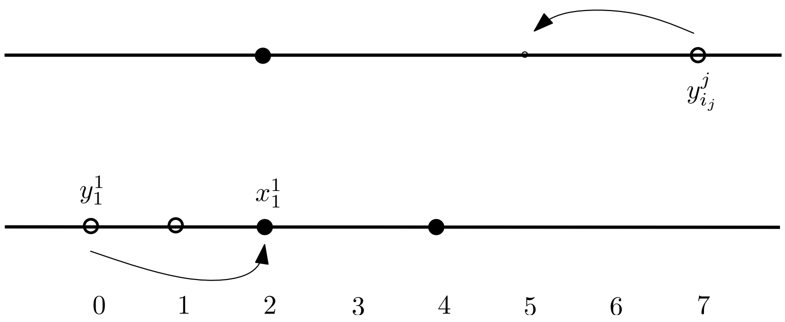


Figure 3: Equalizing transfer

It is then possible for the unique agent in $\mathcal{N}(2)$ *to transfer* 2 *units of income to agent 1 without breaking any OPG inequality, which corresponds to an equalizing transfer (see figure 3). Note that it would have been possible to transfer even more without breaking the OPG inequality. Yet we do not need to do so because, after receiving 2 units of income, agent 1 of category 1 has obtained the income of the target distribution* $\mathbf{x}$.

• *As an example of a breaking transfer, consider the distributions where* $\mathcal{N}(1) = \{1, 2\} = \mathcal{N}(2)$ *and where:*

$$y_1^1 = 0 = y_1^2,\ y_2^1 = 7 = y_2^2,$$

$$x_1^1 = 5,\ x_2^1 = 6,\ x_1^2 = 1 \text{ and } x_2^2 = 2.$$

We have $\mathbf{x} \succsim^{OPG} \mathbf{y}$ *and, again, it happens that* $v_1^c = 7 > y_1^1$. *It is possible for individual 2 in category 2 to transfer* 3 *units of income to individual 1 in category 1. Doing this transfer changes the distribution from* $\mathbf{y}$ *to* $\widehat{\mathbf{x}}$ *where* $\widehat{\mathbf{x}}$ *is defined by:*

$$\widehat{x}_1^1 = 3,\ \widehat{x}_2^1 = 7,\ \widehat{x}_1^2 = 0 \text{ and } \widehat{x}_2^2 = 4.$$

As can be seen, $\mathbf{x} \succsim^{OPG} \widehat{\mathbf{x}}$*. Yet transferring* $3 + \varepsilon$ *(for any* $\varepsilon \in]0, 1/2]$*) would destroy this OPG dominance of the transformed distribution by the target* $\mathbf{x}$. *Indeed consider the distribution* $\widehat{\mathbf{x}}^\varepsilon$ *defined by:*

$$\begin{aligned}
\widehat{x}_1^{\varepsilon 1} &= 3 + \varepsilon \\
\widehat{x}_2^{\varepsilon 2} &= 4 - \varepsilon \\
\widehat{x}_2^{\varepsilon 1} &= \widehat{x}_2^1 = 7 \text{ and,} \\
\widehat{x}_1^{\varepsilon 2} &= \widehat{x}_1^2 = 0
\end{aligned}$$

Then, using the ordered vector $\mathbf{v} = (3+\varepsilon, 3+\varepsilon)$ *of poverty lines, one has:*

$$\begin{aligned} P^{\widehat{\mathbf{x}}^{\varepsilon}}(3+\varepsilon, 3+\varepsilon) &= \max(3+\varepsilon-(3+\varepsilon), 0) + \max(3+\varepsilon-7, 0) \\ &\quad + \max(3+\varepsilon-0, 0) + \max(3+\varepsilon-(4-\varepsilon), 0) \\ &= 3+\varepsilon \ (\textit{if } \varepsilon \in]0, 1/2]) \\ &< P^{\mathbf{x}}(3+\varepsilon, 3+\varepsilon) \\ &= \max(3+\varepsilon-5, 0) + \max(3+\varepsilon-6, 0) \\ &\quad + \max(3+\varepsilon-1, 0) + \max(3+\varepsilon-2, 0) \\ &= 3+2\varepsilon \end{aligned}$$

This example is illustrated on Figure 4.

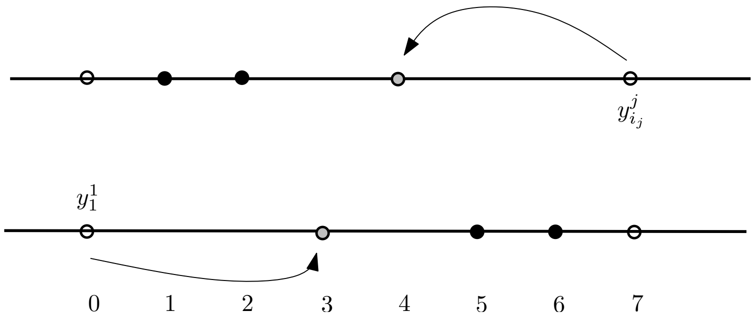


Figure 4: Breaking transfer

• *Finally, as an example of a "half transfer (illustrated on Figure 5), consider the distributions where* $\mathcal{N}(1) = \{1,2\} = \mathcal{N}(2)$ *and where:*

$$y^1_1 = 0 = y^2_1,\ y^1_2 = 6 = y^2_2,$$

$$x^1_1 = 4,\ x^1_2 = 5,\ x^2_1 = 1 \textit{ and } x^2_2 = 2.$$

As can be seen, we have $\mathbf{x} \succsim^{OPG} \mathbf{y}$ *and* $v^c_1 = 6 > x^1_1$*. It is possible for agent 2 in category 2 to transfer* 3 *to the poorest agent in category 1 - which is precisely half of their income difference - without breaking any of the inequalities that define OPG dominance.*

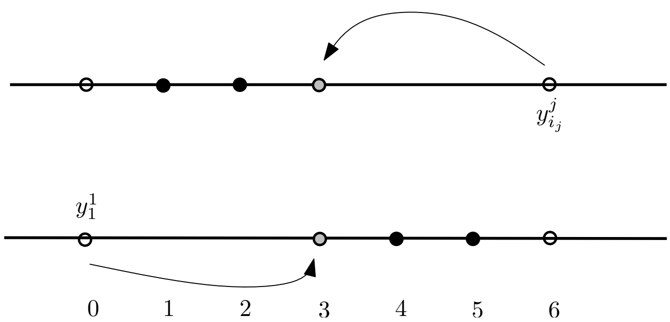


Figure 5: Half transfer

Theorem 2 shows the possibility of performing a BTPIT to the benefit of the poorest agent in category 1 in the dominated distribution $\mathbf{y}$ in such a way that the distribution obtained after the transfer remains dominated by $\mathbf{x}$ as per the OPG criterion. However this Theorem does not rule out the alternative possibility of performing FIP between two individuals in such a way as to preserve the OPG dominance of the distribution obtained after performing this operation by $\mathbf{x}$. In the next theorem, we identify a circumstance where such a possibility of performing a FIP is also present.

Theorem 3 *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be two distributions in* $\mathcal{D}(I)$ *such that* $\mathbf{x} \succsim^{OPG} \mathbf{y}$. *Suppose that* $v_1^c > y_1^1$. *Let* $h_0 \in \{1, ..., k\}$ *be a category such that* $v_{h_0+1}^c < v_{h_0}^c = v_1^c$ *(with the convention that* $v_{k+1}^c = \underline{v}$*). Suppose also that*

- $\forall i \in \mathcal{N}(1)$, $x_i^1 > v_1^c$
- *For any category* h *such that* $h_0 \geq h \geq 2$, *one has* $y_i^h \neq v_1^c$;

Then there exists a distribution $\widehat{\mathbf{x}} \in \mathcal{D}(I)$ *such that* $\widehat{\mathbf{x}}$ *has been obtained from* $\mathbf{y}$ *by a FIP and* $\mathbf{x} \succsim^{OPG} \widehat{\mathbf{x}}$.

Proof. We prove this theorem by appealing to Lemma 3. We notice first that, since $x_i^1 > v_1^c \; \forall i \in \mathcal{N}(1)$ one must have $v_1^c = v_2^c$. Suppose indeed by contradiction that $v_1^c > v_2^c$. This means that, for any number ε such that $v_1^c - v_2^c > \varepsilon > 0$, one has that $(v_1^c - \varepsilon, v_2^c, ..., v_k^c) \in \mathcal{V}$ and

$$\begin{aligned}
& P^{\mathbf{y}}(v_1^c - \varepsilon, v_2^c, ..., v_k^c) - P^{\mathbf{x}}(v_1^c - \varepsilon, v_2^c, ..., v_k^c) \\
= \;& -\varepsilon p^{\mathbf{y}}(1, v_1^c) + \varepsilon p^{\mathbf{x}}(1, v_1^c) + P^{\mathbf{y}}(v_1^c, v_2^c, ..., v_k^c) - P^{\mathbf{x}}(v_1^c, v_2^c, ..., v_k^c) \\
= \;& -\varepsilon p^{\mathbf{y}}(1, v_1^c) \\
< \;& 0
\end{aligned}$$

since $p^{\mathbf{x}}(1, v_1^c) = 0 = P^{\mathbf{y}}(v_1^c, v_2^c, ..., v_k^c) - P^{\mathbf{x}}(v_1^c, v_2^c, ..., v_k^c)$ (by definition of $\mathbf{v}^c$) and $p^{\mathbf{y}}(1, v_1^c) \geq 1$. But this is a contradiction of the fact that $\mathbf{x} \succsim^{OPG} \mathbf{y}$. It then follows that $h_0 \geq 2$ and that the second bullet statement of the theorem (e.g. "For any category h such that $h_0 \geq h \geq 2$, one has $y_i^h \neq v_1^c$") is not empty. Using this fact and Corollary 2, one concludes in the existence of an agent $i_1 \in \mathcal{N}(1)$ for which $y_{i_1}^1 = v_1^c$. Hence the vector of poverty lines $\mathbf{v}^c$ is just like the vector $\mathbf{w}$ in the antecedent clause of Lemma 3. It then follows from Lemma 5 that:

$$\sum_{h=l+1}^{h_0} \overline{p}^{\mathbf{y}}(h, y_{i_1}^1) = \sum_{h=l+1}^{h_0} p^{\mathbf{y}}(h, y_{i_1}^1) < \sum_{h=l+1}^{h_0} p^{\mathbf{x}}(h, y_{i_1}^1) \leq \sum_{h=l+1}^{h_0} \overline{p}^{\mathbf{x}}(h, y_{i_1}^1)$$

for $l = 1, ..., h_0 - 1$, which implies that Inequality (10) in the antecedent clause of Lemma 3 holds. The existence of a distribution $\widehat{\mathbf{x}} \in \mathcal{D}(I)$ such that $\widehat{\mathbf{x}}$ has been obtained from $\mathbf{y}$ by a FIP and $\mathbf{x} \succsim^{OPG} \widehat{\mathbf{x}}$ then immediately follows from this Lemma. ∎

We notice that the receiver of the FIP exhibited in Theorem 3 is not the poorest individual in category 1. It is another agent in category 1 whose income is equal to v_1^c. We now provide an example of a situation where both a BTPIT and a FIP are possible.

