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Perceived Competition

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Abstract

In contrast to standard economic models, recent empirical evidence suggests that agents often operate based on subjective and divergent views of the competitive landscape. We develop a novel framework in which such imperfections are explicitly modeled through subjective *perception networks*, and introduce the concept of *perception-consistent equilibrium* (PCE), in which agents' actions and conjectures respond to the feedback generated by perceived competition. We establish the existence of equilibrium in broad classes of aggregative games. The model typically yields multiple equilibria, including outcomes that feature patterns of localized exclusion. Remarkably, heterogeneity in beliefs induces *perceived competition rents*—payoff differentials that arise purely from subjective misperceptions. We further show that PCE actions correspond to ordinal centrality measures, with eigenvector centrality emerging as a behavioral benchmark in separable payoff environments. Finally, a graph-theoretic taxonomy of PCEs reveals a hierarchical structure that ranks perceived competition rents. We also give conditions under which a unique stable equilibrium exists.

Keywords: Competition, perception-consistent equilibrium, exclusionary equilibria, bounded rationality, ordinal centrality, eigenvector centrality, perceived competition rent.

JEL Classification: C72, D43, Z13.

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1 Introduction

Motivation. Competition is a central concept in economics. Firms compete for consumers and market share, researchers for funding, and, more broadly, agents compete for scarce resources. The standard view treats competition as an *objective* and reciprocal notion: firms are assumed to know exactly which other firms they are competing with. However, since firms cannot monitor every potential rival, they tend to focus on those they *perceive* as the most relevant threats (Porter, 1980; Zajac and Bazerman, 1991; McDonald and Eisenhardt, 2020). These perceptions often extend beyond direct or mutual competitors and may cross traditional market boundaries. As noted by Zaheer and Usai (2004), competitive perceptions are firm-specific and typically asymmetric—"it may well be that firm A perceives firm B to be a competitor while firm B only considers firm C as a competitor" (Zaheer and Usai, 2004, p. 74). Empirical research shows that a firm's position within an industry's competition network can help explain why seemingly similar firms differ in their innovation outcomes. Using panel data on enterprise infrastructure software firms, Thatchenkery and Katila (2021) demonstrate that because a firm's attention is selectively focused on certain competitors—attention that may not be reciprocated—its position (e.g., its network centrality) in the *perceived competition* network can significantly influence its product innovation. Thus, the competitors a firm perceives determine its position in the perceived network, generating heterogeneous competitive feedback across otherwise similar firms.

A general insight emerges from this discussion: both an agent's network position and their perception of competition are crucial in shaping outcomes. This paper examines how perceived competition influences agents' strategic choices and how these choices depend on positions within a perceived network. We formalize imperfect knowledge of global competition through a *perception network*—a directed graph representing perceived competitive relationships in the economy. In such a network, agent i may believe she competes with agent j, while agent j remains unaware of any rivalry with i. The perception network thus highlights the gap between the objective structure of economic competition (global competition) and its subjective interpretation by agents (perceived competition). This conceptualization echoes the insights of Zaheer and Usai (2004) and Thatchenkery and Katila (2021).

We model agents' behavior using a broad class of aggregative games, in which an agent's *objective* payoff depends on her own action and on a global aggregator of the actions of all agents in the economy. However, we assume that each agent i makes decisions based on a *perceived utility function*, where the global aggregate is replaced by a local aggregate over the efforts of agents in i's *perception set*—the subset of agents that i perceives as her competitors.

The discrepancy between the true (global) aggregate effort and the perceived aggregate effort, from agent i's perspective, is captured by a new variable α_i, which enters agent i's perceived utility function. This conjecture α_i is endogenously determined in equilibrium and can be interpreted as the "narrative" agent i constructs about the intensity of global competition in the economy. Accordingly, at equilibrium, α_i reflects agent i's *misperceived intensity of competition*. An alternative interpretation of α_i is that it represents agent i's belief about how representative her perception set is of the broader economy.[1,2]

We introduce a natural equilibrium concept, *Perception-Consistent Equilibrium* (PCE), which captures the intuition behind our model.[3] The concept of a PCE captures both the agents' local sightedness—each agent chooses an action level that maximizes her *perceived* utility—and the fact that the action levels of all individuals in the economy are consistent in equilibrium: conjectures, as captured by α_i, must be confirmed in equilibrium. Given their *local* conjectures, agents are only best responding to the choice of agents' actions in their perception set. Yet, *globally*, actions must be consistent. Hence, at a PCE, agents' perceived utility must be equal to their objective payoff in the underlying game.[4] The perception network is therefore interpreted as part of the feedback structure. At equilibrium, the conjectures α_i form an endogenous reflection of the perception network. A PCE therefore captures a form of bounded rationality in a game in which information is inherently incomplete, and where conjectures have a very specific perceived and local form. Note that if the perception network is complete, a PCE is nothing more than a Nash equilibrium. Indeed, in this case, there are no longer any misperceptions about the environment.

Let us go back to our example on innovation. As in our model with the perception network, Thatchenkery and Katila (2021) defined networks in terms of firms' *subjective perceptions* of who is a competitor (Zaheer and Usai, 2004). Focusing on the software industry, they interviewed former and current chief executive officers (CEOs) of leading firms in this industry, other key executives (chief operating officer, chief technology officer (CTO), Director of Business Development, Vice President of Sales) and found that seemingly homogeneous rival firms tracked very different competitors and thus have a different perception of competition.

[1]Conjecture α_i is bounded above at one. If the equilibrium quantity α_i^* is equal to one, agent i conjectures that she and her perception set represent the whole economy. This coincides, in fact, with a perfect knowledge of competition. Instead, the further α_i is below 1 in equilibrium, the higher is the discrepancy between perceived competition and global competition.

[2]The term "narrative" has not a clear definition in the literature and thus has received very different interpretations (see e.g., Shiller, 2017; Bénabou et al., 2018; Eliaz and Spiegler, 2020; Campbell et al., 2025). Here, a narrative is related to the knowledge each agent has about their competitors and what global competition looks like. For example, if some groups reside in very isolated communities, their narratives about their competitors may be very different from those living in more open societies. In our model, this discrepancy is captured by α_i.

[3]A PCE is a version of self-confirming equilibrium (Fudenberg and Levine, 1993; Battigalli et al., 2015, 2023) based on local conjectures.

[4]Implicitly, there is a dynamical learning process in which each agent best replies to what she *locally* observes in the previous period. A PCE is then the steady state of this dynamic learning process.

Main Contribution. We consider perception networks that are weakly connected and we first show that a PCE exists for a large class of *competitive aggregative games*. Importantly, at any PCE, the agents' efforts represent an *ordinal centrality measure*, a novel notion introduced by Sadler (2022). Many well-known network centralities, such as degree, eigenvector and Katz-Bonacich centrality, represent ordinal centralities. The typical situation at a PCE is exclusionary with the co-existence of agents who choose a positive action level and some market exclusion of local pockets of agents.

We study a subclass of competitive aggregative games by imposing additional structure on the payoff function. Thanks to the added structure on the payoff function, we derive results that go much beyond the mere existence of equilibria. First, we show that perception-consistent equilibria provide a behavioral foundation of the *eigenvector centrality* measure. More precisely, we prove that in any weakly connected perception network, and at any PCE, the effort level of each active agent is proportional to their *eigenvector centrality*. Second, we fully characterize the set of PCEs. We introduce an intuitive decomposition of the perception networks into communities, using techniques based on the Frobenius normal form.[5,6] Communities are ranked according to a partial order, which provides some intuition on the main features of perceived competition. Agents in communities that are dense and hidden from others are more likely to be active in equilibrium. Indeed, inactivity in some parts of the perception network is a salient feature of PCEs; in particular, inclusive equilibria co-exist with exclusionary equilibria that display market exclusion.

The multiplicity of perception-consistent equilibria highlights the behavioral richness of our equilibrium concept. To refine its predictive power, we also study the stability of PCEs. We introduce a natural dynamic process that captures the bounded rationality embedded in the notion of PCE. We establish the uniqueness of a *stable* PCE, which can take the form of either an inclusive or an exclusionary equilibrium. A community is active in a stable PCE if it is "aware" of the largest and densest community in the entire perception network. Whenever an inclusive PCE exists, it is guaranteed to be stable.

Let us illustrate our main results (which agent makes effort in equilibrium and how the position in the network affects this effort) with our leading example on enterprise infrastructure software. As stated above, Thatchenkery and Katila (2021) studied this industry because it is intensely competitive, firms widely differ in their perceptions about relevant competitors, and weak property rights help enable knowledge from competitors to influence the firm's own projects (Thatchenkery, 2017). They obtained three main results that are consistent with our theoretical results: (i) firms that do not perceive each other as competitors are positively

[5] As we will define later, a community is a type of strongly connected sub-network of the perception network.

[6] Indeed, in weakly connected networks, the standard application of the Perron-Frobenius theorem cannot be used. See Online Appendix B.1 for a definition and some results on the Frobenius normal form.

associated with product innovation (e.g., positive effort); (ii) the link between competition and innovation is not simply a matter of who the firm names as rivals, but how these choices position the firm within the wider network of perceived rivals within an industry; (iii) it is the position of the firm in the perceived (or competition) network (measured by network centrality in our model) and not in the collaboration network that matters for innovation.[7]

Contribution to the literature. Our paper contributes to the games-on-network literature.[8] In many situations in which networks matter, agents make both binary decisions (*extensive margin*) and quantity decisions (*intensive margin*). The literature on network games has mostly focused on the intensive margin by assuming that actions are continuous (Jackson and Zenou, 2015). There are, however, some papers that have considered network games with discrete actions—see, for example, Morris (2000), Brock and Durlauf (2001), and Leister et al. (2022)—and these are some of the first papers to consider both extensive and intensive margins.[9] We show that the extensive margin (i.e., who is active) is *community based*, that is, agents belonging to the same community are either all active or all inactive, and depends on the density of the community, whereas the intensive margin is *individual based* and solely determined by the position in the perception network: the higher is the individual eigenvector centrality, the higher is the effort of each agent.

Our model provides a microfoundation of eigenvector centrality, a general result that holds beyond the standard case of strongly connected perception networks, for which eigenvector centrality is usually defined. Other papers have provided a microfoundation of eigenvector centrality (see, in particular, Golub and Lever (2010) and Golub (2025)). For example, Golub and Jackson (2010, 2012) develop models on DeGroot updating in which eigenvector centrality is the right way to characterize an agent's influence. However, this connection arises from a heuristic learning process rather than behavior in a game. Banerjee et al. (2013) provide a microfoundation of eigenvector centrality by showing that it is the limit of diffusion centrality.[10]

As with our model, there are also network games that focus on imperfect information about the network and introduce new equilibrium concepts related to our PCE. In particular, McBride (2006), Lipnowski and Sadler (2019) and Battigalli et al. (2023) consider *self-confirming* and *peer-confirming equilibria*. Lipnowski and Sadler (2019) apply self-confirming

[7]There is a large literature on "understanding the notion of competitor" in marketing, in particular, in strategy and organizational research. Several scholars have highlighted the importance of using *perceptual* rather than *objective* data to capture each firm's idiosyncratic concerns about its competitors (Porac et al., 1995; Reger and Huff, 1993).

[8]For overviews, see Jackson (2008), Jackson and Zenou (2015), and Jackson et al. (2017).

[9]Other papers (e.g., Calvó-Armengol and Zenou (2004); Bramoullé and Kranton (2007)) have considered both but without being able to provide a general characterization of the equilibria. These models are also usually plagued by multiple equilibria.

[10]Some papers have also provided an axiomatic foundation of eigenvector centrality; see e.g., Palacios-Huerta and Volij (2004); Dequiedt and Zenou (2017); Bloch et al. (2023).

equilibria (SCE) and rationalizable SCE to games where feedback about the actions of others is described by a network topology: agents observe only the actions of their peers (i.e., neighbors), but their payoffs may depend on everybody's actions and are not observed ex-post. The main difference with our PCE is that Lipnowski and Sadler (2019) allow agents to make any conjecture about agents who are not their neighbors, while conjectures about neighbors must be correct at equilibrium.[11] In our model, agents' conjectures about their environment are fundamentally incorrect (agents are unaware to what happens outside their perception sets) and perception networks are directed, so agent i may be best responding to j's actions, while the reverse may not be true. Another noteworthy difference is that adding links in Lipnowski and Sadler (2019) restricts the set of permissible profiles/conjectures and thus shrinks the set of equilibria. Instead, adding directed links to perceived sets may, in fact, enlarge (or change) the set of PCEs.[12] Finally, the notion of a perception-consistent equilibrium is equivalent to a bounded rationality refinement of the self-confirming equilibrium (SCE) developed by Battigalli et al. (2023) when the game is written in such a way that the feedback agents receive is made explicit. In fact, PCE is a refinement of SCE whereby agents wrongly believe that they compete locally rather than globally.[13]

Our equilibrium characterization in terms of communities also relates to other network models that partition agents into endogenous community structures, including risk sharing (Ambrus et al., 2014), interaction between market and community (Gagnon and Goyal, 2017), behavioral communities (Jackson and Storms, 2019), information resale and intermediation (Manea, 2021), and technology adoption (Leister et al., 2022). However, the driving forces and policy implications are very different from ours. In particular, all these papers assume a perfect knowledge of the network and use standard equilibrium concepts.

2 The Model

Consider a finite set of agents, denoted by $N = \{1, 2, \cdots, n\}$, and with $n \geq 2$. Agent $i \in N$ chooses action $x_i \in \mathbb{R}_+$, and receives an *objective utility* given by the function $\pi_i(\mathbf{x}; \alpha)$, where $\mathbf{x} := (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n_+$ denotes an action profile, while $\alpha \geq 0$ is a non-negative

[11] Indeed, in Lipnowski and Sadler (2019), it is implicitly assumed in the definition of peer-confirming equilibrium that players know the whole network structure.

[12] When the network is complete, the set of peer-confirming equilibria coincides with the set of Nash equilibria. For the empty network, peer-confirming equilibria coincide with rationalizable equilibria. Increasing the number of links reduces the number of equilibria. In contrast, the set of PCEs may very well increase when links are added.

[13] Our paper is also related to Frick et al. (2022) who develop a model of social interactions and misinferences by agents and to Jackson (2019) who studies agents' misperceptions of interaction patterns in a network game. By developing a different model and a different equilibrium concept, we see our paper as complementary to that of Frick et al. (2022) and Jackson (2019)

parameter.[14] We often write $\pi_i(x_i, \mathbf{x}_{-i}, \alpha)$, where $\mathbf{x}_{-i} = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$ is the action vector of agents other than i at $\mathbf{x}$. Similarly, for each $S \subset N$, we write $\mathbf{x}_S = (x_i)_{i \in S}$. We assume that $x_i \mapsto \pi(x_i, \mathbf{x}_{-i}, \alpha)$ is strictly quasi-concave. We refer to X as the aggregate effort, i.e., $X := \sum_{j \in N} x_j$. For each subset $S \subset N$, let $X_S := \sum_{j \in S} x_j$. The utility function π_i can also be represented as a function of the player's own action x_i and the sum of all players' actions X.

Assumption 2.1. *There exists $h_i : \{(x_i, X, \alpha) : X \geq x_i \geq 0, \alpha \geq 0\} \to \mathbb{R}$, a continuously differentiable function such that*

$$\pi_i(x_i, \mathbf{x}_{-i}; \alpha) = h_i(x_i, X, \alpha), \;\; \forall \mathbf{x}. \tag{1}$$

Moreover,

(a) *There exists $M_i > 0$ such that, for any $\alpha_i \leq \alpha$ and any $\mathbf{x}_{-i}$ such that $\sum_{j \neq i} x_j \geq M_i$, the map $x_i \mapsto h_i(x_i, x_i + \mathbf{x}_{-i}, \alpha_i)$ is strictly decreasing.*

(b) *$X \mapsto h_i(x_i, X, \alpha)$ is strictly decreasing for any $\alpha > 0$, weakly decreasing at $\alpha = 0$, and $\alpha \mapsto h_i(x_i, X, \alpha)$ is strictly increasing.*

(c) *Given $\alpha > 0$ and $X \geq x_i \geq 0$, we have $h_i(x_i, x_i, 0) < h_i(x_i, X, \alpha)$.*

In the sequel, we will refer to α as the *value of competition* and to X as the *intensity of competition*. The different applications we study will make this distinction clear. Let us comment on Assumption 2.1. By point (a), and given the strict quasi-concavity of the payoff function, if the value of competition α is not sufficiently high and the other agents exert excessive effort (as captured by the threshold M_i), then agent i's payoff strictly decreases with her own effort. In other words, beyond the agent-specific threshold M_i, the intensity of competition becomes too high for agent i, who then no longer finds it optimal to match the competition by further increasing x_i. Point (b) further states that agent i's payoff weakly decreases with X, the overall intensity of competition, but strictly increases with α, the value of competition. Typically, when $\alpha = 0$, competition provides no benefit, so exerting effort x_i becomes worthwhile only when the value of competition is positive—regardless of the (possibly adverse) intensity of that competition. This observation underlies point (c), which states that for any given effort level x_i, agent i's utility is strictly lower when the value of competition is zero than when it is positive. This holds irrespective of the effort levels chosen by the other agents.

[14] Depending on the application we study, we also refer to $x_i \in \mathbb{R}^L_+$ as the effort exerted by agent i. This should cause no confusion.

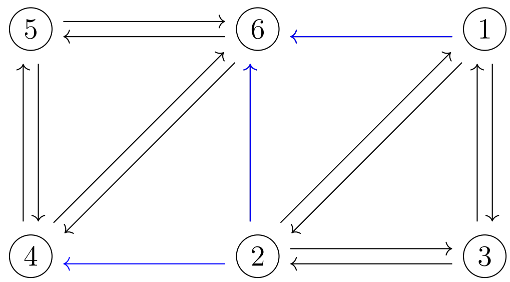


Figure 1: A Perception Network

To illustrate Assumption 2.1, we introduce a functional form that satisfies all of the above requirements. For each agent i, let $\theta_i > 0$ be an individual-specific parameter. Let $\bar{\theta} > 0$ be an aggregate parameter, and let $c > 0$ be the unit cost of effort. We have:

$$\pi_i(x_i, \mathbf{x}_{-i}, \alpha) = \frac{\theta_i + x_i}{\bar{\theta} + X}\alpha - cx_i.$$

It is easy to check that items (a), (b), and (c) of Assumption 2.1 are satisfied—see the proof of Lemma 2 in the Appendix for more details.

2.1 Perception Networks and Perceived Utility

To capture the novelties introduced by the perceived competition framework, we first define agents' perceptions of their competitors as a *directed graph* $(N, \mathbf{G})$, where $\mathbf{G}$ is an $n \times n$ adjacency matrix with entries $g_{ij} \in \{0, 1\}$. For each pair of agents $i, j \in N$, we have $g_{ij} = 1$ if and only if agent i is aware of agent j. Since N is fixed, we refer to the perception structure simply as $\mathbf{G}$. We call $\mathbf{G}$ a *perception network*, and define the *perception set* of agent i as $\mathcal{N}_i := \{j \in N : g_{ij} = 1\}$. Thus, each agent's perception of the competitive environment is captured ex ante solely by their out-degree in the network. The matrix $\mathbf{G}$ can be interpreted as part of the agents' *information feedback* structure—that is, the information available to them after they simultaneously choose their actions. In this framework, $\mathcal{N}_i$ denotes the set of agents whose actions agent i correctly anticipates and directly observes. For instance, if $\mathcal{N}_i \subset N \setminus \{i\}$, then agent i fails to perceive at least one other agent $j \in N$ as a competitor. For any pair $i, j \in N$, we write $i \rightrightarrows j$ if there exists a *directed path* from i to j in $\mathbf{G}$. Throughout the paper, we assume that the perception network $\mathbf{G}$ is *weakly connected*, which means that the undirected graph obtained by ignoring the direction of edges is connected, and that the weakly connected graph $\mathbf{G}$ satisfies the *no-isolation property*, that is, for each $i \in N$, we have that $\mathcal{N}_i \neq \emptyset$ and there exists $j \in N$ such that $i \in \mathcal{N}_j$. Let us illustrate the perceived competition framework with the following example.

Example 1. *A perception network.*
Figure 1 displays a perception network $\mathbf{G}$ with six agents. Let $S_1 = \{1, 2, 3\}$ and $S_2 = \{4, 5, 6\}$. For each $i \in S_2$, their perception sets $\mathcal{N}_i \subset S_2$. Hence, each agent $i \in S_2$ is only aware of agents that are themselves in S_2. There is no path $i \rightrightarrows j$ that would indirectly link agent i to some agent $j \in S_1$. In contrast, agents 1 and 2 are directly connected to some agents in S_2. Indeed, $\mathcal{N}_1 = \{2, 3, 6\}$ and $\mathcal{N}_2 = \{1, 3, 4, 6\}$. Note that $\mathcal{N}_3 \subset S_1$. $\diamond$

Remark 1. *The perception network* $\mathbf{G}$ *is an aggregate construct based on agents' perception sets and therefore not a physical network. We model* $\mathbf{G}$ *as an unweighted directed graph, where* $g_{ij} \in \{0, 1\}$ *captures whether agent* i *is aware of agent* j*. This modeling choice is crucial because weakly connected graphs are non-generic in the space of weighted directed graphs.*

As stated above, agents observe only the *aggregate effort* in their perception set, as captured by their out-degrees. They also observe their own payoff, which is determined by their objective utility function, though typically without a complete understanding of how this payoff is generated. In particular, since the intensity of competition X and the value of competition α are generally unknown, agents must form conjectures about these two quantities. The expected intensity of competition is modeled as a function of each agent's individual conjecture about the network, denoted by $\mathbf{G}_i$, resulting in $X(\mathbf{G}_i)$. For simplicity, we assume that the perception network is exogenously given. Consequently, the conjecture about X is reduced to agents observing the actions of those in their perception set, $\mathbf{x}_{\mathcal{N}_i}$, so that $X(\mathbf{G}_i) = X_{\mathcal{N}_i}$. Agents must also form a conjecture regarding the value of competition, which we denote by α_i for each agent i. This value is agent-specific, capturing each agent's subjective "understanding" of the competitive environment. We will see below that α_i will be determined endogenously in equilibrium.

Given their perceptions—or conjectures—about the global level of competition, quantified by α_i, as well as the efforts exerted by agents in their perception set, $\mathbf{x}_{\mathcal{N}_i}$, the *perceived utility* of agent i is defined as

$$u_i \left(x_i, \mathbf{x}_{\mathcal{N}_i}, \alpha_i\right) := h_i \left(x_i, x_i + X_{\mathcal{N}_i}, \alpha_i\right), \tag{2}$$

Given the perception network $\mathbf{G}$, agents simultaneously choose their actions and form conjectures about the value of competition, based on their *perceived* utility functions.

2.2 Perception-Consistent Equilibrium

We now introduce the equilibrium concept that captures the core intuition behind our model. The notion of a *Perception-Consistent Equilibrium* (PCE) captures both the agents' local sightedness—each agent chooses an action level that maximizes her perceived utility, given the actions of those in her perception set and her simultaneously formed conjecture about the value of competition—and the requirement that, at equilibrium, agents' action levels and conjectures must be mutually consistent across the population. That is, the subjective utility anticipated by each agent at equilibrium must align with the objective utility actually realized.[15]

Agents interact in a simultaneous *competitive agregative game* defined on the perceived network $\mathbf{G}$. Given $\mathbf{G}$, each agent i chooses an action $x_i \geq 0$ and forms a conjecture $\alpha_i \geq 0$ about the value of competition in the economy. Agents' decisions are based on their subjective utility functions and their perception sets $\mathcal{N}_i$.

Definition 1. *Given a perception network* $\mathbf{G}$*, a* ***Perception-Consistent Equilibrium*** *(PCE) is a profile* $(x_i^*, \alpha_i^*)_{i\in N} \in \mathbb{R}_+^n \times \mathbb{R}_+^n$ *such that, for each* $i \in N$*,*

(i) (Perceived best reply) $x_i^* = Argmax_{x_i \geq 0}\, u_i\left(x_i, \mathbf{x}^*_{\mathcal{N}_i}, \alpha_i^*\right)$;

(ii) (Confirmed conjecture) $u_i\left(x_i^*, \mathbf{x}^*_{\mathcal{N}_i}, \alpha_i^*\right) = \pi_i\left(x_i^*, \mathbf{x}^*_{-i}; \alpha\right)$.

We say that $\mathbf{x}^*$ *is a PCE effort profile if there exists* $(\alpha_i^*)_i$ *such that* (i) *and* (ii) *hold.*

Condition (i) states that agent i is best responding to both the actions of the agents in her perception set, as well as to her misperceived intensity of competition. Note that agent i is best responding according to her perceived utility. At equilibrium, agent i correctly anticipates the actions of those in her perception set. These actions implicitly signal to agent i the beliefs her neighbors have regarding the actions chosen by those in their own respective perception sets. Such an implicit information transmission will typically be reflected in the equilibrum conjecture α_i^*—since agent i best responds to the conjunction of $\mathbf{x}^*_{\mathcal{N}_i}$ and α_i^*. Note that α_i^*, while an equilibrium conjecture regarding the value of competition, is also a proxy for the (aggregate) intensity of competition, whose substantial part may be taking place outside of agent i's perception set.

