

Growth and Public Debt: What Are the Relevant Tradeoffs?

Kazuo Nishimura
Carine Nourry
Thomas Seegmuller
Alain Venditti

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Kazuo NISHIMURA[†], Carine NOURRY[‡], Thomas SEEGMULLER[‡] and
Alain VENDITTI^{‡,§}

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Abstract: *The interplay between growth and public debt is addressed considering a Barro-type [1] endogenous growth model where public spending is financed through taxes on income and public debt. Debt is assumed to be a fixed proportion of GDP which is used as a policy parameter by the government. We first show that when debt is a large enough proportion of GDP, two distinct BGPs may co-exist, one being indeterminate. Therefore, local and global indeterminacy may arise and self-fulfilling expectations appear as a crucial ingredient to understand the impact of debt on growth and on macroeconomic fluctuations. We then exhibit two types of important trade-off associated with self-fulfilling expectations. First, we show that the lowest BGP is always decreasing with respect to the ratio of debt/GDP while the highest one is increasing. As a result, depending on the BGP selected by agents' expectations, the relationship between debt and growth is not always negative. Second, we show that the highest BGP, which provides the highest welfare, is always locally indeterminate while the lowest is always locally determinate. Therefore, depending on the expectations of agents, when debt is increasing, large fluctuations associated to self-fulfilling beliefs may occur and be associated at the same time with welfare*

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[†]RIEB, Kobe University & KIER, Kyoto University

[‡]Aix-Marseille University (Aix-Marseille School of Economics)-CNRS-EHESS

[§]EDHEC Business School

losses if there is a coordination on the low steady-state. Finally, a simple calibration exercise allows to provide an understanding of the recent experiences of many OECD countries.

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1 Introduction

The last financial crisis has shed the light on the problem of large public debt in developed countries, in particular in Europe. Actually, in many advanced countries, debt levels have increased dramatically during the two last decades. The control of the growth rate of public spendings has become a major concern for economists and policy-makers while public deficits are reaching extremely high amounts. Indeed, a heavily indebted country may appear as fragile, for many reasons, among which solvency, or simply because it is unlikely to raise sufficient funds to deal with a large negative shock on its economy.

The Maastricht treaty introduced a rule on the maximal amount a country may contract, limiting the debt to 60% of GDP, but over the last decade this limit has been exceeded by almost all European countries. Most advanced countries, not only European ones, are now characterized by large amounts of public debt, low growth rates and large fluctuations of GDP. Given this fact, two types of questions become central:

1. What is the relationship between the level of debt and growth?
2. What is the relationship between macroeconomic stability and debt level?

Whereas the literature has focused recently on the first question, mainly through empirical analyses, little attention has been paid to the second.

This paper proposes to fill this gap and to study the relationships between debt, level of growth and macroeconomic fluctuations in the simple framework of a Barro-type [1] endogenous growth model. Our aim is precisely to discuss the effect of public debt on the endogenous growth rate and on the fluctuations of this growth rate. We focus on government intervention as a source of macroeconomic fluctuations when government spending is financed through taxes on income and public debt. Public spending is useful because it improves households' utility of consumption and production as externalities. In order to focus on the recent public policies, we assume that debt is a fixed proportion of GDP and that this proportion is used as a policy parameter by the government.

The size of public debt as a proportion of GDP and its impact on growth has been widely discussed and debated from an empirical point of view over

the recent years. The well-known paper of Reinhart and Rogoff [22] shows that a gross public debt exceeding 90% of nominal GDP on a sustained basis may have a significant negative impact on the growth rate.¹ While subject to a recent controversy,² this type of result has led the IMF after the starting of the global financial crisis to strongly advise European countries to decrease their debt. The main objective was to boost growth but also to stabilize the economies. However, the recent empirical literature shows that the conclusions are not so clear-cut. While some recent contributions find evidences for a negative relationship between average debt-to-GDP ratios and long-run growth performance (Woo and Kumar [27]), others find several thresholds for the debt-to-GDP ratio and the size of the impact on growth may be small (Baum *et al.* [2]). More generally, the relationship between debt and growth appears to be based on complex non-linear effects and to be heterogeneous across countries.³ In particular, Minea and Parent [13] show that for debt-to-GDP ratios above 115%, the correlation between growth and debt becomes positive.

The impact of debt on the stability of economies has also been discussed. For instance, Sutherland *et al.* [25] argue that the level of government debt has a significant impact on business cycle characteristics. They identify the characteristics of a “low debt” business cycle and a “high debt” business cycle aggregating the countries according to their level of debt. In countries with high debt, the cycle is more pronounced, with phases of expansions longer and larger and recessions also more pronounced. Such differences usually rely on the “vulnerability” of high public debt economies. Government then have less latitude to run the appropriate fiscal policy in case of negative shocks. As explained in Panizza and Presbitero [20], beside all these empirical studies, a precise theoretical analysis of the existence of non-monotonicity or threshold effects in the relationship between public debt, economic growth and aggregate fluctuations is not yet available in the literature. Our paper provides a first step

¹See also Reinhart *et al.* [21] and [23].

²Herndon *et al.* [12] have indeed identified a mistake in the methodology of Reinhart and Rogoff [22]. They show that when properly calculated, the average real GDP growth rate for countries carrying a public-debt-to-GDP ratio of over 90% is significantly larger than the level evaluated by Reinhart and Rogoff.

³See for instance Eberhart and Presbitero [7] and the survey of Panizza and Presbitero [20].

in that direction focusing on the role of global indeterminacy and self-fulfilling expectations.

While not directly concerned with the impact of debt on growth and macroeconomic instability, Collard *et al.* [4]-[5] provide an interesting focus with respect to our research agenda. They analyse the determinants of government debt under the twin assumptions that governments have limited horizons and default only when government income falls short of debt service requirements. They derive a government's maximum sustainable debt ratio, that strongly varies across countries. The difference between actual and maximum sustainable debt ratios then creates a "margin of safety" that allows governments to increase debt if necessary with little corresponding increase in default risk. In light of these results, when we focus on the recent experiences of Spain and Italy who have faced increasing interest rates generated by speculative attacks against their sovereign debt, we clearly see that an objective "margin of safety", based on countries' fundamentals, may not be enough to prevent a large increase of the perceived default risk by investors. This remark suggests a potential for expectation-driven effects. This is precisely the main focus of our paper.

Our framework allows to contribute to these different debates, i.e. the interplay between the ratio of debt over GDP and both growth and expectation-driven macroeconomic fluctuations. On the one hand, uniqueness of the balanced growth path (BGP) is obtained in two different circumstances: (*i*) when the tax rate is weak and debt is a small proportion of GDP, or (*ii*) when the tax rate is strong and debt is a large proportion of GDP. In case (*i*), the long-run growth rate is decreasing in the share of debt as a proportion of GDP,⁴ while it is increasing in case (*ii*). There is also a difference in terms of the local stability properties of the BGP. In case (*i*), the BGP can be locally determinate or indeterminate depending on the level of debt. There exists indeed a critical level below which local determinacy arises and expectations do not affect the dynamics. On the contrary, if the ratio of debt-to-GDP is larger than this level, the BGP becomes locally indeterminate leading to the existence of expectation-driven fluctuations. Under a sufficiently large level of debt, expect-

⁴When uniqueness of BGP holds, this is a standard result. See for instance Saint-Paul [24].

tations on future growth rates are self-fulfilling. In case (ii), this mechanism is amplified and local indeterminacy of BGP always holds.

On the other hand, when the fiscal pressure is strong enough and the share of debt as proportion of GDP admits intermediary but sufficiently large values, two distinct BGPs may co-exist. Global indeterminacy may arise and self-fulfilling expectations appear as a crucial ingredient to understand the impact of debt on growth. We also exhibit two types of important trade-off associated with self-fulfilling expectations. First, we show that the lowest BGP is always decreasing in the ratio of debt over GDP, while the highest one is increasing. As suggested by the recent literature on non-linear relationships between debt and growth, depending on the BGP selected by agents' expectations, the relationship between debt and growth is not necessarily negative. This result contributes to the debate suggested by Panizza and Presbitero [20] and may in particular match the conclusion of Minea and Parent [13].

Second, we clearly exhibit a trade-off between welfare and macroeconomic fluctuations: with a large enough debt-to-GDP ratio, the highest BGP, which provides the highest welfare, is always locally indeterminate while the lowest is always locally determinate. Our results then show non-trivial effects of debt on growth and macroeconomic fluctuations. Depending on the expectations of agents, when debt is increasing large, fluctuations associated to self-fulfilling beliefs may occur and be associated at the same time with welfare losses if there is a coordination on the low steady-state. These results clearly show that, above a threshold, the size of debt has a dramatic impact on the dynamic properties of equilibria.

We also discuss the implications of our conclusions for the main OECD countries considering numerical illustrations based on realistic calibrations for the size of debt, the growth rate and the main fundamentals. We show that the OECD countries can be split into two subsets: a first one with countries characterized by global indeterminacy and two long run growth rates, the lowest being associated to a strong recession, and a second one with countries characterized by a unique long run growth rate which is locally indeterminate. In particular, except Greece and Portugal who are in the second set, all European countries belong to the first set and may be all subject to strong macroeconomic fluctuations based on pessimistic expectations. Our analysis

then provides a basis for understanding the recent experiences of many OECD countries relating macroeconomic instability to self-fulfilling expectations.

The rest of the paper is organized as follows. In the next Section, we present the model and define the intertemporal equilibrium. In Section 3, we discuss the effect of debt in the long run analyzing the existence and multiplicity of steady-states, comparative statics and welfare. Section 4 provides an analysis of the effect of debt in the short run focusing on sunspot fluctuations and global indeterminacy. Section 5 presents numerical illustrations based on the main OECD countries. Section 6 concludes the paper and a final Appendix contains all the proofs.

1.1 Related literature

As explained previously, there are few theoretical contributions that study the link between debt, growth and fluctuations in endogenous growth models. On the one hand, Futagami *et al.* [8] consider a government which focuses on a target for its policy based on the level of debt to the size of the economy, namely the capital stock. They show that multiple BGPs may arise together with local indeterminacy and expectation-driven fluctuations.⁵ However, Minea and Villieu [15] prove that these conclusions are not robust to the consideration of a debt-to-GDP target for the government. They show indeed that in this case, there exists a unique BGP which is locally determinate. In light of our results, their conclusion seems to be quite sensitive to the specification of consumers' preferences. On the other hand, Greiner and Semmler [9, 10] examine how government's financing method affects economies. They show that the link between public deficit and long run growth is not clear cut. However, they do not particularly focus on the impact of the ratio of debt-to-GDP.

Our paper can also be seen as an extension of models with public spending but without public debt, like Cazzavillan [3]. He shows that a strong public good externality in preferences, such that utility has increasing returns, is required to have indeterminacy and endogenous fluctuations. We assume

⁵Minea and Villieu [14] prove the possible existence of multiple BGPs and a form of global (but not local) indeterminacy considering instead a deficit-to-GDP target for the government.

instead that the utility function is characterized by a small public good externality and, thus, decreasing returns. This confirms that the main channel through which self-fulfilling expectations arise in our framework is the level of debt over GDP.

Finally, note that, in two recent papers, we have already studied the consequences of the constant level of debt over GDP on macroeconomic stability, but without endogenous growth. In Nishimura *et al.* [18], we consider a stationary Ramsey economy with heterogeneous agents and without a public good externality in production. We assume that the tax rate is constant while public spending is endogenously adjusted. We show that public debt can be destabilizing or stabilizing, i.e. generating or ruling out damped or persistent macroeconomic fluctuations (period-two cycles), depending on whether the public spending externality and the elasticity of capital labor substitution are low or large. However, indeterminacy can never occur. In Nishimura *et al.* [19], we also consider a stationary Ramsey economy but with homogeneous agents and without any externality in the fundamentals. We assume instead a constant public spending and thus, through the budget constraint of the government, an endogenous adjustment of the tax rate. We then show that a large enough debt can be destabilizing through the occurrence of self-fulfilling expectations and sunspot-driven fluctuations.

2 The model

We consider a discrete time economy ($t = 0, 1, \dots, \infty$), with three types of agents, a constant population of identical infinitely lived households, a large number of identical competitive firms and a government.

2.1 Households

Each consumer is endowed with one unit of labor and an initial stock of private physical capital which depreciates at a constant rate $\delta \in (0, 1)$. We assume that the total population is normalized to one. The representative agent has separable preferences over time which depend on consumption c_t and public spending G_t that is assumed to act as an externality on utility for consumption.

To be consistent with endogenous growth, we assume that the intertemporal utility function is given by

$$\sum_{t=0}^{+\infty} \beta^t \frac{c_t^{1-\theta}}{1-\theta} G_t^\eta \quad (1)$$

with $\beta \in (0, 1)$ the discount factor, $\theta \in (0, 1)$ the inverse of the elasticity of intertemporal substitution in consumption (EIS), and $\eta \geq 0$ the size of the public good externality such that $1 > \theta - \eta \geq 0$. These restrictions allow to consider low enough public spending externalities in utility and some EIS larger than one. Indeed, the most recent estimates by Gruber [11] (see also Mulligan [16], Vissing-Jorgensen and Attanasio [26]) exhibit values around 2. This also means that the degree of homogeneity of the instantaneous utility function $1 - \theta + \eta$ is lower than one, i.e. we do not allow increasing returns due to public spending externalities.

The representative household derives income from wage, capital and government bonds that allow to finance public debt. Denote r_t the real interest rate on physical capital, \bar{r}_t the interest rate on government bonds and w_t the real wage. The representative household pays taxes on labor income, capital income and on the remuneration of bonds' holding, at a constant rate $\tau \in (0, 1)$. He then maximizes (1) facing the budget constraint:

$$c_t + k_{t+1} + b_{t+1} = (1 - \tau)[r_t k_t + w_t] + [1 + (1 - \tau)\bar{r}_t] b_t + (1 - \delta)k_t \quad (2)$$

Utility maximization gives:

$$\left(\frac{c_{t+1}}{c_t}\right)^\theta \left(\frac{G_{t+1}}{G_t}\right)^{-\eta} = \beta R_{t+1} = \beta [1 + (1 - \tau)\bar{r}_{t+1}] \quad (3)$$

with the gross interest rate $R_{t+1} \equiv (1 - \tau)r_{t+1} + 1 - \delta$ and the transversality conditions

$$\lim_{t \rightarrow +\infty} \beta^t c_t^{-\theta} G_t^\eta k_{t+1} = 0 \quad \text{and} \quad \lim_{t \rightarrow +\infty} \beta^t c_t^{-\theta} G_t^\eta b_{t+1} = 0 \quad (4)$$

Obviously, we get at the equilibrium the equality $R_{t+1} = [1 + (1 - \tau)\bar{r}_{t+1}]$ since physical capital k_{t+1} and government bonds b_{t+1} are perfectly substitutable saving assets.