Example 2 *Let $k = 2$, $\mathcal{N}(1) = \{1, 2, 3\}$, $\mathcal{N}(2) = \{1, 2, 3, 4\}$ and* $\mathbf{y}$ *and* $\mathbf{x}$ *be defined by*

$$y_1^1 = 2,\ y_2^1 = y_3^1 = 3,\ y_1^2 = 0,\ y_2^2 = y_3^2 = y_4^2 = 4$$

and:

$$x_1^1 = x_2^1 = x_3^1 = 4,\ x_1^2 = x_2^2 = x_3^2 = x_4^2 = 2.$$

As can be seen, $\mathbf{x} \succsim^{OPG} \mathbf{y}$, *$v_1^c = v_2^c = 3$, and $y_1^1 = 2$. According to Theorem 2, agent 2 in category 2 can transfer some income to agent 1 of category 1. However, Theorem 3 states that a FIP is also possible. Indeed the conditions of this theorem are satisfied since $x_i^1 = 4 > v_1^c$ for all $i \in \mathcal{N}(1)$. also $y_i^2 \neq 3$ for all $i \in \mathcal{N}(2)$. By virtue of Theorem 3, the distribution obtained after exchanging the income 4 of agent 2 in category 2 with the income 3 of agent 2 in category 1 remains dominated by distribution* $\mathbf{x}$*. The situation is illustrated in the Figure below.*

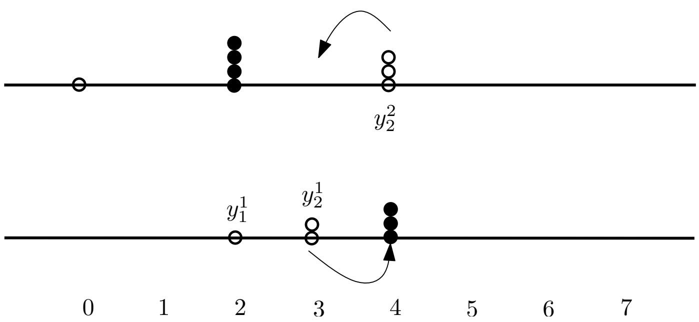


Figure 6: Agent 2 in category 2 can exchange income with agent 2 in category 1.

• **Case** $(B) : v_1^c = y_1^1$

If this case holds, one is led by the very definition of the critical value v_1^c to the existence of a sequence $\{\mathbf{w}^m\}$ of poverty lines vectors (with $\mathbf{w}^m \in \mathcal{V}$ for every m) such that $P^{\mathbf{y}}(\mathbf{w}^m) - P^{\mathbf{x}}(\mathbf{w}^m) = 0$ and $w_1^m = y_1^1 + \varepsilon_1^m$, for $\varepsilon_1^m > 0$, and $\varepsilon_1^m \to 0$. By compactness of $\mathcal{V}$, we may assume without loss of generality[4] that the sequence $\mathbf{w}^m$ of ordered poverty lines vectors converges to some limit $\overline{\mathbf{w}} \in \mathcal{V}$. By continuity of the poverty gap function P, one must have $P^{\mathbf{y}}(\overline{\mathbf{w}}) - P^{\mathbf{x}}(\overline{\mathbf{w}}) = 0$.

Taking this limit vector $\overline{\mathbf{w}} \in \mathcal{V}$ of ordered poverty lines, we first establish the existence, in the initial distribution $\mathbf{y}$, of some agent in a category strictly larger than 1 with an income strictly larger than the lowest income observed in category 1. This agent will be a natural candidate for permuting his/her higher income with that of the poorest agent in category 1. A crucial step for the identification of such an agent is the following Lemma.

Lemma 7 *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be two distributions in $\mathcal{D}(I)$ such that* $\mathbf{x} \succsim^{OPG} \mathbf{y}$. *Suppose that $v_1^c = y_1^1$. Then, there exists $h_0 \geq 2$ such that $y_1^1 = \overline{w}_1 = \overline{w}_2 = ... =$*

[4]Up to taking a subsequence if necessary.

$\overline{w}_{h_0} > \overline{w}_{h_0+1}$. *Moreover there exists* $g_0 \leq h_0$ *such that* $g_0 \geq 2$,

$$\sum_{h=1}^{g_0} \overline{p}^{\mathbf{y}}(h, y_1^1) = \sum_{h=1}^{g_0} \overline{p}^{\mathbf{x}}(h, y_1^1),$$

and:

$$\sum_{h=1}^{l} \overline{p}^{\mathbf{y}}(h, y_1^1) > \sum_{h=1}^{l} \overline{p}^{\mathbf{x}}(h, y_1^1)$$

for all $l < g_0$.

This lemma indeed identifies a category g_0 strictly larger than 1 in which a "potential donor" to the poorest agent in the worst category can be selected. As we now establish, this donor can transfer to the poorest agent category 1 the totality of their income difference (and thus exchange his or her income with that of the recipient) while maintaining the dominance of the distribution $\mathbf{x}$ over the distribution created by the FIP.

Theorem 4 *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be two distributions in* $\mathcal{D}(I)$ *such that* $\mathbf{x} \succsim^{OPG} \mathbf{y}$. *Suppose that* $v_1^c = y_1^1$. *Then there exists a distribution* $\widehat{\mathbf{x}} \in \mathcal{D}(I)$ *such that* $\widehat{\mathbf{x}}$ *has been obtained from* $\mathbf{y}$ *by a FIP and* $\mathbf{x} \succsim^{OPG} \widehat{\mathbf{x}}$.

Proof. We base the argument on Lemma 3. We must therefore prove that the limit vector of poverty lines $\overline{\mathbf{w}}$ satisfies the conditions imposed on the vector $\mathbf{w}$ of this lemma. From Lemma 7, we have that $y_1^1 = \overline{w}_1 = \overline{w}_2 = ... = \overline{w}_{h_0} > \overline{w}_{h_0} + 1$ for some $h_0 \geq 2$. We know also from Lemma 7 that there is a category $g_0 \leq h_0$ satisfying $g_0 \geq 2$ for which one has:

$$\sum_{h=1}^{g_0} \overline{p}^{\mathbf{y}}(h, y_1^1) \leq \sum_{h=1}^{g_0} \overline{p}^{\mathbf{x}}(h, y_1^1),$$

and

$$\sum_{h=1}^{l} \overline{p}^{\mathbf{y}}(h, y_1^1) > \sum_{h=1}^{l} \overline{p}^{\mathbf{x}}(h, y_1^1)$$

for all $l = 1, ..., g_0 - 1$. As a consequence one has:

$$\sum_{h=l+1}^{g_0} \overline{p}^{\mathbf{y}}(h, y_1^1) < \sum_{h=l+1}^{g_0} \overline{p}^{\mathbf{x}}(h, y_1^1)$$

for $l = 1, ..., g_0 - 1$. and the conclusion of the theorem follows from Lemma 3. ■

In the next example, we illustrate Theorem 4

Example 3 *Assume that* $k = 2$ *and* $\mathcal{N}(1) = \mathcal{N}(2) = \{1, 2\}$ *and consider distributions* $\mathbf{y}$ *and* $\mathbf{x}$ *defined respectively by:*

$$y_1^1 = 3, \; y_2^1 = 7 \; y_1^2 = 0, \; y_2^2 = 4,$$

and:

$$x_1^1 = 5, \; x_2^1 = 6, \; x_1^2 = 1, \; x_2^2 = 2.$$

One can check that $\mathbf{x} \succsim^{OPG} \mathbf{y}$ *and* $v_1^c = 3 = y_1^1$. *Theorem 4 states that a FIP between an agent in category* 2 *and agent* 1 *of category 1 is possible without breaking the OPG inequality. Indeed agent* $i_2 = 2$ *(to keep the notations of Lemma 3) can exchange his income with agent* 1 *in category 1.*

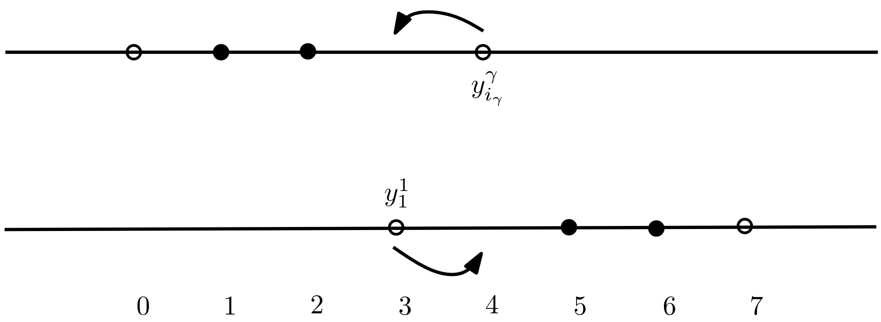


Figure 7: $v_1^c = y_1^1$; an FIP is possible

3.4 Proof of the main result

We now prove the last implication of Theorem 1.

Let $\mathbf{x}$ and $\mathbf{y}$ be as in Theorem 1. By a recursive argument on the finite set of agents, proving implication (iii) of Theorem 1 amounts to showing that $\mathbf{x} \succsim^{OPG} \mathbf{y}$ implies the possibility of going from $\mathbf{y}$ to some distribution $\overline{\mathbf{x}} \in \mathcal{D}(I)$ by a finite sequence of BTPIT and/or FIP in such a way that:

- $\mathbf{x} \succsim^{OPG} \overline{\mathbf{x}}$,
- there exists $h \in \{1, ..., k\}$ for which $x_i^h = \overline{x}_{\overline{i}}^h$ for some i and $\overline{i} \in \mathcal{N}(h)$.