The second part of the definition of a PCE is embedded in Condition (ii), a *consistency* requirement imposed in equilibrium, stating that conjectures must be confirmed: at a PCE, agents' anticipated perceived utilities have to be equal to their *objective* payoffs in the underlying aggregative game. Indeed, agents need to be able to rationalize their decisions, given their anticipations regarding those in their perception sets and their conjectured value of competition. An imbalance between the actual payoff received and the one anticipated through

[15]It is worth noting that the concept of PCE corresponds to a *Self-Confirming Equilibrium* (SCE) with local conjectures—a bounded rationality refinement of the standard SCE. Conjectures and perceptions are defined as described above. At a PCE, each agent maximizes her subjective (perceived) utility (2), which is an expected utility based on deterministic subjective beliefs. Moreover, conjectures about the efforts of agents in their perception set must be correct; see Fudenberg and Levine (1993), Battigalli et al. (2015), and Battigalli et al. (2023). We thank Pierpaolo Battigalli for helpful discussions on the connection between PCE and SCE.

the subjective utility maximization operation cannot stand at equilibrium, and hence conjectures must be confirmed. It is important to emphasize that a match between anticipated and received payoffs isn't synonymous with a correct conjectured value of the competition. As we will show later, a typical PCE conjectured value of competition for agent i is $\alpha_i^* < \alpha$.

Example 2. *An illustration for the definition of PCE.*
We revisit Figure 1 introduced in Example 1. At a PCE, agents $i \in S_2$ directly observe the choices made by agents in S_2, and only those. They receive no direct information on actions chosen in the rest of the perception network, outside of S_2. At a PCE, the conjecture they each make about the value of competition must be identical, and such conjectures are confirmed, hence adjusting the discrepancy between observed and subjective payoffs. Agents $i \in S_2$ can only benchmark their actions to those in S_2, and no further information transmission from $\mathbf{G}$ is taking place. In contrast, agents in S_1 are in a much favorable position. While $\mathcal{N}_3 = \{1, 2\}$, $\mathcal{N}_i \cap S_2 \neq \emptyset$ for $i = 1, 2$. Agents 1 and 2 can benchmark their actions not only against agents in S_1 but also to some of those in S_2. Agent 3 indirectly benefits from this since agent 1 and 2's actions endogenously reflect the part of $\mathbf{G}$ drawns from S_2: there is a path $3 \rightrightarrows j$ for $j = 4, 5, 6$. Indeed, agents 1 and 2 are not linked to agent 5, but there is a path connecting them to her. $\diamond$

A salient feature of PCEs is that multiplicity of equilibria is the norm, and equilibrium action profiles may often display zero efforts for some agents. Let $N_+(\mathbf{x}) := \{i \in N : x_i > 0\}$ be the set of *active agents* at action profile $\mathbf{x}$, i.e. those who choose a positive action level at $\mathbf{x}$. Likewise, let $N_0(\mathbf{x}) := \{i \in N : x_i = 0\}$. A PCE $(x_i^*, \alpha_i^*)_{i \in N}$ is an *inclusive equilibrium* if $N_+(\mathbf{x}^*) = N$. Otherwise, if $N_+(\mathbf{x}^*) \subset N$, we define $(x_i^*, \alpha_i^*)_{i \in N}$ as an *exclusionary equilibrium*. As will be illustrated in Example 3, the co-existence of inclusive and exclusive PCEs is typical in the perceived competition framework. We also introduce an additional qualification on PCEs which is useful for our purpose. We define a *non-zero* PCE effort profile $\mathbf{x}^* \in \mathbb{R}^n_+ \setminus \mathbf{0}$ as an equilibrium that excludes the trivial case where all agents exert zero effort.

Observation 1: *α_i represents an individual conjecture, serving as a proxy for the aggregate behavior formed in parts of the economy beyond agent i's perception set. Consequently, at equilibrium, α_i^* emerges endogenously as a reflection of the perception network $\mathbf{G}$.*

Observation 2: *The conjecture α_i takes on a specific, endogenously determined value α_i^* at equilibrium due to the confirmed conjecture condition. For example, consider the case where $\pi_i(\mathbf{x}, \alpha) = \frac{\theta_i + x_i}{\bar{\theta} + X} - cx_i$, with $\bar{\theta} > \theta_i > 0$ for each agent i. Then, equating objective and subjective utilities yields $\alpha_i = \frac{\bar{\theta} + x_i + X_{\mathcal{N}_i}}{\bar{\theta} + X}\alpha$. This illustrates why we often expect PCE conjectures to satisfy $\alpha_i^* < \alpha$. Lemma 1 below offers a more detailed account of the confirmed conjecture condition.*

2.3 Existence

Given the definition of a PCE, a profile $\mathbf{x}^*$ is a PCE effort profile if there exists a conjecture profile $(\alpha_i)_{i\in N}$ such that conditions (i) (perceived best-reply) and (ii) (confirmed conjectures) hold. The existence problem is therefore twofold. First, does a subjective maximum exist, given the perceived choices of other agents and the conjectured level of competition? Second, does the conjectured level of competition ensure that conjectures are confirmed? And third, given a choice of effort levels, is the equilibrium conjectured value of competition uniquely determined?

We begin with the latter question. Under Assumption 2.1, the uniqueness of a conjecture profile $(\alpha_i)_{i\in N}$ is readily ensured, given any non-zero action profile $\mathbf{x}$. In addition, the resulting value of competition is positive even for these agents i who choose $x_i = 0$ at profile $\mathbf{x}$.[16]

Lemma 1. *Given a profile* $\mathbf{x} \in \mathbb{R}^n_+ \setminus \mathbf{0}$*, there exists for each agent* i *a unique value of* $\alpha_i \in]0, \alpha]$ *such that*

$$h_i(x_i, X, \alpha) = h_i\left(x_i, x_i + X_{\mathcal{N}_i}, \alpha_i\right). \tag{3}$$

We call this value $\alpha_i(\mathbf{x})$*. The mapping* $\mathbf{x} \in \mathbb{R}^n_+ \setminus \mathbf{0} \mapsto \alpha_i(\mathbf{x})$ *is continuously differentiable.*

A profile $\mathbf{x}^*$ is therefore a PCE effort profile if and only if for each agent i, the following subjective best-reply condition holds:

$$x_i^* = Argmax_{x_i \geq 0}\ h_i\left(x_i, x_i + X^*_{\mathcal{N}_i}, \alpha_i(\mathbf{x}^*)\right),\ \forall i \in N. \tag{4}$$

Thanks to Lemma 1, the question of existence of a PCE is now reduced to the search for an equilibrium profile $\mathbf{x}^*$. By the strict quasi-concavity assumption and the assumptions on h_i, the map $BR_i : \mathbf{x} \in \mathbb{R}^n_+ \to \mathbb{R}_+$, defined by

$$Br_i(\mathbf{x}) := Argmax_{y_i \geq 0}\ h_i\left(y_i, y_i + X_{\mathcal{N}_i}, \alpha_i(\mathbf{x})\right), \tag{5}$$

is both single-valued and continuously differentiable. A PCE is, therefore, a fixed point of the best-response mapping defined as $Br : \mathbf{x} \in \mathbb{R}^n_+ \mapsto (Br_1(\mathbf{x}), ..., Br_n(\mathbf{x}))$.

We now introduce two additional conditions on functions $h_i(\cdot,\cdot)$ that will be helpful to guarantee the existence of a PCE.[17]

[16] That the equilibrium value of competition α_i^* is positive for agents with $x_i^* = 0$ is a consequence of Assumption 2.1. Later, we introduce a weaker condition—Assumption 3.1—which no longer guarantees that inactive agents have a positive conjectured value of competition. See Example E.4 for a case in which $\alpha_i^* > 0$ despite $x_i^* = 0$, and Example 3 for one where $x_i^* = 0$ indeed implies $\alpha_i^* = 0$.

[17] We denote by $\partial h_i/\partial x$ the derivative of h_i with respect to its first argument and $\partial h_i/\partial X$ the derivative of h_i with respect to its second argument.

Assumption 2.2 (Homogeneity). *Given $\lambda > 0$, there exists $\alpha(\lambda) > 0$ such that, for all $X > 0$ and all i,*

$$h_i(x_i, \lambda X, \alpha(\lambda)) \leq h_i(x_i, X, \alpha).$$

Assumption 2.2 requires that the intensity of competition cannot be disproportionately more impactful than the value of competition itself. As a result, even when aggregate effort is small, one can further reduce the intensity of competition while still identifying an intermediate value of competition that ensures agent i's payoff is weakly lower than under the original levels of X and α.

Assumption 2.3 (Incentive to exert effort). *Given $\underline{\alpha} > 0$, there exists $\underline{X} > 0$ such that, if $2x_i < X < \underline{X}$, then*

$$\frac{\partial h_i}{\partial x_i}(x_i, X, \alpha) + \frac{\partial h_i}{\partial X}(x_i, X, \alpha) > 0, \ \forall \alpha \geq \underline{\alpha}.$$

Assumption 2.3 states that, for some level of competition $\underline{\alpha} > 0$, and when aggregate effort X is sufficiently small, any agent who is not exerting the highest effort has an incentive to increase their effort.

Theorem 1 (PCE Existence 1). *Suppose that Assumptions 2.1, 2.2 and 2.3 hold. Then, a non-zero Perception-Consistent Equilibrium effort profile $\mathbf{x}^*$ exists.*

Note that a non-zero PCE effort profile $\mathbf{x}^*$ is a fixed point of the best-response map $Br : \mathbf{x} \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\} \mapsto (Br_1(\mathbf{x}), ..., Br_n(\mathbf{x}))$, with $Br_i(\mathbf{x})$ defined in (5). Accordingly, the proof relies on the projection on some compact and convex set of aggregate efforts (excluding $X = 0$) of the best response map. The fixed point $\mathbf{x}^*$ identified there is shown to be a fixed point of the original map Br, by virtue of Assumptions 2.2 and 2.3.

Lemma 1, which establishes the uniqueness of the conjectured value of competition, is a key step toward proving existence—recall, however, that the PCE conjectured value of competition is generally incorrect. Given the informational structure embedded in the concept of a perception-consistent equilibrium, observe that the perception network conveys two key elements at a PCE. First, the directed edges from agent i indicate that her anticipations about the actions of those in her perception set are correct. Second, while agent i observes the actions of those in her perception set $\mathcal{N}_i$, these actions themselves reflect the anticipations formed by agents in $\mathcal{N}_i$, based on their own perception sets. Thus, agent i receives two layers of information: (i) direct observation of actions and (ii) indirect inference of beliefs. One therefore expects an endogenous reflection of the perception graph in the equilibrium action profile.

2.4 A useful Application

Consider a profile of parameters $(\theta_i)_{i\in N} \in \mathbb{R}^n_{++}$, and let parameter $\bar{\theta}$ be such that $\bar{\theta} > \rho\theta_i$, for some $\rho > 1$ and for each $i \in N$.[18] Each agent i is endowed with the following objective utility function,

$$\pi_i(x_i, \mathbf{x}_{-i}; \alpha) := \frac{\theta_i + x_i}{\bar{\theta} + x_i + \sum_{j\neq i} x_j}\alpha - cx_i, \tag{6}$$

Note that function π_i can be represented as in Assumption 2.1, by the map h_i given by

$$h_i(x_i, X, \alpha) := \frac{\theta_i + x_i}{\bar{\theta} + X}\alpha - cx_i.$$

Such a functional form belongs to a larger class of individual objective utility functions $h_i(x_i, X, \alpha) = G_i(x_i)H(X, \alpha) + K_i(X, \alpha)$. In the context of Equation (6), agent i's payoffs has two distinct parts, one linked to own effort, and one linked to an exogenously given individual characteristic, some exogenous quantity $\bar{\theta}$ and the aggregate effort. While with the former, agent i needs to exert effort to obtain a positive payoff, the latter gives some constant returns from the value of competition, weighted down by the current intensity of competition X. In the event where $\mathbf{x} = \mathbf{0}$, each agent i then receives an objective payoff equal to $\frac{\theta_i}{\bar{\theta}}\alpha$. The interpretation of the model is tied to the family of parameters and the relationship between each θ_i and $\bar{\theta}$. We provide two illustrating examples.

First, let $S_1 \cup S_2 = N$. For each $i \in S_1$, $\theta_i = \bar{\theta} - \varepsilon_i > 0$, for ε_i arbitrarily small. For each $i \in S_2$, $\theta_i = \varepsilon_i$, for $\varepsilon_i > 0$, arbitrarily small. In the absence of efforts, agents in S_1 are guaranteed a fixed rent as a fraction arbitrarily close to the value of competition, while agents in S_2 only get a tiny return. Second, consider the case in which $\bar{\theta} = \sum_j \theta_j$ and Tullock models of the form $\pi_i = \frac{f_i(x_i)}{\sum_j f_j(x_j)}\alpha - cx_i$, with $f_i(x_i) = \theta_i + x_i$. In this context, θ_i represents the intrinsic ability of agent i to win a contest, independent of their own effort x_i. In other words, the ratio $\theta_i / \sum_j \theta_j$ reflects the relative ability of agent i to win the contest compared to other players, regardless of effort levels. Specifically, if all players exert zero effort, the probability that agent i wins the contest is precisely $\theta_i / \sum_j \theta_j$.

We now move from the objective to the subjective utility function of agent i. In this setting, agents are assumed to know the parameters θ_i and $\bar{\theta}$. However, given the perception network $\mathbf{G}$, the value of competition α is typically unknown. Each agent i thus forms a conjecture α_i about the value of competition in the economy. The subjective utility function

[18] The assumption that $\bar{\theta} > \rho\theta_i$ is important as it will guarantee that assumption 2.3 holds for small enough $\bar{\theta}$ with the family of functions we introduce in Equation (6).

of agent i is then given by,

$$u_i(x_i, \mathbf{x}_{\mathcal{N}_i}, \alpha_i) := \frac{\theta_i + x_i}{\overline{\theta} + x_i + X_{\mathcal{N}_i}}\alpha_i - cx_i.$$

Let us now incorporate the perception network into the analysis. It is important to note that both exclusionary and inclusive PCEs may arise, as well as equilibria in which all agents exert zero effort. The emergence of these distinct equilibrium types critically depends on the value of $\bar{\theta}$, as illustrated by Lemma 2 and Proposition 1 below.

Lemma 2. *If $\overline{\theta}$ is small enough, there exists a non-zero Perception-Consistent Equilibrium effort profile $\mathbf{x}^*$.*

To prove the existence of an equilibrium, we simply show that Assumptions 2.1, 2.2, and 2.3 are satisfied when $\bar{\theta}$ is sufficiently small. We can then invoke Theorem 1 to guarantee the existence of a non-zero PCE. Intuitively, when $\bar{\theta}$ becomes too large, incentives to exert effort vanish, and agents simply enjoy the fixed rent determined by their respective ratios $\theta_i/\bar{\theta}$. Conversely, when this rent is small enough, it becomes possible to further characterize the type of non-zero PCEs that arise. This is achieved through a useful identity that describes equilibrium effort levels, as established in the next proposition.

Proposition 1. *Pick any non-zero PCE effort profile $\mathbf{x}^*$. Let the sub-profile $\mathbf{x}_+^*$ be defined from $\mathbf{x}^*$ on the set of active agents $N_+(\mathbf{x}^*)$. Then, $\mathbf{x}_+^*$ satisfies the following identity,*

$$\mathbf{G}_+\mathbf{x}_+^* = \frac{c\left(X^* + \overline{\theta}\right)}{\alpha - c\left(X^* + \overline{\theta}\right)}\mathbf{x}_+^* + \frac{\alpha}{\alpha - c\left(X^* + \overline{\theta}\right)}\boldsymbol{\theta} + \frac{c\overline{\theta}\left(\overline{\theta} + X^*\right) - \alpha\overline{\theta}}{\alpha - c\left(X^* + \overline{\theta}\right)}\mathbf{1}, \tag{7}$$

where $\mathbf{G}_+$ is the adjacency matrix obtained from $\mathbf{G}$ and containing only the agents in $N_+(\mathbf{x}^*)$, $\mathbf{1}$ is the column vector of 1 (of dimensionality given by the rows $|\mathbf{x}_+^*|$) and $\boldsymbol{\theta}$ is the column vector of $(\theta_i)_i$ of agents $i \in N_+(\mathbf{x}^*)$.

When the utility function is given by (6), this proposition characterizes the equilibrium effort of each agent, at any perception-consistent equilibrium, which corresponds to a variation of the eigenvector centrality of each agent, expressed as:

$$\mathbf{G}_+\mathbf{x}_+^* = \frac{c\left(X^* + \overline{\theta}\right)}{\alpha - c\left(X^* + \overline{\theta}\right)}\mathbf{x}_+^*.$$

Furthermore, the PCE conjectured value of competition is easily traced back, and we obtain that for each agent i:

$$\alpha_i(\mathbf{x}^*) = \left(\frac{\overline{\theta} + x_i^* + X_{\mathcal{N}_i}^*}{\overline{\theta} + X^*}\right)\alpha.$$

Remark 2. *If $\theta_i = \overline{\theta} = 0$, for all i, the (objective) utility function simplifies to*

$$\pi_i(x_i, \mathbf{x}_{-i}; \alpha) := \frac{x_i}{X}\alpha - cx_i, \tag{8}$$

which corresponds to the standard Tullock model. Note that this utility function does not satisfy Assumption 2.1(a), as it is not well-defined at $x_i = 0$.[19] *At any PCE, the set of active players is such that each active agent's effort is equal to their eigenvector centrality (in the sub-network of active players). Specifically, we can show that:*

$$\mathbf{G}_{+}\mathbf{x}_{+}^{*} = \frac{cX^*}{\alpha - cX^*}\mathbf{x}_{+}^{*}.$$

In Section E.1 of the Online Appendix, we provide an example showing how heterogeneous perceptions of competition can lead to stark differences in equilibrium behavior and outcomes, even under symmetric payoffs. When both agents perceive each other as competitors, they exert identical effort levels. However, if one agent (agent 2) fails to perceive the other (agent 1) as a competitor, her effort level drops significantly, leading to lower payoffs. Despite having identical abilities, agent 1 benefits from a *perceived competition rent* due to better awareness of the competitive environment. This highlights how the structure of the perception network, rather than objective differences, drives disparities in behavior and welfare at a PCE.

3 An Important Extension

3.1 PCE characterization in the general case

Extending the concept of PCE. Assumption 2.1, among other things, requires that the function $h_i(x_i, X, \alpha)$ is strictly increasing in both the agent's own effort (over some range) and in the level of competition. In particular, the latter condition must hold even when agent i exerts no effort. The previously introduced application satisfies this assumption as long as the profile $(\theta_i)_{i \in N}$ consists solely of strictly positive parameters. However, cases in which some or all of the θ_i equal zero also cover applications of interest—such as the Tullock and Cournot models (see Online Appendix D for a discussion of the Cournot case). Thus, in many relevant applications, the payoff function may fail to satisfy Assumption 2.1 when effort is zero. Recall from Lemma 1 that the conjectured level of competition $\alpha_i(\mathbf{x})$ is uniquely determined at a confirmed conjecture for any effort profile $\mathbf{x}$. Moreover, $\alpha_i(\mathbf{x})$ remains positive whenever α is, even for agents for whom $x_i = 0$. Outside the scope of Assumption 2.1, the uniqueness or well-definedness of $\alpha_i(\mathbf{x})$ for arbitrary $\mathbf{x}$ is no longer guaranteed. This motivates a careful

[19] We address this case in Section 3.

extension of the PCE concept to ensure consistency in the conjecture $\alpha_i(\mathbf{x}^*)$ derived at a PCE equilibrium profile $\mathbf{x}^*$.[20]

Let us now weaken Assumption 2.1 to the following more general setting below. We simultaneously impose some restrictions on the behavior of the function $\alpha_i(.)$ around $x_i = 0$, regardless of the aggregate effort exerted by agents in $\mathcal{N}_i$.

Assumption 3.1. *The payoff function can be represented as in Equation (1) stated in Assumption 2.1, with h_i defined on $\{(x_i, X, \alpha) \in \mathbb{R}^3 : X \geq x_i > 0, \alpha \geq 0\}$, and satisfying 2.1 a), b), and c) over this restricted domain, $x_i > 0$:*

$$\pi_i(x_i, \mathbf{x}_{-i}; \alpha) = h_i(x_i, X, \alpha), \ \forall \mathbf{x} \text{ such that } X \geq x_i > 0.$$

Moreover, define the mapping $\alpha_i(\cdot)$ as in Lemma 1, for $\mathbf{x}$ such that $x_i > 0$. We assume that it has the following property: given $\mathbf{x}_{-i} \neq 0$, $\lim_{\mathbf{y}\to(0,\mathbf{x}_{-i})} \alpha_i(\mathbf{y})$ is well-defined, and only depends on $\mathbf{x}_{-i}$: formally there exists a real number, $\alpha_i(0, \mathbf{x}_{-i})$, such that

$$\forall \epsilon > 0, \exists \gamma > 0 : \quad s.t. \ \ y_i + \|\mathbf{y}_{-i} - \mathbf{x}_{-i}\| < \gamma \Rightarrow |\alpha_i(\mathbf{y}) - \alpha_i(0, \mathbf{x}_{-i})| < \epsilon.$$

Equipped with the above restriction on the behavior of conjecture $\alpha_i(.)$ around $x_i = 0$, let us now provide a careful extension of the notion of a PCE.

Definition 2. *Under Assumption 3.1, an action profile $\mathbf{x}^*$ is a PCE if,*

(a) *for all $i \in N_+(\mathbf{x}^*)$, identity (4) holds;*

(b) *for all $i \in N_0(\mathbf{x}^*)$, the following holds: For all $\eta > 0$,*

$$\eta = Argmax_{x_i \geq \eta}\, h_i\left(x_i, x_i + X^*_{\mathcal{N}_i}, \alpha_i(0, \mathbf{x}^*_{-i})\right). \tag{9}$$

Definition 2 provides a natural extension of the PCE concept to cases in which the payoff functions may not be well-behaved around $x_i = 0$, particularly when $x_{\mathcal{N}_i} = 0$. This extension enables the characterization of *exclusionary* PCEs, highlighting the underlying mechanisms governing the behavior of inactive agents. For agents $i \in N_+(\mathbf{x}^*)$, the subjective utility maximization condition remains unchanged, and Lemma 1 continues to apply since $x_i^* > 0$. Item (b) in Definition 2 specifies the behavior of inactive agents—as those for whom the subjective best response is strictly decreasing as x_i approaches zero. More precisely, for any $\eta > 0$ arbitrarily close to zero, agent i's subjective utility over $x_i \geq \eta$ is maximized at η,

[20]Even when the mapping $x_i \mapsto h_i(x_i, X, \alpha)$ is well-defined at $x_i = 0$, it may still violate Assumption 2.1. For instance, in many applications, we have $h_i(0, X, \alpha) \equiv 0$, which contradicts part (b) of Assumption 2.1.

taking as given both $\mathbf{x}^*_{-i}$ and the conjecture $\alpha_i(0, \mathbf{x}^*_{-i})$. In other words, an agent is considered as inactive if she always chooses the smallest feasible action available.[21] Observe that, since in item (b), the quantity $\arg\max_{x_i \geq \eta} h_i\left(x_i, x_i + X^*_{\mathcal{N}_i}, \alpha^*_i\right)$ is decreasing as $\eta \to 0$, it always admits a limit. Item (b) then implies that this limit must be zero. Observe also that, if h_i satisfies Assumption 2.1, including at $x_i = 0$, Definition 2 is then equivalent to Definition 1.

To illustrate the mechanics behind the reformulation of PCE introduced in Definition 2, let us consider again the utility function given by Equation (6), $\pi_i(x_i, \mathbf{x}_{-i}, \alpha) = \frac{\theta_i + x_i}{\bar{\theta} + X}\alpha - cx_i$, with the restriction that $\bar{\theta} = \theta_i = 0$. In this case, Assumption 2.1 is violated, but Assumption 3.1 holds. Suppose that, given $\mathbf{G}$, there exists an exclusionary PCE $\mathbf{x}^*$.