2.2 Firms

A representative firm produces the final good y_t using a Cobb-Douglas technology with constant returns at the private level but which is also affected by

a public good externality, $y_t = Ak_t^s(L_t G_t)^{1-s}$, where $s \in (0, 1/2)$ is the share of capital income in GDP and $A > 0$ is a productivity parameter. As population is normalized to one, $L_t = 1$, we get a standard Barro-type [1] formulation such that $y_t = Ak_t^s G_t^{1-s}$. Profit maximization then gives:

$$r_t = As \left(\frac{G_t}{k_t} \right)^{1-s} \quad \text{and} \quad w_t = A(1-s)k_t \left(\frac{G_t}{k_t} \right)^{1-s} \quad (5)$$

2.3 Government

Public spending G_t is financed by total income taxation and debt, through the following budget constraint:

$$G_t + (1 + \bar{r}_t)b_t = \tau(r_t k_t + w_t + \bar{r}_t b_t) + b_{t+1} \quad (6)$$

where $\tau \in (0, 1)$ is the constant proportional tax rate on households' total income. Total public expenditure is the sum of public spendings G_t and the reimbursement of debt contracted the previous period $(1 + \bar{r}_t)b_t$, and b_{t+1} is the new issue of debt.

In accordance with sustainability constraints like the one imposed by the Maastricht treaty, we assume that debt cannot exceed a fixed proportion of GDP, namely $b_t \leq \alpha y_t$. In this paper, we consider equilibria with positive debt. Along a balanced growth path with positive debt, both b_t and y_t grow at the same rate. Accordingly, we will focus in the following on equilibria with $b_t = \alpha y_t$ along which α can be interpreted as a policy parameter under the control of the government (see de la Croix and Michel [6]). The limit case without debt is of course obtained when $\alpha = 0$.

2.4 Intertemporal equilibrium

Let us consider equations (2), (3), (5) and (6) with $b_t = \alpha y_t$. Using (5) and (6), equation (2) becomes:

$$c_t + k_{t+1} + G_t = y_t + (1 - \delta)k_t \quad (7)$$

Let us also denote $x_t \equiv G_t/k_t$, $d_t \equiv c_t/k_t$, and $\gamma_t \equiv k_{t+1}/k_t$ the growth factor. We derive $y_t = Ak_t x_t^{1-s}$. Therefore, equations (3), (6) and (7) can be written as follows:

$$\left(\frac{d_{t+1}}{d_t} \right)^\theta \left(\frac{x_{t+1}}{x_t} \right)^{-\eta} \gamma_t^{\theta-\eta} = \beta [(1 - \tau)Asx_{t+1}^{1-s} + 1 - \delta] \quad (8)$$

$$x_t + \alpha Ax_t^{1-s} [(1-\tau)Asx_t^{1-s} + 1 - \delta] = \tau Ax_t^{1-s} + \alpha Ax_{t+1}^{1-s} \gamma_t \quad (9)$$

$$d_t + \gamma_t + x_t = Ax_t^{1-s} + 1 - \delta \quad (10)$$

From (10) we derive some expressions for d_t and d_{t+1} . Substituting these expressions into (8) allows finally to define an intertemporal equilibrium as a pair $\{x_t, \gamma_t\}_{t \geq 0}$ solution of the following system of two difference equations:

$$\beta x_{t+1}^\eta \frac{(1-\tau)Asx_{t+1}^{1-s} + 1 - \delta}{[Ax_{t+1}^{1-s} - x_{t+1} + 1 - \delta - \gamma_{t+1}]^\theta} = \frac{\gamma_t^{\theta-\eta} x_t^\eta}{[Ax_t^{1-s} - x_t + 1 - \delta - \gamma_t]^\theta} \quad (11)$$

$$\alpha Ax_{t+1}^{1-s} \gamma_t = x_t - \tau Ax_t^{1-s} + \alpha Ax_t^{1-s} [(1-\tau)Asx_t^{1-s} + 1 - \delta]$$

and the transversality conditions (4) with k_0 given.⁶ Note that there is no predetermined variable in this dynamical system. Indeed, γ_0 depends on k_1 , and from (9), x_0 depends on x_1 . This means that x_t and γ_t are both forward variables. Note also that the initial consumption c_0 has to be chosen in accordance with any initial choice of γ_0 and x_0 since, using (10), c_0 is given by $c_0 = k_0 [Ax_0^{1-s} - x_0 + (1 - \delta) - \gamma_0]$.

Note finally that when there is no debt, i.e. $\alpha = 0$, the second equation of the system (11) leads to a constant value for x and the dynamical system becomes one-dimensional with a difference equation characterizing the dynamics of the growth factor γ_t .

3 Effect of debt in the long run: Steady-states, comparative statics and welfare

A (non trivial) steady-state equilibrium corresponds to a stationary sequence $\{x_t, \gamma_t\} = \{x^*, \gamma^*\} \in \mathbb{R}_{++}^2$ for all t which satisfies the system (11). The stationary value of the growth factor γ^* allows to define a *balanced-growth path* (BGP) along which all the variables c_t , k_t and G_t grow at the common constant rate $g^* = \gamma^* - 1$.

Along a steady-state equilibrium, the first equation of system (11) gives:

$$x(\gamma) = \left(\frac{\gamma^{\theta-\eta} - \beta(1-\delta)}{\beta As(1-\tau)} \right)^{\frac{1}{1-s}} \quad (12)$$

⁶Of course, there is usually an initial condition on public debt. One way to conciliate it with the fiscal rule $b_t = \alpha y_t$ and the government budget (6) from $t = 0$ is to consider an initial condition $b_{-1} \geq 0$ satisfying $G_{-1} + b_{-1}(1 + \bar{r}_{-1}) = \tau(y_{-1} + \bar{r}_{-1}b_{-1}) + \alpha y_0$.

Because we introduce endogenous growth explained by productive public spendings (Barro [1]), this equation establishes a positive link between G/k and the growth factor γ . Indeed, a larger level of public spendings per unit of capital means a larger interest rate, that fosters growth. Substituting this expression into the second equation implies that γ^* is a solution of the following equation:

$$\Delta(\gamma) = \Omega(\gamma) \quad (13)$$

with

$$\Delta(\gamma) \equiv \frac{1}{A} \left(\frac{\gamma^{\theta-\eta} - \beta(1-\delta)}{\beta A s (1-\tau)} \right)^{\frac{s}{1-s}} + \alpha \frac{\gamma^{\theta-\eta}}{\beta}, \text{ and } \Omega(\gamma) \equiv \tau + \alpha \gamma \quad (14)$$

Obviously, the expression of $x(\gamma)$ implies that any admissible solution must satisfy $\gamma > [\beta(1-\delta)]^{\frac{1}{\theta-\eta}} \equiv \gamma_{inf}$. This means that any BGP is characterized by strictly positive ratios of public spendings over capital and interest rate $r = sAx^{1-s}$. Moreover, as along a BGP all the variables c_t , k_t and G_t grow at the common constant rate, the transversality conditions (4) evaluated along such a BGP hold if $\gamma < \beta^{\frac{1}{\theta-\eta-1}} \equiv \gamma_{sup}$, with $\gamma_{sup} > \gamma_{inf}$. Note that the condition $\gamma < \gamma_{sup}$ is also equivalent to $\gamma^{\theta-\eta}/\beta = R > \gamma$ which means that the interest factor is larger than the growth factor.

In the following, we restrict our attention to tax rates that are not unrealistically large:

Assumption 1. $\tau < 1 - s$.

Since we assume that $s < 1/2$, this assumption is always fulfilled for tax rates lower or equal than one half. According to our numerical illustration (see Table 1 in Section 5), this corresponds to the empirical evidence found at least in OECD countries.

Any equilibrium $\gamma \in (\gamma_{inf}, \gamma_{sup})$ has also to satisfy $d(\gamma) = Ax^{1-s} - x + 1 - \delta - \gamma > 0$. This inequality is ensured by the following Lemma:

Lemma 1. *Under Assumption 1, there exists $\hat{A} > 0$ such that if $A > \hat{A}$, we have $d(\gamma) > 0$ for all $\gamma \in (\gamma_{inf}, \gamma_{sup})$.*

Proof. See Appendix 7.1.

Based on this Lemma, the existence and the number of stationary solutions such that $\gamma \in (\gamma_{inf}, \gamma_{sup})$ are derived using only equation (13). As a benchmark case, we start by considering the configuration without debt:

Proposition 1. *Under Assumption 1, let $A > \hat{A}$ and $\alpha = 0$. Then there exists $\hat{\tau} \in (0, 1 - s)$ such that there is a unique steady-state $\gamma^* \in (\gamma_{inf}, \gamma_{sup})$ if and only if $\tau \in (0, \hat{\tau})$.*

Proof. See Appendix 7.2.

This case corresponds to the Barro [1] model. Let us rewrite the budget constraint of the government (9) evaluated at the steady state as follows:

$$\frac{G}{y} + \alpha R = \tau + \alpha\gamma \Leftrightarrow \frac{x^s}{A} + \alpha R = \tau + \alpha\gamma \quad (15)$$

with $R = (1 - \tau)sAx^{1-s} + 1 - \delta$. We also get the following equivalent expression for the capital accumulation equation (10):

$$d + \gamma - 1 + \delta = Ax^{1-s} \left[1 - \frac{x^s}{A} \right] \quad (16)$$

When $\alpha = 0$, we then get $x = (A\tau)^{1/s}$ and $d + \gamma - (1 - \delta) = A^{1/s}\tau^{(1-s)/s}(1 - \tau)$. As it is well-known, a larger tax rate has two effects. On the one hand, it reduces the disposable income, meaning that there are less incentives to invest. On the other hand, it raises the tax base. Under Assumption 1, the second effect dominates. As a result, if τ is large, i.e. close to the labor share $1 - s$, the tax revenue and thus the expression $\tau^{(1-s)/s}(1 - \tau)$ are too large,⁷ and there is no balanced budget satisfying positive consumption and/or the transversality condition. On the contrary, when the tax rate is low enough, i.e. lower than $\hat{\tau}$, uniqueness of the steady state comes from the fact that in the Barro [1] model, there is a positive link between G/k or G/y and γ because public spendings are productive. If γ is low, the ratio G/y is quite close to 0, and there is an excess of the tax revenue, while if γ is large enough, there is a surplus of public spendings. The budget is then balanced between these two configurations and the associated consumption is positive.

When $\alpha > 0$, the uniqueness of the steady-state is not always ensured or may occur under different fiscal pressures. The results are summarized in the following Proposition:

Proposition 2. *Under Assumption 1, let $A > \hat{A}$ and $\alpha > 0$. Then there exist $\hat{\tau} \in (0, 1 - s)$, $\hat{\alpha} > \underline{\alpha} > 0$ and $\tilde{\Theta} \in (0, 1)$ such that the following results hold:*

⁷Note indeed that $\tau^{(1-s)/s}(1 - \tau)$ is maximized when $\tau = 1 - s$.

1. There is a unique steady-state $\gamma^* \in (\gamma_{inf}, \gamma_{sup})$ if one of the following conditions is satisfied:

(a) $\alpha < \hat{\alpha}$ and $\tau \in (0, \hat{\tau})$;

(b) $\alpha > \hat{\alpha}$ and $\tau \in (\hat{\tau}, 1 - s)$;

2. There are two steady-states $\gamma_1^*, \gamma_2^* \in (\gamma_{inf}, \gamma_{sup})$, with $\gamma_1^* < \gamma_2^*$, if $\alpha \in (\underline{\alpha}, \hat{\alpha})$, $\tau \in (\hat{\tau}, 1 - s)$ and $\theta - \eta \in (0, \tilde{\Theta})$.

Proof. See Appendix 7.3.

The existence of a steady-state is obtained in two different configurations: i) when the share of debt over GDP and the tax rate are both low enough, or ii) when they are both large enough. There is therefore a form of positive relationship between debt and tax for the existence of endogenous growth. If the tax rate is large enough, there may also exist two admissible steady-states as long as the share α of public debt over GDP takes intermediate values and the utility function is close enough to a linear one ($\theta - \eta$ weak enough). Contrary to the conclusions of Minea and Villieu [15], we show here that considering the debt-to-GDP ratio as a policy instrument does not necessarily preclude the existence of multiple BGPs. Note however that if we assume as in Minea and Villieu [15] that $\theta = 1$ and $\eta = 0$, then multiplicity is ruled out.

Recall that the transversality condition is equivalent to $R = \gamma^{\theta-\eta}/\beta > \gamma$. The interpretation of uniqueness of the steady state when both τ and α are low enough (Case 1 (a) of Proposition 2) is quite similar to the case without debt, after Proposition 1. The main difference with the case $\alpha = 0$ is that now taxation is also used to finance the difference between debt reimbursement $R_t b_t$ and new debt emission b_{t+1} which becomes $(R - \gamma)b$ at the steady-state, but this is not a dominant effect since debt $b = \alpha y$ is low.

In Case 1 (b) of Proposition 2, we still have uniqueness of the steady state, but the story is different. Let us rewrite equation (15) as follows

$$\alpha(R - \gamma) + \frac{x^s}{A} = \tau \quad (17)$$

The difference between debt reimbursement and debt emission is measured by $R - \gamma > 0$. When γ is close to its upper bound, this difference is close to zero. Since the tax rate is close to $1 - s$, there is an excess of tax revenue and debt emission with respect to public spendings and reimbursement of debt leading to $\alpha(R - \gamma) + x^s/A < \tau$. On the contrary, when γ is close to its lower bound,

the debt reimbursement $R - \gamma$ is quite large. Despite the fact that public spendings are lower, there is a surplus of spending and debt reimbursement implying $\alpha(R - \gamma) + x^s/A > \tau$. Uniqueness of the steady-state is then ensured.

The coexistence of two steady states (Case 2 of Proposition 2) is explained by the same story except that now the size of debt is lower as $\alpha \in (\underline{\alpha}, \hat{\alpha})$. When γ increases from γ_{inf} to γ_{sup} , public spendings raise while the difference between debt reimbursement and debt emission reduces. In other words, these two effects go in opposite directions following a raise of γ . Since the ratio of debt-to-GDP has intermediate values and the tax rate is large enough, there is an excess of funds to finance public spendings and debt reimbursement when the growth factor is either close to γ_{inf} (public spending over GDP low enough) or close to γ_{sup} (difference between debt reimbursement and debt emission low enough). In both cases, we then have $\alpha(R - \gamma) + x^s/A > \tau$ and multiple steady-states may arise.