Indeed, whenever we have brought one agent in one category to the income level that an agent of this category has in the final distribution $\mathbf{x}$, we can remove that agent from that category and restart the procedure. Since the numbers of agents and categories are finite, this completes the proof. Let us therefore construct an algorithm for moving from $\mathbf{y}$ to some distribution $\overline{\mathbf{x}}$ as described above by a finite sequence of BTPIT and/or FIP. We construct this algorithm by first setting $\mathbf{x}(0) := \mathbf{y}$ and by recursively defining $\mathbf{x}(n+1)$ from $\mathbf{x}(n)$ in the following manner. Let $v_1^c(n)$ be the critical value defined as per (14) but applied to $\mathbf{x}(n)$ rather than to $\mathbf{y}$.

$(P1)$ If $v_1^c(n) = x_1^1(n)$ then proceed to a FIP, which is possible by Theorem 4.

$(P2)$ If $v_1^c(n) > x_1^1(n)$ and if

- $\forall i \in \mathcal{N}(1)$, $x_i^1 > v_1^c(n)$,
- For any category h such that $h_0 \geq h \geq 2$, one has $x_i^h(n) \neq v_1^c(n)$,[5]

[5] where h_0 is the category such that $v_1^c(n) = v_{h_0}^c(n) > v_{h_0+1}^c(n)$.

then proceed to a FIP, as described by Theorem 3 (remembering that the recipient of such a FIP is not the poorest individual of category 1 in that case).

(MT) otherwise proceed to the maximal transfer defined by Theorem 2 and Definition 5.

By construction, $\mathbf{x} \succsim^{OPG} \mathbf{x}(n)$ for any n. If there exists some $n^* \in \mathbb{N}_+$ such that, for some category $h \in \{1, ..., k\}$, one has:

$$x_i^h(n^*) = x_j^h$$

for some $i, j \in \mathcal{N}(h)$ then the algorithm ends and is said to be *finite*. If it does not end, then the algorithm generates an infinite (non-stationary) sequence. The only thing that remains to be proved is that the later is impossible and that the algorithm is finite, as then $\overline{\mathbf{x}} := \mathbf{x}(n^*)$ satisfies the property stated above. We prove this by way of contradiction and therefore suppose that our algorithm generates an infinite sequence $(\mathbf{x}(n))_{n\in\mathbb{N}}$. We proceed by first establishing a series of claims (all proved in the Appendix).

Claim 1 *There exists some $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, we are in either case ($P2$) or (MT) so that $x_1^1(n) < v_1^c(n)$.*

An immediate consequence of this claim is that, for any $n \geq n_0$, we can also define the quantities $v_h^c(n)$ through expression (15).

Claim 2 *Let n_0 be the integer whose existence was established in Claim 1. Then, for all $n \geq n_0$ and all categories $h = 1, ..., k$, it is the case that $v_h^c(n+1) \leq v_h^c(n)$.* [6]

In the next claim, we establish the existence of some step in the algorithm beyond which, if anything, only maximal transfers occur.

Claim 3 *There exists $n_1 \in \mathbb{N}$ such that, for any $n \geq n_1$, the distribution $\mathbf{x}(n+1)$ is obtained from $\mathbf{x}(n)$ by means of a maximal transfer.*

We proved that, for any $n \geq n_1$, a maximal transfer of type (MT) occurs at time n. Since the algorithm is infinite, no transfer can be equalizing as per Definition 5. Hence, the maximal transfers at every step must either be a *breaking* or a *half* transfer of Definition 5. We next claim that at every step after n_1, if a breaking transfer is required by the algorithm, then the donor involved in the transfer will never be the donor again in a subsequent transfer.[7]

[6] Note that the conclusion of this claim, namely the fact that the critical value weakly decreases, is not true for operations of type (P1), which on the contrary necessarily weakly increase the critical value $v_1^c(n)$.

[7] While the proof of the claim is slightly cumbersome, the intuition behind it is somewhat clear. Indeed, by its very definition, a breaking transfer is such that the donor cannot give more at this stage without breaking at least one of the OPG dominance inequalities. As n increases, the (ordered poverty gap) difference between distribution $\mathbf{x}(n)$ and distribution $\mathbf{x}$ gets smaller and smaller. Hence it becomes harder and harder to make a transfer without breaking some of the OPG inequality.

Claim 4 *There exists $n_2 \in \mathbb{N}$ such that, for any $n \geq n_2$, the distribution $\mathbf{x}(n+1)$ is obtained from $\mathbf{x}(n)$ through a half-transfer.*

We now establish that none of the donors implied in the half-transfers that remain after all breaking transfers have been performed can be in category 1.

Claim 5 *For any $n \geq n_2$, the distribution $\mathbf{x}(n+1)$ is obtained from $\mathbf{x}(n)$ through a half-transfer whose donor is not in category 1.*

We are now ready to establish a contradition, and thus prove that the algorithm is finite. As proved in Claim 5, if the algorithm is infinite, there is some $n_2 \in \mathbb{N}$ such that, for $n \geq n_2$, $\mathbf{x}(n+1)$ is obtained from $\mathbf{x}(n)$ through a half transfer, the donor of which being not in category 1. Yet, once an agent in category 1 has received a half transfer from an agent of a superior category, his/her income becomes equal to that of the donating agent . Hence, the donating agent can not be selected again by the algorithm to donate to that same agent. Since the number of agents is finite, this completes the proof.

The possibility of constructing an infinite sequence of transfers starting from a OPG dominated distribution $\mathbf{y}$ and going to a dominating one $\mathbf{x}$ is rather serious. It is actually this possibility that has motivated our choice of making a FIP in the algorithm in the case (labelled as $(P2)$ above) where $v_1^c(n) > x_1^1(n)$, $x_i^1 > v_1^c(n)$ for every agent $i \in \mathcal{N}(1)$ and where there exists some $h_0 \in \{1, ..., k\}$ satisfying $v_{h_0+1}^c(n) < v_{h_0}^c(n) = v_1^c(n)$ such that, for any $h \in \{1, ..., h_0\}$ and any $i \in \mathcal{N}(h)$, one has $y_i^h(n) \neq v_1^c(n)$, even though performing a BTPIT transfer would also be possible in that case thanks to Theorem 2. The problem that could be encountered if one were doing a maximal BTPIT in the case $(P2)$ is that of being trapped into an infinite sequence of maximal transfers, as illustrated by Example 2 above. If one were to perform a BTPIT rather than a FIP in this example, the maximal transfer would clearly be a "half-transfer" of 1/2. Performing this transfer would yield the distribution $\widehat{\mathbf{x}}$ defined by:

$$\widehat{x}_1^1 = \widehat{x}_2^1 = 5/2,\ \widehat{x}_3^1 = 3,\ \widehat{x}_1^2 = 0,\ \widehat{x}_2^2 = \widehat{x}_3^2 = \widehat{x}_4^2 = 4$$

Notice that the critical value $v_1^c(\widehat{\mathbf{x}})$ associated to $\widehat{\mathbf{x}}$ is still $3 > \widehat{x}_1^1 = 5/2$. Theorem 2 indicates that agent 3 of category 1 can make a transfer to one of the two poorest agents of that same category. The maximal transfer that agent 3 of category 1 can transfer to either one of the two poorest agents of category 1 is a "half-transfer" of 1/4. If this transfer is performed, then the distribution $\widehat{\widehat{\mathbf{x}}}$ is obtained, with $\widehat{\widehat{\mathbf{x}}}$ defined by:

$$\widehat{\widehat{x}}_1^1 = 5/2, \widehat{\widehat{x}}_2^1 = \widehat{\widehat{x}}_3^1 = 11/4,\ \widehat{\widehat{x}}_1^2 = 0,\ \widehat{\widehat{x}}_2^2 = \widehat{\widehat{x}}_3^2 = \widehat{\widehat{x}}_4^2 = 4$$

But from this distribution $\widehat{\widehat{\mathbf{x}}}$, the critical value $v_1^c(\widehat{\mathbf{x}})$ is 11/4 and this could allow for a half transfer of 1/8 between either agent 2 or 3 of category 1 and the poorest agent 1 of this category and so on. It is easy to see that if one resorting to the transfer allowed by Theorem 2 in that case, one would obtain an infinite sequence of half transfers (with the "half" becoming smaller and smaller). It is to avoid this possibility that our algorithm imposes to perform the FIP allowed by Theorem 3 every time the conditions of case $(P2)$ are verified.

4 Conclusion

In this paper, we provide a workable definition of the meaning of "income equalization" when performed between agents who are vertically differentiated with respect to some other characteristic. The definition of equalization that we provided for this case is that of transferring an amount of income from a richer and highly ranked agent to a poorer and less highly ranked one that does not exceed the income difference between the two agents. Specifically, inequality is unquestionably reduced when such a transfer is performed between a rich and highly ranked agent to a poorer and less highly ranked one. If the transfer does not exceeds half the income difference between the donator and the receiver, then such a transfer is called a BTPIT. If the transfer is larger than half the income difference, then the transfer can be viewed as a combination of BTPIT of less than half the income difference and a FIP. The paper has identified the normative foundations of this notion of equalization. Specifically, it has shown that the smallest transitive ranking of distributions consistent with this notion of equalization is the unanimity of all rankings of the two distributions that would be agreed upon by a utilitarian evaluator who considers that the marginal utility of income for every agent is decreasing with respect to both the income and the type. The paper has also identified an empirically implementable criterion - the Ordered Poverty gap criterion - that is equivalent to this notion of equalization. While Gravel and Moyes (2012) have shown that the dominance of a distribution over another by the ordered poverty gap criterion was equivalent to the possibility of going from a phantom-augmented dominated distribution to the phantom-augmented dominating one by a finite sequence of Pigou-Dalton transfers (between agents of a given type) and/or FIP, these authors did not establish the equivalence without resort to dummy or phantom agents. This paper is therefore, to the very best of our knowledge, the only one that has provided an equivalence between a notion of normative dominance, an elementary principle of equalization, and an empirically implementable criterion that applies to distribution of a cardinally meaningful attribute among vertically differentiated agents. It is our hope that the implementable criterion that we justified in this fashion - originally proposed by Bourguignon (1989) - that we have characterized by this approach will be used with increasing confidence by practitioners when evaluating inequalities among vertically differentiated agents.