By revisiting the first-order conditions of the subjective utility maximization program, we characterize the equilibrium behavior of both active and inactive agents at PCE $\mathbf{x}^*$:

- for all $i \in N_+(\mathbf{x}^*)$,

$$c = \frac{\mathbf{X}^*_{\mathcal{N}_i}}{(x^*_i + \mathbf{X}^*_{\mathcal{N}_i})^2}\alpha_i(\mathbf{x}^*) = \frac{\mathbf{X}^*_{\mathcal{N}_i}}{(x^*_i + \mathbf{X}^*_{\mathcal{N}_i})X^*}\alpha, \tag{10}$$

- for all $i \in N_0(\mathbf{x}^*)$,

$$c \geq \frac{\mathbf{X}^*_{\mathcal{N}_i}}{(\eta + \mathbf{X}^*_{\mathcal{N}_i})^2}\alpha_i(0, \mathbf{x}^*_{-i}) = \frac{(\mathbf{X}^*_{\mathcal{N}_i})^2}{(\eta + \mathbf{X}^*_{\mathcal{N}_i})^2 X^*}\alpha, \text{ for all } \eta > 0, \tag{11}$$

Note that the latter terms on the right-hand sides of Equations (10) and (11) are obtained by observing that, on the hand, $\alpha_i(\mathbf{x}^*) = \frac{x^*_i + X^*_{\mathcal{N}_i}}{X^*}\alpha$ while, on the other hand, $\alpha_i(0, \mathbf{x}^*_{-i}) = \frac{X^*_{\mathcal{N}_i}}{X^*}\alpha$. Note also that if $\mathbf{x}^*_{\mathcal{N}_i} > 0$, Equation (11) is equivalent to $c \geq \frac{\alpha}{X^*}$, and it is automatically satisfied if $\mathbf{x}^*_{\mathcal{N}_i} = 0$ since $c \geq 0$.

Definition 2 clarifies the mechanics for inactive agents for payoff functions that are not well-defined at 0. While the equilibrium conjecture of inactive agents remains uniquely defined, Equation (11) ensures the coherence of inactive agents' behavior around $x^*_i = 0$. Indeed, $\alpha_i(0, \mathbf{x}^*_{-i})$ is uniquely determined.

Theorem 2 (**PCE Existence 2**). *Suppose that Assumptions 3.1, 2.2 and 2.3 hold. Then, a non-zero Perception-Consistent Equilibrium effort profile* $\mathbf{x}^*$ *exists.*

This result provides a more general existence theorem for PCEs, extending and superseding the one established in Theorem 1.

[21] Note that, by quasi-concavity, item (*b*) in Definition 2 is equivalent to the seemingly weaker condition (b'): $\lim_{\eta \to 0} \arg\max_{x_i \geq \eta}\ h_i\left(x_i, x_i + X^*_{\mathcal{N}_i}, \alpha_i(0, \mathbf{x}^*_{-i})\right) = 0$.

3.2 Properties of PCEs in the Separable Case

To better isolate the effects of the perception network on agents' equilibrium efforts, we now depart from the heterogeneous payoff setting and focus on the symmetric case in which $h_i \equiv h$ for all $i \in N$. In addition, we impose further structure on the payoff function. Specifically, we assume that agents can disentangle the impact of their own effort x_i from that of the intensity of competition X.

Definition 3 (**Payoff Separability**). *Agents have symmetric and separable payoff functions if Assumption 3.1 holds, with $h_i \equiv h$, where $h(x_i, X, \alpha) = G(x_i)H(X, \alpha)$. The function $G(.)$ is strictly increasing in x_i with $G(0) \geq 0$. The function $H(.,.)$ is defined on $\{\alpha > 0, X > 0\}$. It is strictly decreasing in X, and strictly increasing in α, and such that*

$$H(x_i, 0) < H(X, \alpha), \text{ for all } X \geq x_i > 0, \alpha > 0.$$

In this case, the map $\alpha_i(\cdot)$ satisfies the following property: there exists a continuous function $a : \mathbb{R}_+ \times \mathbb{R}_{++} \to \mathbb{R}_+$, strictly increasing in its first argument and strictly decreasing in its second, such that $a(X, X) = \alpha$ and $\alpha_i(\mathbf{x}) = a(x_i + X_{\mathcal{N}_i}, X)$.

Equipped with this structure for the conjecture function $\alpha_i(\cdot)$, we now investigate two key properties of PCEs that capture how the perception network shapes agents' equilibrium behavior.

3.2.1 Closedness of the set of active agents

In the separable case, an important network property emerges: at any PCE, if an agent is inactive, then all agents in her perception set must also be inactive. Conversely, if all agents in agent i's perception set are active at equilibrium, then agent i is active as well.

Proposition 2 (**Closedness**). *Suppose that agents have separable payoff functions given by $h(x, X, \alpha) = G(x)H(X, \alpha)$. Then Assumption 3.1 is satisfied. Moreover, if $\mathbf{x}^* \neq \mathbf{0}$ is a PCE effort profile, then:*

(1) $H(X^*, \alpha) > 0$.

(2) *If $G(0) = 0$, the set of active agents is closed, that is, $X^*_{\mathcal{N}_i} > 0 \Rightarrow x^*_i > 0$.*

Part (2) of Proposition 2 implies that the set of active agents at any PCE is a closed set. In particular, if $x^*_j > 0$ for some agent j, and there exists a directed path from agent i to agent j, denoted $i \rightrightarrows j$, then agent i must also be active at equilibrium, i.e., $x^*_i > 0$. In this sense, positive effort propagates *upstream* along directed paths in the perception network.

The closedness of $N_+(\mathbf{x}^*)$ already highlights a striking consequence of the perception network on equilibrium behavior, and captures a core feature of the perceived competition framework: agents' actions reflect those taken along the directed paths that originate from their out-neighborhoods. In particular, this result sheds light on the structure of the equilibrium conjectured values of competition. Rather than serving as mere adjustment parameters, the conjectures α_i^* reflect the volume and structure of information transmitted to agent i through the perception network—both directly via $\mathcal{N}_i$ and indirectly via the directed paths originating in $\mathcal{N}_i$

3.2.2 Ordinal centrality.

We adapt here the definition of ordinal centrality introduced in Sadler (2022). Consider the following partial order $\succeq_{oc}$ on $\mathbf{G}$. For any $S, S' \subset N$, we say that S *dominates* S', denoted $S \succeq S'$, if there exists an injective function $f : S' \to S$ such that $f(i) \succeq i$ for all $i \in S'$. The partial order $\succeq_{oc}$ is said to satisfy *recursive monotonicity* if $i \succeq_{oc} j$ whenever $\mathcal{N}_i \succeq_{oc} \mathcal{N}_j$. Now consider a PCE action profile $\mathbf{x}^*$, and apply the ordering $\succeq_{oc}$ to the set of active agents $N_+(\mathbf{x}^*)$. The profile $\mathbf{x}^*$ is said to be *ordered according to* $\succeq_{oc}$—that is, to follow an ordinal centrality—if $i \succeq_{oc} j \iff x_i^* \geq x_j^*$, which is in turn implied by the condition $X^*_{\mathcal{N}_i} \geq X^*_{\mathcal{N}_j}$.

To establish a link between PCEs and ordinal centrality, we first introduce a responsiveness condition on utility functions, which provides an ordering over marginal changes in payoffs induced by variations in individual actions.

Definition 4 (**The Marginal Feedback Condition**). *For any non-zero PCE $\mathbf{x}^*$, define the mapping $\Phi_{\mathbf{x}^*} : y \in \mathbb{R}_+ \to \Phi_{\mathbf{x}^*}(y) = \frac{\partial H}{\partial X}(y, a(y, X^*))$. We say that $h(x_i, X, \alpha) = G(x_i)H(X, \alpha)$ satisfies the $\mathbf{x}^*$-marginal feedback condition at the non-zero PCE action profile $\mathbf{x}^*$ if $\Phi_{\mathbf{x}^*}$ is strictly increasing, and for each i and each $\Delta > 0$,*

$$G(x_i)\Phi_{\mathbf{x}^*}(y) > G(x_i + \Delta)\Phi_{\mathbf{x}^*}(y + \Delta) \quad \text{for all } x_i \leq y. \tag{12}$$

The function $h(x_i, X, \alpha)$ satisfies the marginal feedback condition if it satisfies the $\mathbf{x}^*$-marginal feedback condition at each non-zero PCE $\mathbf{x}^*$.

Definition 4 introduces a responsiveness condition imposed on the function h at equilibrium, capturing how marginal payoffs change when effort levels increase jointly. Importantly, the function $\Phi_{\mathbf{x}^*}(y)$ is evaluated at a fixed intensity of competition X^*—that is, the aggregate effort determined at the PCE profile $\mathbf{x}^*$—which remains constant when computing $\Phi_{\mathbf{x}^*}(y+\Delta)$. As a result, agent i can be incentivized to increase effort only if the corresponding rise in the local intensity of competition is offset by a sufficiently strong increase in the conjectured value of competition. In Section E.3 of the Online Appendix, we use our example of two agents with payoffs given by (6) to illustrate the Marginal Feedback Condition.

Proposition 3 (PCE Efforts and Ordinal Centrality). *Let $h(x, X, \alpha) = G(x)H(X, \alpha)$ satisfy payoff separability and the marginal feedback condition. Assume, in addition, that $G(x)$ is a concave function. Then, for each non-zero PCE action profile $\mathbf{x}^*$, given $i, j \in N_+(\mathbf{x}^*)$, we have*

$$X^*_{\mathcal{N}_i} > X^*_{\mathcal{N}_j} \Rightarrow x^*_i > x^*_j, \quad \text{and } X^*_{\mathcal{N}_i} = X^*_{\mathcal{N}_j} \Rightarrow x^*_i = x^*_j. \tag{13}$$

*In particular, the partial ordering $\succsim$ defined on $N_+(\mathbf{x}^*)$ as $i \succsim j \Leftrightarrow x^*_i \geq x^*_j$ is an **ordinal centrality.***

Proposition 2 suggests that the more directed paths an agent possesses, the more likely she is to belong to the set of active agents at equilibrium. Proposition 3 and the associated notion of ordinal centrality go further by establishing a meaningful ranking of actions within the perception network. While this result does not yet identify the precise mechanism by which certain agents contribute more at a PCE, it shows that perception sets that are more "central" will be subject to more intense competition than those that are less central. It is important to note, however, that ordinal centrality does not necessarily imply closedness. In Section E.4 of the Online Appendix, we provide a counterexample—based on the payoff function given in Equation (6)—demonstrating that an action profile ordered according to ordinal centrality may nonetheless fail to satisfy the closedness property.

4 PCE Mechanisms and Characterization

In this section, we delve deeper into the mechanisms that determine perception-consistent equilibria (PCEs). We introduce an additional assumption on payoff functions, with two main objectives in mind. First, we seek to identify a general mechanism underlying the emergence of PCEs. Second, given the potential multiplicity of equilibria, we aim to develop a methodology for characterizing the full set of PCEs. Addressing these two goals will allow us to systematically answer the following questions: (i) the *extensive margin*—which agents are active at a PCE; (ii) the *intensive margin*—how equilibrium effort levels are determined across agents; and (iii) the *degree of multiplicity*—how many PCEs exist for a given perception network.

Definition 5 (A subclass of separable multiplicative payoff functions). *Consider the class of payoff functions introduced in Definition 3 and assume that they can be written as*

$$h(x_i, X, \alpha) = G(x_i)H(X, \alpha) = x_i f\left(\frac{X}{\alpha}\right).$$

Such payoff functions satisfy Assumptions 2.2, 2.3, and 3.1. Two important examples are the standard Tullock model, presented in equation (8)*, and the Cournot model, given in equation* (D.1) *in Online Appendix D.*[22]

4.1 A Behavioral Foundation for Eigenvector Centrality

Let $\rho(\mathbf{G})$ be the *spectral radius* of the matrix $\mathbf{G}$. That is, $\rho(\mathbf{G}) = \max\{|\lambda_1|, \dots, |\lambda_n|\}$, where $\lambda_1, \cdots, \lambda_n$ are the eigenvalues of $\mathbf{G}$. By the Perron-Frobenius theorem (see Lemma B1 in Online Appendix B.1), $\rho(\mathbf{G})$ is an eigenvalue of $\mathbf{G}$, associated with a positive eigenvector and uniquely defined up to multiplication by a positive constant.

Theorem 3 (Eigenvector Centrality). *Let $h(x_i, X, \alpha) \equiv x_i f(X/\alpha)$ (Definition 5). Then, a non-zero action profile $\mathbf{x}^*$ is a PCE profile if and only if*

$$\mathbf{G}\mathbf{x}^* = \frac{-f(X^*/\alpha) - f'(X^*/\alpha)[X^*/\alpha]}{f(X^*/\alpha)}\,\mathbf{x}^*. \tag{14}$$

Theorem 3 provides a *microfoundation (or behavioral foundation) of eigenvector centrality.* In particular, it shows that, for any weakly connected network, at any PCE, the effort of each agent is proportional to her eigenvector centrality, in the sub-network of active agents.[23] In other terms, $\mathbf{x}^*$ is an eigenvector of $\mathbf{G}$ associated to the eigenvalue $\frac{-f(X^*/\alpha)-f'(X^*/\alpha)[X^*/\alpha]}{f(X^*/\alpha)}$. For instance, in the Tullock model, $\mathbf{x}^*$ is an eigenvector associated to the eigenvalue $\frac{cX^*}{\alpha - cX^*}$ and to eigenvalue $\frac{2X^* - \bar{\beta} + c}{\bar{\beta} - c - X^*}$ in the Cournot model (see equation (D.3) in Online Appendix D).

Before delving further into the microfoundations of eigenvector centrality, let us first highlight the improvement over Proposition 3. Owing to the additional structure imposed on the function h, we can identify the ordinal centrality measure that ranks equilibrium actions. This connection emerges from both conditions required for an action profile to constitute a PCE. To see this, consider the first-order condition of the agents' optimization problem. Each agent maximizes their perceived utility given by Equation (2), which in this setting takes the form $x_i f\left(\frac{x_i + X_{\mathcal{N}_i}}{\alpha_i}\right)$. At equilibrium, the first-order condition for an active agent becomes

$$f\left(\frac{x_i^* + X^*_{\mathcal{N}_i}}{\alpha_i(\mathbf{x}^*)}\right) + \frac{x_i^*}{\alpha_i(\mathbf{x}^*)} f'\left(\frac{x_i^* + X^*_{\mathcal{N}_i}}{\alpha_i(\mathbf{x}^*)}\right) = 0.$$

[22] Note that the marginal feedback condition obviously holds under the restriction introduced in Definition 5. Indeed, we have $G(x) = x$ and $H(X, \alpha) = f(X/\alpha)$. Thus $a(y, X^*) = \alpha \frac{y}{X^*}$, and $\Phi(y) = \frac{1}{y}\frac{Xf'(X/\alpha)}{\alpha}$, which is strictly decreasing in y. Moreover $G(x)\Phi(y) = \frac{x}{y}\frac{Xf'(X/\alpha)}{\alpha}$. Since $f' < 0$, we have $G(x)\Phi(y) > G(x+\Delta)\Phi(y+\Delta)$ for $x < y$.

[23] Eigenvector centrality is usually defined for strongly connected networks. Indeed, in this case, it is a well-defined measure of centrality captured by the Perron-Frobenius vector associated with the adjacency matrix (Jackson, 2008). In Online Appendix B.3, we provide a more general definition of eigenvector centrality for networks that are not necessarily strongly connected.

The confirmed conjecture requirement imposes:

$$\alpha_i(\mathbf{x}^*) = \frac{x_i^* + X_{\mathcal{N}_i}^*}{x_i^* + \mathbf{x}_{-i}}\alpha.$$

Substituting this into the first-order condition yields:

$$f\left(\frac{X^*}{\alpha}\right) + \frac{x_i^*}{x_i^* + X_{\mathcal{N}_i}^*}\frac{X^*}{\alpha}f'\left(\frac{X^*}{\alpha}\right) = 0.$$

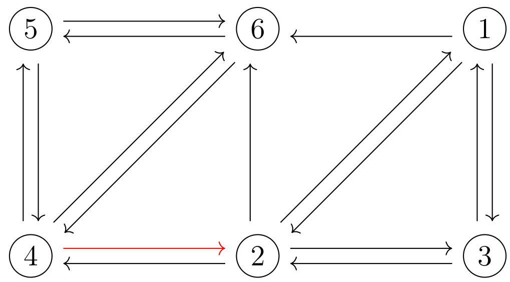
From here, Equation (14) follows directly, using the identity $\mathbf{G}\mathbf{x}^* = (\mathbf{x}_{\mathcal{N}_i}^*)_{i\in N}$. A detailed proof is provided in the Appendix.

Positive feedback effect at equilibrium. A PCE is shaped by two features of the perception network. First, directed edges ensure agent i's anticipations about her perception set are accurate. Second, the actions she observes from neighbors $\mathcal{N}_i$ also embed those neighbors' own beliefs. Thus, agent i extracts both direct observations and indirect signals about the broader network. She then forms a conjecture α_i capturing perceived competition (e.g., expected local resources in Tullock models). Theorem 3 shows that aggregating these signals—with conjectures confirmed at equilibrium—yields an ordering of actions consistent with eigenvector centrality. The intuition is that alignment between actions and network position emerges endogenously: each agent optimizes based on her neighborhood and inferred beliefs of others. The more central agent i and her neighbors are, the more accurate α_i, reinforcing centrality's role in shaping equilibrium behavior.

Positive feedback without strategic complementarities. The connection between agents' actions and eigenvector centrality is typically derived from models with strategic complementarities. In such settings, more central agents tend to interact with other central agents—see, for instance, the canonical model of Ballester et al. (2006) or the ordinal centrality foundations in Sadler (2022). In contrast, Theorem 3 yields a similar connection without relying on strategic complementarities. The perceived positive feedback at equilibrium in our setting is purely relative. Specifically, agents in the perception set of agent i exert a less negative impact on her than those outside the set—an asymmetry that is anticipated within the model. In contrast, the influence of agents outside the perception set is typically misperceived. This discrepancy contributes to the emergence of a seemingly positive feedback effect at equilibrium, even in the absence of strategic complementarities.

Uniqueness of equilibrium in strongly connected perception networks. A perception network is said to be *strongly connected* if, for every pair of agents $i, j \in N$, there exists a directed path from i to j. In this case, there exists a unique perception-consistent equilibrium

Figure 2: A strongly connected perception network



$\mathbf{x}^*$, which is necessarily an inclusive PCE.[24] Figure 2 illustrates a strongly connected version of the perception network introduced in Figure 3a. Crucially, the unique inclusive PCE action profile $\mathbf{x}^*$ differs from the action profile at the unique (and symmetric) Nash equilibrium of the complete network. As established by Theorem 3, equilibrium actions at a PCE are determined by agents' eigenvector centrality. For instance, under the PCE $\mathbf{x}^*$, we observe that $x_6^* < x_4^*$.

4.2 On the Determinants of the set of PCEs

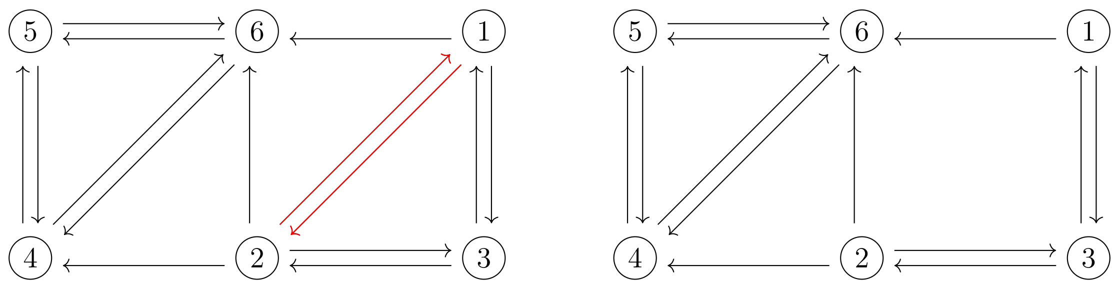
The eigenvector centrality result enables us to determine the correct ordering of equilibrium actions at any PCE. However, multiplicity of PCEs is often the norm, with inclusive and exclusionary PCEs potentially coexisting. This section focuses on characterizing the full set of PCEs. Leveraging the additional separability assumption on payoffs introduced in Definition 5, we employ a novel graph-theoretic decomposition to identify all such equilibria. Building on the eigenvector centrality insight, we begin by illustrating the potential multiplicity of PCEs.

Example 3. *On the multiplicity of PCEs.*
Consider the two perception networks displayed in Figure 3 and the Tullock model defined in Equation (8), with $\pi_i(x_i, x_{-i}, \alpha) = \frac{x_i}{X}\alpha - cx_i$. We show the dramatic difference between the set of PCE between these two perception networks.

Figure 3a: The perception sets are defined as follows: $\mathcal{N}_1 = \{2, 3, 6\}$, $\mathcal{N}_2 = \{1, 3, 4, 6\}$, $\mathcal{N}_3 = \{1, 2\}$, $\mathcal{N}_4 = \{5, 6\}$, $\mathcal{N}_5 = \{4, 6\}$, $\mathcal{N}_6 = \{4, 5\}$. Observe that the subsets $\{1, 2, 3\}$ and $\{4, 5, 6\}$ each resemble complete networks when considered in isolation, suggesting that competition should be intense within both subgroups. However, due to the asymmetric structure of the perception sets, awareness flows only in one direction—through agents 1 and 2—which significantly alters equilibrium behavior compared to the case of a complete network. Agents 1 and 2 observe the action of agent 6 and, implicitly, the beliefs that agent 6 holds about

[24] The uniqueness of equilibrium in strongly connected networks follows as a corollary to Theorem 3. See Corollary 2 in the Appendix.

Figure 3: Two similar perception networks with different densities



(a) A dense perception network

(b) A less dense perception network

those in her perception set. However, agents 1 and 2 do not directly observe those agents themselves. This asymmetry gives rise to a unique PCE: an exclusionary equilibrium in which agents $\{1, 2, 3\}$ are *active*, while agents $\{4, 5, 6\}$ receive no resources—despite the absence of any explicit barrier to access. Importantly, this exclusionary outcome cannot be attributed to fundamentals of the economy; rather, it stems purely from differences in perception. The feedback structure embedded in the perception network creates stark inequality in outcomes. Notably, the subgraphs induced by $\{1, 2, 3\}$ and $\{4, 5, 6\}$ have identical spectral radii. This similarity in network density, coupled with asymmetries in perception, is central to understanding the uniqueness of the exclusionary PCE—a preview of the mechanisms explored in the next section.

Figure 3b: The perception sets of agents 1 and 2 are now given by $\mathcal{N}_1 = \{3, 6\}$ and $\mathcal{N}_2 = \{3, 4, 6\}$. This modification significantly alters the set of PCEs. Specifically, the reduction in connectivity within the sub-network $\{1, 2, 3\}$ can be interpreted as a relative weakening of the competitive pressure exerted by this group. In turn, this shift amplifies the relative strength of the competitive feedback from agents $\{4, 5, 6\}$, despite the fact that their perception sets remain unchanged. As a result, an inclusive PCE now exists in which all agents receive a positive share of the resource. Perhaps unexpectedly, an exclusionary PCE also persists—similar in structure to the one identified in Figure 3a, though the equilibrium actions of agents $\{1, 2, 3\}$ differ. See Example F3 for the explicit computation of both inclusive and exclusionary PCEs for Figures 3a and 3b. $\diamond$

4.2.1 Communities

Example 3 illustrates the multiplicity of equilibria, including the coexistence of inclusive and exclusionary PCEs. Motivated by this, we aim to develop a systematic method for characterizing the entire set of PCEs. We introduce a novel approach under the restriction that payoffs take the form $h(x_i, X, \alpha) = x_i f\left(\frac{X}{\alpha}\right)$. Our method relies on a new graph-theoretic decomposition that (i) partitions the perception network into communities and (ii) ranks these communities according to a specific partial order. The partitioning is grounded in an intuitive application of the *Frobenius normal form*, which is applicable to weakly connected networks.[25] The resulting community ranking provides key insights into the structure of perceived competition within the network.

Let us start with some important definitions. For any subset of agents $M \subseteq N$, let $\mathbf{G}_M$ denote the restriction of matrix $\mathbf{G}$ to M. Given $\mathbf{G}$, we say that $(M, \mathbf{G}_M)$ is a *strongly connected sub-network* of $\mathbf{G}$ if for all $i, j \in M$, we have both $i \rightrightarrows j$ and $j \rightrightarrows i$.

Definition 6. *Given $M \subset N$, $(M, \mathbf{G}_M)$ is a **strongly connected component** (or **community**) of $(N, \mathbf{G})$ if:*

(i) it is a strongly connected sub-network;

(ii) for each $I \subset N \setminus M$, $(M \cup I, \mathbf{G}_{M \cup I})$ is not a strongly connected network.

Let $\mathcal{C}(\mathbf{G})$ denote the set of communities in $(N, \mathbf{G})$, i.e.,

$$\mathcal{C}(\mathbf{G}) := \{M \subset N : (M, \mathbf{G}_M) \text{ is a community of } (N, \mathbf{G})\}.$$

An element of $\mathcal{C}(\mathbf{G})$ is a subset of agents of cardinality at least equal to two, and such that the corresponding sub-network is strongly connected. Note that under the no-isolation assumption, $\mathcal{C}(\mathbf{G})$ is never empty and communities form a partition of the set of agents.