The following Figure summarizes the conclusions of Proposition 2:

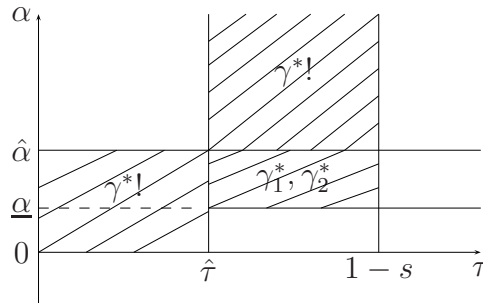


Figure 1: Uniqueness and multiplicity of steady-states.

In the following Corollary, we examine how the different types of steady-states vary according to variations of the debt-output ratio. However, discussing the impact of a policy parameter on the growth factor is not enough in an optimal intertemporal dynamic model. We need also to focus on the welfare that can be easily computed along the long-run balanced growth path characterized by the stationary values of the growth factor γ , consumption d and government spending x . We have indeed

$$W(\gamma) = \frac{k_0^{1-(\theta-\eta)} d(\gamma)^{1-\theta} x(\gamma)^\eta}{1-\theta} \frac{1}{1-\beta\gamma^{1-(\theta-\eta)}} \quad (18)$$

We get the following result:

Corollary 1. *Under Assumption 1, let $A > \hat{A}$ and $\alpha > 0$. The following results hold:*

1. If $\alpha < \hat{\alpha}$ and $\tau \in (0, \hat{\tau})$, $d\gamma^*/d\alpha < 0$;
2. If $\alpha > \hat{\alpha}$ and $\tau \in (\hat{\tau}, 1 - s)$, $d\gamma^*/d\alpha > 0$;
3. If $\alpha \in (\underline{\alpha}, \hat{\alpha})$, $\tau \in (\hat{\tau}, 1 - s)$ and $\theta - \eta \in (0, \tilde{\Theta})$, $d\gamma_1^*/d\alpha < 0$ and $d\gamma_2^*/d\alpha > 0$.

Moreover, there exist $\hat{\beta} \in (0, 1)$ and $\hat{\delta} \in (0, 1)$ such that if $\beta \in (\hat{\beta}, 1)$ and $\delta \in (0, \hat{\delta})$, then $W'(\gamma) > 0$.

Proof. See Appendix 7.4.

Focusing on Case 3 of this Corollary, the following figure shows how the multiplicity of steady-states is affected by a variation of α . The curve $\Delta(\gamma) - \Omega(\gamma)$ is going up when α increases. This illustrates the comparative statics for the lower and higher BGPs γ_1^* and γ_2^* .

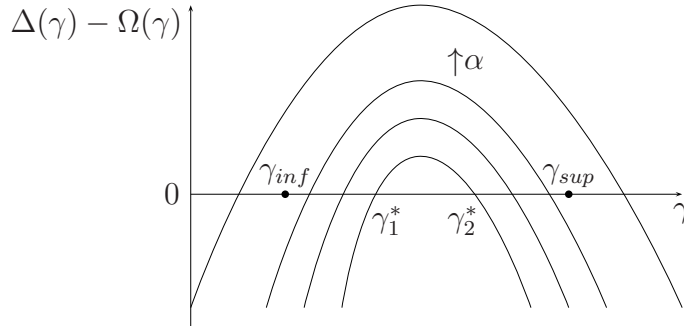


Figure 2: Multiplicity of steady-states and comparative statics.

We now briefly explain how we can understand the effect of α on these two steady states. As we have seen previously, the coexistence of the two steady states can be explained focusing on the intertemporal budget constraint of the government. There is an excess of the tax revenue and debt emission with respect to public spendings and debt reimbursement when γ is either lower than γ_1^* or larger than γ_2^* , whereas the opposite is true when γ is between γ_1^* and γ_2^* . To understand the effect of α , recall that any BGP is characterized by $R > \gamma$ and the difference between debt reimbursement and debt emission per unit of GDP is measured by $\alpha(R - \gamma)$. We immediately conclude that following any increase of the ratio of debt-to-GDP α , the regions with a surplus of public funds shrink, meaning that γ_1^* goes down, while γ_2^* goes up.

The comparative statics results of Corollary 1 provide quite different conclusions than the empirical results of Reinhart and Rogoff [22]. The relationship between debt and the growth rate indeed strongly depends on both the

level of the ratio of debt-to-GDP and the level of the tax rate. First, the growth rate is always a decreasing function of the share of debt as a proportion of GDP when this share is small enough. Second, for intermediary values of α , multiple steady-states arise and the lower growth factor is negatively affected by α while the higher one is positively affected. And third, above a threshold level $\hat{\alpha}$, and when the tax rate is large enough, the unique long-run growth rate becomes an increasing function of α . These results suggest a complex relationship between debt and growth in line with the recent literature surveyed by Panizza and Presbitero [20]. Moreover, they provide a theoretical justification for the positive correlation exhibited by Minea and Parent [13] with large enough debt-to-GDP ratios.

Our results then show that the negative impact of debt on growth as exhibited by Reinhart and Rogoff [22] and some of the recent literature has to be taken with caution. Above the threshold value $\underline{\alpha}$, when multiplicity holds, they only get part of the story as they miss the second possible equilibrium growth rate, which is an increasing function of α , and thus the possible strong impact of agents' expectations that select the long run equilibrium. As a result, they miss the fact that when the tax rate and the share of debt over GDP are high enough, a quite large amount of public good can be financed and increasing debt allows to boost even more the production process through the productive externality. The growth factor increases as a result. The fact that this positive impact is associated to a high enough tax rate can be linked to the concept of "debt tolerance" formulated by Reinhart *et al.* [23] that characterizes countries with strong fiscal structures, i.e. able to raise a large amount of taxes (see also Collard *et al.* [4]-[5]).

Corollary 1 also shows that along a long-run balanced growth path, welfare is an increasing function of the stationary growth factor γ . This holds whatever the degree of the public good externality in the utility. We then exhibit a first important trade-off associated with self-fulfilling expectations. We show indeed that depending on the long-run equilibrium selected by agents' expectations, increasing the share of debt as a proportion of GDP may have a positive or a negative impact on both the growth rate and welfare.

4 Effect of debt in the short run: Sunspot fluctuations and global indeterminacy

Focusing on the stability properties of the steady-states, we analyse now whether the ratio of debt-to-GDP is destabilizing, through the occurrence of local indeterminacy and sunspot equilibria. As we have seen, two steady-states may coexist. Hence, in order to simplify the local stability analysis and to focus on a precise steady-state, we first provide a normalization procedure of a BGP.

4.1 A normalized steady-state

In order to get long run positive growth, we want to focus on the existence of a steady-state value of γ which is larger than 1. To simplify this analysis, we use the constant A to get the existence of a normalized steady-state $\gamma^* > 1$. This procedure is useful to ensure that γ^* remains invariant with respect to parameters' changes, in particular the share α of debt over GDP. This invariance will allow us to provide a bifurcation analysis in order to study how the dynamical properties of the steady-state are affected when α is modified. As shown in the following Proposition, this property is obtained by choosing adequately the value of A that will adjust accordingly to keep γ^* constant.

Proposition 3. *Under Assumption 1, let $\theta - \eta < \beta$. Then there exists $\underline{\gamma} \in [1, \gamma_{sup})$ such that any given value $\gamma^* \in (\underline{\gamma}, \gamma_{sup})$ is an admissible invariant solution of equation (13) if and only if $\alpha \in (0, \alpha_{Max})$ and $A = A^*$ with*

$$\alpha_{Max} = \frac{\tau}{\gamma^* \left(\frac{\gamma^{*\theta} - \eta - 1}{\beta} - 1 \right)} (> \hat{\alpha}) \quad \text{and} \quad A^* = \frac{\left[\frac{\gamma^{*\theta} - \eta - \beta(1-\delta)}{\beta s(1-\tau)} \right]^s}{\left[\tau - \alpha \gamma^* \left(\frac{\gamma^{*\theta} - \eta - 1}{\beta} - 1 \right) \right]^{1-s}} (> \hat{A}) \quad (19)$$

Proof. See Appendix 7.5.

We may then easily characterize the stability properties of the normalized steady-state by linearizing the dynamic system (11) around the steady-state (x^*, γ^*) with x^* as given by (12).

Lemma 2. *Under Assumption 1, let $\theta - \eta < \beta$, $A = A^*$, $\alpha \in (0, \alpha_{Max})$ and $\gamma^* \in (\underline{\gamma}, \gamma_{sup})$. The characteristic polynomial is given by $P(\lambda) \equiv \lambda^2 - T\lambda + D = 0$, where:*

$$\begin{aligned}
D &= \left(1 + \frac{(\theta-\eta)d^*(\alpha)}{\theta\gamma^*}\right) \frac{B_2(\alpha)}{\alpha\gamma^*(1-s)} + \frac{B_1(\alpha)d^*(\alpha)}{\theta\gamma^*(1-s)} \equiv D(\alpha) \\
T &= 1 + \frac{(\theta-\eta)d^*(\alpha)}{\theta\gamma^*} + \frac{[B_1(\alpha)+B_3]d^*(\alpha)}{\theta\gamma^*(1-s)} + \frac{B_2(\alpha)}{\alpha\gamma^*(1-s)} \equiv T(\alpha)
\end{aligned} \tag{20}$$

with

$$\begin{aligned}
d^*(\alpha) &= \left(\frac{\gamma^{*\theta-\eta-\beta(1-\delta)}}{\beta s(1-\tau)}\right) \left[1 - \tau + \alpha\gamma^* \left(\frac{\gamma^{*\theta-\eta-1}}{\beta} - 1\right)\right] + 1 - \delta - \gamma^* \geq 0 \\
B_1(\alpha) &= \eta - \frac{\theta}{d^*(\alpha)} \left(\frac{\gamma^{*\theta-\eta-\beta(1-\delta)}}{\beta s(1-\tau)}\right) \left[1 - s - \tau + \alpha\gamma^* \left(\frac{\gamma^{*\theta-\eta-1}}{\beta} - 1\right)\right] \\
B_2(\alpha) &= s\tau + \alpha\gamma^* \left(1 - s\frac{\gamma^{*\theta-\eta-1}}{\beta}\right) + \alpha(1-s)\frac{\gamma^{*\theta-\eta-\beta(1-\delta)}}{\beta} > 0 \\
B_3 &= (1-s)\frac{\gamma^{*\theta-\eta-\beta(1-\delta)}}{\gamma^{*\theta-\eta}} > 0
\end{aligned}$$

Proof. See Appendix 7.6.

As explained at the end of Section 2, our dynamical system (11) has no predetermined variable. Therefore, local determinacy is obtained if the steady-state is totally unstable (a source). In this case, there is no transitional dynamics and the equilibrium jumps on the steady-state from date 0. On the contrary, when the steady-state is either a saddle-point or totally stable (a sink), there is local indeterminacy with the existence of a continuum of equilibrium paths. As a result, sunspot fluctuations also occur.

We show now that the stability properties of the normalized steady-state crucially depend on the value of the ratio of debt-to-GDP.

Proposition 4. *Under Assumption 1, let $\theta - \eta < \beta$, $A = A^*$ and $\gamma^* \in (\underline{\gamma}, \gamma_{sup})$. Then there exist $\bar{\delta} \in (0, 1)$, $\underline{\beta} \in (0, 1)$, $\bar{\alpha} \in (0, \alpha_{Max})$ and $\bar{\Theta} \in (0, \beta)$ such that if $\delta \in (0, \bar{\delta})$, $\beta \in (\underline{\beta}, 1)$ and $\theta - \eta \in (0, \bar{\Theta})$, the normalized steady-state γ^* is:*

- i) a source, i.e. locally determinate, when $\alpha \in (0, \bar{\alpha})$,*
- ii) a saddle-point, i.e. locally indeterminate when $\alpha \in (\bar{\alpha}, \alpha_{Max})$.*

Proof. See Appendix 7.7.

When the share of debt over GDP is large enough, the local indeterminacy of the steady-state implies that there exist expectation-driven fluctuations. While Corollary 1 showed that a higher level of debt over GDP may foster the long run growth rate, this result exhibits a negative impact based on the fact that, over a precise threshold, public debt has a destabilizing role on

the economy and leads to the existence of fluctuations based on self-fulfilling prophecies. As a result, debt may generate endogenous recessions associated to recurrent decreases of growth with the possible occurrence of negative growth rates. Note also that our results are in line with Sutherland *et al.* [25] since large debt fosters macroeconomic instability.

It is worth mentioning that Proposition 4 provides quite different conclusions than models without public debt like Cazzavillan [3]. The assumption $\theta - \eta > 0$ implies that the homogeneity of the utility function is lower than one and restricts our attention to a weak public good externality in the utility function. Proposition 4 shows that local indeterminacy may arise when the share of debt over GDP is large enough, even without externality in utility ($\eta = 0$). On the contrary, Cazzavillan [3] proves that local indeterminacy requires a sufficiently large externality in utility, i.e. a degree of homogeneity larger than one. This confirms that the existence of sunspot fluctuations is here fundamentally derived from a large enough debt. Moreover, contrary to the conclusions of Minea and Villieu [15], considering the debt-to-GDP ratio as a policy instrument does not necessarily preclude the indeterminacy of the BGP.

4.2 A second steady-state

We have shown in Proposition 3 that the normalized steady-state is obtained if $\alpha \in (0, \alpha_{Max})$ and $A = A^*$, with $\alpha_{Max} > \hat{\alpha}$ and $A^* > \hat{A}$. Following case 2 in Proposition 2, a second admissible steady-state may arise depending on the value of α . Although some general conditions for the existence of a second steady-state have been stated in Proposition 2, we need here to complete these conditions as we start from the existence of a normalized steady-state that remains constant for any $\alpha \in [0, \alpha_{Max})$. We now locate precisely the second steady-state $\tilde{\gamma}$ with respect to γ^* and 1. The configuration with $\tilde{\gamma} < 1$ is clearly associated to endogenous recession. We derive the following Proposition:

Proposition 5. *Under Assumption 1, let $\theta - \eta < \beta$, $A = A^*$ and $\gamma^* \in (\underline{\gamma}, \gamma_{sup})$. Then there exist $\hat{\alpha} \in (\bar{\alpha}, \alpha_{Max})$, $\underline{\alpha} \in (0, \bar{\alpha})$ and $\alpha_1 \in (\bar{\alpha}, \hat{\alpha})$ such that when $\alpha \in (\underline{\alpha}, \hat{\alpha})$, there is a second admissible steady-state $\tilde{\gamma} \in (\gamma_{inf}, \gamma_{sup})$ which satisfies:*

- i) $\tilde{\gamma} \in (\gamma^*, \gamma_{sup})$ when $\alpha \in (\underline{\alpha}, \bar{\alpha})$,
- ii) $\tilde{\gamma} \in (1, \gamma^*)$ when $\alpha \in (\bar{\alpha}, \alpha_1)$,
- iii) $\tilde{\gamma} \in (\gamma_{inf}, 1)$ when $\alpha \in (\alpha_1, \hat{\alpha})$.