5 Appendix

5.1 Proof of the results of section 3.2

5.1.1 Lemma 1.

For the first statement, assume by contraposition that $y_1^1 > x_1^1$. Consider then the vector of poverty lines $(y_1^1, \underline{v}, ..., \underline{v}) \in \mathcal{V}$. One has:

$$\begin{aligned} P^{\mathbf{y}}(y_1^1, \underline{v}, ..., \underline{v}) &= 0 \text{ and:} \\ P^{\mathbf{x}}(y_1^1, \underline{v}, ..., \underline{v}) &\geq y_1^1 - x_1^1 > 0 \end{aligned}$$

so that $x \succsim^{OPG} y$ does not hold, as required. The second statement holds by a mirror argument.

5.1.2 Lemma 2.

For any type $\overline{h} = 1, ..., k$, the vector of poverty lines $v^{\overline{h}}$ defined by:

$$\mathbf{v}^{\overline{h}} = (\underbrace{\overline{v}, ..., \overline{v}}_{\overline{h}}, \underbrace{\underline{v}, ..., \underline{v}}_{k-\overline{h}})$$

clearly belongs to $\mathcal{V}$. Hence, since $x \succsim^{OPG} y$, one has:

$$\begin{aligned} P^{\mathbf{x}}(\mathbf{v}^{\overline{h}}) &\leq P^{\mathbf{y}}(\mathbf{v}^{\overline{h}}) \\ &\Longleftrightarrow \\ \sum_{h=1}^{\overline{h}} n(h)\overline{v} - \sum_{h=1}^{\overline{h}} \sum_{i \in \mathcal{N}(h)} x_i^h &\leq \sum_{h=1}^{\overline{h}} n(h)\overline{v} - \sum_{h=1}^{\overline{h}} \sum_{i \in \mathcal{N}(h)} y_i^h \\ &\Longleftrightarrow \\ \sum_{h=1}^{\overline{h}} \sum_{i \in \mathcal{N}(h)} x_i^h &\geq \sum_{h=1}^{\overline{h}} \sum_{i \in \mathcal{N}(h)} y_i^h \quad \blacksquare \end{aligned}$$

5.1.3 Lemma 3.

The proof of the existence of a category $\gamma \in \{2, ..., g_0\}$ defined by:

$$\gamma = \min\{g : g \geq 2 \text{ and } \exists i \ \in \mathcal{N}(g) \text{ such that } y_i^g > y_{i_1}^1\}$$

is an immediate consequence of Inequality (10), which implies that there exists one agent with income strictly larger than $y_{i_1}^1$ in one of the categories of $\{2, ..., g_0\}$. We first prove inequality (13). If either $v_1 \leq y_{i_1}^1$ or $v_\gamma \geq y_{i_\gamma}^\gamma$, inequality (13) trivially holds (because $\min\{y_{i_\gamma}^\gamma, v_1\} - \max\{y_{i_1}^1, v_\gamma\} \leq 0$ in this case). Hence we suppose that $v_1 > y_{i_1}^1$ and $v_\gamma < y_{i_\gamma}^\gamma$. We establish the result by considering three different cases.

Case (i): $y_{i_1}^1 \leq v_{g_0} \leq v_1 \leq y_{i_\gamma}^\gamma$. By definition of $y_{i_\gamma}^\gamma$, one has that:

$$\overline{p}^{\mathbf{y}}(h, w) = \overline{p}^{\mathbf{y}}(h, w')$$

for $h = 2, ..., g_0$ and any w and $w' \in [y_{i_1}^1, y_{i_\gamma}^\gamma[$. Indeed, the number of poor in categories $2, .., g_0$ at distribution y does not change when we move the poverty line applicable to all these categories from $y_{i_1}^1$ to $y_{i_\gamma}^\gamma$). Combining this with inequality (10) and the fact that $\overline{p}^{\mathbf{x}}(h,w)$ is non-decreasing with respect to w, we have:

$$\begin{aligned}
& \sum_{g=2}^{g_0}\sum_{i\in\mathcal{N}(g)}[\max(v_1-y_i^g,0)-\max(v_1-x_i^g,0)] \\
\leq & \sum_{g=2}^{g_0}\sum_{i\in\mathcal{N}(g)}[\max(v_g-y_i^g,0)-\max(v_g-x_i^g,0)] \\
+ & \sum_{g=2}^{g_0}[\overline{p}^{\mathbf{y}}(g,v_g)-\overline{p}^{\mathbf{x}}(g,v_g)](v_1-v_g) \\
= & \sum_{g=2}^{g_0}\sum_{i\in\mathcal{N}(g)}[\max(v_g-y_i^g,0)-\max(v_g-x_i^g,0] \\
+ & \sum_{h=2}^{g_0}\sum_{g=h}^{g_0}[\overline{p}^{\mathbf{y}}(g,v_g)-\overline{p}^{\mathbf{x}}(g,v_g)](v_{h-1}-v_h) \\
\leq & \sum_{g=2}^{g_0}\sum_{i\in\mathcal{N}(g)}[\max(v_g-y_i,0)-\max(v_g-x_i,0]+v_{g_0}-v_1
\end{aligned}$$

Hence:

$$\begin{aligned}
P^{\mathbf{y}}(\mathbf{v})-P^{\mathbf{x}}(\mathbf{v}) &= P^{\mathbf{y}}(v_1,...,v_1,v_{g_0+1},...,v_k)-P^{\mathbf{x}}(v_1,...,v_1,v_{g_0+1},...,v_k) \\
&\quad -\sum_{g=2}^{g_0}\sum_{i\in\mathcal{N}(g)}[\max(v_1-y_i^g,0)-\max(v_1-x_i^g,0)] \\
&\quad +\sum_{g=2}^{g_0}\sum_{i\in\mathcal{N}(g)}[\max(v_g-y_i^g,0)-\max(v_g-x_i^g,0)] \\
&\geq P^{\mathbf{y}}(v_1,...,v_1,v_{g_0+1},...,v_k) \\
&\quad -P^{\mathbf{x}}(v_1,...,v_1,v_{g_0+1},...,v_k)+v_1-v_{g_0} \\
&\geq v_1-v_{g_0} \text{ (because } (v_1,...,v_1,v_{s_0+1},...,v_k)\in\mathcal{V}) \\
&\geq v_1-v_\gamma \\
&= \min\{y_{i_\gamma}^\gamma,v_1\}-\max\{y_{i_1}^1,v_\gamma\}.
\end{aligned}$$

as required.

Case (ii): $v_1 > y_{i_\gamma}^\gamma$ and $v_{g_0}\geq y_{i_1}^1$.
In this case, there exists some $\underline{h}\in\{1,...,\gamma-1\}$ such that $v_1\geq ...\geq v_{\underline{h}} > y_{i_\gamma}^\gamma \geq v_{\underline{h}+1}\geq ...\geq v_k$. Let $\tilde{v}=(y_{i_\gamma}^\gamma,...,y_{i_\gamma}^\gamma,v_{\underline{h}+1},...,v_k)$. Then $\tilde{v}$ belongs to case (i) and, consequently:

$$P^{\mathbf{y}}(\tilde{\mathbf{v}})-P^{\mathbf{x}}(\tilde{\mathbf{v}})\geq \tilde{v}_1-\tilde{v}_\gamma=y_{i_1}^1-v_\gamma.$$

Moreover denoting $\hat{v}:=(v_1,...,v_{\underline{h}},v_{\underline{h}+1}^c,...,v_k^c)$, one has, by definition of v^c:

$$\begin{aligned}
(P^{\mathbf{y}}(\tilde{\mathbf{v}})-P^{\mathbf{x}}(\tilde{\mathbf{v}}))-(P^{\mathbf{y}}(\mathbf{v})-P^{\mathbf{x}}(\mathbf{v})) &= (P^{\mathbf{y}}(\mathbf{v}^c)-P^{\mathbf{x}}(\mathbf{v}^c))-(P^{\mathbf{y}}(\hat{\mathbf{v}})-P^{\mathbf{x}}(\hat{\mathbf{v}})) \\
&= P^{\mathbf{x}}(\hat{\mathbf{v}})-P^{\mathbf{x}}(\hat{\mathbf{v}}) \\
&\leq 0
\end{aligned}$$

Hence we have:

$$P^{\mathbf{y}}(\mathbf{v})-P^{\mathbf{x}}(\mathbf{v})\geq P^{\mathbf{y}}(\tilde{\mathbf{v}})-P^{\mathbf{x}}(\tilde{\mathbf{v}})\geq y_{i_\gamma}^\gamma-v_\gamma=\min\{y_{i_\gamma}^\gamma,v_1\}-\max\{y_{i_1}^1,v_\gamma\}$$

as required.