To illustrate this definition, consider the perception network in Figure 3(a) with $N = \{1, 2, 3, 4, 5, 6\}$ and define $M_1 = \{1, 2, 3\}$ and $M_2 = \{4, 5, 6\}$. We can see that both $(M_1, \mathbf{G}_{M_1})$ and $(M_2, \mathbf{G}_{M_2})$ are strongly connected components: each is a strongly connected sub-network, and it is not possible to enlarge any of these two sub-networks to form a larger strongly connected network. Hence, $\mathcal{C}(\mathbf{G}) = \{M_1, M_2\}$. Note that the set of communities is unchanged in the perception network of Figure 3(b).

We introduce a natural partial ordering $\succeq$ on the set of communities $\mathcal{C}(\mathbf{G})$.[26] The binary relation $\succeq$ is defined as:

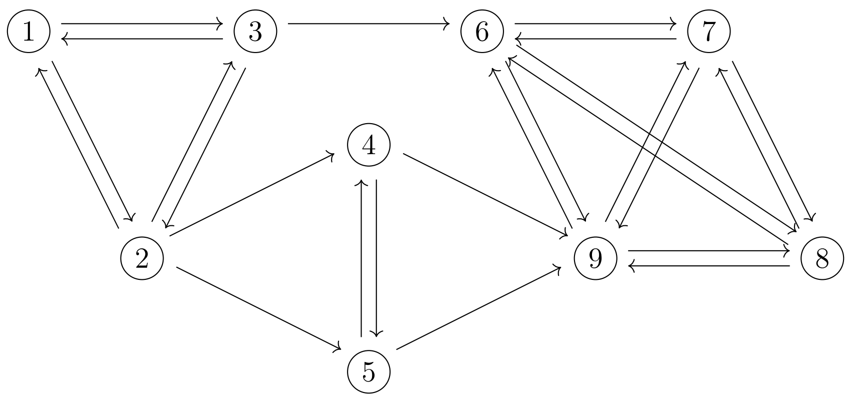
$$M' \succeq M \text{ if and only if } \exists i' \in M',\ i \in M \text{ such that } i' \rightrightarrows i.$$

In other terms, $M' \succeq M$ if there exists a path from M' to M. As usual, if $M' \succeq M$ and $M' \neq M$, we write $M' \succ M$. A community M is $\succeq$-*maximal* if no community M' exists such

[25] See Online Appendix B.1 for a detailed discussion of the Frobenius normal form.

[26] A *partial ordering* is a reflexive, antisymmetric, and transitive binary relation.

Figure 4: Network structure in Example 4



that $M' \succ M$. That is, there is no community M' for which one of its members is *aware* of some members of community M. In that case, we say that there is no M' that *is aware* of M. If not, we say that M is *hidden* from M'. The maximal elements for this partial ordering are communities that few agents are aware of. This "advantage" can be intuitively captured by making the following observation: thanks to the closedness of the set of active agents, given a PCE $\mathbf{x}^*$, we have

$$M \subset N_+(\mathbf{x}^*) \Rightarrow M' \subset N_+(\mathbf{x}^*),\ \forall M' \succeq M.$$

Let us now illustrate the concept of communities and the $\succeq$-ordering.

Example 4. *An illustration of community rankings.*

Figure 3 revisited. There are two communities: $M_1 = \{1, 2, 3\}$ and $M_2 = \{4, 5, 6\}$. Since there is a link from 1 to 6 and no path from M_2 to M_1 , we have $M_1 \succ M_2$, and, clearly, community M_1 is $\succeq$-*maximal.*

A larger perception network. Consider now Figure 4 with $N = \{1, 2, \cdots, 9\}$. There are three communities: $M_1 = \{1, 2, 3\}$, $M_2 = \{4, 5\}$, and $M_3 = \{6, 7, 8, 9\}$. To check that these three connected sub-networks satisfy (*ii*) in Definition 6, note that there is no path from M_3 to either M_1 or M_2, and that there is no path from M_2 to M_1. However, there is a link from 2 to 4, so that $M_1 \succ M_2$. There is also a link from 4 to 9, so that $M_2 \succ M_3$. Finally, we obtain $M_1 \succ M_2 \succ M_3$. Community M_1 is $\succeq$-*maximal.* ◇

4.2.2 Characterization of Perception-Consistent Equilibria.

Thus far, we have shown that any weakly connected network $\mathbf{G}$ can be associated with an ordering over its set of strongly connected components. Equipped with this decomposition of $\mathbf{G}$, we now turn to the mechanism that determines PCEs. Specifically, we characterize the set of active agents at a PCE based on their position within the perception network and the "density" of the strongly connected component (or community) to which they belong.

Definition 7. *Given a community M, define*

$$\bar{M} := \left\{ i' \in N : \exists i \in M \text{ with } i' \rightrightarrows i \right\} = \bigcup_{M' : M' \succeq M} M'.$$

We refer to $\bar{M}$ as the candidate set with root M. *A perception-consistent equilibrium* $\mathbf{x}^*$ *of* $(N, \mathbf{G})$ *such that* $N_+(\mathbf{x}^*) = \bar{M}$ *is called an* equilibrium with root M.

A *candidate set* is a group of agents that could plausibly constitute the set of active players at equilibrium. Specifically, if some agent i is active in a set M, then all agents in the closure $\bar{M}$ must also be active. This follows from the closedness property of the set of active agents at a PCE: any agent that is path-connected to an active agent must also be active. Recall that the spectral radius of a matrix $\mathbf{G}$ is defined as $\rho(\mathbf{G}) = \max\{|\lambda_1|, \ldots, |\lambda_n|\}$, where λ_ℓ denotes the eigenvalues of $\mathbf{G}$. We then have the following useful identity: $\rho(\mathbf{G}) = \max_{M \in \mathcal{C}(\mathbf{G})} \rho(\mathbf{G}_M)$, as shown in Equation (B.2) in Online Appendix B. This equality highlights the connection between the ordering over communities and the Frobenius normal form of the perception network $\mathbf{G}$.

Proposition 4. ***PCE and Community Ranking.*** *There is at most one perception-consistent equilibrium* $\mathbf{x}^*$ *with root M. It exists if and only if*

$$\rho(\mathbf{G}_M) > \max\left\{ \rho(\mathbf{G}_{M'}) : \; M' \in \mathcal{C}(\mathbf{G}), \; M' \succ M \right\}. \tag{15}$$

In particular, for any $\succeq$-maximal M, there always exists an equilibrium with root M.

This proposition identifies the mechanism that determines which agents are active and which are inactive at a PCE. Once the sets of active and inactive agents are established, the eigenvector centrality result uniquely determines the actions chosen by the active agents. Importantly, for any community M, there exists *at most one* PCE rooted in M. Proposition 4 provides a necessary and sufficient condition for the existence of such a PCE, based on a comparison of leading eigenvalues. This condition is automatically satisfied for *maximal* communities, since the set $\{M' \in \mathcal{C}(\mathbf{G}) : M' \succ M\}$ is empty. However, for non-maximal elements of $\mathcal{C}(\mathbf{G})$, verifying the existence of a PCE with root M requires checking the non-trivial inequality (15). Proposition 4 thus sheds light on the differences in the set of PCEs illustrated in Example 3.

Remark 3. *The mechanism behind PCEs, as revealed by Proposition 4, is independent of the specific application. That is, the sets of active and inactive agents remain invariant across* *all models within the class of payoff functions defined by* $h(x_i, X, \alpha) = x_i f\left(\frac{X}{\alpha}\right)$. *However, the equilibrium quantities chosen by the active agents do depend on the particular form of the aggregator* f. *These quantities are fully determined by the eigenvector centrality characterization, as the scalar* $\lambda(X^*/\alpha) \equiv \frac{-f(X^*/\alpha)-f'(X^*/\alpha)[X^*/\alpha]}{f(X^*/\alpha)}$ *is directly shaped by the choice of* f. $\diamond$

Theorem 3 establishes that an agent becomes more central—and thus chooses higher actions—the more she is connected to other central agents. However, central agents are not necessarily confined to communities that are hierarchically superior. In fact, highly central agents may be positioned anywhere within the community hierarchy. Consequently, it is not straightforward to determine, *a priori*, whether and how actions and payoffs can be ranked across communities. Addressing this question requires imposing additional structure on the set of perception networks. We provide a partial yet sharp answer, offering a detailed account of the perceived competition rents accrued by agents.

Definition 8. *Consider a class of perception networks* $\mathcal{G}$, *referred to as* ***regular communities graphs***, *defined as follows. For any* $\mathbf{G} \in \mathcal{G}$, *the set of communities* $\mathcal{C}(\mathbf{G})$ *consists of* ℓ *communities, ordered as* $M_1 \succ M_2 \succ \cdots \succ M_\ell$. *For each* $k = 1, \ldots, \ell$, *community* M_k *is a regular subgraph of order* γ_k *satisfying the following conditions for every agent* $i \in M_k$:

(i) $|\mathcal{N}_i \cap M_{k+1}| = \lambda_k$ *for* $1 \leq k \leq \ell - 1$, *where* $1 \leq \lambda_k \leq |M_{k+1}|$;

(ii) $|\mathcal{N}_i \cap M_h| = 0$ *for all* $h \neq k, k+1$.

Note that every community M_k—except for the last one, M_ℓ—is associated with two positive integers: γ_k and λ_k. The former (γ_k) denotes the number of intra-community directed links for each agent $i \in M_k$, i.e., the *intra-community perception* of M_k; the latter (λ_k) represents the number of directed links from each $i \in M_k$ to agents in M_{k+1}, i.e., the *inter-community perception* of M_k. Community M_ℓ is only assigned the value γ_ℓ, as it has no directed links to agents outside of M_ℓ. Given these structural restrictions, it follows that for any regular communities graph $\mathbf{G} \in \mathcal{G}$ and any community M_k, the spectral radius of the subgraph induced by M_k satisfies $\rho(\mathbf{G}_{M_k}) = \gamma_k$.

Definition 9. *A PCE action profile* $\mathbf{x}^* \equiv (\mathbf{x}^*_{M_1}, ..., \mathbf{x}^*_{M_k}, \mathbf{0}_{N\setminus\mathcal{C}(\mathbf{G})})$ *with root* M_k *is an inner-community symmetric equilibrium if for each community* M_h, $h \leq k$, *and each* $i, j \in M_h$, $x_i^* = x_j^* \equiv \bar{x}_{M_h}$. *Likewise, by the inner-community symmetricity, for each* $i \in M_h$, *we simply write* $\pi_i(\mathbf{x}^*, \alpha) \equiv \bar{\pi}_{M_h}(\mathbf{x}^*, \alpha)$.

By Proposition 4, a PCE profile $\mathbf{x}^*$ with root M_k exists if and only if $\gamma_{M_k} > \gamma_{M_h}$ for each $h < k$. By Definitions 8 and 9, for any $\mathbf{G} \in \mathcal{G}$, a PCE with root M_k must be an inner-community symmetric equilibrium $\mathbf{x}^* \equiv (\mathbf{x}^*_{M_1}, ..., \mathbf{x}^*_{M_k}, \mathbf{0}_{N\setminus\mathcal{C}(\mathbf{G})})$.

Proposition 5. ***Perceived Competition Rents.*** *Let $\mathbf{G} \in \mathcal{G}$ be a regular communities graph with ℓ communities and such that $M_1 \succ M_2 \succ ... \succ M_\ell$. Assume that $\gamma_k > \gamma_h$ for all $1 \leq h < k \leq l$, so that there exists an inner-community symmetric PCE $\mathbf{x}^*$, with root M_k. Then, for $1 \leq h < k \leq l$, we have:*

$$\bar{\pi}_{M_h}(\mathbf{x}^*, \alpha) = \prod_{z=h}^{k} \left[\frac{\lambda_z}{\gamma_k - \gamma_z} \right] \bar{\pi}_{M_k}(\mathbf{x}^*, \alpha)$$

In addition, assume that $\lambda_h = |M_{h+1}|$. Then, for $1 \leq h < k \leq l$,

$$\bar{\pi}_{M_h}(\mathbf{x}^*, \alpha) > \bar{\pi}_{M_k}(\mathbf{x}^*, \alpha)$$

Proposition 5 provides a detailed account of perceived competition rents not only within a given PCE but also across different PCEs. Agent i's preferred PCE is always the one rooted in the community to which she belongs. For example, consider a maximal community M_1. By Proposition 4 and closedness, agents in M_1 are active in all PCEs, as their superior perception of the competitive environment justifies their participation. However, their payoffs decline with the size of the active set, since, all else equal, agents in M_1 prefer facing less rather than more competition. Crucially, payoff advantages for agents in M_1 are guaranteed only at the PCE with root M_1. If other PCEs exist, they must be rooted in different communities M_k with $k > 1$, again by Proposition 4. Moreover, if a PCE $\mathbf{x}_k^*$ has root M_k, then $\rho(\mathbf{G}_{M_k}) > \rho(\mathbf{G}_{M_1})$. Proposition 5 thus links payoffs across active communities, reflecting both intra- and inter-community perceptions along the upstream chain from M_k to M_1. Accordingly, whether M_1 maintains payoff dominance at other PCEs depends on the structure of the perception network. Nonetheless, inequality remains a core feature of PCEs: for any PCE $\mathbf{x}^*$ rooted in M_k, and for any $i \in M_k$ up to M_1, we have $\pi_i(\mathbf{x}_i^*) > \pi_i(\mathbf{x}_j^*) = 0$ for all $j \in M_\ell$ with $\ell > k$.

In Section F of the Online Appendix, we illustrate Proposition 5 and its corollary by revisiting Example 4 using the Tullock model as an application.

4.3 Dynamics, Stability and Eigenvector Centrality

This section aims to refine the set of equilibria by characterizing those perception-consistent equilibria (PCE) that are *stable*. This refinement serves two main purposes. First, it allows us to identify which equilibria are robust to perturbations, thereby providing a *dynamic microfoundation* for the concept of PCE. Second, such refinement is essential for extending the eigenvector centrality microfoundation to general networks. As noted earlier, the efforts of active agents are proportional to their eigenvector centrality within the sub-network of active players. This observation naturally prompts the following question: *Is there a connection between eigenvector centrality in the full network and PCE?* The answer is affirmative. We show that there exists exactly one PCE in which agents' efforts are proportional to their eigenvector centrality in the full network. This unique PCE is precisely the one we identify as *stable.*

4.3.1 Understanding PCE: A Dynamical Viewpoint

Before proceeding to the analysis of stable equilibria, we note that a PCE has a very natural dynamic foundation, which we provide below. Doing so nicely bridges the previous part to the stability analysis. Agents (correctly) assume that the game is aggregative as per equation (1) (objective utility), but misperceive the intensity of competition. In this setting, a natural adaptive process, which mimics classical best-response dynamics, is the following:

- Agents initially have beliefs on both the aggregate effort in their perception set and the parameter. These two quantities are denoted by $X^0_{\mathcal{N}_i}$ and α^0_i.
- At period $t \geq 1$, agent i chooses her effort level, x^t_i, by maximizing the map

$$x_i \in [0, +\infty[\mapsto h_i\left(x_i, x_i + \mathbf{x}^{t-1}_{\mathcal{N}_i}, \alpha^{t-1}_i\right).$$

Then, she observes the realization of the total effort in her perception set, $X^t_{\mathcal{N}_i}$, and updates the parameter α^t_i by identifying the quantities $h_i(x^t_i, X^t, \alpha)$ and $h_i\left(x^t_i, x^t_i + X^t_{\mathcal{N}_i}, \alpha^t_i\right)$. For instance, suppose that the quantity which maximizes $x_i \mapsto h_i\left(x_i, x_i + \mathbf{x}^{t-1}_{\mathcal{N}_i}, \alpha^{t-1}_i\right)$ is $x^t_i = 1$. Suppose also that $X^t = 4$ and $X^t_{\mathcal{N}_i} = 2$. Agent i observes that the payoff she obtains is $h_i(1, 4, \alpha)$. Since she believes that her payoff is of the form $h_i\left(x^t_i, x^t_i + \mathbf{x}^t_{\mathcal{N}_i}, \alpha^t_i\right)$, and observes that $x^t_i + X^t_{\mathcal{N}_i} = 3$, she concludes that $\alpha^t_i = 3\alpha/4$.[27]. Then, before choosing her effort at time $t+1$, she updates her conjecture about her payoff function, as $x_i \mapsto h_i\left(x_i, (x_i + 2), 3\alpha/4\right) = x_i f\left(\frac{3\alpha}{4(x_i+2)}\right)$.

This adaptive process $(\mathbf{x}^t)_{t \geq 0}$ can be written as $x^{t+1}_i = Br_i(\mathbf{x}^t)$, where Br_i is the perceived best-response map of agent i, defined in (5):

$$x^{t+1}_i = Argmax_{x_i \geq 0}\ y_i\, f\left(\frac{\alpha(x^t_i + X^t_{\mathcal{N}_i})}{X^t(x_i + X^t_{\mathcal{N}_i})}\right).$$

For instance, in the standard Tullock model, defined in (8), we can explicitly compute the best-response map, and we obtain the dynamics

$$x^{t+1}_i = \begin{cases} -X^t_{\mathcal{N}_i} + \left(\frac{\alpha}{cX^t} X^t_{\mathcal{N}_i}\left(x^t_i + X^t_{\mathcal{N}_i}\right)\right)^{1/2} & \text{if } \frac{\alpha}{cX^t}(x^t_i + X^t_{\mathcal{N}_i}) \geq X^t_{\mathcal{N}_i} \\ 0 & \text{if } \frac{\alpha}{cX^t}(x^t_i + X^t_{\mathcal{N}_i}) < X^t_{\mathcal{N}_i} \end{cases}$$

[27]We can be that specific because, in this section, we treat the multiplicative form of Definition 5. In the general case, she would only conclude that α^t_i solves the equation $h_i(1, 4, \alpha) = h_i(1, 3, \alpha_i)$.

4.3.2 Stable Perception-Consistent Equilibria

As usual, the stability of equilibria is defined through a meaningful dynamical system, whose rest points are exactly the equilibria we are interested in. A stable equilibrium is then defined as a stable rest point of the dynamics. In our framework the natural dynamics is what we will call *perceived best-response dynamics*:

$$\dot{x}_i(t) = -x_i(t) + Br_i\left(\mathbf{x}(t)\right). \tag{16}$$

It captures the idea that agents smoothly adapt their actions in the direction of their perceived best-response. Choosing the appropriate state space, the stationary points of this ordinary differential equation are precisely the perception-consistent equilibria of our problem, and stability for a given PCE $\mathbf{x}^*$ means that the solutions of (16) starting from initial conditions close enough to $\mathbf{x}^*$ converge back to $\mathbf{x}^*$. Formally:

Definition 10. *A perception-consistent equilibrium* $\mathbf{x}^*$ *is said to be* ***asymptotically stable*** *for (16) if there exists an open neighborhood* U *of* $\mathbf{x}^*$ *such that*

$$\lim_{t\to+\infty} \sup_{\mathbf{x}_0 \in U\cap\mathbf{S}} \|\phi(\mathbf{x}_0, t) - \mathbf{x}^*\| = 0,$$

where $\mathbf{S}$, defined in (22) in the proof of Theorem 4, contains all the relevant states of the problem we consider, and $(\phi(\mathbf{x}, t))_{\mathbf{x}\in\mathbf{S}, t\geq 0}$ is the semi-flow associated to (16) on $\mathbf{S}$. Specifically, $\phi(\mathbf{x}, t)$ is equal to the position of the (unique) solution of system (16), starting at state $\mathbf{x}$.

Definition 10 states that a PCE $\mathbf{x}^*$ is asymptotically stable if it *uniformly* attracts all solutions starting in an open neighborhood of itself. This is a standard concept of stability used in economics (Benaïm and Hirsch, 1999; Weibull, 2003), and in network games in particular (Bramoullé et al., 2016; Bervoets and Faure, 2019).

We now characterize the PCEs that are asymptotically stable with respect to the best-response dynamics (16). It turns out that being asymptotically stable depends entirely on the sub-network of active players in this PCE, in a very simple and intuitive way. Given a PCE $\mathbf{x}^*$, we call $\rho(\mathbf{x}^*)$ the largest eigenvalue of the sub-network $(N_+(\mathbf{x}^*), \mathbf{G}_{N_+(\mathbf{x}^*)})$.

Theorem 4. [***Stability***] *There is a unique asymptotically stable equilibrium* $\mathbf{x}^*$. *It satisfies* $\rho(\mathbf{x}^*) = \rho(\mathbf{G})$. *Moreover, agents' effort levels at* $\mathbf{x}^*$ *are proportional to their eigenvector centrality in the whole network* $(N, \mathbf{G})$.

Formally speaking, this result holds under the mild assumption that $(N, \mathbf{G})$ has a *unique dominant component*, as properly defined in condition (UDC) in Online Appendix B.3.[28]

[28] A weakly connected network has a unique dominant component if $\forall M, M' \in \mathcal{C}(\mathbf{G})$, $\rho(\mathbf{G}_M) = \rho(\mathbf{G}_{M'}) = \rho(\mathbf{G}) \Rightarrow M \succeq M'$ or $M' \succeq M$.

When this assumption is satisfied, there is exactly one PCE for which the largest eigenvalue of the set of active players is equal to $\rho(\mathbf{G})$. The intuition behind the characterization in terms of largest eigenvalues is as follows. We must show that it is the only asymptotically stable equilibrium. Suppose that $\mathbf{x}^*$ is a PCE such that $\rho(\mathbf{x}^*)$ is strictly smaller than $\rho(\mathbf{G})$. Then, one can find a community M in which agents are inactive at $\mathbf{x}^*$, while having $\rho(\mathbf{G}_M) = \rho(\mathbf{G})$. Now, suppose that we slightly perturb $\mathbf{x}^*$ so that, instead of playing zero, agents in M play $\epsilon \mathbf{u}_i$, where $\mathbf{u}$ is the normalized positive eigenvector associated with $\rho(\mathbf{G})$. Since, for agents in M, this initial condition is associated with an eigenvalue that is strictly larger than the eigenvalue associated with $\mathbf{x}^*$, the agents in M will want to increase their effort and not come back to zero. Thus, it is clear that $\mathbf{x}^*$ cannot be stable.[29] We conclude the proof by showing that the (unique) PCE for which $\rho(\mathbf{x}^*) = \rho(\mathbf{G})$ is stable using standard methods. The last part of the theorem directly follows from the definition of eigenvector centrality.

Theorem 4 provides a simple and efficient analytic method for checking which PCEs are stable by looking for communities with the highest spectral radii. We illustrate this below.

Example 5. *Uniqueness of the stable PCE.*
We consider again Figure 3(a) and (b). Recall that there are two communities: $M_1 = \{1, 2, 3\}$ and $M_2 = \{4, 5, 6\}$, with $M_1 \succ M_2$. The only difference is that the perception network in Figure 3(a) has two extra links between agents 1 and 2. This is an important difference because the largest eigenvalue of the $\succeq$-maximal community, M_1, changes: it is equal to 2 in Figure 3(a), whereas it is equal to $\sqrt{2}$ in Figure 3(b). In Figure 3(a), there is a unique exclusionary PCE –and, therefore, stable. In Figure 3(b), there are two PCEs: $\mathbf{x}^*$ with root $M_1 = \{1, 2, 3\}$, and $\mathbf{y}^*$ with root $M_2 = \{4, 5, 6\}$. Since $\rho(\mathbf{G}_{M_1}) = \sqrt{2} < \rho(\mathbf{G}_{M_2}) = 2 = \rho(\mathbf{G})$, only $\mathbf{y}^*$ is stable. ◇

In summary, for any perception network, we can determine the unique stable perception-consistent equilibrium. Firstly, we establish the $\succeq$-ordering as defined when we introduced the notion of communities. Secondly, we determine the different perception-consistent equilibria by checking, for each community, if its spectral radius is strictly greater than that of the communities that dominate it as per the $\succeq$-ordering (Proposition 4). For each PCE, we can ascertain the effort of each agent, which is proportional to her eigenvector centrality (Theorem 3) in the set of active agents. Finally, the unique stable perception-consistent equilibrium is the PCE for which the corresponding root has the same largest eigenvalue as the whole perception network (Theorem 4). We conclude this section with the following corollary of Theorem 4.

[29] For ease of presentation, *asymptotically stable* PCEs are referred to as *stable* PCEs.

Corollary 1.