Proof. See Appendix 7.8.

Proposition 5 then shows, in accordance with Proposition 2, that there exist two admissible steady-states as long as the share α of public debt over GDP is large enough but not too big. Moreover, when α is too low, the second steady-state $\tilde{\gamma}$ is not admissible as it is larger than γ_{sup} .

Considering Proposition 4, we immediately derive the following conclusion:

Corollary 2. *Under Assumption 1, let $\theta - \eta < \beta$, $A = A^*$, $\gamma^* \in (\underline{\gamma}, \gamma_{sup})$, $\delta \in (0, \bar{\delta})$, $\beta \in (\underline{\beta}, 1)$ and $\theta - \eta \in (0, \bar{\Theta})$. Then the normalized steady-state γ^* is globally determinate if and only if $\alpha \in [0, \underline{\alpha}]$.*

As $\underline{\alpha} < \bar{\alpha}$, the normalized steady-state γ^* is indeed unique and locally determinate when $\alpha \in [0, \underline{\alpha}]$. This result is again in line with Sutherland *et al.* [25] since low debt prevents the occurrence of business cycles. However, as α increases from $\underline{\alpha}$, $\tilde{\gamma}$ becomes admissible with $\tilde{\gamma} > \gamma^*$, until $\alpha = \bar{\alpha}$ and $\tilde{\gamma} = \gamma^*$. If α increases again, then $\tilde{\gamma} < \gamma^*$ until α becomes too large so that $\tilde{\gamma}$ is no longer admissible as it is lower than γ_{inf} .

As we have normalized the steady-state γ^* which exists and remains constant for any $\alpha \in [0, \alpha_{Max})$, we may immediately derive the local stability properties of the second steady-state $\tilde{\gamma}$:

Corollary 3. *Under Assumption 1, let $A = A^*$, $\gamma^* \in (\underline{\gamma}, \gamma_{sup})$, $\beta \in (\underline{\beta}, 1)$, $\delta \in (0, \bar{\delta})$ and $\theta - \eta \in (0, \bar{\Theta})$. Then the second admissible steady-state $\tilde{\gamma} \in (\gamma_{inf}, \gamma_{sup})$ is:*

- i) a saddle-point, i.e. locally indeterminate if $\alpha \in (\underline{\alpha}, \bar{\alpha})$,
- ii) a source, i.e. locally determinate, if $\alpha \in (\bar{\alpha}, \hat{\alpha})$.

It follows that when $\alpha \in (\underline{\alpha}, \hat{\alpha})$, global indeterminacy arises.

Propositions 4 and 5 together with Corollary 3 show that the ratio of debt-to-GDP has a strong impact on the stability properties of the equilibrium, in particular when multiple balanced growth paths occur. In this case ($\alpha > \underline{\alpha}$), we exhibit an important trade-off as the largest steady-state is always

locally indeterminate, i.e. characterized by expectation-driven fluctuations, while the lowest steady-state is always locally determinate. The larger the ratio of debt-to-GDP is, the lower the locally determinate steady-state. It can even be less than one, i.e. characterized by an economic recession. Because of multiplicity, there is also a global indeterminacy suggesting that depending on agents' expectations, the actual equilibrium may converge towards the lowest or the highest growth factor, eventually with large fluctuations. As we know from Corollary 1 that the indeterminate steady state has the highest welfare, global indeterminacy can have strong implications in terms of welfare loss if agents coordinate their expectations on the lowest growth factor.

These results, that put a strong emphasis on self-fulfilling prophecies, reinforce the consistency of our analysis, already highlighted in the previous section, with the empirical facts exhibited by the recent literature. As shown by Sutherland *et al.* [25], increasing the ratio of debt-to-GDP affects the business cycle of countries leading to the existence of expectation-driven fluctuations. Depending on agents' beliefs, the impact on growth can be negative in the short or longer term through the possible occurrence of recessions. Moreover, as suggested by Panizza and Presbitero [20], there exist some non-linear and threshold effects associated to the existence of multiple equilibria and global indeterminacy. Finally, the existence of self-fulfilling prophecies can explain the fact that while some countries have a potential maximal sustainable debt ratio higher than their actual one, their perceived default risk can increase dramatically without any fundamental reason (see Collard *et al.* [4]-[5]).

4.3 Economic interpretations

To provide an intuition for these results, we recall that local indeterminacy means that several dynamic paths converge to a BGP and the selection of one dynamic path depends on agents' expectations. In other words, a way of providing an intuition for local indeterminacy of the largest BGP is to explain whether expectations may be self-fulfilling.

Let us consider the intertemporal budget constraint of the government and assume that the households expect a larger growth factor at the next period. According to the constraint on debt, they expect a larger level of future debt emission. This broader funding opportunity may be used to either reimburse

the cost of debt or enhance productive public spendings. At this stage, we note that, especially for $\theta - \eta$ low enough, the larger the growth rate, the lower the difference between R and γ , as measured by $\gamma^{\theta-\eta}/\beta - \gamma$. We derive that along a BGP with a high enough level of growth, the difference between the cost of debt reimbursement and debt emission as measured by $R - \gamma$ is lower, meaning that a larger share of government revenue is devoted to productive public spendings. This raises the ratio G/k , which leads to an increase of the interest rate r . Since growth comes from a Barro-type [1] formulation, current growth raises. Therefore, a higher expected growth rate induces larger current growth, meaning that expectations are self-fulfilling.

It is important to note that this does not happen if one focuses on a BGP with a too low growth factor. Indeed, in such a case, a too large share of taxation and debt emission is used to debt reimbursement, rather than to improve the level of productive public spending. Finally, our mechanism requires a sufficiently large ratio of public debt-to-GDP. This allows public spendings to strongly depend on the expected level of production. Otherwise, expectations would not sufficiently affect current public spendings and growth.

5 A simple calibration on OECD countries

Let us now confront our theoretical conclusions to the recent experiences of OECD countries with a simple numerical exercise. Using a standard calibration consistent with quarterly data, we assume that $(\delta, \beta) = (0.025, 0.98)$. Following Gruber [11], Mulligan [16], Vissing-Jorgensen and Attanasio [26] who provide estimates for the elasticity of intertemporal substitution in consumption larger than unity and smaller than 2, we assume that $\theta = 0.54$. Concerning the size η of the public good externality into the utility function there is of course no direct empirical evidence. However, Ni [17] provides some estimations of the substitutability of government purchases for private consumption that support the interval $\eta \in (0.22, 0.44)$. We will consider in the following $\eta = 0.4$ so that $\theta - \eta = 0.14$. With these values, the maximum admissible growth factor is $\gamma_{sup} \approx 1.02377$.

The following Table provides numbers in 2014 for the fiscal pressure τ , the share of capital income into GDP s and the share of gross debt into GDP α .

Concerning the growth factor of GDP, as γ^* is the long-run value, we consider the average over 2010 – 2014 for the main OECD countries in order to avoid the recession that occurred in 2009.⁸

Countries	τ (% of GDP)	s	α (% of GDP)	γ_{10-14} (average 2010-14)
France	45.3	0.31	95	1.0104
Germany	37.6	0.31	74.7	1.0196
Greece	33.8	0.4	177.1	0.952
Ireland	28.3	0.43	109.7	1.0144
Italy	44.4	0.31	132.1	0.9948
Japan	28.6	0.4	230	1.0152
Portugal	32.5	0.36	130.2	0.9908
Spain	32.9	0.39	97.7	0.995
UK	35.2	0.3	89.4	1.0174
US	24.3	0.36	101.5	1.022

Table 1: Main statistics in 2014 (Source: Eurostat and OECD)

As can be seen, there are huge variations of the fiscal pressure τ and the ratio of debt-to-GDP α across countries. Also, while the share of capital income into GDP is most commonly fixed at 0.3, there are quite large differences across countries with a value often larger than 0.3. For each illustration we will use the parameters' values of the country we want to focus on. Note that four countries are characterized on average by a recession over the last five years.

The numbers of Table 1 implies for all countries that the normalized steady state γ^* corresponds to the higher BGP. Depending on the value of the share α , there is thus the possibility of a second BGP lower than γ^* . Indeed, the critical values for the share α are provided in the following Table:

⁸The critical bound α_{Max} reaches unrealistically large values (larger than 20). In practice we focus on some interval $\alpha \in [0, 3)$ to be able to cover the case of Japan.

Countries	α (% of GDP)	$\underline{\alpha}$ (% of GDP)	$\bar{\alpha}$ (% of GDP)	α_1 (% of GDP)	$\hat{\alpha}$ (% of GDP)
France	95	69.1	70.3	71.3	179
Germany	74.7	56.7	57	58.5	148.8
Greece	177.1	81	86.6	82.7	133.8
Ireland	109.7	71.8	72.4	73.3	112
Italy	132.1	69.2	71.8	71.3	175.8
Japan	230	64.2	64.7	65.7	113
Portugal	130.2	63.4	65.8	65	128
Spain	97.7	72.3	74.4	74	130
UK	89.4	50.8	51.2	52.4	139
US	101.5	45.7	45.8	47	96.2

Table 2: Critical values of α

This Table immediately shows that depending on whether $\alpha \in (\bar{\alpha}, \hat{\alpha})$ or $\alpha > \hat{\alpha}$, the OECD countries can be splitted into two sets: a first one in which there exists a second negative long run growth rate, i.e. associated to an endogenous recession, and a second set in which the normalized growth rate the unique admissible steady state. The first set is composed of France, Germany, Ireland, Italy, Spain and UK while the second set contains Greece, Japan, Portugal and the US. Moreover, in each of these two sets, we can define two sub-sets depending on whether the normalized growth rate is positive or negative.

As a first illustration consider the case of France with a quite strong fiscal pressure corresponding to $\tau = 0.453$. We choose a normalized steady-state $\gamma^* = 1.0104$. As $\alpha \in (\alpha_1, \hat{\alpha})$, we conclude that γ^* is locally indeterminate and that global indeterminacy also arises as there exists a second steady-state $\tilde{\gamma} = 0.8342$ corresponding to a recession which is saddle-point stable. Our results then show that there are quite large fluctuations associated to self-fulfilling beliefs that can be related to the recent macroeconomic instability observed in France over the last five years. This suggests also that depending on the expectations of agents, there is a possibility of a large recession if there is a coordination on the low steady-state. Very similar conclusions are obtained for Germany, with $\gamma^* = 1.0196$ and $\tilde{\gamma} = 0.854$, Ireland, with $\gamma^* = 1.0144$ and $\tilde{\gamma} = 0.7247$, or UK, with $\gamma^* = 1.0174$ and $\tilde{\gamma} = 0.762$, suggesting that

even though the ratio of debt-to-GDP is much smaller, the possible occurrence of a strong recession is not ruled-out. It is worth noting that this recession can be stronger for Ireland and UK than for France or Germany as their second growth factor is significantly lower. It is also interesting to remark that for France and Ireland, a decrease of their debt ratio in order to meet the Maastricht constraint would be enough to ensure $\alpha < \underline{\alpha}$ and thus to guarantee the uniqueness of the long-run growth equilibrium without any fluctuations. For Germany and UK, it would be necessary to satisfy a slightly stronger constraint than the one imposed by Maastricht. It is finally important to note that, although their macroeconomic indicators are quite different, in particular concerning the unemployment rate, France and Germany, and more generally all the countries of this sub-set, face a similar risk of important recession due to a too large debt.

Spain, with $\gamma^* = 0.995$ and $\tilde{\gamma} = 0.797$, and Italy, with $\gamma^* = 0.9948$ and $\tilde{\gamma} = 0.741$, are also in a similar configuration except that both steady states are lower than one. As the normalized BGP is lower than 1, we have $\bar{\alpha} > \alpha_1$. We get $\alpha \in (\bar{\alpha}, \hat{\alpha})$ leading to a local indeterminacy for γ^* , determinacy for $\tilde{\gamma}$ and thus the existence of global indeterminacy with the possibility of a strong recession if there is a coordination on the low steady-state. But here, short-run fluctuations are also characterized by some strong instability due to the high occurrence of recessions and booms. The configuration is more difficult than for the first sub-set of countries since recessions are here more frequent, but again, a significant decrease of their debt would allow these countries to improve a lot their situation: lowering the debt-to-GDP ratio around 70–71% for Italy (thus a decrease of 45%) and 73% for Spain (thus a decrease of 25%) would increase the lower equilibrium growth factor up to a value significantly larger than 1 and would allow an eventual coordination of expectations on this growth equilibrium. Moreover, meeting the Maastricht constraint would allow both countries to stabilize their economy.

Consider now the case of the US. We focus on the average value $\gamma^* = 1.022$. Contrary to France, as the share of debt over GDP α is larger than $\hat{\alpha}$ there is a unique steady-state which is locally indeterminate. There are still expectation-driven fluctuations but the possibility of a strong recession is ruled-out as there is no global indeterminacy. Similar conclusions are obtained for Japan with

$\gamma^* = 1.0152$. It is worth noting that both countries would need to do strong efforts of debt reduction in order to eliminate all possibility of macroeconomic instability: a decrease of more than 50% for the US and around 75% for Japan.

In the case of Greece with $\gamma^* = 0.9523$, we find as in the US that $\alpha > \hat{\alpha}$ and the unique steady-state γ^* is locally indeterminate with expectation-driven fluctuations but in this case this is related to instability into a strong recession. The same kind of configuration occurs with Portugal which is characterized by $\gamma^* = 0.9908$ and $\alpha > \hat{\alpha}$, but with the possibility of growth factors larger than 1 during some quarters, as this was the case of the last few years. For both countries, a significant reduction of their debt would allow them to improve a lot their situation: decreasing the debt-to-GDP ratio by a half driving it to around 82% for Greece and 64% for Portugal would lead to the existence of a second equilibrium growth factor significantly larger than 1. The existence of large fluctuations would not be ruled out but the occurrence of optimistic expectations could allow a coordination of the high growth equilibrium. Moreover, here again, meeting the Maastricht constraint would allow both countries to stabilize completely their economy.