Case (iii): $v_{g_0} < y_{i_1}$.(without any assumption on the relative standing of v_1 vis-à-vis $y^{\gamma}_{i_\gamma}$)
In this case, there exists some $\overline{h} < \{1, ..., g_0 - 1\}$ such that $v_{\overline{h}} \geq y^1_{i_1} > v_{\overline{h}+1} \geq ... \geq v_k$. We first note that:

$$\begin{aligned} & \sum_{g=\overline{h}+1}^{k} \sum_{i\in\mathcal{N}(g)} [\max(v_g - y^g_i, 0) - \max(v_1 - x^g_i, 0)] \\ \geq & \sum_{g=\overline{h}+1}^{k} \sum_{i\in\mathcal{N}(g)} [\max(w_g - y^g_i, 0) - \max(w_1 - x^g_i, 0)] \end{aligned}$$

because assuming otherwise would imply that:

$$\begin{aligned} P^{\mathbf{y}}(w_1, ..., w_{\overline{h}}, v_{\overline{h}+1}, ..., v_k) - P^{\mathbf{x}}(w_1, ..., w_{\overline{h}}, v_{\overline{h}+1}, ..., v_k) &< P^{\mathbf{y}}(\mathbf{w}) - P^{\mathbf{x}}(\mathbf{w}) \\ &= 0, \end{aligned}$$

contradicting the statement that $x \succsim^{OPG} y$ (since $(w_1, ..., w_{\overline{h}}, v_{\overline{h}+1}, ..., v_k)$ belongs to $\mathcal{V}$). Let $\tilde{v} := (v_1, ..., v_{\overline{h}}, w_{\overline{h}+1}, ..., w_k) \in \mathcal{V}$. Observe with care that the vector $\tilde{v}$ so defined corresponds either to case (i) (if $v_1 \leq y^{\gamma}_{i_\gamma}$) or to case (ii) (if $v_1 > y^{\gamma}_{i_\gamma}$). Observe also that $\max\{y^1_{i_1}, \tilde{v}_\gamma\} \leq \max\{y^1_{i_1}, v_\gamma\}$. Indeed if $\tilde{v}_\gamma \leq y^1_{i_1}$ there is nothing to prove. If on the other hand $\tilde{v}_\gamma > y^1_{i_1}$, then $\tilde{v}_\gamma = v_\gamma$ by definition of $\overline{h}$ and the inequality $\max\{y^1_{i_1}, \tilde{v}_\gamma\} \leq \max\{y^1_{i_1}, v_\gamma\}$ also holds. Collecting these observations, we obtain that

$$\begin{aligned} P^{\mathbf{y}}(\mathbf{v}) - P^{\mathbf{x}}(\mathbf{v}) &\geq P^{\mathbf{y}}(\tilde{\mathbf{v}}) - P^{\mathbf{x}}(\tilde{\mathbf{v}}) \\ &\geq \min\{y^{\gamma}_{i_\gamma}, v_1\} - \max\{\tilde{v}_\gamma, y^1_{i_1}\} \text{ (by cases (i) or (ii))} \\ &\geq \min\{y^{\gamma}_{i_\gamma}, v_1\} - \max\{y^1_{i_1}, v_\gamma\} \end{aligned}$$

which proves (13) in that last case.

Let us now establish the existence of a distribution $\overline{\mathbf{x}} \in D(I)$ that has been obtained from y by a FIP and that is such that $x \succsim^{OPG} \overline{\mathbf{x}}$. Let $\overline{\mathbf{x}}$ be the distribution obtained from y by means of a FIP from agent $i^\gamma \in N(\gamma)$ to agent $i^1 \in \mathcal{N}(1)$. Let us show that $\mathbf{x} \succsim^{OPG} \overline{\mathbf{x}}$. Consider any vector $v \in \mathcal{V}$ of ordered poverty lines. If $v_\gamma \geq y^{\gamma}_{i^\gamma}$ or $v_1 \leq y^1_{i^1}$, it is clear that $P^{\overline{\mathbf{x}}}(v) = P^{\mathbf{y}}(v) \geq P^{\mathbf{x}}(v)$. If on the other hand $v_\gamma < y^{\gamma}_{i^\gamma}$ and $v_1 > y^1_{i^1}$, one can obtain by straightforward computations[8]:

$$\begin{aligned} P^{\overline{\mathbf{x}}}(\mathbf{v}) &= P^{\mathbf{y}}(\mathbf{v}) - \max\{y^{\gamma}_{i^\gamma} - v_\gamma, 0\} - \max\{y^1_{i^1} - v_1, 0\} \\ &\quad + \max\{y^1_{i^1} - v_\gamma, 0\} + \max\{y^{\gamma}_{i^\gamma} - v_1, 0\} \\ &= P^{\mathbf{y}}(\mathbf{v}) - y^{\gamma}_{i^\gamma} + v_\gamma + \max\{y^1_{i^1} - v_\gamma, 0\} + \max\{y^{\gamma}_{i^\gamma} - v_1, 0\} \\ &= P^{\mathbf{y}}(\mathbf{v}) - (y^{\gamma}_{i^\gamma} - \max\{y^{\gamma}_{i^\gamma} - v_1, 0\}) + (v_\gamma - \max\{y^1_{i^1} - v_\gamma, 0\}) \\ &= P^{\mathbf{y}}(\mathbf{v}) - \min\{y^{\gamma}_{i^\gamma}, v_1\} + \max\{y^1_{i^1}, v_\gamma\}. \end{aligned}$$

Using the inequality (13) proved above, this implies that:

$$P^{\overline{\mathbf{x}}}(\mathbf{v}) - P^{\mathbf{x}}(\mathbf{v}) \geq P^{\mathbf{y}}(\mathbf{v}) - P^{\mathbf{x}}(\mathbf{v}) - \min\{y^{\gamma}_{i^\gamma}, v_1\} + \max\{y^1_{i^\gamma}, v_\gamma\} \geq 0.$$

which proves the result. ∎

[8]Some of them using the fact that $\max\{a, b\} = b + \max\{a - b, 0\}$ and $\min\{c, d\} = c - \max\{c - d, 0\}$

5.2 Proof of the lemmas, claims and corollaries of Section 3.3

5.2.1 Lemma 4.

Define the two vectors of poverty lines $\mathbf{v}^-$ and $\mathbf{v}^+$ by:

$$\begin{aligned} v_h^- &= \min(v_h, v_h^c) \text{ and,} \\ v_h^+ &= \max(v_h, v_h^c). \end{aligned}$$

It is clear that $\mathbf{v}^-$ and $\mathbf{v}^+$ both belong to $\mathcal{V}$. By definition of $\mathbf{v}^-$ and $\mathbf{v}^+$, one has:

$$\begin{aligned} & P^{\mathbf{x}}(\mathbf{v}^+) - P^{\mathbf{y}}(\mathbf{v}^+) + P^{\mathbf{x}}(\mathbf{v}^-) - P^{\mathbf{y}}(\mathbf{v}^-) \\ = \; & P^{\mathbf{x}}(\mathbf{v}^c) - P^{\mathbf{y}}(\mathbf{v}^c) + P^{\mathbf{x}}(\mathbf{v}) - P^{\mathbf{y}}(\mathbf{v}) \end{aligned} \tag{21}$$

By definition of $\mathbf{v}^c$, one has $P^{\mathbf{x}}(\mathbf{v}^c) - P^{\mathbf{y}}(\mathbf{v}^c) = 0$. Assume therefore by contradiction that $P^{\mathbf{x}}(\mathbf{v}) - P^{\mathbf{y}}(\mathbf{v}) = 0$ so that, using equality (21), one has:

$$P^{\mathbf{x}}(\mathbf{v}^+) - P^{\mathbf{y}}(\mathbf{v}^+) + P^{\mathbf{x}}(\mathbf{v}^-) - P^{\mathbf{y}}(\mathbf{v}^-) = 0$$

As there exists h_0 such that $v_{h_0} < v_{h_0}^c$, we must have $v_{h_0}^- = v_{h_0} < v_{h_0}^c$. Moreover $v_1^- = \min(v_1, v_1^c) > y_1^1$. Consequently, by the recursive definition of $\mathbf{v}^c$, we must have that:

$$P^{\mathbf{x}}(\mathbf{v}^-) - P^{\mathbf{y}}(\mathbf{v}^-) < 0$$

But this implies that:

$$P^{\mathbf{x}}(\mathbf{v}^+) - P^{\mathbf{y}}(\mathbf{v}^+) > 0$$

a contradiction of the fact that $\mathbf{x} \succsim^{OPG} \mathbf{y}$ and that $\mathbf{v}^+$ belongs to $\mathcal{V}$. ■

5.2.2 Corollary 1.

Using the recursive definition of the vector v^c provided by (4), it is clear that:

$$v_h^c \geq \min_{v_h} \left\{ \exists v_{-h} : \; v_1 > y_1^1, \; \mathbf{v} = (v_h, v_{-h}) \in \mathcal{V} \text{ and } P^{\mathbf{x}}(\mathbf{v}) = P^{\mathbf{y}}(\mathbf{v}) \right\}.$$

for all h. To prove that:

$$v_h^c \leq \min_{v_h} \left\{ \exists v_{-h} : \; v_1 > y_1^1, \; \mathbf{v} = (v_h, v_{-h}) \in \mathcal{V} \text{ and } P^{\mathbf{x}}(\mathbf{v}) = P^{\mathbf{y}}(\mathbf{v}) \right\}.$$

we simply notice that, thanks to Lemma 4, any vector $\mathbf{v} \in \mathcal{V}$ such that $v_1 > y_1^1$ and $P^{\mathbf{x}}(\mathbf{v}) - P^{\mathbf{y}}(\mathbf{v}) = 0$ must also satisfy $v_h \geq v_h^c$ for all $h \in \{2, ..., k\}$. ■

5.2.3 Lemma 5.

We first note that, for any sufficiently small strictly positive number ε, the vector of poverty lines:

$$(v_1^c, ..., v_{h_0-1}^c, \underbrace{v_{h_0}^c + \varepsilon, ..., v_{h_0}^c + \varepsilon}_{l - h_0}, \underbrace{v_{h_0}^c, ..., v_{h_0}^c}_{\overline{h} - l}, v_{h_0+\overline{h}+1}^c, ..., v_k^c)$$

belongs to the set $\mathcal{V}$. Hence, since $x \succsim^{OPG} y$, one has:

$$\begin{aligned}
& P^{\mathbf{x}}(\mathbf{v}^c) + \varepsilon[\sum_{h=h_0}^{h_0+l} \overline{p}^{\mathbf{x}}(h, v_h^c)] \\
= \; & P^{\mathbf{x}}(v_1^c, ..., v_{h_0-1}^c, v_{h_0}^c + \varepsilon, ..., v_{h_0}^c + \varepsilon, v_{h_0}^c, ..., v_{h_0}^c, v_{h_0+\overline{h}+1}^c, ..., v_k^c) \\
\leq \; & P^{\mathbf{y}}(v_1^c, ..., v_{h_0-1}^c, v_{h_0}^c + \varepsilon, ..., v_{h_0}^c + \varepsilon, v_{h_0}^c, ..., v_{h_0}^c, v_{h_0+\overline{h}+1}^c, ..., v_k^c) \\
= \; & P^{\mathbf{y}}(\mathbf{v}^c) + \varepsilon[\sum_{h=h_0}^{h_0+l} \overline{p}^{\mathbf{y}}(h, v_h^c)]
\end{aligned}$$