(i) Suppose that the perception network $\mathbf{G}$ *admits an inclusive PCE* $\mathbf{x}^*$*. Then* $\rho(\mathbf{x}^*) = \rho(\mathbf{G})$*, and* $\mathbf{x}^*$ *is asymptotically stable.*

(ii) Suppose that the perception network $\mathbf{G}$ *admits* k *distinct PCEs, denoted by* $\mathbf{x}^{*1}, \ldots, \mathbf{x}^{*k}$*, such that* $|N_+(\mathbf{x}^{*1})| < \cdots < |N_+(\mathbf{x}^{*k})|$*. Then the unique asymptotically stable PCE is* $\mathbf{x}^{*k}$*—that is, the equilibrium that maximizes the number of active agents across all PCEs. Note that* $|N_+(\mathbf{x}^{*k})|$ *is not necessarily equal to* n*.*

In Appendix G, we examine various welfare and policy implications. Section G.1 begins with key insights on aggregate payoffs (welfare) and effort levels. Under the Tullock contest function, we show that while aggregate utility declines with increasing network density, aggregate effort rises. We then analyze three distinct policy interventions. In Section G.2, we consider the addition of a single link in the network—effectively providing a pair of agents with more information about their competitors—and demonstrate that this increases the relative effort between them. Next, in Section G.3, we study the classic key-player policy of Ballester et al. (2006). In the context of the Tullock model, we show that removing any agent never increases total equilibrium effort. Finally, Section G.4 investigates a social mixing policy, where two initially separate networks are merged. We show that the total effort in any new stable PCE of the merged network exceeds the combined total efforts of the two original networks. Thus, connecting the neighborhoods enhances aggregate effort.

5 Conclusion

In the standard Industrial Organization literature, firms are typically assumed to know all their competitors. Yet, in practice, interviews with CEOs and managers reveal that seemingly similar firms often hold very different views of their competitive environment. In particular, perceptions of market intensity vary widely across businesses due to differences in competencies, beliefs, and circumstances (Kemp and Hanemaaijer, 2004; Giaglis and Fouskas, 2011).

Although extensive work in cognitive psychology, strategic management, and marketing has emphasized the role of perceived competition and networks (see, e.g., Zaheer and Usai, 2004; Thatchenkery and Katila, 2021), no formal model has captured these concepts or their implications for market equilibrium. We propose such a model, in which agents possess only imperfect knowledge of their competitors, represented by a perceived network. We introduce a new equilibrium concept, the perception-consistent equilibrium (PCE). At a PCE, agents know only their local competitors and never learn about their global competitors, yet their subjective payoffs must coincide with those under global competition—without implying a Nash equilibrium of the full-information game. We show that PCEs feature local pockets of inactivity and market exclusion, with agents' effort levels ordered by their eigenvector centrality. This highlights the importance of individual (perceived) network position and community membership in shaping outcomes.

More broadly, the concept of perceived competition helps explain situations where competition is not viewed as reciprocal and agents focus solely on local rivals, even though competition operates globally.

Appendix: Proofs of all results in the main text

Proof of Lemma 1. By Assumption 2.1, note that $\lim_{\alpha_i \to 0} h_i\left(x_i, x_i + X_{\mathcal{N}_i}, \alpha_i\right) < h_i(x_i, X, \alpha) \leq h_i\left(x_i, x_i + X_{\mathcal{N}_i}, \alpha\right)$. Since the map $\alpha \mapsto h_i(x_i, x_i + X_{\mathcal{N}_i}, \alpha)$ is strictly increasing and that $\lim_{\alpha_i \to 0} h_i\left(x_i, x_i + X_{\mathcal{N}_i}, \alpha_i\right) < h_i\left(x_i, x_i + X_{\mathcal{N}_i}, \alpha\right)$, existence and uniqueness of $\alpha_i \in]0, \alpha]$ verifying Equation (3) follows. Regularity of the mapping α_i is inherited from the fact that h_i is continuously differentiable. □

Proof of Theorem 1. We prove existence under the set of assumptions listed in the Theorem, Assumptions 2.1, 2.2, and 2.3. Given $\mathbf{x} \neq \mathbf{0}$, recall that $\alpha_i(\mathbf{x})$ is the solution of the equation $h_i\left(x_i, x_i + X_{\mathcal{N}_i}, \alpha_i\right) = h_i\left(x_i, X, \alpha\right)$, and that a non-zero PCE is a fixed point of the map $Br : \mathbf{x} \in \mathbb{R}_+^n \setminus \{\mathbf{0}\} \mapsto (Br_1(\mathbf{x}), ..., Br_n(\mathbf{x}))$ (see (5)).

By point *c*) of Assumption 2.1, there exists $\overline{x}_i$ such that $Br_i(\mathbf{x}) < \overline{x}_i$, for any $\mathbf{x} \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$. Consequently, taking $\overline{X} := \sum_i \overline{x}_i$ we obtain that the image of the map Br is contained in $\left\{\mathbf{x} \in \mathbb{R}_+^n : \sum_i x_i \leq \overline{X}\right\}$. Let $\underline{X} > 0$ be constructed as follows: by Assumption 2.2, there exists a quantity $\alpha(1/n) > 0$ such that, for all $X > 0$,

$$h_i\left(x_i, \frac{1}{n}X, \alpha(1/n)\right) \leq h_i(x_i, X, \alpha), \;\; \forall x_i \leq X. \tag{17}$$

Let $\underline{\alpha} := \alpha(1/n)$. By Assumption 2.3, there exists $\underline{X}$ such that

$$\frac{\partial h_i}{\partial x_i}\left(x_i, X, \alpha\right) + \frac{\partial h_i}{\partial X}\left(x_i, X, \alpha\right) > 0, \;\; \forall (x_i, X) \text{ such that } \; 2x_i \leq X \leq \underline{X}, \;\; \forall \alpha > \underline{\alpha}. \tag{18}$$

Now define $K := \left\{\mathbf{x} \in \mathbb{R}_+^n : \sum_{i=1}^n x_i \in [\underline{X}, \overline{X}]\right\}$, and let π_K be the projection on K [30]. Define $\hat{B}_r : K \mapsto K$ as $\hat{B}_r(\mathbf{x}) = \Pi_K(Br(\mathbf{x}))$. K is compact and convex, and $\hat{B}r$ is continuous from

[30] i.e. $\pi_K(\mathbf{x})$ is the (unique) point in K minimizing the Euclidean distance from $\mathbf{x}$ to K.

K to itself. Hence $\hat{Br} : K \to K$ admits a fixed point $\mathbf{x}^*$. We now need to check that $\mathbf{x}^*$ is a PCE.

Suppose by contradiction that $\mathbf{x}^* \neq Br(\mathbf{x}^*)$, and let $\hat{\mathbf{x}} := Br(\mathbf{x}^*)$. Then $0 \leq \hat{x}_i < x_i^*$ for all i. Let i be such that $X^*_{\mathcal{N}_i} \geq X^*_{\mathcal{N}_j}$ for all j. Note that

$$h_i(x_i^*, x_i^* + X^*_{\mathcal{N}_i}, \underline{\alpha}) \leq h_i\left(x_i^*, \frac{1}{n}X^*, \underline{\alpha}\right) \leq h_i(x_i^*, X^*, \alpha) = h_i(x_i^*, x_i^* + X^*_{\mathcal{N}_i}, \alpha_i(\mathbf{x}^*)).$$

The first inequality follows from the fact that, since the graph has the no-isolation property, we have $X^*_{\mathcal{N}_i} \geq \max_j X^*_{\mathcal{N}_j} \geq \max_j x_j^* \geq \frac{1}{n}X^*$. The second inequality follows from (17), and the equality from the definition of $\alpha_i(\mathbf{x}^*)$. Consequently, $\underline{\alpha} \leq \alpha_i(\mathbf{x}^*)$. Also note that $2\hat{x}_i \leq \hat{x}_i + X^*_{\mathcal{N}_i} \leq X^* = \underline{X}$. Applying (18) to $(\hat{x}_i, \hat{x}_i + X^*_{\mathcal{N}_i}, \alpha_i(\mathbf{x}^*))$, we obtain

$$\frac{\partial h_i}{\partial x_i}\left(\hat{x}_i, \hat{x}_i + X^*_{\mathcal{N}_i}, \alpha_i(\mathbf{x}^*)\right) + \frac{\partial h_i}{\partial X}\left(\hat{x}_i, \hat{x}_i + X^*_{\mathcal{N}_i}, \alpha_i(\mathbf{x}^*)\right) > 0,$$

This contradicts the fact that $\hat{x}_i = Br_i(\mathbf{x}^*)$. Hence $\mathbf{x}^*$ is a PCE. □

Proof of Lemma 2. The map $x_i \mapsto \pi_i(x_i, \mathbf{x}_{-i}, \alpha)$ is strictly concave:

$$\frac{\partial \pi_i}{\partial x_i}(x_i, \mathbf{x}_{-i}; \alpha) = \frac{\overline{\theta} - \theta_i + \mathbf{x}_{-i}}{(\overline{\theta} + x_i + \mathbf{x}_{-i})^2}\alpha - c,$$

which is strictly decreasing in x_i. Moreover, point (a) of Assumption 2.1 follows because $\frac{\partial \pi_i}{\partial x_i}$ is negative for large enough $\mathbf{x}_{-i}$. Point (b) is immediate. Also, for point (c),

$$h_i(x_i, x_i, 0) = -cx_i < \frac{\theta_i + x_i}{\overline{\theta} + x_i + \mathbf{x}_{-i}}\alpha - cx_i = h_i(x_i, X, \alpha),$$

which concludes the proof that Assumption 2.1 holds.

We now check that Assumption 2.2 holds: let $\lambda > 0$, and pick $\alpha(\lambda) := \alpha \min\{\lambda, 1\}$. Note that $\alpha(\lambda) = \min_{X>0} \frac{\overline{\theta}+\lambda X}{\overline{\theta}+X}$. Thus

$$\frac{\theta_i + x_i}{\overline{\theta} + \lambda X}\alpha(\lambda) < \frac{\theta_i + x_i}{\overline{\theta} + X}\alpha, \text{ for all } X > 0.$$

This proves that $h_i(x_i, \lambda X \alpha(\lambda)) < h(x_i, X, \alpha)$ for all $X > 0$. Assumption 2.2 holds.

Finally, we have

$$\frac{\partial h_i}{\partial x_i}(x_i, X, \alpha) + \frac{\partial h_i}{\partial X}(x_i, X, \alpha) = \frac{\overline{\theta} - \theta_i + \mathbf{x}_{-i}}{(\overline{\theta} + X)^2}\alpha - c \geq \frac{(\rho - 1)\overline{\theta} + \mathbf{x}_{-i}}{(\overline{\theta} + X)^2}\alpha - c \geq \frac{K}{\overline{\theta} + X}\alpha - c,$$

for some positive constant K. Hence Assumption 2.3 holds provided $\overline{\theta}$ is small enough. $\square$

Proof of Proposition 1. We now show that, at a PCE, the equilibrium effort of each agent is given by (7). First, note that by solving $\pi_i(x_i, \mathbf{x}_{-i}; \alpha) = u_i(x_i, \mathbf{x}_{\mathcal{N}_i}, \alpha_i)$, we obtain that $\alpha_i(\mathbf{x})$ is the solution of the equation

$$\frac{\theta_i + x_i}{\overline{\theta} + x_i + X_{\mathcal{N}_i}}\alpha_i - cx_i = \frac{\theta_i + x_i}{\overline{\theta} + X}\alpha - cx_i.$$

Consequently,

$$\alpha_i(\mathbf{x}) = \alpha\frac{\overline{\theta} + x_i + X_{\mathcal{N}_i}}{\overline{\theta} + X}.$$

Let $\mathbf{x}^*$ be a PCE. Then, for all i, x_i^* satisfies the first-order condition

$$x_i^* = Argmax_{x_i \geq 0}\ h_i\left(x_i, x_i + X^*_{\mathcal{N}_i}, \alpha_i(\mathbf{x}^*)\right).$$

By solving this program for $i \in N_+(\mathbf{x}^*)$, we obtain

$$\alpha\frac{\left(\overline{\theta} - \theta_i + X^*_{\mathcal{N}_i}\right)}{\overline{\theta} + X^*} = c\left(\overline{\theta} + x_i^* + X^*_{\mathcal{N}_i}\right).$$

Solving this equation, we get

$$X^*_{\mathcal{N}_i} = \frac{c\left(X^* + \overline{\theta}\right)}{\alpha - c\left(X^* + \overline{\theta}\right)}x_i^* + \frac{\alpha\left(\theta_i - \overline{\theta}\right) + c\overline{\theta}\left(\overline{\theta} + X^*\right)}{\alpha - c\left(X^* + \overline{\theta}\right)}.$$

In vector-matrix form, we obtain:

$$\mathbf{G}_+\mathbf{x}_+^* = \frac{c\left(X^* + \overline{\theta}\right)}{\alpha - c\left(X^* + \overline{\theta}\right)}\mathbf{x}_+^* + \frac{\alpha}{\alpha - c\left(X^* + \overline{\theta}\right)}\boldsymbol{\theta} + \frac{c\overline{\theta}\left(\overline{\theta} + X^*\right) - \alpha\overline{\theta}}{\alpha - c\left(X^* + \overline{\theta}\right)}\mathbf{1},$$

which corresponds to equation (7). $\square$

Proof of Theorem 2. We first define the map Br in the more general settings of Assumption 3.1. If $x_i > 0$ then, for any $\mathbf{x}_{-i}$ the map $Br_i(x_i, \mathbf{x}_{-i})$ can be defined exactly the same way as in (5). Now, if $x_i = 0$ and $\mathbf{x}_{-i} \neq 0$, the quantity $\alpha_i(0, \mathbf{x}_{-i})$ is well-defined. By strict quasi-concavity, the set $Argmax_{y_i > 0}\ h_i\left(y_i, y_i + X_{\mathcal{N}_i}, \alpha_i(0, \mathbf{x}_{-i})\right)$ is either a singleton or empty. Thus we can extend the map Br_i in $(0, \mathbf{x}_{-i})$ as follows;

$$Br_i(0, \mathbf{x}_{-i}) = \begin{cases} Argmax_{y_i > 0}\ h_i\left(y_i, y_i + X_{\mathcal{N}_i}, \alpha_i(0, \mathbf{x}_{-i})\right) & \text{if it is nonempty,} \\ 0 & \text{otherwise.} \end{cases}$$

The map Br is continuous on $\mathbb{R}^N_+ \setminus \{\mathbf{0}\}$. By Definition 2, a non-zero PCE is a fixed point of the map Br. We can the replicate the end of the proof of Theorem 1. The only difference lies in the definition of $\alpha_i(0, \mathbf{x}_{-i})$. However, this does not raise any issue, because $x_i^* > 0$ for all i.

□

Proof of Proposition 2. For any $\mathbf{y}$ such that $y_i > 0$, we have $H(y_i + \mathbf{y}_{\mathcal{N}_i}, \alpha_i(y_i, \mathbf{y}_{-i})) = H(y_i + \mathbf{y}_{-i}, \alpha)$. Hence, as y_i goes to zero and $\mathbf{y}_{-i}$ goes to $\mathbf{x}_{-i}$, we have that $\alpha_i(y_i, \mathbf{y}_{-i})$ converges to $\alpha_i(0, \mathbf{x}_{-i})$, the unique solution of the equation $H(X_{\mathcal{N}_i}, \alpha_i) = H(\mathbf{x}_{-i}, \alpha)$. Hence, Assumption 3.1 holds.

1) Let $i \in N_+(\mathbf{x}^*)$. Then the first-order condition for i is

$$G'(x_i^*)H(x_i^* + X^*_{\mathcal{N}_i}, \alpha_i(\mathbf{x}^*)) + G(x_i^*)\frac{\partial H}{\partial X}(x_i^* + X^*_{\mathcal{N}_i}, \alpha_i(\mathbf{x}^*)) = 0.$$

Since $H(X^*, \alpha) = H(x_i^* + X^*_{\mathcal{N}_i}, \alpha_i(\mathbf{x}^*))$, we obtain

$$G'(x_i^*)H(X^*, \alpha) + G(x_i^*)\frac{\partial H}{\partial X}(x_i^* + X^*_{\mathcal{N}_i}, \alpha_i(\mathbf{x}^*)) = 0$$

Since $G'(x_i^*) > 0$, $G(x_i^*) > 0$, and $\frac{\partial H}{\partial X} < 0$, we necessarily have $H(X^*, \alpha) > 0$. This concludes the proof of first point.

2) Suppose that $X^*_{\mathcal{N}_i} > 0$, and $x_i^* = 0$. By Point $b)$ of Assumption 2.1, $\alpha_i(0, \mathbf{x}_{-i}) > 0$. Moreover $\lim_{\eta \to 0} Argmax_{x_i \geq \eta} G(x_i)H(x_i + X^*_{\mathcal{N}_i}, \alpha_i(0, \mathbf{x}^*_{-i})) = 0$, i.e.

$$G'(\eta)H(\eta + X^*_{\mathcal{N}_i}, \alpha_i(0, \mathbf{x}^*_{-i})) + G(\eta)\frac{\partial H}{\partial X}(\eta + X^*_{\mathcal{N}_i}, \alpha_i(0, \mathbf{x}^*_{-i})) \leq 0.$$

for all $\eta > 0$. Since $H\left(X^*_{\mathcal{N}_i}, \alpha_i(0, \mathbf{x}^*_{-i})\right) = H(X^*, \alpha)$, sending η to zero, we obtain $G'(0)H(X^*, \alpha) \leq 0$, a contradiction. □

Proof of Proposition 3. Suppose by contradiction that $X^*_{\mathcal{N}_i} > X^*_{\mathcal{N}_j}$ and $x_j^* \geq x_i^*$. Define $\Delta := x_j^* - x_i^* \geq 0$ and $\Delta' := X^*_{\mathcal{N}_j} - X^*_{\mathcal{N}_i} < 0$. Given the confirmed-conjecture implication that $H(X^*, \alpha) = H(x_i^* + X^*_{\mathcal{N}_i}, \alpha_i(\mathbf{x}^*))$, and by the first-order conditions of the subjective utility maximization, we obtain that:

$$0 = G'(x_i^*)H(X^*, \alpha) + G(x_i^*)\Phi(x_i^* + X^*_{\mathcal{N}_i}) = G'(x_j^*)H(X^*, \alpha) + G(x_j^*)\Phi(x_j^* + X^*_{\mathcal{N}_j})$$

By concavity of G, we have $G'(x_i^*)H(X^*, \alpha) \geq G'(x_j^*)H(X^*, \alpha)$. Therefore

$$G(x_i^*)\Phi(x_i^* + X^*_{\mathcal{N}_i}) \leq G(x_j^*)\Phi(x_j^* + X^*_{\mathcal{N}_j})$$

On the other hand,

$$\begin{aligned} G(x_j^*)\Phi(x_j^* + X_{\mathcal{N}_j}^*) &= G(x_i^* + \Delta)\Phi(x_i^* + X_{\mathcal{N}_i}^* + \Delta + \Delta') \\ &< G(x_i^* + \Delta)\Phi(x_i^* + X_{\mathcal{N}_i}^* + \Delta) \\ &\leq G(x_i^*)\Phi(x_i^* + X_{\mathcal{N}_i}^*), \end{aligned}$$

where the strict inequality follows from the fact that Φ is strictly increasing, while the last inequality follows from Equation (12). We reach a contradiction. Hence $X_{\mathcal{N}_i}^* > X_{\mathcal{N}_j}^* \Rightarrow x_i^* > x_j^*$.

We now prove the second implication: suppose by contradiction that $X_{\mathcal{N}_i}^* = X_{\mathcal{N}_j}^*$ and $x_j^* > x_i^*$. Then $\Delta := x_j^* - x_i^* > 0$ and $\Delta' := X_{\mathcal{N}_j}^* - X_{\mathcal{N}_i}^* = 0$. By first-order conditions and concavity of G, we have again $G(x_i^*)\Phi(x_i^* + X_{\mathcal{N}_i}^*) \leq G(x_j^*)\Phi(x_j^* + X_{\mathcal{N}_j}^*)$. On the other hand,

$$\begin{aligned} G(x_j^*)\Phi(x_j^* + X_{\mathcal{N}_j}^*) &= G(x_i^* + \Delta)\Phi(x_i^* + X_{\mathcal{N}_i}^* + \Delta) \\ &< G(x_i^*)\Phi(x_i^* + X_{\mathcal{N}_i}^*), \end{aligned}$$

a contradiction. Hence $X_{\mathcal{N}_i}^* = X_{\mathcal{N}_j}^*$ implies that $x_j^* = x_i^*$
We now show that $\succsim$ satisfies *recursive monotonicity*. Suppose that there exists an injective map $f : \mathcal{N}_j \to \mathcal{N}_i$ such that $f(k) \succsim k$, for all $k \in \mathcal{N}_j$, i.e. $x_{f(k)}^* \geq x_k^*$ for all $k \in \mathcal{N}_j$. Then $X_{\mathcal{N}_i}^* \geq \sum_{k\in\mathcal{N}_j} x_{f(k)}^* \geq \sum_{k\in\mathcal{N}_j} x_k^* = X_{\mathcal{N}_j}^*$. Thus by (13), we obtain that $x_i^* \geq x_j^*$, i.e. that $i \succsim j$. Recursive monotonicity holds. □

Proof of Theorem 3. We first show that, if $\mathbf{x}^*$ is a PCE then, for all i

$$X_{\mathcal{N}_i}^* = \frac{-f(X^*/\alpha) - f'(X^*/\alpha)[X^*/\alpha]}{f(X^*/\alpha)} x_i^*. \tag{19}$$

Since $X^* > 0$, the set of active agents at $\mathbf{x}^*$ is non-empty.
• Let i be active: $x_i^* > 0$. The derivative of the map $x_i \mapsto h_i(x_i, x_i + X_{\mathcal{N}_i}, \alpha_i)$ is equal to

$$f\left(\frac{x_i + X_{\mathcal{N}_i}}{\alpha_i}\right) + \frac{x_i}{\alpha_i} f'\left(\frac{x_i + X_{\mathcal{N}_i}}{\alpha_i}\right).$$

Here $\alpha_i(\mathbf{x}) = \alpha\frac{x_i + X_{\mathcal{N}_i}}{x_i + \mathbf{x}_{-i}}$. By point (a) of Definition 2, for all $i \in N_+(\mathbf{x}^*)$, we have

$$f\left(\frac{X^*}{\alpha}\right) + \frac{x_i^*}{x_i^* + X_{\mathcal{N}_i}^*}\frac{X^*}{\alpha} f'\left(\frac{X^*}{\alpha}\right) = 0.$$

Since $f'(X^*/\alpha) < 0$, we necessarily have $-f'(X^*/\alpha)X^*/\alpha \geq f(X^*/\alpha) > 0$. As a result

$$X^*_{\mathcal{N}_i} = \frac{-f(X^*/\alpha) - f'(X^*/\alpha)X^*/\alpha}{f(X^*/\alpha)} x_i^*.$$

• Let now i be inactive: $x_i^* = 0$. By closedness of the set of active players, we have $X^*_{\mathcal{N}_i} = 0$. Hence (19) also holds.