6 Conclusion

In this paper, we have considered a simple Barro-type [1] endogenous growth model where public spending is financed through taxes on income and public debt, and is useful because it improves households' utility of consumption and production as externalities. The ratio of debt over GDP is fixed and is used as a policy parameter by the government as recently in many European countries.

We have proved that when debt is a large enough proportion of GDP, two distinct BGPs may co-exist, one being indeterminate. We have exhibited two types of important trade-off associated with self-fulfilling expectations. First, we have shown that the lowest BGP is always a decreasing function of the ratio of debt/GDP while the highest one is an increasing function. As a result, depending on the BGP selected by agents' expectations, the relationship between debt and growth is not necessarily negative.

Second, local and global indeterminacy may arise and self-fulfilling expectations appear as a crucial ingredient to understand the impact of debt on

growth and on macroeconomic fluctuations. There is clearly a trade-off between welfare and macroeconomic fluctuations: the highest BGP, which provides the highest welfare, is always locally indeterminate while the lowest is always locally determinate. Our results then show non-trivial effects of debt on growth and macroeconomic fluctuations. Depending on the expectations of agents, large fluctuations associated to self-fulfilling beliefs may occur and be associated at the same time with welfare losses if there is a coordination on the low steady-state.

Our paper then provides a theoretical analysis that improves our understanding of the complex non-linear and threshold effects between public debt, growth and macroeconomic fluctuations. We have also discussed the implications of our findings for the main OECD countries considering numerical illustrations based on realistic calibrations for the size of debt, the growth rate and the main fundamentals. We have shown that the existence of multiple equilibria and global indeterminacy can provide a basis for understanding the recent experiences of many OECD countries relating the occurrence of endogenous recessions and macroeconomic instability to self-fulfilling expectations.

7 Appendix

7.1 Proof of Lemma 1

Substituting (12) in $d(\gamma)$, we get:

$$d(\gamma) = \frac{\gamma^{\theta-\eta}-\beta(1-\delta)}{\beta s(1-\tau)} - \left[\frac{\gamma^{\theta-\eta}-\beta(1-\delta)}{\beta A s(1-\tau)} \right]^{\frac{1}{1-s}} + 1 - \delta - \gamma \quad (21)$$

We derive that $d(\gamma_{inf}) = 1 - \delta - [\beta(1 - \delta)]^{\frac{1}{\theta-\eta}} > 0$. Moreover,

$$d(\gamma_{sup}) = \frac{\beta^{\frac{1}{\theta-\eta}-1}-(1-\delta)}{s(1-\tau)} \left[1 - \frac{1}{A^{\frac{1}{1-s}}} \left(\frac{\beta^{\frac{1}{\theta-\eta}-1}-(1-\delta)}{s(1-\tau)} \right)^{\frac{s}{1-s}} - s(1-\tau) \right] > 0 \quad (22)$$

if and only if $A > A_1(\tau)$, with

$$A_1(\tau) \equiv \frac{[\beta^{\frac{1}{\theta-\eta}-1}-(1-\delta)]^s}{[s(1-\tau)]^s[1-s(1-\tau)]^{1-s}} \quad (23)$$

Note that $A_1'(\tau) > 0$. Since $\tau < 1 - s$, we get $A_1(1 - s) > A_1(\tau)$ for any $\tau < 1 - s$ with

$$A_1(1 - s) = \frac{[\beta^{\frac{1}{\theta-\eta}-1}-(1-\delta)]^s}{s^{2s}[(1-s)(1+s)]^{1-s}} \equiv \widehat{A}_1 \quad (24)$$

Computing now the first and second derivatives of $d(\gamma)$, we get:

$$\begin{aligned}
d'(\gamma) &= (\theta - \eta) \frac{\gamma^{\theta-\eta-1}}{\beta s(1-\tau)} \left[1 - \frac{1}{(1-s)A^{\frac{1}{1-s}}} \left(\frac{\gamma^{\theta-\eta-\beta(1-\delta)}}{\beta s(1-\tau)} \right)^{\frac{s}{1-s}} \right] - 1 \\
d''(\gamma) &= (\theta - \eta)(\theta - \eta - 1) \frac{\gamma^{\theta-\eta-2}}{\beta s(1-\tau)} \left[1 - \frac{1}{(1-s)A^{\frac{1}{1-s}}} \left(\frac{\gamma^{\theta-\eta-\beta(1-\delta)}}{\beta s(1-\tau)} \right)^{\frac{s}{1-s}} \right] \\
&\quad - (\theta - \eta)^2 \left(\frac{\gamma^{\theta-\eta-1}}{\beta s(1-\tau)} \right)^2 \frac{s}{(1-s)^2 A^{\frac{1}{1-s}}} \left(\frac{\gamma^{\theta-\eta-\beta(1-\delta)}}{\beta s(1-\tau)} \right)^{\frac{2s-1}{1-s}}
\end{aligned} \tag{25}$$

Since $\theta - \eta < 1$, we easily derive that $d''(\gamma) < 0$ for all $\gamma \in (\gamma_{inf}, \gamma_{sup})$ if the term between brackets on the first line of the expression of $d''(\gamma)$ is positive. As this term is a decreasing function of γ we conclude that it is positive for all $\gamma \in (\gamma_{inf}, \gamma_{sup})$ if it is positive when evaluated at γ_{sup} . This is obtained if $A > A_2(\tau)$, with

$$A_2(\tau) \equiv \frac{[\beta^{\frac{1}{\theta-\eta-1}-(1-\delta)}]_s}{[s(1-\tau)]^s(1-s)^{1-s}} \tag{26}$$

Since $\tau < 1 - s$, we get $A_2(1 - s) > A_2(\tau)$ for any $\tau < 1 - s$ with

$$A_2(1 - s) = \frac{[\beta^{\frac{1}{\theta-\eta-1}-(1-\delta)}]_s}{s^{2s}(1-s)^{1-s}} \equiv \widehat{A}_2 \tag{27}$$

and $\widehat{A}_2 > \widehat{A}_1$. Let $\widehat{A}_2 = \widehat{A}$. We conclude that if $A > \widehat{A}$, then $d''(\gamma) < 0$ for all $\gamma \in (\gamma_{inf}, \gamma_{sup})$. Therefore, the concavity of $d(\gamma)$ over $(\gamma_{inf}, \gamma_{sup})$ together with $d(\gamma_{inf}) > 0$ and $d(\gamma_{sup}) > 0$ ensure that when $A > \widehat{A}$, $d(\gamma) > 0$ for all $\gamma \in (\gamma_{inf}, \gamma_{sup})$ and all $\tau < 1 - s$. \square

7.2 Proof of Proposition 1

When $\alpha = 0$, we derive from equation (13) that $\Omega(\gamma) = \tau$ is constant and

$$\Delta(\gamma) = \frac{1}{A} \left(\frac{\gamma^{\theta-\eta-\beta(1-\delta)}}{\beta A s(1-\tau)} \right)^{\frac{s}{1-s}}$$

is increasing from $\Delta(\gamma_{inf}) = 0$ to

$$\Delta(\gamma_{sup}) = \frac{1}{A} \left[\frac{\beta^{\frac{1}{\theta-\eta-1}-(1-\delta)}}{A s(1-\tau)} \right]^{\frac{s}{1-s}}$$

There exists a unique steady-state $\gamma^* \in (\gamma_{inf}, \gamma_{sup})$ if and only if the inequality $\Delta(\gamma_{sup}) > \tau$ is satisfied, i.e.

$$\frac{1}{A} \left[\frac{\beta^{\frac{1}{\theta-\eta-1}-(1-\delta)}}{A s} \right]^{\frac{s}{1-s}} > \tau(1-\tau)^{\frac{s}{1-s}} \equiv \varphi(\tau) \tag{28}$$

$\varphi(\tau)$ is increasing for all $\tau < 1 - s$, and reaches its maximum value for $\tau = 1 - s$, meaning that $\varphi(\tau) < \varphi(1 - s) = (1 - s)s^{\frac{s}{1-s}}$. Since we need to assume $A > \widehat{A}$

in order to ensure $d(\gamma) > 0$, where \widehat{A} is given by (27), it follows that inequality cannot hold when $\tau = 1 - s$. As it obviously holds when $\tau = 0$, we conclude that there exists $\hat{\tau} \in (0, 1 - s)$ such that $\Delta(\gamma_{sup}) > \tau$ for $\tau \in (0, \hat{\tau})$, whereas $\Delta(\gamma_{sup}) < \tau$ for $\tau \in (\hat{\tau}, 1 - s)$. \square

7.3 Proof of Proposition 2

By direct inspection of equation (13), we see that $\Omega(\gamma)$ is linearly increasing in γ , with $\Omega'(\gamma) = \alpha > 0$, and

$$\Delta'(\gamma) = (\theta - \eta)\gamma^{\theta-\eta-1} \left[\frac{s}{A(1-s)} \frac{[\gamma^{\theta-\eta-\beta(1-\delta)}]^{1-s}}{[\beta As(1-\tau)]^{1-s}} + \frac{\alpha}{\beta} \right] > 0 \quad (29)$$

Using $s \in (0, 1/2)$, we also easily derive that $\Delta(\gamma)$ is concave, i.e. $\Delta''(\gamma) < 0$. Since $\Delta(\gamma_{inf}) = \alpha(1 - \delta)$ and $\Omega(\gamma_{inf}) = \tau + \alpha[\beta(1 - \delta)]^{\frac{1}{\theta-\eta}}$, we have $\Delta(\gamma_{inf}) < (>)\Omega(\gamma_{inf})$ if and only if $\alpha < (>)\widehat{\alpha}$, with:

$$\widehat{\alpha} \equiv \frac{\tau}{1 - \delta - [\beta(1 - \delta)]^{\frac{1}{\theta-\eta}}} \quad (30)$$

Similarly, using

$$\Delta(\gamma_{sup}) = \frac{1}{A} \left[\frac{\beta^{\frac{1}{\theta-\eta-1}} - (1-\delta)}{As(1-\tau)} \right]^{\frac{s}{1-s}} + \alpha\beta^{\frac{1}{\theta-\eta-1}}$$

and $\Omega(\gamma_{sup}) = \tau + \alpha\beta^{\frac{1}{\theta-\eta-1}}$, we get $\Delta(\gamma_{sup}) > \Omega(\gamma_{sup})$ if and only if:

$$\frac{1}{A} \left[\frac{\beta^{\frac{1}{\theta-\eta-1}} - (1-\delta)}{As} \right]^{\frac{s}{1-s}} > \tau(1 - \tau)^{\frac{s}{1-s}} \equiv \varphi(\tau) \quad (31)$$

Using the same argument as in the proof of Proposition 1, we conclude that when $A > \widehat{A}$, there exists $\hat{\tau} \in (0, 1 - s)$ such that $\Delta(\gamma_{sup}) > \Omega(\gamma_{sup})$ for $\tau \in (0, \hat{\tau})$, whereas $\Delta(\gamma_{sup}) < \Omega(\gamma_{sup})$ for $\tau \in (\hat{\tau}, 1 - s)$.

We can then prove case 1 of the Proposition. Assume first that $\alpha < \widehat{\alpha}$, i.e. $\Delta(\gamma_{inf}) < \Omega(\gamma_{inf})$. The existence of a unique steady-state $\gamma^* \in (\gamma_{inf}, \gamma_{sup})$ is ensured if $\Delta(\gamma_{sup}) > \Omega(\gamma_{sup})$ i.e. if $A > \widehat{A}$ and $\tau \in (0, \hat{\tau})$. Assume now that $\alpha > \widehat{\alpha}$, i.e. $\Delta(\gamma_{inf}) > \Omega(\gamma_{inf})$. The existence of a unique steady-state $\gamma^* \in (\gamma_{inf}, \gamma_{sup})$ is ensured if $\Delta(\gamma_{sup}) < \Omega(\gamma_{sup})$ i.e. if $A > \widehat{A}$ and $\tau \in (\hat{\tau}, 1 - s)$.

Let us finally focus on case 2 where $\alpha < \widehat{\alpha}$, i.e. $\Delta(\gamma_{inf}) < \Omega(\gamma_{inf})$, and $A > \widehat{A}$ with $\tau \in (\hat{\tau}, 1 - s)$, i.e. $\Delta(\gamma_{sup}) < \Omega(\gamma_{sup})$. There are two steady-states if there exists $\widehat{\gamma} \in (\gamma_{inf}, \gamma_{sup})$ defined by $\Delta'(\widehat{\gamma}) = \Omega'(\widehat{\gamma})$ that satisfies $\Delta(\widehat{\gamma}) > \Omega(\widehat{\gamma})$. The equality $\Delta'(\gamma) = \Omega'(\gamma)$ is equivalent to

$$g(\gamma) \equiv (\theta - \eta) \frac{\gamma^{\theta-\eta-1} s}{\beta A(1-s)} \left(\frac{\gamma^{\theta-\eta-\beta(1-\delta)}}{\beta A s(1-\tau)} \right)^{\frac{2s-1}{1-s}} \frac{1}{A s(1-\tau)} = \alpha \left[1 - (\theta - \eta) \frac{\gamma^{\theta-\eta-1}}{\beta} \right] \equiv h(\gamma)$$

Note that $g'(\gamma) < 0$ and $h'(\gamma) > 0$. Moreover, we have

$$g(\gamma_{inf}) = +\infty > h(\gamma_{inf}) = \alpha \left[1 - (\theta - \eta) \frac{[\beta(1-\delta)]^{\frac{\theta-\eta-1}{\beta}}}{\beta} \right]$$

Similarly, we have $h(\gamma_{sup}) = \alpha[1 - (\theta - \eta)]$ and

$$g(\gamma_{sup}) = (\theta - \eta) \frac{s}{(1-s)A^{\frac{1}{1-s}}} \left(\frac{\beta^{\frac{1}{\theta-\eta-1} - (1-\delta)}}{s(1-\tau)} \right)^{\frac{s}{1-s}} \frac{1}{\beta^{\frac{1}{\theta-\eta-1} - (1-\delta)}}$$

Since $A > \widehat{A}$, we get

$$g(\gamma_{sup}) < \frac{(\theta-\eta)s}{\beta^{\frac{1}{\theta-\eta-1} - (1-\delta)}}$$

Therefore, we get $g(\gamma_{sup}) < h(\gamma_{sup})$ if

$$\frac{(\theta-\eta)s}{\beta^{\frac{1}{\theta-\eta-1} - (1-\delta)}} < \alpha[1 - (\theta - \eta)]$$

or equivalently

$$\alpha > \frac{(\theta-\eta)s}{\left[\beta^{\frac{1}{\theta-\eta-1} - (1-\delta)} \right] [1 - (\theta - \eta)]} \equiv \underline{\alpha}$$

It follows that there exists $\widetilde{\Theta} \in (0, 1)$ such that $\underline{\alpha} < \widehat{\alpha}$ if $\theta - \eta \in (0, \widetilde{\Theta})$.