which, when combined with the fact that $P^{\mathbf{x}}(\mathbf{v}^c) - P^{\mathbf{y}}(\mathbf{v}^c) = 0$ by definition of $\mathbf{v}^c$, implies inequality (16). Similarly, one can also remark that the vector of poverty lines:

$$(v_1^c, ..., v_{h_0-1}^c, \underbrace{v_{h_0}^c, ..., v_{h_0}^c}_{l-h_0}, \underbrace{v_{h_0}^c - \varepsilon, ..., v_{h_0}^c - \varepsilon}_{\overline{h}-l}, v_{h_0+\overline{h}+1}^c, ..., v_k^c)$$

belongs to the set $\mathcal{V}$ for a small enough ε. By the recursive definition of $\mathbf{v}^c$, we have:

$$\begin{aligned}
= \; & P^{\mathbf{x}}(v_1^c, ..., v_{h_0-1}^c, v_{h_0}^c, ..., v_{h_0}^c, v_{h_0}^c - \varepsilon, ..., v_{h_0}^c - \varepsilon, v_{h_0+\overline{h}+1}^c, ..., v_k^c) \\
< \; & P^{\mathbf{y}}(v_1^c, ..., v_{h_0-1}^c, v_{h_0}^c, ..., v_{h_0}^c, v_{h_0}^c - \varepsilon, ..., v_{h_0}^c - \varepsilon, v_{h_0+\overline{h}+1}^c, ..., v_k^c)
\end{aligned}$$

and, therefore:

$$\begin{aligned}
& P^{\mathbf{x}}(\mathbf{v}^c) - \varepsilon[\sum_{h=h_0+l}^{h_0+\overline{h}} p^{\mathbf{x}}(h, v_h^c)] \\
= \; & P^{\mathbf{x}}(v_1^c, ..., v_{h_0-1}^c, v_{h_0}^c, ..., v_{h_0}^c, v_{h_0}^c - \varepsilon, ..., v_{h_0}^c - \varepsilon, v_{h_0+\overline{h}+1}^c, ..., v_k^c) \\
< \; & P^{\mathbf{y}}(v_1^c, ..., v_{h_0-1}^c, v_{h_0}^c, ..., v_{h_0}^c, v_{h_0}^c - \varepsilon, ..., v_{h_0}^c - \varepsilon, v_{h_0+\overline{h}+1}^c, ..., v_k^c) \\
= \; & P^{\mathbf{y}}(\mathbf{v}^c) - \varepsilon[\sum_{h=h_0+l}^{h_0+\overline{h}} p^{\mathbf{y}}(h, v_h^c)]
\end{aligned}$$

which, when combined with $P^{\mathbf{x}}(\mathbf{v}^c) - P^{\mathbf{y}}(\mathbf{v}^c) = 0$, implies inequality (17). ■

5.2.4 Corollary 2.

If $v_1^c > v_2^c$, one can apply Lemma 5 to the case where $h_0 = 1$ and $\overline{h} = 0$. In this case, Inequalities (16) and (17) write:

$$\overline{p}^{\mathbf{x}}(1, v_1^c) \leq \overline{p}^{\mathbf{y}}(1, v_1^c)$$

and:

$$p^{\mathbf{x}}(1, v_1^c) > p^{\mathbf{y}}(1, v_1^c).$$

Hence, there must exist an agent $i \in \mathcal{N}(1)$ such that $y_i^1 = v_1^c$. More generally, if $v_1^c = v_2^c = ... = v_{k+1}^c > v_{k+2}^c$, one applies Lemma 5 to the case where $h_0 = 1$ (taking $l = \overline{h}$ in (16) and $l = 0$ in (17)) which gives

$$\sum_{h=1}^{\overline{h}+1} \overline{p}^{\mathbf{x}}(1, v_1^c) \leq \sum_{h=1}^{\overline{h}+1} \overline{p}^{\mathbf{y}}(1, v_1^c)$$

and:

$$\sum_{h=1}^{\overline{h}+1} p^{\mathbf{x}}(1, v_1^c) > \sum_{h=1}^{\overline{h}+1} p^{\mathbf{y}}(1, v_1^c).$$

One then obtains the existence of some $j \in \{1, ..., \overline{h}+1\}$ and some $i \in \mathcal{N}(j)$ such that $y_i^j = v_j^c = v_1^c$. ■

5.2.5 Lemma 6.

Choose the strictly positive number ε_1 in such a way as to satisfy:

$$\varepsilon_1 < \min \mid a - b \mid \tag{22}$$

for all pairs of distinct numbers a, $b \in I(x, y)$ and:

$$\varepsilon_1 < v_1^c - y_1^1 \tag{23}$$

Consider then any numbers $v_2, ..., v_k$ such that $\mathbf{v} = (y_1^1 + \varepsilon_1, v_2, ..., v_k) \in \mathcal{V}$ and let $h_0 \in \{1, ..., k\}$ be such that $v_h > y_1^1$ for all $h \in \{1, ..., h_0\}$ and $v_h \leq y_1^1$ for all $j \in \{h_0 + 1, ..., k\}$ (if there are such j). One can then write the vector $\mathbf{v}$ as:

$$\mathbf{v} = (y_1^1 + \varepsilon_1, y_1^1 + \varepsilon_2, ..., y_1^1 + \varepsilon_{h_0}, v_{h_0+1}, v_{h_0+2}, ..., v_k)$$

for some (possibly empty) list $\varepsilon_2, ..., \varepsilon_{h_0}$ satisfying $\varepsilon_1 \geq \varepsilon_2 ... \geq \varepsilon_{h_0} > 0$. Let us prove that:

$$P^{\mathbf{y}}(\mathbf{v}) \geq P^{\mathbf{x}}(\mathbf{v}) + \varepsilon_1$$

Clearly, for ε_1 satisfying (22) and (23), one has:

$$P^{\mathbf{y}}(\mathbf{v}) = P^{\mathbf{y}}(y_1^1, ..., y_1^1, v_{h_0+1}, v_{h_0+2}, ..., v_k) + \sum_{h=1}^{h_0} \varepsilon_h \overline{p}^{\mathbf{y}}(h, y_1^1),$$

and:

$$P^{\mathbf{x}}(\mathbf{v}) = P^{\mathbf{x}}(y_1^1, ..., y_1^1, v_{h_0+1}, v_{h_0+2}, ..., v_k) + \sum_{h=1}^{h_0} \varepsilon_h \overline{p}^{\mathbf{x}}(h, y_1^1),$$

and, therefore:

$$\begin{aligned} P^{\mathbf{y}}(\mathbf{v}) - P^{\mathbf{x}}(\mathbf{v}) &= P^{\mathbf{y}}(y_1^1, ..., y_1^1, v_{h_0+1}, ..., v_k) - P^{\mathbf{x}}(y_1^1, ..., y_1^1, v_{h_0+1}, ..., v_k) \\ &\quad + \sum_{h=1}^{h_0} \varepsilon_h [\overline{p}^{\mathbf{y}}(h, y_1^1) - \overline{p}^{\mathbf{x}}(h, y_1^1)] \end{aligned} \tag{24}$$

If

$$P^{\mathbf{y}}(y_1^1, ..., y_1^1, v_{h_0+1}, ..., v_k) > P^{\mathbf{x}}(y_1^1, ..., y_1^1, v_{h_0+1}, ..., v_k)$$

then there is nothing to prove. Indeed, from Lemma 1 and the assumption that $y_i^1 \neq x_i^1$ for all $i \in \mathcal{N}(1)$, one has that $\overline{p}^{\mathbf{y}}(1, y_1^1) \geq 1$ and $\overline{p}^{\mathbf{x}}(1, y_1^1) = 0$. Hence:

$$\begin{aligned} &P^{\mathbf{y}}(y_1^1, ..., y_1^1, v_{h_0+1}, v_{h_0+2}, ..., v_k) - P^{\mathbf{x}}(y_1^1, ..., y_1^1, v_{h_0+1}, v_{h_0+2}, ..., v_k) \\ &+\varepsilon_1 [\overline{p}^{\mathbf{y}}(1, y_1^1) - \overline{p}^{\mathbf{x}}(1, y_1^1)] \\ &> \varepsilon_1 \end{aligned}$$

for any ε_1 satisfying (22) and (23). Because of this, one can choose ε_1 sufficiently small so as make the numbers $\varepsilon_2, ..., \varepsilon_{h_0}$ sufficiently small for the inequality:

$$\begin{aligned} &P^{\mathbf{y}}(y_1^1, ..., y_1^1, v_{h_0+1}, v_{h_0+2}, ..., v_k) - P^{\mathbf{x}}(y_1^1, ..., y_1^1, v_{h_0+1}, v_{h_0+2}, ..., v_k) \\ &+\varepsilon_1 [\overline{p}^{\mathbf{y}}(1, y_1^1) - \overline{p}^{\mathbf{x}}(1, y_1^1)] + \sum_{h=2}^{h_0} \varepsilon_h [\overline{p}^{\mathbf{y}}(h, y_1^1) - \overline{p}^{\mathbf{x}}(h, y_1^1)] \\ \geq \quad &\varepsilon_1 \end{aligned}$$

to hold. Suppose now that:

$$P^{\mathbf{y}}(y_1^1, ..., y_1^1, v_{h_0+1}, v_{h_0+2}, ..., v_k) = P^{\mathbf{x}}(y_1^1, ..., y_1^1, v_{h_0+1}, v_{h_0+2}, ..., v_k)$$

In that case, it follows from (24) that:

$$P^{\mathbf{y}}(\mathbf{v})-P^{\mathbf{x}}(\mathbf{v})=\sum_{h=1}^{h_0}\varepsilon_h[\overline{p}^{\mathbf{y}}(h,y_1^1)-\overline{p}^{\mathbf{x}}(h,y_1^1)]$$

This equality can equivalently be written as:

$$P^{\mathbf{y}}(\mathbf{v})-P^{\mathbf{x}}(\mathbf{v})=\sum_{h=1}^{h_0}[\varepsilon_h-\varepsilon_{h+1}]\sum_{g=1}^{h}[\overline{p}^{\mathbf{y}}(g,y_1^1)-\overline{p}^{\mathbf{x}}(g,y_1^1)] \tag{25}$$

using the convention that $\varepsilon_{h_0+1}=0$. Notice that, by definition of v_1^c, one must have $P^{\mathbf{y}}(\mathbf{v})-P^{\mathbf{x}}(\mathbf{v})>0$ if ε_1 satisfies (23). Notice also that, for all $h\in\{1,...,h_0\}$, one has:

$$\sum_{g=1}^{h}[\overline{p}^{\mathbf{y}}(g,y_1^1)-\overline{p}^{\mathbf{x}}(g,y_1^1)]\geq 1 \tag{26}$$