We now prove the reverse implication. Suppose that $\mathbf{x}^* \in \mathbb{R}^n_+$ is different from zero and satisfies identity (19), with $\frac{-f(X^*/\alpha)-f'(X^*/\alpha)X^*/\alpha}{f(X^*/\alpha)} \geq 0$. For each active agent i, we have

$$f\left(\frac{X^*}{\alpha}\right) + \frac{x_i^*}{x_i^* + X^*_{\mathcal{N}_i}} \frac{X^*}{\alpha} f'\left(\frac{X^*}{\alpha}\right) = 0.$$

Hence x_i^* satisfies the subjective best-reply condition given in Equation (4).
On the other hand, for an inactive agent i, we have $X^*_{\mathcal{N}_i} = 0$, and $\mathbf{x}^*_{-i} = X^*$. We have $\alpha_i(0, \mathbf{x}_{-i}) = \alpha \frac{X_{\mathcal{N}_i}}{\mathbf{x}_{-i}}$. Since $\frac{-f(X^*/\alpha)-f'(X^*/\alpha)X^*/\alpha}{f(X^*/\alpha)} \geq 0$, we have

$$f\left(\frac{X^*}{\alpha}\right) + \frac{X^*}{\alpha} f'\left(\frac{X^*}{\alpha}\right) \leq 0.$$

It implies that the derivative of $x_i \mapsto h_i(x_i, x_i, \alpha_i(0, \mathbf{x}_{-i}))$ is strictly negative for all $x_i > 0$. Consequently $\text{Argmax}_{x_i \geq \eta} h_i\left(x_i, x_i, \alpha_i(0, \mathbf{x}^*_{-i})\right) = \eta$, which concludes the proof. □

Corollary 2. *Let* $(N, \mathbf{G})$ *be a strongly connected network. Then, there exists a unique Perception-Consistent Equilibrium.*

Proof. Suppose that $(N, \mathbf{G})$ is a strongly connected network. Then $\mathbf{G}$ is irreducible and, by the Perron-Frobenius Theorem, there exists a positive eigenvector $\mathbf{y}$ associated to $\rho(\mathbf{G})$. Moreover, any non-negative eigenvector of $\mathbf{G}$ is a multiple of $\mathbf{y}$. By Theorem 3, $\mathbf{x}^*$ is a PCE if and only if it is a non-negative eigenvector of $\mathbf{G}$, associated to eigenvalue $\frac{-f(X^*)-f'(X^*)X^*}{f(X^*)}$. Hence $\mathbf{x}^*$ is a PCE if and only if $\mathbf{x}^*$ is a multiple of $\mathbf{y}$ and $\rho(\mathbf{G}) = \frac{-f(X^*)-f'(X^*)X^*}{f(X^*)}$. Such a vector exists and is uniquely defined. □

Proof of Proposition 4. First note that if $\mathbf{x}^*$ is a PCE with root M then its restriction to $\bar{M}$ is a positive eigenvector of $\mathbf{G}_{\bar{M}}$. By definition of $\bar{M}$, the matrix $\mathbf{G}_{\bar{M}}$ admits a Frobenius normal form as follows:

$$\mathbf{G}_{\bar{M}} = \begin{bmatrix} A_1 & A_{12} & ... & ... & A_{1s+1} \\ 0 & A_2 & A_{23} & ... & A_{2s+1} \\ ... & ... & ... & ... & ... \\ 0 & ... & 0 & A_s & A_{ss+1} \\ 0 & ... & ... & 0 & A_{s+1} \end{bmatrix}, \text{ with } \; A_{s+1} = \mathbf{G}_M.$$

Note that the set $V := \cup_{i=1}^{s} V_i$ is closed and, by definition of $\bar{M}$ we necessarily have $\{M' \in \mathcal{C}(\mathbf{G}) : \; M' \subset V\} = \{M' \in \mathcal{C}(\mathbf{G}) : \; M' \succ M\}$. Hence

$$\rho(\mathbf{G}_V) = \max_{i=1,\dots,s} \rho(A_i) = \max_{M' \in \mathcal{C}(\mathbf{G}):\, M' \succ M} \rho(\mathbf{G}_{M'}).$$

If $\rho(\mathbf{G}_M) > \max_{M' \in \mathcal{C}(\mathbf{G}): M' \succ M} \rho(\mathbf{G}_{M'})$ then

$$\rho(A_{s+1}) = \rho(\mathbf{G}_M) > \max_{M' \in \mathcal{C}(\mathbf{G}): M' \succ M} \rho(\mathbf{G}_{M'}) = \max_{i=1,\dots,s} \rho(A_i),$$

Consequently, we are in the conditions of Lemma B6 in Online Appendix B.[31] Thus $\mathbf{G}_{\bar{M}}$ then admits a unique positive eigenvector $\mathbf{y} = (y_i)_{i \in \bar{M}}$, such that $\sum_{i \in \bar{M}} y_i$ solves the equation $\frac{-f(y/\alpha) - y f'(y/\alpha)/\alpha}{f(y/\alpha)} = \rho$. Let then $\mathbf{x}^*$ be defined as $x_i^* = y_i$ if $i \in \bar{M}$ and $x_i^* = 0$ if $i \in N \setminus \bar{M}$. By construction, $\mathbf{x}^*$ is a PCE with root M and there can be no other one.
To prove the converse, suppose now that $\rho(\mathbf{G}_M) \leq \max_{M' \in \mathcal{C}(\mathbf{G}): M' \succ M} \rho(\mathbf{G}_{M'})$. Then

$$\rho(A_{s+1}) = \rho(\mathbf{G}_M) \leq \max_{M' \in \mathcal{C}(\mathbf{G}): M' \succ M} \rho(\mathbf{G}_{M'}) = \max_{i=1,\dots,s} \rho(A_i).$$

But then, by Lemma B4 in Online Appendix B, we obtain that $\mathbf{G}_{\bar{M}}$ admits no positive eigenvector, and hence $\mathbf{x}^*$ could not be a PCE with root M. This concludes the proof.[32] $\square$

Proof of Proposition 5. Let $\mathbf{G} \in \mathcal{G}$ and $\mathbf{x}^*$ be a PCE with root M_k. Given the structure on $\mathbf{G}$, $\mathbf{x}^*$ is an inner-community symmetric PCE with root M_k, i.e. $\mathbf{x}^* \equiv (\mathbf{x}^*_{M_1}, ..., \mathbf{x}^*_{M_k}, \mathbf{0}_{N \setminus \mathcal{C}(\mathbf{G})})$.

Step 1. Consider the subjective utility maximization program of agent $i \in M_k$. By the first-order condition, we have

[31] Lemma B6 asserts that if a nonnegative matrix admits a Frobenius normal form with some specific restrictions on its upper block, then it admits a unique positive eigenvector, and it is therefore a strongly nonnegative matrix, as defined in Definition B2. See Online Appendix B for details.
[32] Lemma B4 asserts that a nonnegative matrix admits a positive eigenvector if and only if it is a strongly nonnegative matrix.

$$f\left(\frac{x_i^*+x_{\mathcal{N}_i}^*}{\alpha_i^*}\right)+\frac{x_i^*}{\alpha_i^*}f'\left(\frac{x_i^*+x_{\mathcal{N}_i}^*}{\alpha_i^*}\right)=0$$

By the confirmed conjecture requirement at a PCE, we have that $\alpha_i^* = \frac{x_i^*+x_{\mathcal{N}_i}^*}{X^*}\alpha$. Substituting into the first-order condition of agent i, and given the within-community symmetric PCE, we obtain,

$$f\left(\frac{X^*}{\alpha}\right)+\frac{\bar{x}_{M_k}X^*}{\bar{x}_{M_k}(1+\gamma_{M_k})\alpha}f'\left(\frac{X^*}{\alpha}\right)=0,\ \text{ i.e. }\ X^*=-\frac{f\left(\frac{X^*}{\alpha}\right)}{f'\left(\frac{X^*}{\alpha}\right)}(1+\gamma_{M_k})\alpha$$

Step 2. Consider now community M_{k-1}. By the first-order condition and the confirmed conjecture requirement, we have

$$f\left(\frac{X^*}{\alpha}\right)+\frac{\bar{x}_{M_{k-1}}X^*}{\left[(1+\gamma_{M_{k-1}})\bar{x}_{M_{k-1}}+\lambda_{M_{k-1}}\bar{x}_{M_k}\right]\alpha}f'\left(\frac{X^*}{\alpha}\right)=0$$

Substituting for X^*, and solving for $\bar{x}_{M_{k-1}}$ as a function of $\bar{x}_{M_k}$ we obtain

$$\bar{x}_{M_{k-1}}=\frac{\lambda_{M_{k-1}}}{(1+\gamma_{M_k})-(1+\gamma_{M_{k-1}})}\bar{x}_{M_k}=\frac{\lambda_{M_{k-1}}}{\rho(\mathbf{G}_{M_k})-\rho(\mathbf{G}_{M_{k-1}})}\bar{x}_{M_k}$$

Finally, since $\pi_i(\mathbf{x}^*,\alpha)=x_i^*f\left(\frac{X^*}{\alpha}\right)$ for each $i\in N$, we also obtain that

$$\bar{\pi}_{M_{k-1}}(\mathbf{x}^*,\alpha)=\frac{\lambda_{M_{k-1}}}{\rho(\mathbf{G}_{M_k})-\rho(\mathbf{G}_{M_{k-1}})}\bar{\pi}_{M_k}(\mathbf{x}^*,\alpha)$$

Step 3. We can continue iterating this process, up to community M_1, by repeating similar steps. We omit the additional details, and conclude that, for each community M_h, M_{h+1}, with $h+1\leq k$, and for each $i\in M_h$, and $j\in M_{h+1}$, we have:

$$\bar{\pi}_{M_h}(\mathbf{x}^*,\alpha)=\frac{\lambda_{M_h}}{\rho(\mathbf{G}_{M_k})-\rho(\mathbf{G}_{M_h})}\bar{\pi}_{M_{h+1}}(\mathbf{x}^*,\alpha)$$

Finally, note also that by iterating across communities, we obtain that for each community M_h with $1\leq h<k$, and each $i\in M_h$,

$$\bar{x}_{M_h}=\prod_{z=h}^{k}\left[\frac{\lambda_{M_z}}{\rho(\mathbf{G}_{M_k})-\rho(\mathbf{G}_{M_z})}\right]\bar{x}_{M_k} \tag{20}$$

Therefore, for each M_h with $M_h \succ ... \succ M_k$ we obtain that,

$$\bar{\pi}_{M_h}(\mathbf{x}^*, \alpha) = \prod_{z=h}^{k} \left[\frac{\lambda_{M_z}}{\rho(\mathbf{G}_{M_k}) - \rho(\mathbf{G}_{M_z})} \right] \bar{\pi}_{M_k}(\mathbf{x}^*, \alpha) \tag{21}$$

Equation (20) and (21) can be used to link any PCE with root M_k actions and profits across communities that are further apart in the ranking of communities. Hence for each community M_h and M_p, with $p \leq k$ and such that $M_h \succ ... \succ M_p$,

$$\bar{\pi}_{M_h}(\mathbf{x}^*, \alpha) = \prod_{z=h}^{p} \left[\frac{\lambda_{M_z}}{\rho(\mathbf{G}_{M_k}) - \rho(\mathbf{G}_{M_z})} \right] \bar{\pi}_{M_p}(\mathbf{x}^*, \alpha)$$

□

Proof of Theorem 4. Let us first show that the system (16) is well-behaved on the set

$$\mathbf{S} := \{\mathbf{x} \neq \mathbf{0} : \; x_i \geq 0 \;\forall i, \;\; f(X/\alpha) > 0\} \tag{22}$$

in the sense that, for any initial condition in $\mathbf{S}$, there exists a unique solution $(\mathbf{x}(t))_{t\geq 0}$ which forever remains in $\mathbf{S}$.
Given a PCE $\mathbf{x}^*$, we necessarily have $\mathbf{x}^* \in \mathbf{S}$, because the expression $\frac{-f(X/\alpha) - Xf'(X/\alpha)/\alpha}{f(X/\alpha)}$ is either undefined or negative when $f(X/\alpha) \leq 0$. As a consequence $\mathbf{S}$ contains all the relevant states of the problem we consider. □

Lemma 3. *System (16) induces a semiflow on* $\mathbf{S}$.

Proof. We need to check that the vector field B points inward on the boundary of $\mathbf{S}$. Suppose that $\mathbf{x} \in \mathbf{S}$, with $f(X/\alpha) = 0$. Then $Br_i(\mathbf{x}) = Argmax_{y_i \geq 0}\; y_i f\left(\frac{X}{\alpha}\frac{y_i + X_{\mathcal{N}_i}}{x_i + X_{\mathcal{N}_i}}\right)$. Since $f(X/\alpha) = 0$ and f is decreasing, the map $y_i \mapsto y_i f\left(\frac{X}{\alpha}\frac{y_i + X_{\mathcal{N}_i}}{x_i + X_{\mathcal{N}_i}}\right)$ is equal to zero when $y_i = x_i$, is negative when $y_i > x_i$ and positive when $y_i < x_i$. Hence $Br_i(x_{-i}) < x_i$ for all i such that $x_i > 0$. This implies that

$$\sum_i Br_i(\mathbf{x}) - x_i < 0,$$

i.e. $\dot{X} < 0$ at state $\mathbf{x}$. □

The following result will be useful to prove that a point is not asymptotically stable. It directly follows from the definition of asymptotic stability.

Lemma 4. *Let* $\mathbf{x}^*$ *be a PCE such that* $\rho(\mathbf{x}^*) < \rho(\mathbf{G})$. *Then* $\mathbf{x}^*$ *is not asymptotically stable.*

Proof. By definition of asymptotic stability, we need to prove that there exists an open neighborhood U_0 of $\mathbf{x}^*$ with the property that, for any open neighborhood U of $\mathbf{x}^*$ and any $T > 0$, there exists $\mathbf{x} \in U$ such that $\phi(\mathbf{x}, t) \notin U_0$, for any $t \geq T$.

Recall that $\mathbf{x}^*$ is an eigenvector of $\mathbf{G}$, associated to eigenvalue $\rho(\mathbf{x}^*)$, given by

$$\rho(\mathbf{x}^*) = \frac{-f(X^*/\alpha) - f'(X^*/\alpha)X^*/\alpha}{f(X^*/\alpha)}.$$

In what follows, let $\rho^* := \rho(\mathbf{x}^*)$ and $\rho := \rho(\mathbf{G})$.

Let $M^* \in \mathcal{C}$ be the root of $\mathbf{x}^*$. For any $M \succeq M^*$ we necessarily have $\rho(\mathbf{G}_M) < \rho^*$. Let $C := N \setminus \bar{M}^* \neq \emptyset$. By construction, $\mathbf{G}_C$ is a nonnegative matrix with largest eigenvalue ρ, and we call $\mathbf{u}$ the eigenvector associated to ρ, whose components sum to one.
For $\epsilon > 0$, define $\mathbf{a}^\epsilon = (a_i^\epsilon)_i$ as follows:

$$a_i^\epsilon = \epsilon u_i \ \forall i \in C, \quad \text{and} \quad a_i^\epsilon = x_i^* \ \forall i \in \bar{M}^*,$$

We claim that, for any $i \in C$, $\mathbf{a}_i^\epsilon < Br_i(\mathbf{a}^\epsilon)$. By definition of C, we have $g_{ij} = 0$ for any $i \in C$ and any $j \in \bar{M}^*$. Consequently

$$(\mathbf{G}\mathbf{a}^\epsilon)_i = \sum_{j \in C} g_{ij} \mathbf{a}_j^\epsilon = (\mathbf{G}_C \mathbf{a}^\epsilon)_i = \rho \epsilon u_i.$$

Then for $\epsilon > 0$ and $i \in C$, $Br_i(\mathbf{a}^\epsilon)$ is the only zero of the map

$$y_i \mapsto H_i^\epsilon(y_i) = f\left(\frac{A^\epsilon(y_i + \rho\epsilon u_i)}{\alpha(1+\rho)\epsilon u_i}\right) + \frac{y_i A^\epsilon}{\alpha(1+\rho)\epsilon u_i} f'\left(\frac{A^\epsilon(y_i + \rho\epsilon u_i)}{\alpha(1+\rho)\epsilon u_i}\right).$$

where $A^\epsilon = \sum_i a_i^\epsilon = X^* + \epsilon$. On the other hand

$$H_i^\epsilon(a_i^\epsilon) = f(A^\epsilon/\alpha) + \frac{A^\epsilon}{\alpha(1+\rho)} f'(A^\epsilon/\alpha).$$

Since $\rho > \rho^*$, we have

$$f(X^*/\alpha) + \frac{X^*}{\alpha(1+\rho)} f'(X^*/\alpha) > 0.$$

By continuity, for small enough $\epsilon > 0$, we have $H_i^\epsilon(a_i^\epsilon) > 0$, meaning that $\mathbf{a}_i^\epsilon < Br_i(\mathbf{a}^\epsilon)$. This concludes the proof that $\mathbf{x}^*$ is not asymptotically stable for dynamics (16). □

We now prove the following lemma, which completes the proof of Theorem 4:

Lemma 5. *Let $\mathbf{x}^*$ be a PCE such that $\rho(\mathbf{x}^*) = \rho(\mathbf{G})$. Then $\mathbf{x}^*$ is asymptotically stable.*

Proof. We prove this lemma in the particular case of Tullock contest. A general proof without relying on the explicit formulation of map $f(\cdot)$ would be extremely tedious and lengthy. We believe that illustrating the spirit of the proof on a concrete example is more illuminating. If $\mathbf{B}(\mathbf{x}) := -\mathbf{x}+Br(\mathbf{x})$ is differentiable in an open neighborhood of a PCE, then a simple sufficient condition for an interior equilibrium to be asymptotically stable is that the eigenvalues of the Jacobian matrix of $\mathbf{B}(.)$, evaluated at $\mathbf{x}^*$, have negative real parts. Unfortunately the map $\mathbf{B}$ is not differentiable at a non-interior PCE, and we then cannot use this result. However we can compute the directional derivatives of $\mathbf{B}$ at any PCE: let $\mathbf{u} \neq \mathbf{0}$ be such that $u_i \geq 0\ \forall i$. Then the directional derivative of $\mathbf{B}$ in $\mathbf{x}^*$ along $\mathbf{u}$, namely the quantity

$$D_{\mathbf{u}}\mathbf{B}(\mathbf{x}^*) := \lim_{h\to 0, h>0} \frac{\mathbf{B}(\mathbf{x}^* + h\mathbf{u})}{h}$$

exists, and we can compute it: given $h > 0$,

$$B_i(\mathbf{x}^*+h\mathbf{u}) = -(x_i^*+hu_i+(\mathbf{G}(\mathbf{x}^*+h\mathbf{u}))_i)+\left(\frac{\alpha}{c(X^*+hU)}(\mathbf{G}(\mathbf{x}^*+h\mathbf{u}))_i)(x_i^*+hu_i+(\mathbf{G}(\mathbf{x}^*+h\mathbf{u}))_i)\right)^{1/2}$$

The term in the square root can be written

$$\begin{aligned}
& \frac{\alpha}{cX^*}\left(1-h\frac{U}{X^*}\right)\left[(\mathbf{G}\mathbf{x}^*)_i(x_i^*+(\mathbf{G}\mathbf{x}^*)_i)+h\left[(\mathbf{G}\mathbf{x}^*)_i(u_i+(\mathbf{G}\mathbf{u})_i)+(\mathbf{G}\mathbf{u})_i(x_i^*+(\mathbf{G}\mathbf{x}^*)_i)\right]\right]+\mathcal{O}(h^2) \\
= & \frac{\alpha}{cX^*}(\mathbf{G}\mathbf{x}^*)_i(x_i^*+(\mathbf{G}\mathbf{x}^*)_i)\left(1-h\frac{U}{X^*}\right)\left[1+h\left[\frac{u_i+(\mathbf{G}\mathbf{u})_i}{x_i^*+(\mathbf{G}\mathbf{x}^*)_i}+\frac{(\mathbf{G}\mathbf{u})_i}{(\mathbf{G}\mathbf{x}^*)_i}\right]\right]+\mathcal{O}(h^2) \\
= & \frac{\alpha}{cX^*}(\mathbf{G}\mathbf{x}^*)_i(x_i^*+(\mathbf{G}\mathbf{x}^*)_i)\left[1+h\left[-\frac{U}{X^*}+\frac{u_i+(\mathbf{G}\mathbf{u})_i}{x_i^*+(\mathbf{G}\mathbf{x}^*)_i}+\frac{(\mathbf{G}\mathbf{u})_i}{(\mathbf{G}\mathbf{x}^*)_i}\right]\right]+\mathcal{O}(h^2)
\end{aligned}$$

Observing that $(\frac{\alpha}{cX^*}(\mathbf{G}\mathbf{x}^*)_i(x_i^*+(\mathbf{G}\mathbf{x}^*)_i))^{1/2} = x_i^*+(\mathbf{G}\mathbf{x}^*)_i$, the square root of the above quantity is equal to

$$(x_i^*+(\mathbf{G}\mathbf{x}^*)_i)\left[1+\frac{h}{2}\left[\frac{-U}{X^*}+\frac{u_i+(\mathbf{G}\mathbf{u})_i}{x_i^*+(\mathbf{G}\mathbf{x}^*)_i}+\frac{(\mathbf{G}\mathbf{u})_i}{(\mathbf{G}\mathbf{x}^*)_i}\right]\right]+\mathcal{O}(h^2)$$

Hence, since $(x_i^*+(\mathbf{G}\mathbf{x}^*)_i) = \frac{\alpha}{\alpha-cX^*}x_i^*$, we obtain

$$\begin{aligned}
B_i(\mathbf{x}^*+h\mathbf{u}) &= -(hu_i+h(\mathbf{G}\mathbf{u})_i)+\frac{h}{2}\left[\frac{-\alpha U}{X^*(\alpha-cX^*)}x_i^*+(u_i+(\mathbf{G}\mathbf{u})_i)+\frac{\alpha}{cX^*}(\mathbf{G}\mathbf{u})_i\right]+\mathcal{O}(h^2) \\
&= \frac{h}{2}\left[\frac{-U\alpha}{X^*(\alpha-cX^*)}x_i^*-u_i+\frac{\alpha-cX^*}{cX^*}(\mathbf{G}\mathbf{u})_i\right]+\mathcal{O}(h^2)
\end{aligned}$$

Consequently

$$\lim_{h\to+\infty, h>0}\frac{B_i(\mathbf{x}^*+h\mathbf{u})}{h} = \frac{1}{2}\left[\frac{-\alpha U}{X^*(\alpha-cX^*)}x_i^*-u_i+\frac{\alpha-cX^*}{cX^*}(\mathbf{G}\mathbf{u})_i\right] = \frac{1}{2}\left(\mathbf{D}F(\mathbf{x}^*)\mathbf{u}\right)_i,$$

which proves that

$$D_{\mathbf{u}}\mathbf{B}(\mathbf{x}^*) = \frac{1}{2}\left(-I_N + \frac{1+\rho(\mathbf{x}^*)}{X^*}L(\mathbf{x}^*) + \frac{1}{\rho(\mathbf{x}^*)}\mathbf{G}\right)\cdot \mathbf{u},$$

where $\mathbf{L}(\mathbf{x}^*)$ is the matrix where every column is equal to $\mathbf{x}^*$.

Let $\mathbf{D}(\mathbf{x}^*) := \frac{1}{2}\left(-I_N + \frac{1+\rho(\mathbf{x}^*)}{X^*}L(\mathbf{x}^*) + \frac{1}{\rho(\mathbf{x}^*)}\mathbf{G}\right)$. We first show that all eigenvalues of $\mathbf{D}(\mathbf{x}^*)$ have a negative real part. Suppose that $\mathbf{D}(\mathbf{x}^*)\cdot\mathbf{u} = \lambda\cdot\mathbf{u}$, with $\mathbf{u}\neq 0$. Call $U := \sum_{i\in N} u_i$. Then we have

$$-\mathbf{u} - \frac{1+\rho}{X^*}U\mathbf{x}^* + \frac{1}{\rho}\mathbf{G}\mathbf{u} = 2\lambda\mathbf{u}$$

which gives

$$\left(\mathbf{I}_N - \frac{1}{\rho(1+2\lambda)}\mathbf{G}\right)\mathbf{u} = -\frac{1+\rho}{X^*(1+2\lambda)}U\mathbf{x}^*.$$

Suppose that $Re(\lambda) > 0$ or that λ is pure imaginary. Then $|1+\lambda| > 1$ and the matrix $\mathbf{G}/(\rho(1+2\lambda))$' spectral radius is strictly smaller than one. As a consequence $\mathbf{I}_N - \frac{1}{\rho(1+2\lambda)}\mathbf{G}$ is invertible and

$$\left(\mathbf{I}_N - \frac{1}{\rho(1+2\lambda)}\mathbf{G}\right)^{-1} = \sum_{p=0}^{+\infty}\frac{1}{\rho^p(1+2\lambda)^p}\mathbf{G}^p.$$

Consequently

$$\begin{aligned}\mathbf{u} &= -\frac{1+\rho}{X^*(1+2\lambda)}U\left(\mathbf{I}_N - \frac{1}{\rho(1+2\lambda)}\mathbf{G}\right)^{-1}\mathbf{x}^*\\ &= -\frac{1+\rho}{X^*(1+2\lambda)}U\sum_{p=0}^{+\infty}\frac{1}{\rho^p(1+2\lambda)^p}\mathbf{G}^p\mathbf{x}^*\\ &= -\frac{1+\rho}{X^*(1+2\lambda)}U\sum_{p=0}^{+\infty}\frac{1}{(1+2\lambda)^p}\mathbf{x}^*\\ &= -\frac{1+\rho}{2X^*\lambda}U\mathbf{x}^*\end{aligned}$$

Since $\mathbf{u}\neq 0$, this equality implies that $U\neq 0$ and summing the coordinates of $\mathbf{u}$ we obtain that $2\lambda = -(1+\rho) < 0$, a contradiction.
Suppose now that $\lambda = 0$. Then we have

$$\left(\mathbf{I}_N - \frac{1}{\rho}\mathbf{G}\right)\mathbf{u} = -\frac{1+\rho}{X^*}U\mathbf{x}^*.$$

Suppose that $U\neq 0$. Then, multiplying both sides of the equality by $\sum_{k=0}^{K}\frac{1}{\rho^k}\mathbf{G}^k$, we obtain the identity

$$\left(\mathbf{I}_N - \frac{1}{\rho^{K+1}}\mathbf{G}^{K+1}\right)\mathbf{u} = -\frac{1+\rho}{X^*}U\sum_{k=0}^{K}\frac{1}{\rho^k}\mathbf{G}^k\mathbf{x}^* = -\frac{1+\rho}{X^*}UK\mathbf{x}^*$$

The modulus of the left-hand is bounded above by $2|\mathbf{u}|$, while the modulus of the right-hand side term grows to infinity with K, which is a contradiction. Hence $U = 0$. This means that

$$\mathbf{G}\mathbf{u} = \rho\mathbf{u},$$

i.e. that $\mathbf{u}$ is in fact an eigenvector associated to the largest eigenvalue of $\mathbf{G}$. Since $\sum_i u_i = 0$, this contradicts the fact that $(N, \mathbf{G})$ is a simple network.