Let us then assume that $\theta - \eta \in (0, \widetilde{\Theta})$ and $\alpha \in (\underline{\alpha}, \widehat{\alpha})$. We know that there exists $\widehat{\gamma} \in (\gamma_{inf}, \gamma_{sup})$ such that $\Delta'(\widehat{\gamma}) = \Omega'(\widehat{\gamma})$. We need finally to show that $\Delta(\widehat{\gamma}) > \Omega(\widehat{\gamma})$. When $\alpha = \underline{\alpha}$ we get $\Delta'(\gamma_{sup}) = \Omega'(\gamma_{sup})$ and there is no steady-state. When $\alpha = \widehat{\alpha}$, we get $\Delta(\gamma_{inf}) = \Omega(\gamma_{inf})$ and $\Delta(\gamma_{sup}) < \Omega(\gamma_{sup})$. In this case there exists two steady-states but the lower one is equal to γ_{inf} . Therefore, since $\partial\Delta(\gamma)/\partial\alpha - \partial\Omega(\gamma)/\partial\alpha = \gamma^{\theta-\eta}/\beta - \gamma > 0$, it follows that there exists a unique $\underline{\alpha} \in (\underline{\alpha}, \widehat{\alpha})$ such that there are two steady-states γ_1^* and γ_2^* for $\alpha \in (\underline{\alpha}, \widehat{\alpha})$. \square

7.4 Proof of Corollary 1

To determine the comparative statics of each type of steady-state with respect to α , we differentiate equation (13) to get:

$$\frac{d\gamma^*}{d\alpha} = \frac{\partial\Delta(\gamma^*)/\partial\alpha - \partial\Omega(\gamma^*)/\partial\alpha}{\Omega'(\gamma^*) - \Delta'(\gamma^*)} \quad (32)$$

Since $\partial\Delta(\gamma)/\partial\alpha - \partial\Omega(\gamma)/\partial\alpha = \gamma^{\theta-\eta}/\beta - \gamma > 0$, the sign of $d\gamma^*/d\alpha$ is given by the sign of $\Omega'(\gamma^*) - \Delta'(\gamma^*)$, i.e. the difference between the slopes of $\Omega(\gamma)$ and $\Delta(\gamma)$ evaluated at each steady-state. Using Proposition 2, we easily derive

that $d\gamma^*/d\alpha < 0$ in case 1(a), $d\gamma^*/d\alpha > 0$ in case 1(b), and $d\gamma_1^*/d\alpha < 0$ and $d\gamma_2^*/d\alpha > 0$ in case 3(b).

Consider now the expression of $W(\gamma)$ as given by (18). We get:

$$W'(\gamma) = \frac{k_0^{1-(\theta-\eta)} d(\gamma)^{-\theta} x(\gamma)^\eta \left[(1-\theta)d'(\gamma) + \eta d(\gamma) \frac{x'(\gamma)}{x(\gamma)} \right] [1-\beta\gamma^{1-(\theta-\eta)}] + [1-(\theta-\eta)]\beta\gamma^{-(\theta-\eta)} d(\gamma)}{1-\theta} \frac{1}{[1-\beta\gamma^{1-(\theta-\eta)}]^2}$$

with

$$\begin{aligned} d'(\gamma) &= (\theta - \eta) \frac{\gamma^{\theta-\eta-1}}{\beta s(1-\tau)} \left[1 - \frac{1}{(1-s)A^{\frac{1}{1-s}}} \left(\frac{\gamma^{\theta-\eta-\beta(1-\delta)}}{\beta s(1-\tau)} \right)^{\frac{s}{1-s}} \right] - 1 \\ x'(\gamma) &= \frac{1}{(1-s)A^{\frac{1}{1-s}}} \left(\frac{\gamma^{\theta-\eta-\beta(1-\delta)}}{\beta s(1-\tau)} \right)^{\frac{s}{1-s}} (\theta - \eta) \frac{\gamma^{\theta-\eta-1}}{\beta s(1-\tau)} > 0 \end{aligned}$$

so that

$$W'(\gamma) > \frac{k_0^{1-(\theta-\eta)} d(\gamma)^{-\theta} x(\gamma)^\eta (1-\theta)d'(\gamma) [1-\beta\gamma^{1-(\theta-\eta)}] + [1-(\theta-\eta)]\beta\gamma^{-(\theta-\eta)} d(\gamma)}{1-\theta} \frac{1}{[1-\beta\gamma^{1-(\theta-\eta)}]^2}$$

We have shown in the proof of Lemma 1 that when $A > \widehat{A}$, $d'(\gamma)$ is a monotone decreasing function with $d'(\gamma_{inf}) > d'(\gamma) > d'(\gamma_{sup})$. It follows that

$$W'(\gamma) > \frac{k_0^{1-(\theta-\eta)} d(\gamma)^{-\theta} x(\gamma)^\eta (1-\theta)d'(\gamma_{sup}) [1-\beta\gamma^{1-(\theta-\eta)}] + [1-(\theta-\eta)]\beta\gamma^{-(\theta-\eta)} d(\gamma)}{1-\theta} \frac{1}{[1-\beta\gamma^{1-(\theta-\eta)}]^2}$$

with

$$d'(\gamma_{sup}) = \frac{(\theta-\eta)}{s(1-\tau)} \left[1 - \frac{1}{(1-s)A^{\frac{1}{1-s}}} \left(\frac{\beta^{\frac{1}{\theta-\eta}-1} (1-\delta)}{s(1-\tau)} \right)^{\frac{s}{1-s}} \right] - 1 \equiv \Psi(\beta, \delta) - 1$$

and $\Psi(\beta, \delta) > 0$. Let us denote

$$f(\gamma) \equiv (1-\theta)d'(\gamma_{sup}) [1-\beta\gamma^{1-(\theta-\eta)}] + [1-(\theta-\eta)]\beta\gamma^{-(\theta-\eta)} d(\gamma)$$

Using $\gamma \in (\gamma_{inf}, \gamma_{sup})$, assume first that $d'(\gamma_{sup}) \geq 0$. We derive

$$\begin{aligned} f(\gamma) &> (1-\theta) \left[d'(\gamma_{sup}) \left(1 - \beta\gamma_{sup}^{1-(\theta-\eta)} \right) + \beta\gamma_{sup}^{-(\theta-\eta)} d(\gamma) \right] \\ &> (1-\theta)\beta^{\frac{1}{1-(\theta-\eta)}} > 0 \end{aligned}$$

for any $\gamma \in (\gamma_{inf}, \gamma_{sup})$. Assume now that $d'(\gamma_{sup}) < 0$. We derive

$$\begin{aligned} f(\gamma) &> (1-\theta) \left[d'(\gamma_{sup}) \left(1 - \beta\gamma_{inf}^{1-(\theta-\eta)} \right) + \beta\gamma_{sup}^{-(\theta-\eta)} d(\gamma) \right] \\ &> (1-\theta) \left\{ \Psi(\beta, \delta) \left(1 - \beta^{\frac{1}{\theta-\eta}} (1-\delta)^{\frac{1-(\theta-\eta)}{\theta-\eta}} \right) - 1 + \beta^{\frac{1}{\theta-\eta}} (1-\delta)^{\frac{1-(\theta-\eta)}{\theta-\eta}} \right. \\ &\quad \left. + \beta^{\frac{1}{1-(\theta-\eta)}} d(\gamma) \right\} \end{aligned}$$

When $\beta = 1$ and $\delta = 0$, we get

$$f(\gamma) > (1-\theta)d(\gamma) > 0$$

Therefore, there exist $\hat{\beta} \in (0, 1)$ and $\hat{\delta} \in (0, 1)$ such that if $\beta \in (\hat{\beta}, 1)$ and $\delta \in (0, \hat{\delta})$, then $f(\gamma) > 0$ for any $\gamma \in (\gamma_{inf}, \gamma_{sup})$. The result follows. \square

7.5 Proof of Proposition 3

A stationary solution $\gamma^* \in (1, \gamma_{sup})$ satisfies equation (13) if

$$h(\gamma^*, \alpha) \equiv \left(\frac{\gamma^{*\theta-\eta-\beta(1-\delta)}}{\beta s(1-\tau)} \right)^{\frac{s}{1-s}} A^{\frac{-1}{1-s}} - \left[\tau - \alpha \gamma^* \left(\frac{\gamma^{*\theta-\eta-1}}{\beta} - 1 \right) \right] = 0 \quad (33)$$

This equation can hold only if $\alpha < \alpha_{Max}$, with:

$$\alpha_{Max} = \frac{\tau}{\gamma^* \left(\frac{\gamma^{*\theta-\eta-1}}{\beta} - 1 \right)} \quad (34)$$

Note that if $\gamma^* = \gamma_{inf}$ then $\alpha_{Max} = \hat{\alpha}$. Moreover, as $\gamma^* > 1$, straightforward computations show that if $\theta - \eta < \beta$, α_{Max} is an increasing function of γ^* . It follows that for any $\gamma^* \in (1, \gamma_{sup})$, $\alpha_{Max} > \hat{\alpha}$.

Let us then assume that $\theta - \eta < \beta$ and $\alpha < \alpha_{Max}$. There is a unique $A = A^*$ solving equation (33), where A^* is given by (19). We immediately see that $\lim_{\gamma \rightarrow \gamma_{inf}} A^* = 0$, whereas

$$\lim_{\gamma \rightarrow \gamma_{sup}} A^* = \frac{\left[\beta^{\frac{1}{\theta-\eta-1}} - (1-\delta) \right]^s}{s^s (1-\tau)^s \tau^{1-s}} \equiv A_{\gamma_{sup}}^* \quad (35)$$

Since for any $\tau < 1 - s$ we have $A_{\gamma_{sup}}^* > \hat{A}$, we conclude that there exists $\hat{\gamma} \in (\gamma_{inf}, \gamma_{sup})$ such that when $\gamma^* \in (\hat{\gamma}, \gamma_{sup})$, $A^* > \hat{A}$. Let us then denote $\underline{\gamma} = \max\{1, \hat{\gamma}\}$ and let us choose $\gamma^* \in (\underline{\gamma}, \gamma_{sup})$ for which $A^* > \hat{A}$. Since $\hat{A} > \tilde{A}$, it follows from Lemma 1 that $d(\gamma^*) > 0$. \square

7.6 Proof of Lemma 2

Linearizing the dynamic system (11) around the normalized steady-state, we obtain:

$$\begin{aligned} \frac{\theta \gamma^*}{d^*(\alpha)} \frac{\Delta \gamma_{t+1}}{\gamma^*} + [B_1(\alpha) + B_3] \frac{\Delta x_{t+1}}{x^*} &= \left[\theta - \eta + \frac{\theta \gamma^*}{d^*(\alpha)} \right] \frac{\Delta \gamma_t}{\gamma^*} + B_1(\alpha) \frac{\Delta x_t}{x^*} \\ \alpha \gamma^* (1-s) \frac{\Delta x_{t+1}}{x^*} &= -\alpha \gamma^* \frac{\Delta \gamma_t}{\gamma^*} + B_2(\alpha) \frac{\Delta x_t}{x^*} \end{aligned}$$

with

$$\begin{aligned} d^*(\alpha) &= \left(\frac{\gamma^{*\theta-\eta-\beta(1-\delta)}}{\beta s(1-\tau)} \right) \left[1 - \tau + \alpha \gamma^* \left(\frac{\gamma^{*\theta-\eta-1}}{\beta} - 1 \right) \right] + 1 - \delta - \gamma^* \geq 0 \\ B_1(\alpha) &= \eta - \frac{\theta}{d^*(\alpha)} \left(\frac{\gamma^{*\theta-\eta-\beta(1-\delta)}}{\beta s(1-\tau)} \right) \left[1 - s - \tau + \alpha \gamma^* \left(\frac{\gamma^{*\theta-\eta-1}}{\beta} - 1 \right) \right] \\ B_2(\alpha) &= s\tau + \alpha \gamma^* \left(1 - s \frac{\gamma^{*\theta-\eta-1}}{\beta} \right) + \alpha (1-s) \frac{\gamma^{*\theta-\eta-\beta(1-\delta)}}{\beta} \\ B_3 &= (1-s) \frac{\gamma^{*\theta-\eta-\beta(1-\delta)}}{\gamma^{*\theta-\eta}} > 0 \end{aligned} \quad (36)$$

Note that $B_2(\alpha)$ can also be written as

$$B_2(\alpha) = s\tau + \alpha \left[\gamma^* - (1-s)(1-\delta) + (1-2s)\frac{\gamma^{*\theta-\eta}}{\beta} \right] > 0$$

We then derive the following linear system

$$\begin{pmatrix} \frac{\Delta\gamma_{t+1}}{\gamma^*} \\ \frac{\Delta x_{t+1}}{x^*} \end{pmatrix} = J \begin{pmatrix} \frac{\Delta\gamma_t}{\gamma^*} \\ \frac{\Delta x_t}{x^*} \end{pmatrix}$$

with

$$J = \begin{pmatrix} 1 + \frac{(\theta-\eta)d^*(\alpha)}{\theta\gamma^*} + \frac{[B_1(\alpha)+B_3]d^*(\alpha)}{\theta\gamma^*(1-s)} & \frac{B_1(\alpha)d^*(\alpha)}{\theta\gamma^*} - \frac{B_2(\alpha)[B_1(\alpha)+B_3]d^*(\alpha)}{\theta\alpha\gamma^{*2}(1-s)} \\ -\frac{1}{1-s} & \frac{B_2(\alpha)}{\alpha\gamma^*(1-s)} \end{pmatrix}$$

Since T and D represent respectively the trace and the determinant of J , the result follows after straightforward simplifications. \square

7.7 Proof of Proposition 4

To study the local stability properties of the normalized steady-state γ^* , we need to establish two technical Lemmas. Consider indeed equation (13) with $A = A^*$ that can be simplified as follows

$$\begin{aligned} h(\gamma, \alpha) &= \left[\tau - \alpha\gamma^* \left(\frac{\gamma^{*\theta-\eta-1}}{\beta} - 1 \right) \right] \left(\frac{\gamma^{\theta-\eta-\beta(1-\delta)}}{\gamma^{*\theta-\eta-\beta(1-\delta)}} \right)^{\frac{s}{1-s}} \\ &\quad - \left[\tau - \alpha\gamma \left(\frac{\gamma^{\theta-\eta-1}}{\beta} - 1 \right) \right] = 0 \end{aligned} \quad (37)$$

We obviously get $h(\gamma^*, \alpha) = 0$. The first Lemma characterizes the slope of the function $h(\gamma, \alpha)$ when $\gamma = \gamma^*$. In the following we denote $h'_1(\gamma, \alpha) = \partial h(\gamma, \alpha)/\partial \gamma$ and $h'_2(\gamma, \alpha) = \partial h(\gamma, \alpha)/\partial \alpha$.