Indeed, by definition of v_1^c, one has for every strictly positive $\delta\leq\varepsilon_1$:

$$\begin{aligned}
& P^{\mathbf{y}}(\underbrace{y_1^1+\delta,...,y_1^1+\delta}_{h},\underbrace{y_1^1,...,y_1^1}_{h_0-h},v_{h_0+1},v_{h_0+2},...,v_k) \\
> \;& P^{\mathbf{x}}(\underbrace{y_1^1+\delta,...,y_1^1+\delta}_{h},\underbrace{y_1^1,...,y_1^1}_{h_0-h},v_{h_0+1},v_{h_0+2},...,v_k)
\end{aligned}$$

Yet,

$$\begin{aligned}
& P^{\mathbf{y}}(\underbrace{y_1^1+\delta,...,y_1^1+\delta}_{h},\underbrace{y_1^1,...,y_1^1}_{h_0-h},v_{h_0+1},v_{h_0+2},...,v_k) \\
= \;& P^{\mathbf{y}}(y_1^1,...,y_1^1,v_{h_0+1},v_{h_0+2},...,v_k)+\delta\sum_{g=1}^{h}\overline{p}^{\mathbf{y}}(g,y_1^1)
\end{aligned}$$

and:

$$\begin{aligned}
& P^{\mathbf{x}}(\underbrace{y_1^1+\delta,...,y_1^1+\delta}_{h},\underbrace{y_1^1,...,y_1^1}_{h_0-h},v_{h_0+1},v_{h_0+2},...,v_k) \\
= \;& P^{\mathbf{x}}(x_1^1,...,x_1^1,v_{h_0+1},v_{h_0+2},...,v_k)+\delta\sum_{g=1}^{h}\overline{p}^{\mathbf{x}}(g,y_1^1)
\end{aligned}$$

Hence, under the assumption that:

$$P^{\mathbf{y}}(y_1^1,...,y_1^1,v_{h_0+1},v_{h_0+2},...,v_k)=P^{\mathbf{x}}(y_1^1,...,y_1^1,v_{h_0+1},v_{h_0+2},...,v_k)$$

one has:

$$\begin{aligned}
& P^{\mathbf{y}}(y_1^1+\delta,...,y_1^1+\delta,y_1^1,...,y_1^1,v_{h_0+1},v_{h_0+2},...,v_k) \\
& -P^{\mathbf{x}}(y_1^1+\delta,...,y_1^1+\delta,y_1^1,...,y_1^1,v_{h_0+1},v_{h_0+2},...,v_k) \\
= \;& \delta\sum_{g=1}^{h}[\overline{p}^{\mathbf{y}}(g,y_1^1)-\overline{p}^{\mathbf{x}}(g,y_1^1)]>0
\end{aligned}$$

which establishes Inequality (26). Together with (25), this leads to the conclusion that:

$$P^{\mathbf{y}}(\mathbf{v})-P^{\mathbf{x}}(\mathbf{v})\geq\varepsilon_1-\epsilon_{h_0+1}=\varepsilon_1.$$

as required. ■

5.2.6 Lemma 7.

Let $\overline{\mathbf{w}}$ be the vector of ordered poverty lines that is the limit of the sequence $\{\mathbf{w^m}\}$ of ordered poverty lines vectors satisfying $w_1^m = y_1^1 + \varepsilon_1^m$ with $\varepsilon_1^m \to 0$ and $P^{\mathbf{x}}(\mathbf{w^m}) - P^{\mathbf{y}}(\mathbf{w^m}) = 0$ for all m that was mentioned in Case (B) of section 3.3. We first show that $\overline{w}_1 = \overline{w}_2$. By contradiction, suppose that $\overline{w}_2 < \overline{w}_1$. Then, there exists a large enough m for which $w_2^m < \overline{w}_1 = y_1^1$. Also, for a large enough m, one has that:

$$w_1^m \overline{p}^{\mathbf{x}}(1, w_1^m) - \sum_{i \in \overline{\mathcal{P}}^{\mathbf{x}}(1, w_1^m)} x_i^1 = w_1 \overline{p}^{\mathbf{x}}(1, y_1^1) - \sum_{i \in \overline{\mathcal{P}}^{\mathbf{x}}(1, y_1^1)} x_i^1 = 0$$

thanks to Lemma 1 and the fact that $y_1^1 \neq x_1^1$. Moreover one has:

$$\begin{aligned} w_1^m \overline{p}^{\mathbf{y}}(1, w_1^m) - \sum_{i \in \overline{\mathcal{P}}^{\mathbf{y}}(1, w_1^m)} y_i^1 &= (\varepsilon_1^m + y_1^1)\overline{p}^{\mathbf{y}}(1, y_1^1) - \sum_{i \in \overline{\mathcal{P}}^{\mathbf{y}}(1, w_1)} y_i^1 \\ &> y_1^1 \overline{p}^{\mathbf{y}}(1, y_1^1) - \sum_{i \in \overline{\mathcal{P}}^{\mathbf{y}}(1, w_1)} y_i^1 = 0 \end{aligned}$$

because $\overline{p}^{\mathbf{y}}(1, y_1^1) \geq 1$. Hence:

$$\begin{aligned} P^{\mathbf{y}}(\mathbf{w}^m) - P^{\mathbf{x}}(\mathbf{w}^m) &= w_1^m \overline{p}^{\mathbf{y}}(1, w_1^m) - \sum_{i \in \overline{\mathcal{P}}^{\mathbf{y}}(1, w_1^m)} y_i^1 + \sum_{h=2}^{k} P^{\mathbf{y}}(h, w_h^m) \\ &\quad -[w_1^m \overline{p}^{\mathbf{x}}(1, w_1^m) - \sum_{i \in \overline{\mathcal{P}}^{\mathbf{x}}(1, w_1^m)} x_i^1 + \sum_{h=2}^{k} P^{\mathbf{x}}(h, w_h^m)] \\ &= 0 \\ &> y_1^1 \overline{p}^{\mathbf{y}}(1, y_1^1) - \sum_{i \in \overline{\mathcal{P}}^{\mathbf{y}}(1, w_1)} y_i^1 + \sum_{h=2}^{k} P^{\mathbf{y}}(h, w_h^m) \\ &\quad -[w_1 \overline{p}^{\mathbf{x}}(1, y_1^1) - \sum_{i \in \overline{\mathcal{P}}^{\mathbf{x}}(1, y_1^1)} x_i^1 + \sum_{h=2}^{k} P^{\mathbf{x}}(h, w_h^m)] \\ &= P^{\mathbf{y}}(y_1^1, w_2^m, ..., w_k^m) - P^{\mathbf{x}}(y_1^1, w_2^m, ..., w_k^m) \end{aligned}$$

a contradiction. We now show that $w_2^m > y_1^1$. Indeed, for m large enough, one has:

$$P^{\mathbf{y}}(\mathbf{w}^m) = P^{\mathbf{y}}(y_1^1, w_2^m, ..., w_k^m) + \varepsilon_1^m p^{\mathbf{y}}(1, y_1^1)$$

and:

$$P^{\mathbf{x}}(\mathbf{w}^m) = P^{\mathbf{x}}(y_1^1, w_2^m, ..., w_k^m) + \varepsilon_1^m p^{\mathbf{x}}(1, y_1^1)$$

Moreover, we know from Lemma 1 and the fact that $y_i^1 \neq x_i^1$ that $\overline{p}^{\mathbf{y}}(1, y_1^1) \geq 1 > 0 = p^{\mathbf{x}}(1, y_1^1)$. Hence, one has:

$$\begin{aligned} P^{\mathbf{y}}(\mathbf{w}^m) - P^{\mathbf{x}}(\mathbf{w}^m) &= 0 \\ &\geq P^{\mathbf{y}}(y_1^1, w_2^m, ..., w_k^m) - P^{\mathbf{x}}(y_1^1, w_2^m, ..., w_k^m) + \varepsilon_1^m \end{aligned}$$

Because of this (and the fact that $\varepsilon_1^m > 0$), assuming that $w_2^m \leq y_1^1$ and, therefore, that the vector of poverty lines $(y_1^1, w_2^m, ..., w_k^m)$ belongs to $\mathcal{V}$ would be contradictory with the fact that $x \succsim^{OPG} y$. We also know that $w_{h_0+1}^m < y_1^1$. Let $l := \min\{h \geq 1 : w_h^m \leq y_1^1\}$. As we have just shown $3 \leq l \leq h_0 + 1$. For $h = 1, ..., l$, we have $w_h^m = y_1^1 + \varepsilon_h^m$ with $\varepsilon_1^m \geq \varepsilon_2^m \geq ... \geq \varepsilon_l^m > 0$. We know already that $\overline{p}^{\mathbf{y}}(1, y_1^1) \geq 1 > 0 = \overline{p}^{\mathbf{x}}(1, y_1^1)$. Suppose that the main claim of the Lemma was false. In that case, we would have:

$$\sum_{g=1}^{h} p^{\mathbf{y}}(g, y_1^1) > \sum_{g=1}^{h} p^{\mathbf{x}}(g, y_1^1)$$

for all $h = 1, ..., h_0$. Yet:

$$
\begin{aligned}
0 &= P^{\mathbf{y}}(\mathbf{w}^m) - P^{\mathbf{x}}(\mathbf{w}^m) \\
&= P^{\mathbf{y}}(y_1^1, ..., y_1^1, w_{l+1}^m, ..., w_k^m) - P^{\mathbf{x}}(y_1^1, ..., y_1^1, w_{l+1}^m, ..., w_k^m) \\
&+ \sum_{h=1}^{l}[\overline{p}^{\mathbf{y}}(h, y_1^1) - \overline{p}^{\mathbf{x}}(h, y_1^1)]\varepsilon_h^m \\
&= P^{\mathbf{y}}(y_1^1, ..., y_1^1, w_{s+1}^m, ..., w_k^m) - P^{\mathbf{x}}(y_1^1, ..., y_1^1, w_{l+1}^m, ..., w_k^m) \\
&+ \sum_{h=1}^{l-1}(\varepsilon_h^m - \varepsilon_{h+1}^m)\sum_{g=1}^{h}[\overline{p}^{\mathbf{y}}(g, y_1^1) - \overline{p}^{\mathbf{x}}(g, y_1^1)] + \varepsilon_l^m \sum_{g=1}^{l}[p^{\mathbf{y}}(g, y_1^1) - p^{\mathbf{x}}(g, y_1^1)] \\
&\geq \sum_{h=1}^{s-1}(\varepsilon_h^m - \varepsilon_{h+1}^m)\sum_{g=1}^{h}[\overline{p}^{\mathbf{y}}(g, y_1^1) - \overline{p}^{\mathbf{x}}(g, y_1^1)] + \varepsilon_l^m \sum_{g=1}^{l}[\overline{p}^{\mathbf{y}}(g, y_1^1) - \overline{p}^{\mathbf{x}}(g, y_1^1)] \\
&\geq \varepsilon_l^m > 0
\end{aligned}
$$

a contradiction.