We proved that the real part of every eigenvalue of $\mathbf{D}F(x^*)$ is strictly negative.

As we proved above, for $\mathbf{x} \in \mathbf{x}$, we have

$$\mathbf{B}(\mathbf{x}) = \mathbf{D}(\mathbf{x}^*) \cdot (\mathbf{x} - \mathbf{x}^*) + \|\mathbf{x} - \mathbf{x}^*\|^2 g(\|\mathbf{x} - \mathbf{x}^*\|)$$

Denote by $(\lambda_1, ..., \lambda_1, \lambda_2, ..., \lambda_2, ..., \lambda_P, ..., \lambda_P)$ the eigenvalues of $\mathbf{D}(\mathbf{x}^*)$, and call n_p the multiplicity of eigenvalue λ_p. Let us first put $\mathbf{D}(\mathbf{x}^*)$ in its Jordan form:

$$\mathbf{D}F(\mathbf{x}^*) = \mathbf{P}\mathbf{J}\mathbf{P}^{-1},$$

where $\mathbf{J}$ is diagonal by blocks, i.e.

$$\mathbf{J} = Diag\left(\mathbf{J}_1, ..., \mathbf{J}_P\right) := \begin{pmatrix} \mathbf{J}_1 & \mathbf{0} & ... & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_2 & ... & \mathbf{0} \\ ... & ... & ... & ... \\ \mathbf{0} & ... & \mathbf{0} & \mathbf{J}_P \end{pmatrix}, \quad \text{with } \mathbf{J}_p = \begin{pmatrix} \lambda_p & 1 & 0 & ... & 0 \\ 0 & \lambda_p & 1 & ... & 0 \\ ... & ... & ... & ... & \\ 0 & ... & 0 & \lambda_p & 1 \\ 0 & ... & ... & 0 & \lambda_p \end{pmatrix}$$

Define now $\mathbf{Q} := Diag\left(\mathbf{Q}_1, ..., \mathbf{Q}_P\right)$, with $\mathbf{Q}_p = Diag(1, \epsilon, ..., \epsilon^{n_p - 1})$. We then have

$$\mathbf{Q}_p^{-1}\mathbf{J}_p\mathbf{Q}_p = \begin{pmatrix} \lambda_p & \epsilon & 0 & ... & 0 \\ 0 & \lambda_p & \epsilon & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & ... & 0 & \lambda_p & \epsilon \\ 0 & ... & ... & 0 & \lambda_p \end{pmatrix}$$

Thus, defining $\mathbf{R} := \mathbf{PQ}$ we obtain

$$\mathbf{R}^{-1}\mathbf{D}(\mathbf{x}^*)\mathbf{R} = \mathbf{Q}^{-1}\mathbf{J}\mathbf{Q} = \mathbf{D}(\lambda) + \epsilon\mathbf{B},$$

where $\mathbf{D}(\lambda)$ is the diagonal matrix filled with the eigenvalues of $\mathbf{D}(\mathbf{x}^*)$.
Now define $V : \mathbf{S} \to \mathbb{R}^+$ as follows:

$$V(\mathbf{x}) := \left|\mathbf{R}^{-1}(\mathbf{x}-\mathbf{x}^*)\right|^2 = \left\langle \mathbf{R}^{-1}(\mathbf{x}-\mathbf{x}^*) \mid \overline{\mathbf{R}^{-1}}(\mathbf{x}-\mathbf{x}^*)\right\rangle$$

We have

$$\begin{aligned}\dot{V}(\mathbf{x}) &= \left\langle \mathbf{R}^{-1}\dot{\mathbf{x}} \mid \overline{\mathbf{R}^{-1}}(\mathbf{x}-\mathbf{x}^*)\right\rangle + \left\langle \overline{\mathbf{R}^{-1}}\dot{\mathbf{x}} \mid \mathbf{R}^{-1}(\mathbf{x}-\mathbf{x}^*)\right\rangle \\ &= \left\langle (\mathbf{D}(\lambda)+\epsilon\mathbf{B})\mathbf{R}^{-1}(\mathbf{x}-\mathbf{x}^*) \mid \overline{\mathbf{R}^{-1}}(\mathbf{x}-\mathbf{x}^*)\right\rangle + \left\langle (\overline{\mathbf{D}(\lambda)}+\epsilon\overline{\mathbf{B}})\overline{\mathbf{R}^{-1}}(\mathbf{x}-\mathbf{x}^*) \mid \mathbf{R}^{-1}(\mathbf{x}-\mathbf{x}^*)\right\rangle \\ &+ \|\mathbf{x}-\mathbf{x}^*\|^2 h(\|\mathbf{x}-\mathbf{x}^*\|),\end{aligned}$$

where $h(a) \to_{a\to 0} 0$. Hence we have

$$\begin{aligned}\dot{V}(\mathbf{x}) &= \left\langle (\mathbf{D}(\lambda)+\overline{\mathbf{D}(\lambda)})\mathbf{R}^{-1}(\mathbf{x}-\mathbf{x}^*) \mid \overline{\mathbf{R}^{-1}}(\mathbf{x}-\mathbf{x}^*)\right\rangle + 2\epsilon Re\left(\left\langle \mathbf{B}\mathbf{R}^{-1}(\mathbf{x}-\mathbf{x}^*) \mid \overline{\mathbf{R}^{-1}}(\mathbf{x}-\mathbf{x}^*)\right\rangle\right) \\ &+ \|\mathbf{x}-\mathbf{x}^*\|^2 h(\|\mathbf{x}-\mathbf{x}^*\|)\end{aligned}$$

Let $\alpha := \max_{p=1,\ldots,P} Re(\lambda_p) < 0$. We have

$$\left\langle (\mathbf{D}(\lambda)+\overline{\mathbf{D}(\lambda)})\mathbf{R}^{-1}(\mathbf{x}-\mathbf{x}^*) \mid \overline{\mathbf{R}^{-1}}(\mathbf{x}-\mathbf{x}^*)\right\rangle \leq 2\alpha|\mathbf{R}^{-1}(\mathbf{x}-\mathbf{x}^*)|^2 = 2\alpha V(\mathbf{x}).$$

As a consequence, choosing ϵ small enough and $\mathbf{x}$ close enough of $\mathbf{x}^*$ we obtain that

$$\dot{V}(\mathbf{x}) \leq \alpha V(\mathbf{x}),$$

which proves that $V(\mathbf{x}(t))$ goes to zero exponentially fast, as t goes to infinity, and this concludes the proof. □

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(Not-for-Publication) Online Appendix

Perceived competition

By Olivier Bochet[1], Mathieu Faure[2], Yan Long[3] and Yves Zenou[4]

A Further comments on the set of Perception-Consistent Equilibria

We first define a restriction on weakly connected networks that is useful to get some additional traction on the characterization of the full set of perception-consistent equilibrium.

Definition A1. *A weakly connected network* $(N, \mathbf{G})$ *is* ***simple*** *if for any* $M_1, M_2 \in \mathcal{C}$ *such that* $\rho(\mathbf{G}_{M_1}) = \rho(\mathbf{G}_{M_2}) = \rho$, *we have* $\max\{\rho(\mathbf{G}_M) : \ M \succ M_1 \text{ or } M \succ M_2\} \geq \rho$.

Hence, simple networks are such that, for any two distinct communities with the same spectral radius, a community must exist whose spectral radius is at least as large, and which is aware of one of them. In other words, (i) we exclude weakly connected networks for which two PCEs with different roots have the same spectral radius, but (ii) we allow the existence of two communities with the same spectral radius if one of them is not part of a PCE. In particular, we exclude networks in which two $\succeq$-maximal communities have the same spectral radius. For example, the network displayed in Figure C (in Online Appendix) is not simple because the two $\succeq$-maximal communities $M_1 = \{2, 3\}$ and $M_2 = \{4, 5\}$ have the same spectral radius (i.e., $\rho(\mathbf{G}_{M_1}) = \rho(\mathbf{G}_{M_2}) = 1$). On the other hand, the networks in Figures 3(a) and (b) (Example 3) and in Figure 4 (Example 4) are simple. Observe, in particular, that in Figure 3(a), the communities $M_1 = \{1, 2, 3\}$ and $M_2 = \{4, 5, 6\}$ have the same spectral radius (i.e., $\rho(\mathbf{G}_{M_1}) = \rho(\mathbf{G}_{M_2}) = 2$). However, because M_2 is not the root of a PCE, this network is simple.

We show that, if the perception network is simple, perception-consistent equilibria always admit a root, meaning that Proposition 4 provides a full characterization of the set of equilibria in that case.

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Proposition A1. *Let* $(N, \mathbf{G})$ *be a simple network, and let* $\mathbf{x}^*$ *be a perception-consistent equilibrium of* $(N, \mathbf{G})$*. Then,* $\mathbf{x}^*$ *admits a root. Moreover, equilibrium efforts are proportional to eigenvector centrality in the sub-network of active players.*

Proof of Proposition A1. Suppose that $\mathbf{x}$ does not admit a root. Then, by Propositions C3 and C4, we have $N_+(\mathbf{x}) = \cup_{i=1}^n \bar{M}_i$, with $n \geq 2$, $M_1, ..., M_n$ being distinct elements of $\mathcal{C}(\mathbf{G})$ and

$$\rho\left(\mathbf{G}_{M_i}\right) = \rho > \max_{M' \in \mathcal{C}(\mathbf{G}): M' \succ M_i} \rho(\mathbf{G}_{M'}),\ \ \forall i = 1, ..., n,$$

which contradicts the fact that $\mathbf{G}$ is simple. □

The last statement of Proposition A1 must be understood as follows: if $\mathbf{x}^*$ is a PCE, then the effort of active agents is proportional to their eigenvector centrality *in the sub-network they generate*. It is important to understand that this result does not say anything about the eigenvector centrality of agents in the whole network, since inactive agents are not taken into account. A direct consequence of Proposition A1 is that there is a *finite* number of equilibria in simple networks because, for any community M, there is at most one PCE with root M. Actually, the set of perception-consistent equilibria is finite if and only if the perception network is simple. This is formally stated below in Proposition A2. In Online Appendix C, we show that we can still describe the set of perception-consistent equilibria in a simple way when the perception network is no longer simple (see Proposition C5).

Proposition A2. *Let* $(N, \mathbf{G})$ *be a weakly connected network. The following are equivalent:*

- (i) *The set of perception-consistent equilibria is finite.*
- (ii) *For any pair* $\left(\mathbf{x}^{1*}, \mathbf{x}^{2*}\right)$ *of perception-consistent equilibria,* $\rho\left(\mathbf{G}_{N_+(\mathbf{x}^{1*})}\right) \neq \rho\left(\mathbf{G}_{N_+(\mathbf{x}^{2*})}\right)$.
- (iii) $(N, \mathbf{G})$ *is a simple network.*

Proof of Proposition A2. $(i) \Rightarrow (ii)$: suppose that (ii) does not hold. Then there exists two PCE $\mathbf{x}_1, \mathbf{x}_2$ such that $\rho(\mathbf{x}_1) = \rho(\mathbf{x}_2) =: \rho$. For $\lambda \in [0, 1]$ and define $\mathbf{x}^\lambda := \lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2$. Then $X^\lambda = X_1 = X_2$. Hence

$$\mathbf{G}\mathbf{x}^\lambda = \lambda \mathbf{G}\mathbf{x}_1 + (1-\lambda)\mathbf{G}\mathbf{x}_2 = \lambda\rho\mathbf{x}_1 + (1-\lambda)\rho\mathbf{x}_2 = \rho\mathbf{x}^\lambda,$$

and $\mathbf{x}^\lambda$ is a PCE. Thus there is a continuum of PCE, contradicting (i).

$(ii) \Rightarrow (i)$: this implication follows from the fact that the set of eigenvalues of subgraphs of $\mathbf{G}$ is finite.

$(ii) \Rightarrow (iii)$: Suppose that (iii) does not hold. Then there exists M_1, M_2 such that $\rho(\mathbf{G}_{M_1}) = \rho(\mathbf{G}_{M_1'})$, $\max_{M' \in \mathcal{C}(\mathbf{G}): M' \succ M_1} \rho(\mathbf{G}_{M'}) < \rho(\mathbf{G}_{M_1})$ and $\max_{M' \in \mathcal{C}(\mathbf{G}): M' \succ M_2} \rho(\mathbf{G}_{M'}) < \rho(\mathbf{G}_{M_2})$.

The last two strict inequalities mean that there exists a PCE with root M_1, and a PCE with root M_2, contradicting (ii).

$(iii) \Rightarrow (ii)$: Assume that (ii) does not hold, and let M_1 (resp. M_2) be the root of $\mathbf{x}_1$ (resp. $\mathbf{x}_2$). Being both PCE, it follows that we have $\max_{M' \in \mathcal{C}(\mathbf{G}) : M' \succ M_1} \rho(\mathbf{G}_{M'}) < \rho(\mathbf{G}_{M_1})$ and $\max_{M' \in \mathcal{C}(\mathbf{G}) : M' \succ M_2} \rho(\mathbf{G}_{M'}) < \rho(\mathbf{G}_{M_2})$, contradicting (iii).

Finally we obtain $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$ and the proof is complete. □

Next, we look at a refinement of weakly connected networks, *semi-connected networks*. A network is semi-connected if for each pair of agents i, j, there exists either a directed path connecting i to j, or a path connecting j to i. Under the no-isolation assumption, and since a semi-connected perception network is also a simple network, the cardinality of the set of PCEs in a semi-connected network is finite and bounded above by $n/2$.

Corollary A1. *If $(N, \mathbf{G})$ is a semi-connected network then there are at most k perception-consistent equilibria, where k is the number of communities.*

Proof of Corollary A1. If the network is semi-connected then the communities are totally ordered: $M_1 \succ M_2 \succ ... \succ M_k$. Hence the number of PCE is equal to

$$Card \left\{ s = 1, ..., k : \ \rho(\mathbf{G}_{M_s}) > \max_{\ell = 1, ..., s-1} \rho(\mathbf{G}_{M_\ell}) \right\}.$$

□

B Non-negative matrices and centrality

B.1 The Frobenius normal form

A matrix is called **nonnegative** if all its elements are nonnegative. Here we consider only nonnegative square matrices of order n, i.e., matrices that have n rows and n columns. A non-negative matrix A is called **irreducible** if its associated directed graph is strongly connected. For convenience, any one-by-one matrix is regarded as irreducible.

Lemma B1. *(Perron-Frobenius Theorem) Let $\mathbf{A}$ be an irreducible matrix. Then*

(i) $\mathbf{A}$ has a positive eigenvalue $\rho(\mathbf{A})$ such that the value of $\rho(\mathbf{A})$ is not less than the absolute value of any other eigenvalue of $\mathbf{A}$;

(ii) the eigenvalue $\rho(\mathbf{A})$is simple and corresponds to a positive eigenvector $\mathbf{x}(\mathbf{A})$;

(iii) any non-negative eigenvector is a multiple of $\mathbf{x}(\mathbf{A})$.

The vector $\mathbf{x}(\mathbf{A})$ *and the number* $\rho(\mathbf{A})$ *that appear in this lemma are called the* ***Perron-Frobenius vector*** *and the* ***Perron-Frobenius eigenvalue*** *of* $\mathbf{A}$*, respectively.*

The following lemma extends some conclusions of the Perron-Frobenius Theorem to nonnegative matrices, hence including those that may not be irreducible. For instance, this is also important for matrices whose associated directed graphs may be weakly connected; a typical case we consider in the perceived competition framework.

Lemma B2. *Let* $\mathbf{A}$ *be a nonnegative matrix; then*

a) $\mathbf{A}$ *has a nonnegative eigenvalue* $\rho(\mathbf{A})$ *such that the value of* $\rho(\mathbf{A})$ *is not less than the absolute value of any other eigenvalue of* $\mathbf{A}$*.*

b) To eigenvalue $\rho(\mathbf{A})$ *corresponds a nonnegative eigenvector* $\mathbf{x}(\mathbf{A})$*.*

c) If there exists a positive eigenvector, then it is necessarily associated to eigenvalue $\rho(\mathbf{A})$*.*

Note that if $\mathbf{x}$ is a nonnegative eigenvector of $\mathbf{A}$, $\mathbf{x}$ is not necessarily associated with $\rho(\mathbf{A})$. Also, there could exist eigenvectors with both negative and positive entries, associated to $\rho(\mathbf{A})$. Lemma B2 simply guarantees the existence of a nonnegative eigenvector associated to $\rho(\mathbf{A})$. Note, however, that if there exists a positive eigenvector, then it is automatically an eigenvector of $\rho(\mathbf{A})$. This case is then similar to case (ii) of Lemma B1.

Lemma B3. *Any nonnegative matrix* $\mathbf{A}$ *can be put in an upper-triangular block form as follows:*[5]

$$\mathbf{A} = \begin{bmatrix} A_1 & A_{12} & ... & ... & ... & ... & ... & A_{1r} \\ 0 & A_2 & A_{23} & ... & ... & ... & ... & A_{2r} \\ ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & ... & 0 & A_s & A_{ss+1} & ... & ... & A_{sr} \\ 0 & ... & ... & 0 & A_{s+1} & 0 & ... & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & ... & ... & ... & ... & 0 & A_{r-1} & 0 \\ 0 & ... & ... & ... & ... & ... & 0 & A_r \end{bmatrix} \tag{B.1}$$

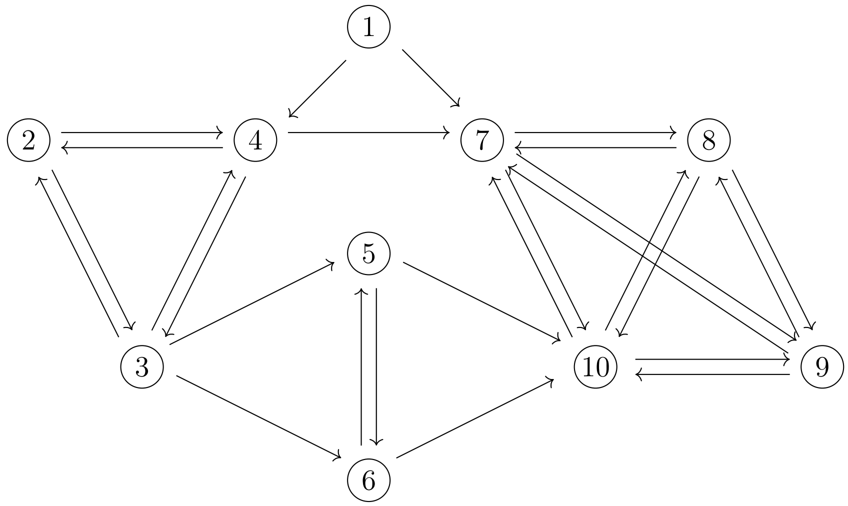
such that:

(i) each block matrix A_i*,* $i = 1, .., r$*, is square and irreducible;*

(ii) for any $i = 1, ...s$*, there exists* $j \in \{i+1, ..., r\}$ *such that the block matrix* A_{ij} *is not zero.*

[5]Up to a permutation of indices.

Figure B1: A Perception Network illustrating the Frobenius Normal Form



This upper triangular block form is known as the **Frobenius normal form**. It is unique up to a permutation. Importantly, note that we have that $\rho(\mathbf{A}) = \max_{i=1\ldots r} \rho(A_r)$. Let us define V_i to be the set of nodes corresponding to the block matrix A_i, $i = 1, .., r$.

Definition B2. *A nonnegative matrix* $\mathbf{A}$ *is* ***strongly nonnegative*** *if we have*

$$\rho(A_r) = \rho(A_{r-1}) = ... = \rho(A_{s+1}) > \max_{i=1,\ldots,s} \{\rho(A_i)\}$$

Obviously, any irreducible matrix is strictly nonegative because the Frobenius normal form then consists of one block. The next results can be found in Rothblum (2014)[6] or Hu and Qi (2016).[7]

Lemma B4. *A nonnegative matrix* $\mathbf{A}$ *admits a positive eigenvector if and only if* $\mathbf{A}$ *is strongly nonnegative.*

Note that, if $\mathbf{A}$ is an irreducible nonnegative matrix, then the conclusion of Lemma B4 directly implies point (*ii*) of Lemma B1, i.e., the Perron Frobenius Theorem.

We illustrate the construction of the Frobenius normal form for the perception network $(N, \mathbf{G})$ displayed in Figure B1 above, itself a variation of Figure 4 found in the main text. There are $N = \{1, 2, \cdots, 10\}$, and we have three communities: $M_1 = \{2, 3, 4\}$, $M_2 = \{5, 6\}$, and $M_3 = \{7, 8, 9, 10\}$ such that $M_1 \succ M_2 \succ M_3$.

[6] Rothblum, U. (2014). Nonnegative matrices and stochastic matrices. In: L. Hogben (Ed.), *Handbook of Linear Algebra, Second Edition*, Chap. 10. CRC Press, pages 1–26.

[7] Hu, S. and Qi, L. (2016). A necessary and sufficient condition for existence of a positive Perron vector. *SIAM Journal of Matrix Analysis and Applications* 37(4), 1747–1770.

Let $\mathbf{C}(m)$ be the adjacency matrix of the complete m-agents network.[8] Keeping the indexing of agents as it is, we have

$$\mathbf{G} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & A_{12} & 0 & A_{14} \\ 0 & A_2 & A_{23} & A_{24} \\ 0 & 0 & A_3 & A_{34} \\ 0 & 0 & 0 & A_4 \end{bmatrix},$$

where $A_1 = 0$, $A_2 = \mathbf{G}_{M_1} = \mathbf{C}(3)$, $A_3 = \mathbf{G}_{M_2} = \mathbf{C}(2)$, and $A_4 = \mathbf{G}_{M_3} = \mathbf{C}(4)$, while $A_{12} = [0\,0\,1]$, $A_{13} = [0\,0]$, $A_{14} = [1\,0\,0\,0]$, etc. In particular, A_{ij} is distinct from the null matrix, except for A_{13} (there is no link from group 1, i.e., agent 1, to community M_2, i.e., agents $\{5, 6\}$). Consequently, we have $s = 3$ and $r = 4$. Note that $\rho(A_4) = 3$ while $\rho(A_1) = 0, \rho(A_2) = 2$ and $\rho(A_3) = 1$. Hence, $\mathbf{G}$ is strongly nonnegative and it thus admits a positive eigenvector. Now, suppose that we remove agent 9 from $\mathbf{G}$ to obtain $(\hat{N}, \hat{\mathbf{G}})$ with $\hat{N} < n$ and $\hat{\mathbf{G}} \subset \mathbf{G}$. The Frobenius normal form of $\hat{\mathbf{G}}$ has the same structure as the one obtained for $\mathbf{G}$, except that $\rho(A_4) = 2 = \max_{i=1,\dots,3} \rho(A_i)$. Hence, the matrix is no longer strictly nonnegative. Therefore, $\hat{\mathbf{G}}$ doesn't admit a positive eigenvector.

It might be useful to clarify the relationship between the Frobenius normal form and the $\succeq$-ordering on communities that we defined in Section 4.2. In the Frobenius normal form of $\mathbf{G}$, any A_i corresponds to the submatrix of a strongly connected component, which can either be a community, or a singleton. Note that, by the no-isolation assumption, A_i cannot be a size one matrix for $i = s+1, ..., r$; it then necessarily corresponds to a community for these indexes. If $M' \succ M$, then there exists some i, i' such that $i' < i$, $\mathbf{G}_M = A_i$ and $\mathbf{G}_{M'} = A_{i'}$. In other words the indexes in the Frobenius normal form are inversely ordered in accordance with the $\succeq$ ordering.

Observe that the Frobenius normal form does not directly help us to characterize the perception-consistent equilibria (i.e., determining which agents are active and which are not). Yet, it will be very useful for some of our proofs because of Lemma B4, which can be applied to any closed set, as we will see in the proof section. We explain below the relationship

[8] That is, $C(m)_{ii} = 0$, $C(m)_{ij} = 1$ for $i \neq j$

between the ordering over communities we constructed and the Frobenius normal form of the perception network $\mathbf{G}$.