Lemma 7.1. *Assume that $\theta - \eta < \beta$, $A = A^*$ and $\gamma^* \in (\underline{\gamma}, \gamma_{sup})$. Then there exists $\bar{\alpha} \in (0, \alpha_{Max})$ such that*

$$h'_1(\gamma^*, \alpha) \underset{\leq}{\geq} 0 \Leftrightarrow \alpha \underset{\leq}{\geq} \bar{\alpha} \quad (38)$$

Proof. Straightforward computations using the fact that $h(\gamma, \alpha) = 0$ along a steady-state yields

$$\begin{aligned} h'_1(\gamma, \alpha) &= \frac{\tau(\theta-\eta)s}{1-s} \frac{\gamma^{\theta-\eta-1}}{\gamma^{\theta-\eta-\beta(1-\delta)}} \\ &\quad - \frac{\alpha \left\{ \left[1 - (\theta-\eta) \frac{\gamma^{\theta-\eta-1}}{\beta} \right] [\gamma^{\theta-\eta-\beta(1-\delta)}] + \gamma^{\theta-\eta} \left(\frac{\gamma^{\theta-\eta-1}}{\beta} - 1 \right) \frac{(\theta-\eta)s}{1-s} \right\}}{\gamma^{\theta-\eta-\beta(1-\delta)}} \end{aligned}$$

Consider the term between braces multiplying α

$$\varphi(\gamma) = \left[1 - (\theta - \eta) \frac{\gamma^{\theta-\eta-1}}{\beta} \right] [\gamma^{\theta-\eta} - \beta(1-\delta)] + \gamma^{\theta-\eta} \left(\frac{\gamma^{\theta-\eta-1}}{\beta} - 1 \right) \frac{(\theta-\eta)s}{1-s}$$

We get $\varphi(\gamma) > 0$ if

$$(\theta - \eta) \frac{\gamma^{\theta-\eta-1}}{\beta} < 1 \Leftrightarrow \gamma > \left(\frac{\theta-\eta}{\beta} \right)^{\frac{1}{1-(\theta-\eta)}}$$

Under $\theta - \eta < \beta$ we get $[(\theta - \eta)/\beta]^{\frac{1}{1-(\theta-\eta)}} < 1$. Since $\gamma^* > 1$, it follows that $h'_1(\gamma^*, \alpha) \geq 0$ if and only if $\alpha \leq \bar{\alpha}$ with

$$\bar{\alpha} = \frac{\frac{\tau(\theta-\eta)s}{1-s} \gamma^{*\theta-\eta-1}}{\left[1 - (\theta-\eta) \frac{\gamma^{*\theta-\eta-1}}{\beta} \right] [\gamma^{*\theta-\eta} - \beta(1-\delta)] + \gamma^{*\theta-\eta} \left(\frac{\gamma^{*\theta-\eta-1}}{\beta} - 1 \right) \frac{(\theta-\eta)s}{1-s}} \quad (39)$$

Straightforward computations show that $\bar{\alpha} < \alpha_{Max}$. \square

The second technical Lemma provides a property of the discriminant of the characteristic polynomial that applies for any $\alpha \in [0, \alpha_{Max}]$.

Lemma 7.2. *Assume that $A = A^*$ and $\gamma^* \in (\underline{\gamma}, \gamma_{sup})$. Then there exist $\bar{\Theta} \in (0, \beta)$, $\delta_1 \in (0, 1)$ and $\beta_1 \in (0, 1)$ such that when $\theta - \eta \in (0, \bar{\Theta})$, $\delta \in (0, \delta_1)$ and $\beta \in (\beta_1, 1)$, both roots of the characteristic polynomial are real and positive for any $\alpha \in [0, \alpha_{Max}]$.*

Proof. Let us first compute the discriminant of the characteristic polynomial. Straightforward computations give

$$\begin{aligned} \Delta &= \left[1 + \frac{(\theta-\eta)d^*(\alpha)}{\theta\gamma^*} + \frac{[B_1(\alpha)+B_3]d^*(\alpha)}{\theta\gamma^*(1-s)} + \frac{B_2(\alpha)}{\alpha\gamma^*(1-s)} \right]^2 \\ &\quad - 4 \left[\left(1 + \frac{(\theta-\eta)d^*(\alpha)}{\theta\gamma^*} \right) \frac{B_2(\alpha)}{\alpha\gamma^*(1-s)} + \frac{B_1(\alpha)d^*(\alpha)}{\theta\gamma^*(1-s)} \right] \end{aligned}$$

Assume first that $B_1(\alpha) \geq 0$. We then get $B_1(\alpha)d^*(\alpha)\alpha + \theta B_2(\alpha) > 0$ and

$$\begin{aligned} \Delta &> \left[1 + \frac{(\theta-\eta)d^*(\alpha)}{\theta\gamma^*} + \frac{[B_1(\alpha)+B_3]d^*(\alpha)}{\theta\gamma^*(1-s)} + \frac{B_2(\alpha)}{\alpha\gamma^*(1-s)} \right]^2 \\ &\quad - 4 \left(1 + \frac{(\theta-\eta)d^*(\alpha)}{\theta\gamma^*} \right) \left(\frac{B_2(\alpha)}{\alpha\gamma^*(1-s)} + \frac{B_1(\alpha)d^*(\alpha)}{\theta\gamma^*(1-s)} \right) \\ &> \left[1 + \frac{(\theta-\eta)d^*(\alpha)}{\theta\gamma^*} + \frac{[B_3-B_1(\alpha)]d^*(\alpha)}{\theta\gamma^*(1-s)} - \frac{B_2(\alpha)}{\alpha\gamma^*(1-s)} \right]^2 + 4 \frac{B_3d^*(\alpha)}{\theta\gamma^*(1-s)} \left[\frac{B_1(\alpha)d^*(\alpha)\alpha + \theta B_2(\alpha)}{\theta\alpha\gamma^*(1-s)} \right] \\ &> 0 \end{aligned}$$

Assume now that $B_1(\alpha) < 0$. We then get

$$\begin{aligned} \Delta &> \left[1 + \frac{(\theta-\eta)d^*(\alpha)}{\theta\gamma^*} + \frac{[B_1(\alpha)+B_3]d^*(\alpha)}{\theta\gamma^*(1-s)} + \frac{B_2(\alpha)}{\alpha\gamma^*(1-s)} \right]^2 - 4 \left(1 + \frac{(\theta-\eta)d^*(\alpha)}{\theta\gamma^*} \right) \frac{B_2(\alpha)}{\alpha\gamma^*(1-s)} \\ &> \left[1 + \frac{(\theta-\eta)d^*(\alpha)}{\theta\gamma^*} + \frac{[B_1(\alpha)+B_3]d^*(\alpha)}{\theta\gamma^*(1-s)} - \frac{B_2(\alpha)}{\alpha\gamma^*(1-s)} \right]^2 + 4 \frac{B_2(\alpha)}{\alpha\gamma^*(1-s)} \frac{[B_1(\alpha)+B_3]d^*(\alpha)}{\theta\gamma^*(1-s)} \end{aligned}$$

with

$$B_1(\alpha) + B_3 = \frac{\theta \left[\frac{\gamma^{*\theta-\eta-\beta(1-\delta)}}{\beta(1-\tau)} + 1 - \delta - \gamma^* \right] + d^*(\alpha) \left[(1-s) \frac{\gamma^{*\theta-\eta-\beta(1-\delta)}}{\gamma^{*\theta-\eta}} - (\theta-\eta) \right]}{d^*(\alpha)} \quad (40)$$

Let us consider the term between brackets that is multiplied by θ . Obviously this term is positive if and only if

$$\gamma^{*\theta-\eta} [1 - \gamma^{*\theta-\eta-1} \beta(1-\tau)] - \beta(1-\delta)\tau > 0$$

Since $\gamma < \gamma_{sup}$ we get

$$1 - \gamma^{*\theta-\eta-1} \beta(1-\tau) > 1 - \tau > 0$$

Moreover, since $\gamma > \gamma_{inf}$ we get

$$\gamma^{*\theta-\eta} [1 - \gamma^{*\theta-\eta-1} \beta(1-\tau)] - \beta(1-\delta)\tau > \beta(1-\delta)(1-\tau) [1 - \gamma^{*\theta-\eta-1} \beta] > 0$$

It follows therefore that

$$\frac{\gamma^{*\theta-\eta-\beta(1-\delta)}}{\beta(1-\tau)} + 1 - \delta - \gamma^* > 0 \quad (41)$$

for any $\gamma \in (\gamma_{inf}, \gamma_{sup})$.

Consider now equation (40) when $\theta - \eta = 0$. We get

$$B_1(\alpha) + B_3 = \frac{\theta \left[\frac{1-\beta(1-\delta)}{\beta(1-\tau)} + 1 - \delta - \gamma^* \right] + d^*(\alpha)(1-s)[1-\beta(1-\delta)]}{d^*(\alpha)}$$

with obviously

$$\frac{1-\beta(1-\delta)}{\beta(1-\tau)} + 1 - \delta - \gamma^* > 0$$

Recalling that $\underline{\gamma} > \gamma_{inf}$, it follows that $B_1(\alpha) + B_3 > 0$ for any $\alpha \in [0, \alpha_{Max})$ as $\gamma^* \in (\underline{\gamma}, \gamma_{sup})$. Therefore, for any given $\gamma^* \in (\underline{\gamma}, \gamma_{sup})$, there exists $\bar{\Theta} \in (0, \beta]$, such that when $\theta - \eta \in (0, \bar{\Theta})$, $B_1(\alpha) + B_3 > 0$ and thus $\Delta > 0$ for any $\alpha \in [0, \alpha_{Max})$.

Let us focus now on the sign of $D(\alpha)$ and $T(\alpha)$ as given by (20). Since $B_1(\alpha) + B_3 > 0$ and $B_2(\alpha) > 0$ for any $\alpha \in [0, \alpha_{Max})$, we immediately get $T(\alpha) > 0$ for any $\alpha \in [0, \alpha_{Max})$. To derive the sign of $D(\alpha)$, we need to study the sign of $B_1(\alpha)d^*(\alpha)\alpha + \theta B_2(\alpha)$. Obvious computations give

$$\begin{aligned} B_1(\alpha)d^*(\alpha)\alpha + \theta B_2(\alpha) &= -\alpha^2(\theta - \eta)\gamma^* \left(\frac{\gamma^{*\theta-\eta-1}}{\beta} - 1 \right) \frac{\gamma^{*\theta-\eta-\beta(1-\delta)}}{\beta s(1-\tau)} \\ &+ \alpha \left\{ \eta \left[\gamma^* \left(\frac{\gamma^{*\theta-\eta-1}}{\beta s} - 1 \right) - \frac{(1-\delta)(1-s)}{s} \right] \right. \\ &+ \frac{\theta}{1-\tau} \left[\gamma^* \left(1 - s \frac{\gamma^{*\theta-\eta-1}}{\beta} \right) - \frac{\gamma^{*\theta-\eta-\beta(1-\delta)}}{\beta s} (1-s)^2 \right. \\ &\left. \left. - \tau \left[\gamma^* \left(1 - s \frac{\gamma^{*\theta-\eta-1}}{\beta} \right) - \frac{\gamma^{*\theta-\eta-\beta(1-\delta)}}{\beta s} [1 - s(1-s)] \right] \right] \right\} \\ &+ \theta s \tau \equiv \phi(\alpha) \end{aligned}$$

The polynomial $\phi(\alpha)$ is concave with $\phi(0) = \theta s \tau > 0$ and $\lim_{\alpha \rightarrow \infty} \phi(\alpha) = -\infty$, so that if $\phi(\alpha_{Max}) > 0$, then $\phi(\alpha) > 0$ for any $\alpha \in [0, \alpha_{Max}]$. When $\alpha = \alpha_{Max}$, we derive from Lemma 2:

$$\begin{aligned} d^*(\alpha_{Max}) &= \left(\frac{\gamma^{*\theta-\eta-\beta(1-\delta)}}{\beta s(1-\tau)} \right) + 1 - \delta - \gamma^* \geq 0 \\ B_1(\alpha_{Max}) &= \eta - \frac{\theta(1-s)}{d^*(\alpha_{Max})} \left(\frac{\gamma^{*\theta-\eta-\beta(1-\delta)}}{\beta s(1-\tau)} \right) \\ B_2(\alpha_{Max}) &= (1-s)\alpha_{Max} \left[\gamma^* + \frac{\gamma^{*\theta-\eta-\beta(1-\delta)}}{\beta} \right] \end{aligned} \quad (42)$$

so that

$$\begin{aligned} \phi(\alpha_{Max}) &= \alpha_{Max} \left\{ \eta \left[\frac{\gamma^{*\theta-\eta-\beta(1-\delta)}}{\beta s(1-\tau)} + 1 - \delta - \gamma^* \right] \right. \\ &\quad \left. + \frac{\theta(1-s)}{s(1-\tau)} \left[\gamma^* s(1-\tau) - \frac{\gamma^{*\theta-\eta}[1-s(1-\tau)]}{\beta} + (1-\delta)[1-s(1-\tau)] \right] \right\} \\ &> \alpha_{Max} \left\{ \eta \left[\frac{\gamma^{*\theta-\eta-\beta(1-\delta)}}{\beta s(1-\tau)} + 1 - \delta - \gamma^* \right] \right. \\ &\quad \left. + \frac{\theta(1-s)}{s(1-\tau)} \left[s(1-\tau) - \frac{\gamma^{*\theta-\eta}[1-s(1-\tau)]}{\beta} + (1-\delta)[1-s(1-\tau)] \right] \right\} \end{aligned}$$

Note first from (41) that

$$\frac{\gamma^{*\theta-\eta-\beta(1-\delta)}}{\beta s(1-\tau)} + 1 - \delta - \gamma^* > \frac{\gamma^{*\theta-\eta-\beta(1-\delta)}}{\beta(1-\tau)} + 1 - \delta - \gamma^* > 0$$