5.2.7 Claim 1

Once an agent of a category higher than 1 has been involved in a FIP defined as in Theorem 4, his/her income becomes weakly smaller than that of the poorest income observed in category 1. Since the income of an agent in any category $h \geq 2$ never increases through the algorithm described above, and since the number of agents in categories higher than 1 is finite, it follows that the the number of FIP of type $(P1)$ in the algorithm is bounded above by the number of agents in categories $2, ..., k$. ■

5.2.8 Claim 2

By definition of the algorithm and the critical vector $\mathbf{v^c(n)}$, one has $P^{\mathbf{x}(n)}(\mathbf{v^c(n)}) = P^{\mathbf{x}}(\mathbf{v^c(n)})$ which directly implies that :

$$P^{\mathbf{x}(n)}(\mathbf{v^c(n)}) = P^{\mathbf{x}(n+1)}(\mathbf{v^c(n)}) = P^{\mathbf{x}}(\mathbf{v^c(n)}) \tag{27}$$

We first observe that if the distribution $\mathbf{x}(n+1)$ is obtained from $\mathbf{x}(n)$ through a maximal transfer (MT), the donor's income is equal to $v_1^c(n)$ and therefore the recipient being the poorest agent in category 1, we necessarily have $x_1^1(n+1) < v_1^c(n)$. On the other hand, if distribution $\mathbf{x}(n+1)$ is obtained from $\mathbf{x}(n)$ through a FIP of type $(P2)$, the recipient has an income equal to $v_1^c(n) > x_1^1(n)$, so that $x_1^1(n) = x_1^1(n+1) < v_1^c(n)$. Hence, in either case, one has $x_1^1(n+1) < v_1^c(n)$. Now by definition of $v_1^c(n+1)$ as an infimum, identity (27) and the fact that $x_1^1(n+1) < v_1^c(n)$, we have $v_1^c(n+1) \leq v_1^c(n)$.

Now combining $P^{\mathbf{x}}(v^c(n)) = P^{\mathbf{x}(n+1)}(v^c(n))$ and $P^{\mathbf{x}}(v^c(n+1)) = P^{\mathbf{x}(n+1)}(v^c(n+1))$ on the one hand and Corollary 1 on the other, it follows that $v_h^c(n+1) \leq v_h^c(n)$ holds for all h as well.

5.2.9 Claim 3

We first observe that $\overline{p}^{\mathbf{x}(n)}(1, v_1^c(n))$ is weakly decreasing for $n \geq n_0$, where n_0 is the integer whose existence was established in Claim 1. Indeed $v_1^c(n)$ is weakly decreasing for $n \geq n_0$ and an agent in category 1 can be designated as the donor at step n only if the algorithm prescribes a maximal transfer and his/her income is equal to $v_1^c(n)$. This proves that the number of agents in category 1 of distribution $\mathbf{x}(n)$ whose income is weakly smaller than $v_1^c(n)$ necessarily weakly decreases as n increases.

Assume now that at some stage $n \geq n_0$ we are in case (P2). In that case, the receiving agent's income is equal to $v_1^c(n)$. Hence $\overline{p}^{\mathbf{x}(n+1)}(1, v_1^c(n+1)) < \overline{p}^{\mathbf{x}(n)}(v_1^c(n))$.

As a result, there can be at most $n(1)$ operations of type (P2) in the algorithm after step n_0.

5.2.10 Claim 4

Let n_1 be as in Claim 3. We need to establish that there can only be finitely many breaking transfers after stage n_1. Consider any $n \geq n_1$ and suppose that $\mathbf{x}(n+1)$ is obtained from $\mathbf{x}(n)$ through a breaking transfer of amount $\alpha > 0$ from agent $j_h \in \mathcal{N}(h)$ (with $h \geq 1$) to agent $1 \in \mathcal{N}(1)$. Let $r_+^h \in \mathcal{N}(1)$ and $r_-^h \in \mathcal{N}(h)$ be as in Definition 1: that is $x^1_{r^1_+}(n+1) = x^1_1(n) + \alpha$ and $x^h_{r^h_-}(n+1) = x^h_{j_h}(n) - \alpha$.

Let $\delta > 0$. By definition of a breaking transfer, there exists $\mathbf{v}(\delta) \in \mathcal{V}$ such that:

$$P^{\mathbf{x}(n+1)^\delta}(\mathbf{v}(\delta)) < P^{\mathbf{x}}(\mathbf{v}(\delta)),$$

where $\mathbf{x}(n+1)^\delta$ denotes the distribution that would be obtained if the transfer at time n was equal to $\alpha + \delta$ rather than α. By compactness of the set $\mathcal{V}$, we may assume without loss of generality that $\lim_{\delta \to 0} \mathbf{v}(\delta) = \mathbf{v}^* \in \mathcal{V}$. By continuity, we then have $P^{\mathbf{x}(n+1)}(\mathbf{v}^*) = P^{\mathbf{x}}(\mathbf{v}^*)$. Note that without loss of generality, we can assume that $v_1(\delta) \geq x^1_1(n) + \alpha$. This implies that $v^*_1 \geq x^1_1(n) + \alpha$.

We now show that $v^*_1 > x^1_1(n+1)$. By contradiction assume that $v^*_1 \leq x^1_1(n+1)$. Then, since $x^1_1(n+1) \leq x^1_1(n) + \alpha$, we necessarily have $v^*_1 = x^1_1(n) + \alpha = x^1_1(n+1)$, that is, the poorest agent in category 1 remains the poorest agent after receiving α at step n. Thus at next step (step $n+1$) the algorithm identifies him as the recipient again. Let $h' \geq 1$ and $j_{h'} \in \mathcal{N}(h')$ be the donor at next step $n+1$ and suppose he/she transfers $\delta > 0$. Since $v^*_{h'} \leq v^*_1 = x^1_1(n+1) < x^{h'}_{j_{h'}}(n+1)$, we have

$$P^{\mathbf{x}(n+2)}(\mathbf{v}(\delta)) = P^{\mathbf{x}(n+1)^\delta}(\mathbf{v}(\delta)) < P^{\mathbf{x}}(\mathbf{v}(\delta)),$$

a contradiction.

Since $v^*_1 > x^1_1(n+1)$, we must have $v^c_h(n+1) \leq v^*_h$ for any h by Corollary 1. By Claim 2, this implies that $v^c_h(m) \leq v^*_h$ for any $m \geq n+1$.
We now claim that for $m > n$, if the donor at step m is in category h, his/her income can not be equal to $x^h_{r^h_-}(n+1)$. Given the finiteness of the population, this will conclude the proof, because it will exclude the donor at stage n from donating again at a future step. Suppose, to the contrary, that there exists $m \geq n+1$ and $l_h \in \mathcal{N}(h)$ such that $v^c_1(m) = v^c_h(m) = x^h_{l_h}(m) = x^h_{r^h_-}(n+1)$, and that agent l_h transfers $\delta_0 > 0$ to agent $1 \in \mathcal{N}(1)$ at stage m. We then have

$$x^1_1(m) < x^h_{l_h}(m) = x^h_{r^h_-}(n+1) = v^c_1(m) \leq v^*_1.$$

Assume without loss of generality that δ_0 is small enough so that $x^1_1(m) \leq v_1(\delta_0) - \delta_0$. Then

$$P^{\mathbf{x}(m+1)}(\mathbf{v}(\delta_0)) - P^{\mathbf{x}(m)}(\mathbf{v}(\delta_0)) = P^{\mathbf{x}(n+1)^{\delta_0}}(\mathbf{v}(\delta_0)) - P^{\mathbf{x}(n+1)}(\mathbf{v}(\delta_0))$$

(both quantities are equal to $-\delta_0 + \max\{0, v_h(\delta_0) - (x^h_{r^h_-(n+1)} - \delta_0)\} - \max\{0, v_h(\delta_0) - x^h_{r^h_-(n+1)}\}$).
Since $P^{\mathbf{x}(m)}(\mathbf{v}(\delta_0)) \leq P^{\mathbf{x}(n+1)}(\mathbf{v}(\delta_0))$, we have

$$P^{\mathbf{x}(m+1)}(\mathbf{v}(\delta_0)) \leq P^{\mathbf{x}(n+1)^{\delta_0}}(\mathbf{v}(\delta_0)) < P^{\mathbf{x}}(v(\delta_0)),$$

a contradiction.

5.2.11 Claim 5

Let $n \geq n_2$. We proved already that the operation at stage n is necessarily a half transfer. Let h_0 be the category such that $v_1^c(n) = v_{h_0}^c(n) > v_{h_0+1}^c(n)$. Suppose by contradiction that the algorithm designates $i \in \mathcal{N}(1)$ to be the donor at stage n. By definition of a maximal transfer, it implies that $\forall h \in \{2, ..., h_0\}$, $\forall i \in \mathcal{N}(h)$, one must have $x_i^h(n) \neq v_1^c(n)$ because, otherwise, an agent in category $h > 1$ would be the donor. Notice also that, by the very definition of a maximal transfer, one must have $x_i^1 > v_1^c(n)$ for any $i \in \mathcal{N}(1)$ because assuming otherwise would make the transfer equalizing, which it can not be. Consequently the conditions of (P2) hold, which is a contradiction.

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