Second note that when $\delta = 0$, the inequality characterizing $\phi(\alpha_{Max})$ becomes

$$\phi(\alpha_{Max}) > \alpha_{Max} \left\{ \eta \left[\frac{\gamma^{*\theta-\eta-\beta}}{\beta s(1-\tau)} + 1 - \gamma^* \right] + \frac{\theta(1-s)}{s(1-\tau)} \left[1 - \frac{\gamma^{*\theta-\eta}[1-s(1-\tau)]}{\beta} \right] \right\}$$

with

$$1 - \frac{\gamma^{*\theta-\eta}[1-s(1-\tau)]}{\beta} > 0 \Leftrightarrow \gamma^* < \left(\frac{\beta}{1-s(1-\tau)} \right)^{\frac{1}{\theta-\eta}}$$

Note then that

$$\left(\frac{\beta}{1-s(1-\tau)} \right)^{\frac{1}{\theta-\eta}} > \gamma_{sup} \Leftrightarrow 1 - s(1-\tau) < \beta^{\frac{1}{1-(\theta-\eta)}}$$

which is satisfied when $\beta = 1$. Therefore, there exists $\delta_1 \in (0, 1)$ and $\beta_1 \in (0, 1)$ such that if $\delta \in (0, \delta_1)$, $\beta \in (\beta_1, 1)$ and $\theta - \eta \in (0, \bar{\Theta})$, then $\phi(\alpha_{Max}) > 0$ and thus $B_1(\alpha)d^*(\alpha)\alpha + \theta B_2(\alpha) > 0$ for any $\alpha \in [0, \alpha_{Max}]$. Under these conditions, and since $d^*(\alpha), B_2(\alpha) \geq 0$, it follows that $D(\alpha)$ is also positive for any $\alpha \in [0, \alpha_{Max}]$. \square

We may now study the local stability properties of $\gamma^* \in (\underline{\gamma}, \gamma_{sup})$ with $\alpha \in (0, \alpha_{Max})$. Consider the characteristic polynomial. Straightforward computations from Lemma 2 give

$$P(1) = -\frac{d^*(\alpha)}{\theta\gamma^*(1-s)} \left[(\theta - \eta) \left(1 - s - \frac{B_2(\alpha)}{\alpha\gamma^*} \right) + B_3 \right]$$

Considering (36), we get

$$\begin{aligned}
(\theta - \eta) \left(1 - s - \frac{B_2(\alpha)}{\alpha\gamma^*}\right) + B_3 &= (\theta - \eta) \left[1 - s - \frac{s\tau}{\alpha\gamma^*} - \left(1 - s \frac{\gamma^{*\theta-\eta-1}}{\beta}\right)\right. \\
&\quad \left. - \frac{1-s}{\gamma^*} \frac{\gamma^{*\theta-\eta-\beta(1-\delta)}}{\beta}\right] + (1-s) \frac{\gamma^{*\theta-\eta-\beta(1-\delta)}}{\gamma^{*\theta-\eta}}
\end{aligned} \tag{43}$$

which is clearly a monotone increasing function of α . So if there exists a positive value of α for which this expression is equal to zero, that value must be unique. Solving equation (43) equal to zero gives after simplifications

$$\begin{aligned}
\alpha &= \frac{\frac{(\theta-\eta)s\tau}{(1-s)\gamma}}{(\theta-\eta) \left[1 - \frac{1}{1-s} \left(1 - s \frac{\gamma^{*\theta-\eta-1}}{\beta}\right) - \frac{1}{\gamma^*} \frac{\gamma^{*\theta-\eta-\beta(1-\delta)}}{\beta}\right] + \frac{\gamma^{*\theta-\eta-\beta(1-\delta)}}{\gamma^{*\theta-\eta}}} \\
&= \frac{\frac{\tau(\theta-\eta)s}{1-s} \gamma^{*\theta-\eta-1}}{\left[1 - (\theta-\eta) \frac{\gamma^{*\theta-\eta-1}}{\beta}\right] [\gamma^{*\theta-\eta-\beta(1-\delta)}] + \gamma^{*\theta-\eta} \left(\frac{\gamma^{*\theta-\eta-1}}{\beta} - 1\right) \frac{(\theta-\eta)s}{1-s}} = \bar{\alpha}
\end{aligned}$$

with $\bar{\alpha}$ as given by Lemma 7.1 (see (39)). Therefore, since $d^*(\alpha) > 0$, we conclude that $P(1) \geq 0$ if and only if $\alpha \leq \bar{\alpha}$. We also get $\lim_{\lambda \rightarrow \pm\infty} P(\lambda) = +\infty$. Consider finally the expression of the trace T as given by (20) in Lemma 2. Using the expressions given in (36), we compute

$$T'(\alpha) = \frac{d^{*'}(\alpha)}{\theta\gamma^*(1-s)} \left[\frac{\gamma^{*\theta-\eta}[1-s-s(\theta-\eta)]-\beta(1-\delta)(1-s)}{\gamma^{*\theta-\eta}} \right] - \frac{s\tau}{\alpha^2\gamma^*(1-s)}$$

with

$$d^{*'}(\alpha) = \left(\frac{\gamma^{*\theta-\eta-\beta(1-\delta)}}{\beta s(1-\tau)} \right) \gamma^* \left(\frac{\gamma^{*\theta-\eta-1}}{\beta} - 1 \right) > 0$$

Since $\theta - \eta < 1$, we get $1 - s - s(\theta - \eta) > 1 - 2s$. Moreover, $s \in (0, 1/2)$ implies $1 - s - s(\theta - \eta) > 0$. As $\gamma^* < \gamma_{sup}$, we derive

$$\begin{aligned}
&\gamma^{*\theta-\eta} [1 - s - s(\theta - \eta)] - \beta(1 - \delta)(1 - s) \\
&< \beta^{\frac{\theta-\eta}{\theta-\eta-1}} \left[1 - s - s(\theta - \eta) - \beta^{\frac{1}{1-(\theta-\eta)}} (1 - \delta)(1 - s) \right]
\end{aligned}$$

We then conclude that when $\beta = 1$ and $\delta = 0$ the right-hand-side of this equation is negative. Therefore there exist $\beta_2 \in (0, 1)$ and $\delta_2 \in (0, 1)$ such that if $\beta \in (\beta_2, 1)$ and $\delta \in (0, \delta_2)$, then $T'(\alpha) < 0$. It follows that the minimal value of $T(\alpha)$ is obtained when $\alpha = \alpha_{Max}$. From equation (34) in the proof of Proposition 3 and using (42) we then derive:

$$\begin{aligned}
T(\alpha_{Max}) - 2 &= \frac{(\theta-\eta)d^*(\alpha_{Max})}{\theta\gamma^*} + \frac{[B_1(\alpha)+B_3]d^*(\alpha_{Max})}{\theta\gamma^*(1-s)} + \frac{B_2(\alpha_{Max})}{\alpha_{Max}\gamma^*(1-s)} - 1 \\
&= \frac{(\theta-\eta)d^*(\alpha_{Max})}{\theta\gamma^*} + \frac{[B_1(\alpha_{Max})+B_3]d^*(\alpha_{Max})}{\theta\gamma^*(1-s)} + \frac{\gamma^{*\theta-\eta-\beta(1-\delta)}}{\gamma^*\beta}
\end{aligned}$$

We have shown in the proof of Lemma 7.2 that if $\theta - \eta \in (0, \bar{\Theta})$, then $B_1(\alpha) + B_3 > 0$. Therefore, $T(\alpha) > 2$ for any $\alpha \in [0, \alpha_{Max}]$. Let $\bar{\delta} = \min\{\delta_1, \delta_2\}$

and $\underline{\beta} = \max\{\beta_1, \beta_2\}$. When $\delta \in (0, \bar{\delta})$, $\beta \in (\underline{\beta}, 1)$ and $\theta - \eta \in (0, \bar{\Theta})$, we conclude finally that for any $\alpha \in [0, \bar{\alpha})$, the normalized steady-state γ^* is locally determinate (a source) as both characteristic roots are larger than one, and for any $\alpha \in (\bar{\alpha}, \alpha_{Max})$, γ^* is locally indeterminate (a saddle-point) as one root is less than one. \square

7.8 Proof of Proposition 5

We derive from equation (37) that when $\alpha = 0$, the unique admissible steady-state is γ^* . On the contrary, as soon as $\alpha > 0$, we get $\lim_{\gamma \rightarrow +\infty} h(\gamma, \alpha) = -\infty$. Since, when $\alpha \in (0, \bar{\alpha})$ we have $h'_1(\gamma^*, \alpha) > 0$, there necessarily exists a second solution $\tilde{\gamma} > \gamma^*$ of $h(\gamma, \alpha) = 0$. When $\alpha = \bar{\alpha}$ we get $\tilde{\gamma} = \gamma^*$. When $\alpha > \bar{\alpha}$, straightforward computations yield

$$h(\gamma_{inf}, \alpha) = - \left[\tau - \alpha \left(1 - \delta - [\beta(1 - \delta)]^{\frac{1}{\theta - \eta}} \right) \right]$$

and thus $h(\gamma_{inf}, \alpha) < 0$ if and only if

$$\alpha < \frac{\tau}{1 - \delta - [\beta(1 - \delta)]^{\frac{1}{\theta - \eta}}} \equiv \hat{\alpha} \in (\bar{\alpha}, \alpha_{Max}) \quad (44)$$

Therefore, when $\alpha \in (\bar{\alpha}, \hat{\alpha})$, the second solution exists and satisfies $\tilde{\gamma} \in (\gamma_{inf}, \gamma^*)$. Note that (44) provides the same expression as the bound given by (30).

We need now to check whether this second solution is admissible, i.e. if $d(\tilde{\gamma}) > 0$. From Lemma 1, we know that, as long as $\tilde{\gamma} \in (\gamma_{inf}, \gamma_{sup})$, $d(\tilde{\gamma}) > 0$. We then have to provide conditions on α to ensure that $\tilde{\gamma} \in (\gamma_{inf}, \gamma_{sup})$. Let us consider equation (37). We have shown that $h'_1(\gamma^*, \alpha) \neq 0$ and thus $h'_1(\tilde{\gamma}, \alpha) \neq 0$ as long as $\alpha \neq \bar{\alpha}$. More precisely, we have

$$h'_1(\tilde{\gamma}, \alpha) \geq 0 \Leftrightarrow \alpha \geq \bar{\alpha} \quad (45)$$

Applying the Implicit Function Theorem, we conclude that $\tilde{\gamma} = \tilde{\gamma}(\alpha)$ with $\tilde{\gamma}'(\alpha) = -h'_2(\tilde{\gamma}, \alpha)/h'_1(\tilde{\gamma}, \alpha)$ and $\tilde{\gamma}(\bar{\alpha}) = \gamma^*$. Straightforward computations using the fact that $h(\gamma, \alpha) = 0$ along a steady-state yields

$$h'_2(\gamma, \alpha) = \frac{\tau}{\alpha} \left[1 - \left(\frac{\gamma^{\theta - \eta} - \beta(1 - \delta)}{\gamma^{*\theta - \eta} - \beta(1 - \delta)} \right)^{\frac{s}{1 - s}} \right]$$

When $\alpha < \bar{\alpha}$ we have $h'_1(\gamma^*, \alpha) > 0$ and thus $h'_1(\tilde{\gamma}, \alpha) < 0$ with $\gamma^* < \tilde{\gamma}$. It follows that $h'_2(\tilde{\gamma}, \alpha) < 0$ and thus $\tilde{\gamma}'(\alpha) < 0$. When $\alpha \in (\bar{\alpha}, \alpha_{Max})$ we have

$h'_1(\gamma^*, \alpha) < 0$ and thus $h'_1(\tilde{\gamma}, \alpha) > 0$ with $\gamma^* > \tilde{\gamma}$. It follows that $h'_2(\tilde{\gamma}, \alpha) > 0$ and thus $\tilde{\gamma}'(\alpha) < 0$. Therefore, for any $\alpha \in (0, \bar{\alpha}) \cup (\bar{\alpha}, \alpha_{Max})$, $\tilde{\gamma}(\alpha)$ is a monotone decreasing function.

From this property, assuming $\beta \in (\underline{\beta}, 1)$, $\delta \in (0, \bar{\delta})$ and $\theta - \eta \in (\underline{\Theta}, s)$, we finally conclude that there exist $\underline{\alpha} \in (0, \bar{\alpha})$ and $\alpha_1 \in (\bar{\alpha}, \hat{\alpha})$ such that the second steady-state exists and is admissible, i.e. such that $\tilde{\gamma} \in (\gamma_{inf}, \gamma_{sup})$, if $\alpha \in (\underline{\alpha}, \hat{\alpha})$, and it satisfies:

- $\tilde{\gamma} \in (\gamma^*, \gamma_{sup})$ if $\alpha \in (\underline{\alpha}, \bar{\alpha})$,
- $\tilde{\gamma} \in (1, \gamma^*)$ if $\alpha \in (\bar{\alpha}, \alpha_1)$,
- $\tilde{\gamma} \in (\gamma_{inf}, 1)$ if $\alpha \in (\alpha_1, \hat{\alpha})$.

Using equation (37), the bounds $\underline{\alpha} \in (0, \bar{\alpha})$ and $\alpha_1 \in (\bar{\alpha}, \hat{\alpha})$ are respectively solutions of $h(\gamma_{sup}, \alpha) = 0$ and $h(1, \alpha) = 0$, and are equal to

$$\underline{\alpha} = \frac{\tau \left[\left(\frac{\frac{\theta-\eta}{\beta^{\theta-\eta-1}-\beta(1-\delta)}}{\gamma^{*\theta-\eta-\beta(1-\delta)}} \right)^{\frac{s}{1-s}} - 1 \right]}{\gamma^* \left(\frac{\gamma^{*\theta-\eta-1}}{\beta} - 1 \right) \left(\frac{\frac{\theta-\eta}{\beta^{\theta-\eta-1}-\beta(1-\delta)}}{\gamma^{*\theta-\eta-\beta(1-\delta)}} \right)^{\frac{s}{1-s}}}, \quad \alpha_1 = \frac{\tau \left[\left(\frac{\frac{1-\beta(1-\delta)}{\gamma^{*\theta-\eta-\beta(1-\delta)}}}{\gamma^{*\theta-\eta-\beta(1-\delta)}} \right)^{\frac{s}{1-s}} - 1 \right]}{\gamma^* \left(\frac{\gamma^{*\theta-\eta-1}}{\beta} - 1 \right) \left(\frac{\frac{1-\beta(1-\delta)}{\gamma^{*\theta-\eta-\beta(1-\delta)}}}{\gamma^{*\theta-\eta-\beta(1-\delta)}} \right)^{\frac{s}{1-s}} - \frac{1-\beta}{\beta}}$$

Note finally that $h'_1(\gamma^*, \alpha)$ and $h'_1(\tilde{\gamma}, \alpha)$ have opposite sign and that this sign changes as α crosses $\bar{\alpha}$. □

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