B.2 Relationship between the $\succeq$ ordering and the Frobenius normal form

Lemma B5. *Let $(N, \mathbf{G})$ be a weakly connected network. Consider the Frobenius normal form (B.1) associated with $\mathbf{G}$. For any $i = 1, ..., r$ either $|V_i| = 1$ or $V_i \in \mathcal{C}(\mathbf{G})$. As a consequence*

$$\rho(\mathbf{G}) = \max_{i=1,...,r} \rho(A_i) = \max_{M \in \mathcal{C}(\mathbf{G})} \rho(\mathbf{G}_M) \tag{B.2}$$

Proof of Lemma B5. Suppose that $|V_i| > 1$. By construction of the Frobenius normal form, (V_i, A_i) is a strongly connected component of $(N, \mathbf{G})$. Hence V_i belongs to the set of communities $\mathcal{C}(\mathbf{G})$. Since $\rho(\mathbf{G}) = \max_{i=1...r} \rho(A_i)$ and $\rho(A_i) = 0$ if $|V_i| = 1$ this concludes the proof of (B.2). □

For any closed set $N' \subset N$, note that $\mathcal{C}(\mathbf{G}_{N'}) = \{M \in \mathcal{C}(\mathbf{G}) : M \subset N'\}$. Hence we have

$$\rho(\mathbf{G}_{N'}) = \max_{M' \in \mathcal{C}(\mathbf{G}):\, M \subseteq N'} \rho(\mathbf{G}_{M'}) \tag{B.3}$$

Lemma B6. *Suppose that $\mathbf{A}$ is a nonnegative matrix that admits a Frobenius normal form (B.1) with $r = s + 1$ and $\rho(A_{s+1}) > \max_{i=1,...,s} \{\rho(A_i)\}$. Then $\mathbf{A}$ admits a **unique** positive eigenvector.*[9]

Proof of Lemma B6. We only need to show that, if $\mathbf{x}$ and $\mathbf{y}$ are two positive eigenvector of $\mathbf{A}$ then $\mathbf{x} = \alpha \mathbf{y}$ for some $\alpha > 0$. We can write $\mathbf{A}$ as follows:

$$\mathbf{A} = \begin{bmatrix} A' & B \\ 0 & A_{s+1} \end{bmatrix}, \text{ where } A' = \begin{bmatrix} A_1 & A_{12} & ... & ... & A_{1s} \\ 0 & A_2 & A_{23} & ... & A_{2s} \\ ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... \\ 0 & ... & ... & 0 & A_s \end{bmatrix} \text{ and } B = \begin{bmatrix} A_{1s+1} \\ A_{2s+1} \\ ... \\ ... \\ A_{ss+1} \end{bmatrix}.$$

Let us write $\mathbf{x}$ as $(\mathbf{x}', \mathbf{x}_{[s+1]})$, according to the decomposition of $\mathbf{A}$ we just wrote and let $\rho := \rho(A_{s+1}) = \rho(\mathbf{A})$. We have

$$\begin{bmatrix} \mathbf{x}' \\ \mathbf{x}_{[s+1]} \end{bmatrix} = \rho^{-1} \begin{bmatrix} \mathbf{A}' \cdot \mathbf{x}' + \mathbf{B} \cdot \mathbf{x}_{[s+1]} \\ \mathbf{A}_{s+1} \cdot \mathbf{x}_{[s+1]} \end{bmatrix},$$

[9] Uniqueness is up to multiplication by a constant.

so that, in particular, $\left(\mathbf{I} - \rho^{-1}\mathbf{A}'\right)\mathbf{x}' = \rho^{-1}\mathbf{B}\mathbf{x}_{[s+1]}$. Since $\rho(\mathbf{A}') < \rho$ by construction, the matrix $\mathbf{I} - \rho^{-1}\mathbf{A}'$ is invertible and we have

$$\mathbf{x}' = \rho^{-1}\left(\mathbf{I} - \rho^{-1}\mathbf{A}'\right)^{-1}\mathbf{B}\mathbf{x}_{[s+1]} \tag{B.4}$$

Now the matrix $\mathbf{A}_{s+1}$ being irreducible and $\mathbf{x}_{[s+1]}, \mathbf{y}_{[s+1]}$ both being positive eigenvectors of $\mathbf{A}_{s+1}$ we must have $\mathbf{x}_{[s+1]} = \alpha\mathbf{y}_{[s+1]}$ Since identity (B.4) holds for both $\mathbf{x}$ and $\mathbf{y}$, we obtain that $\mathbf{x}' = \alpha\mathbf{y}'$, concluding the proof. □

B.3 Eigenvector centrality in weakly connected networks

Eigenvector centrality has been informally introduced by Bonacich (1972)[10] to measure popularity in friendship networks. Given a weighted network $(N, \mathbf{G})$, it was originally defined as any non-negative vector $\mathbf{e}$ having the property that the centrality of agent i is proportional to the average centrality of her neighbors:

$$\lambda e_i = \sum_j \mathbf{G}_{ij} e_j, \ \forall i. \tag{B.5}$$

In the particular case of strongly connected networks, this vector is well-defined because there is a unique solution to the system (B.5), given by the eigenvector associated to the largest eigenvalue λ of $\mathbf{G}$. More generally, there is a consensus consisting in regarding eigenvector centrality as being the unique normalized eigenvector associated to the largest eigenvalue of the network –see e.g., Jackson (2008).[11]

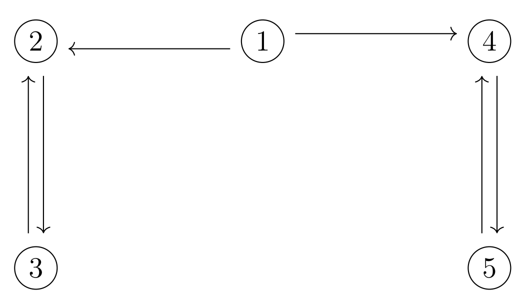


Figure B2: Continuum of PCEs when non unique dominant component

In weakly connected networks, however, eigenvector centrality cannot be defined in the same way because the largest eigenvalue of a weakly connected network is not always simple.

[10]Bonacich, P. (1972). Factoring and weighting approaches to status scores and clique identification. *Journal of Mathematical Sociology* 2(1), 113–120.

[11]The normalized eigenvector associated to the largest eigenvalue of the network is the eigenvector whose components sum to one.

For instance, consider the network in Figure B2, where $\rho(\mathbf{G}) = 1$. The eigenspace associated to $\rho(\mathbf{G})$ is generated by normalized vectors $(1/3, 1/3, 1/3, 0, 0)$ and $(1/3, 0, 0, 1/3, 1/3)$. Hence, any convex combination of these two vectors is a non-negative eigenvector, which means that eigenvector centrality is not well-defined for this network.

Consequently, we focus on an (arguably large) subset of weakly connected graphs, in which the notion of eigenvector centrality can be naturally extended.

Definition B3. [***Unique Dominant Component***] *A weakly connected network has* a unique dominant component *if,*

$$\forall M, M' \in \mathcal{C}(\mathbf{G}),\ \ \rho(\mathbf{G}_M) = \rho(\mathbf{G}_{M'}) = \rho(\mathbf{G}) \Rightarrow M \succeq M' \ \text{or}\ \ M' \succeq M. \tag{UDC}$$

Obviously, any *simple network* has a unique dominant component. A simple adaptation of the proof of Proposition A2 shows that a weakly connected network admits a unique normalized eigenvector associated to $\rho(\mathbf{G})$ if and only if it has a unique dominant component.

Definition B4. [***Eigenvector centrality***] *Suppose that $(N, \mathbf{G})$ has a unique dominant component. Then, the eigenvector centrality of agent i is the i-th component of the normalized eigenvector associated to $\rho(\mathbf{G})$.*

In some networks, it may be the case that some agents exhibit a null eigenvector centrality, and one may wonder what it means, and whether or not this definition makes sense when this happens. As we show now, this definition is indeed meaningful, because our definition of eigenvector centrality is robust to any small perturbations, in the following sense:

Lemma B7. *Suppose that $(N, \mathbf{G})$ has a unique dominant component and call $\mathbf{e}$ the normalized eigenvector associated to $\rho(\mathbf{G})$. Let $(\mathbf{G}^n)_n$ be a sequence of irreducible matrices such that $\lim_{n\to+\infty} \mathbf{G}^n_{ij} = \mathbf{G}_{ij}$. Then $\mathbf{e}^n \to \mathbf{e}$, where $\mathbf{e}^n$ is the normalized eigenvector associated to $\rho(\mathbf{G}^n)$.*

In other words, the sequence of centrality measures always converge to the same vector, regardless of *how* $\mathbf{G}^n$ converges to $\mathbf{G}$. The implication of this observation is that eigenvector centrality is unambiguously defined in networks having a unique dominant component.
Consider now the network depicted above. Here $N = \{1, 2, \cdots, 5\}$. Both $M_1 = \{2, 3\}$, and $M_2 = \{4, 5\}$ are $\succ$-maximal communities. Moreover we have $\rho(\mathbf{G}_{M_1}) = \rho(\mathbf{G}_{M_2}) = 1$. Consequently, this network does not have a unique dominant component. Hence, the set of perception-consistent equilibria is not finite:

$$PCE = \left\{ \frac{\alpha}{12c}(1, \lambda, \lambda, 1-\lambda, 1-\lambda) :\ \lambda \in [0, 1] \right\}.$$

thus, defining an eigenvector centrality for such a network would imply making an arbitrary choice. Indeed, it can be shown that, for any $\lambda \in [0,1]$, one can find a sequence of strongly connected weighted networks $(N, \mathbf{G}^n)$ such that $\mathbf{e}^n$ converges to $\frac{1}{3}(1, \lambda, \lambda, 1-\lambda, 1-\lambda)$.

C Beyond simple networks

In this section we assume that $\mathbf{G}$ is a weakly connected directed graph satisfying the no isolation assumption. As mentioned in the text, the PCE set can be infinite in non-simple networks.

C.1 Structure of the equilibrium set

We say that two distinct communities M_1 and M_2 are **disconnected** if neither $M_1 \succ M_2$ nor $M_2 \succ M_1$.

Proposition C3. *Let $\mathbf{x}^*$ be a PCE. Then, there exists a family of pairwise disconnected communities $\{M_i\}_{i=1,...,n}$ such that*

$$N_+(\mathbf{x}^*) = \cup_{i=1}^n \bar{M}_i. \tag{C.1}$$

Proof of Proposition C3. Since $N_+(\mathbf{x}^*)$ is a closed set of $\mathbf{G}$, we have that $\mathbf{x}^*$ is a positive eigenvector of $\mathbf{G}_{N_+(\mathbf{x})}$, associated to eigenvalue $\rho > 0$. By Lemma B4, that implies that $\mathbf{G}_{N_+(\mathbf{x})}$ is strongly nonnegative, and thus can be written

$$\mathbf{G}_{N_+(\mathbf{x})} = \begin{bmatrix} A_1 & A_{12} & ... & ... & ... & ... & ... & A_{1r} \\ 0 & A_2 & A_{23} & ... & ... & ... & ... & A_{2r} \\ ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & ... & 0 & A_s & A_{ss+1} & ... & ... & A_{sr} \\ 0 & ... & ... & 0 & A_{s+1} & 0 & ... & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & ... & ... & ... & ... & 0 & A_{r-1} & 0 \\ 0 & ... & ... & ... & ... & ... & 0 & A_r \end{bmatrix} \tag{C.2}$$

where $r > s$, $\rho(A_r) = ... = \rho(A_{s+1}) = \rho$, and $\rho(A_i) < \rho$ for $i = 1, ..., s$. Each A_{s+i} being such that $|V_{s+i}| \geq 2$ for $i = 1, ..., r-s$, we have $V_{s+i} \in \mathcal{C}(\mathbf{G})$. Hence, taking $n := r - s$, there exists $M_1, ..., M_n \in \mathcal{C}(\mathbf{G})$ such that $A_{s+i} = \mathbf{G}_{M_i}$ for $i = 1, ..., n$.
We now show that $N_+(\mathbf{x}^*) = \cup_{i=1}^n \bar{M}_i$. Since $N_+(\mathbf{x}^*)$ is closed and $M_i \subset N_+(\mathbf{x})$ we have $\bar{M}_i \subset N_+(\mathbf{x}^*)$. Hence $\cup_{i=1}^n \bar{M}_i \subset N_+(\mathbf{x})$. Now pick $j \in N_+(\mathbf{x}^*)$. By property (ii) of the

Frobenius normal form (see Definition B3), there exists some $i \in \{1, ..., n\}$ such that $j \rightrightarrows M_i$, meaning that $j \in \bar{M}_i$. This concludes the proof. $\square$

Proposition C4. *Let $(M_i)_{i=1,...,n}$ be a family of pairwise disconnected communities. There exists a perception-consistent equilibrium (PCE) $\mathbf{x}^*$ with $N_+(\mathbf{x}^*) = \cup_{i=1}^n \bar{M}_i$ if and only if*

$$\rho\left(\mathbf{G}_{M_1}\right) = ... = \rho\left(\mathbf{G}_{M_n}\right) > \max_{i=1,...,n} \max_{M' \in \mathcal{C}(\mathbf{G}):\, M' \succ M_i} \rho\left(\mathbf{G}_{M'}\right). \tag{C.3}$$

Proof of Proposition C4. The Frobenius normal form of $\mathbf{G}_{\cup_{i=1}^n \bar{M}_i}$ can be written as

$$\mathbf{G}_{\cup_{i=1}^n \bar{M}_i} = \begin{bmatrix} A_1 & A_{12} & ... & ... & ... & ... & ... & A_{1s+n} \\ 0 & A_2 & A_{23} & ... & ... & ... & ... & A_{2s+n} \\ ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & ... & 0 & A_s & A_{ss+1} & ... & ... & A_{ss+n} \\ 0 & ... & ... & 0 & \mathbf{G}_{M_1} & 0 & ... & 0 \\ 0 & ... & ... & .... & ... & ... & ... & ... \\ 0 & ... & ... & ... & ... & 0 & \mathbf{G}_{M_{n-1}} & 0 \\ 0 & ... & ... & ... & ... & ... & 0 & \mathbf{G}_{M_n} \end{bmatrix}. \tag{C.4}$$

By Lemma B4, this matrix admits a positive eigenvector (and therefore there exists a PCE $\mathbf{x}^*$ such that $N_+(\mathbf{x}^*) = \cup_{i=1}^n \bar{M}_i$) if and only if

$$\rho\left(\mathbf{G}_{M_1}\right) = ... = \rho\left(\mathbf{G}_{M_n}\right) > \max_{i=1,...,s} \rho(A_i).$$

Note that $V = \cup_{i=1,...,s} V_i$ is a closed set and thus

$$\{M' \in \mathcal{C}(\mathbf{G}) :\ M' \subset V\} = \{M' \in \mathcal{C}(\mathbf{G}) : M' \succ M_i \text{ for some } i = 1, ..., n\}.$$

Hence

$$\max_{i=1,...,s} \rho(A_i) = \rho\left(\mathbf{G}_V\right) = \max_{i=1,...,n} \max_{M' \in \mathcal{C}(\mathbf{G}):\, M' > M_i} \rho\left(\mathbf{G}_{M'}\right)$$

This concludes the proof. $\square$

In full generality, even if the set of perception-consistent equilibria is no longer finite, we can still describe it in a simple fashion: it is always a finite union of convex sets. Recall that the set of equilibria that admit a root is finite: there is at most one PCE with root M, for $M \in \mathcal{C}$. Let $\{\rho_1, ..., \rho_P\}$ be the set of positive eigenvalues of $\mathbf{G}$. The set of equilibria with a root can be written as

$$\bigcup_{p=1}^{P} S_p, \quad \text{where } S_p := \{\mathbf{x}^* : \ \mathbf{x}^* \ \text{ is a PCE with root } M \text{ such that } \ \rho(\mathbf{G}_M) = \rho_p\},$$

Proposition C5. *Given any network* $\mathbf{G}$ *the set of perception-consistent equilibria can be written as*

$$PCE = \bigcup_{p=1}^{P} \Lambda_p,$$

where Λ_p *is the convex polytope generated by* S_p*:* $\Lambda_p = Conv\,(S_p)$*.*

Proof of Proposition C5. We first show that $\bigcup_{p=1}^{P} \Lambda_p \subset PCE$. It amounts to showing that, if $S_p = \left\{\mathbf{x}^1, ..., \mathbf{x}^n\right\}$, and $\lambda_1, ..., \lambda_p$ are nonnegative numbers that sum to one then $\mathbf{x} := \sum_{j=1} \lambda_j \mathbf{x}^j$ is a PCE. We have

$$\mathbf{G}\mathbf{x} = \sum_{j=1}^{n} \lambda_j \mathbf{G}\mathbf{x}^j = \sum_{j=1}^{p} \lambda_j \rho_p \mathbf{x}^j = \rho_p \mathbf{x}.$$

and this concludes this implication.

We now turn to the other inculsion. Let $\mathbf{x}$ be a PCE. Then, by Proposition C4, there exists $p \in \{1, ..., P\}$ and a family of pairwise disconnected communities $\{M_i\}_{i=1,...,n}$ such that $N_+(\mathbf{x}) = \cup_{i=1}^{n} \bar{M}_i$, and $\rho\left(\mathbf{G}_{M_i}\right) = \rho_p > \max_{M' \succ M_i} \rho\left(\mathbf{G}_{M'}\right), \ \forall i = 1, ..., n$. Call $\mathbf{x}^i$ the equilibrium with root M_i, for $i = 1, ..., n$. We first define the following objects:

$$\tilde{M}_i := \bar{M}_i \setminus \left(\cup_{j \neq i} \bar{M}_j\right); \ \ \tilde{M} := \cup_{i=1}^{n} \bar{M}_i \setminus \left(\cup_{i=1}^{n} \tilde{M}_i\right); \ \ \lambda_i := \frac{\sum_{j \in \tilde{M}_i} x_j}{\sum_{i \in \tilde{M}_i} x_j^i}.$$

Note that, by construction, the family $\left\{\tilde{M}, \tilde{M}_1, ..., \tilde{M}_n\right\}$ constitutes a partition of $\cup_{i=1}^{n} \bar{M}_i$. Call $\mathbf{A}_i := \mathbf{G}_{\tilde{M}_i}$ and $\mathbf{A} := \mathbf{G}_{\tilde{M}}$. Then we can write

$$\mathbf{G}_{\cup_{i=1}^{n} \bar{M}_i} = \begin{bmatrix} \mathbf{A} & \mathbf{B}_1 & ... & ... & \mathbf{B}_n \\ 0 & \mathbf{A}_1 & 0 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & ... & 0 & \mathbf{A}_{n-1} & 0 \\ 0 & ... & ... & 0 & \mathbf{A}_n \end{bmatrix}$$

Be aware that this is not a Frobenius normal form because matrices $\mathbf{A}$ and $\mathbf{A}_i$ are in general not irreducible. However we know the following: $\rho(\mathbf{A}_i) = \rho_p$ for $i = 1, ..., n$ and $\rho(\mathbf{A}) < \rho_p$. Moreover, for $j = 1, ..., p$, $\mathbf{x}^i_{|\tilde{M}_i}$ is, by definition, a positive eigenvector of matrix $\mathbf{A}_i$. This is also true for $\mathbf{x}_{|\tilde{M}_i}$. The Frobenius normal form of $\mathbf{A}_i$ verifies the conditions of Lemma B6, (A_{s+1} corresponding here to M_i). As a result $\mathbf{x}_{|\tilde{M}_i}$ and $\mathbf{x}^i_{|\tilde{M}_i}$ are proportionnal:

$$\mathbf{x}_{|\tilde{M}_i} = \alpha_i \mathbf{x}^i_{|\tilde{M}_i}. \tag{C.5}$$

Since $\mathbf{x}_{|\cup_{i=1}^n \bar{M}_i}$ is an eigenvector of $\mathbf{G}_{\cup_{i=1}^n \bar{M}_i}$ associated to ρ_p we have

$$\rho_p \mathbf{x}_{|\tilde{M}} = \mathbf{A}\mathbf{x}_{|\tilde{M}} + \sum_{i=1}^n \mathbf{B}_i \mathbf{x}_{|\tilde{M}_i},$$

and thus, since $\mathbf{I} - \rho_p^{-1}\mathbf{A}$ is invertible,

$$\rho_p \mathbf{x}_{|\tilde{M}} = \left(\mathbf{I} - \rho_p^{-1}\mathbf{A}\right)^{-1} \sum_{i=1}^n \mathbf{B}_i \mathbf{x}_{|\tilde{M}_i} = \left(\mathbf{I} - \rho_p^{-1}\mathbf{A}\right)^{-1} \sum_{i=1}^n \alpha_i \mathbf{B}_i \mathbf{x}^i_{|\tilde{M}_i}.$$

On the other hand $\mathbf{x}^i_{|\tilde{M}\cup\tilde{M}_j}$ is an eigenvector of $\mathbf{G}_{|\tilde{M}\cup\tilde{M}_j}$ associated to ρ_p. Hence

$$\rho_p \mathbf{x}^i_{|\tilde{M}} = \mathbf{A}\mathbf{x}^i_{|\tilde{M}} + \mathbf{B}_i \mathbf{x}_{|\tilde{M}_i},$$

that is,

$$\rho_p \mathbf{x}^i_{|\tilde{M}} = \left(\mathbf{I} - \rho_p^{-1}\mathbf{A}\right)^{-1} \mathbf{B}_i \mathbf{x}_{|\tilde{M}_i}.$$

Finally we get

$$\rho_p \mathbf{x}_{|\tilde{M}} = \sum_{i=1}^n \alpha_i \rho_p \mathbf{x}^i_{|\tilde{M}_i},$$

i.e. $\mathbf{x}_{|\tilde{M}} = \sum_{i=1}^n \alpha_i \mathbf{x}^i_{|\tilde{M}_i}$. Combining this equality with (C.5) and the fact that $\mathbf{x}^m_{|\tilde{M}_i} = 0$ when $i \neq m$, we obtain that

$$\mathbf{x} = \sum_{i=1}^n \alpha_i \mathbf{x}^i$$

Now $\mathbf{x}$ and $\mathbf{x}^i$ being all associated to the same eigenvalue ρ_p we necessarily have $X = X^i$ for $i = 1, ..., n$. As a result $\sum_{i=1}^n \alpha_i = 1$ and this concludes the proof. □

Remark C1. *When* $\mathbf{G}$ *is a simple network, then every component is degenerate, i.e., they reduce to a singleton. In full generality, in a given component, the largest eigenvalue of the subgraph of active players is invariant.*

D Perceived competition under the Cournot model

Consider a standard homogeneous good Cournot oligopoly game with n firms competing in quantities. The (objective) profit function for each firm i is given by (assuming $\alpha = 1$)

$$\pi_i(x_i, \mathbf{x}_{-i}; \alpha) = h(x_i, X, \alpha) := \left(\max\left\{\bar{\beta} - X, 0\right\} - c\right) x_i, \text{ where } \bar{\beta} > c > 0. \tag{D.1}$$

Firms only observe the quantities produced by their neighbors, $X_{\mathcal{N}_i}$. Given α_i, the *perceived utility* of firm i is then equal to

$$u_i(x_i, \mathbf{x}_{\mathcal{N}_i}, \alpha_i) = h(x_i, x_i + X_{\mathcal{N}_i}, \alpha_i) := \left(\bar{\beta} - \frac{x_i + X_{\mathcal{N}_i}}{\alpha_i}\right)_+ x_i - cx_i. \tag{D.2}$$

Firm i believes that the price is an affine transformation of the total demand, which is correct. However, she does not observe the actual demand X, but only the demand in her perception set, $x_i + X_{\mathcal{N}_i}$. When observing that the realized price she faces in the market actually differs from her perceived price (the one she anticipated from observing the demand in her perception set), she remains unaware that this discrepancy is generated by other firms producing the same product in the economy. Instead, she believes it is because the slope affecting the market price is incorrect.

Observe that, as in the standard Cournot model, there is precisely one price for the homogeneous good defined in (D.1). Note that the market price is not misperceived by firms: it is observed. What firms may misperceive is how the market price emerges. Indeed, their perceptions of the slope affecting it are typically wrong.

PCE with Cournot competition. First, given α_i, each firm i chooses a quantity x_i^* that maximizes her perceived utility (D.2). This leads to:

$$\bar{\beta} - \frac{x_i^* + X_{\mathcal{N}_i}^*}{\alpha_i} - \frac{x_i^*}{\alpha_i} = c.$$

We combine this identity with the consistency condition of equality between objective payoff and subjective utility, which requires that

$$\alpha_i = \frac{x_i^* + X_{\mathcal{N}_i}^*}{X^*}.$$

We then obtain:

$$\bar{\beta} - X^* - \frac{x_i^* X^*}{x_i^* + X_{\mathcal{N}_i}^*} = c.$$

By solving this equation, we get:

$$X^*_{\mathcal{N}_i} = \left(\frac{2X^* - \bar{\beta} + c}{\bar{\beta} - c - X^*}\right) x_i^*,$$

and thus

$$\mathbf{G}\mathbf{x}^* = \left(\frac{2X^* - \bar{\beta} + c}{\bar{\beta} - c - X^*}\right) \mathbf{x}^*. \tag{D.3}$$

E Example with objective payoff given by (6)

E.1 Ignorance is not bliss: Perceived competition rents

Consider a model with two agents whose objective utility function is given by (6) and such that payoffs are symmetric, that is, $\theta_1 = \theta_2 = \theta$ and $\bar{\theta} = 2\theta$.

(i) First assume that the perception network is in fact the complete network. The first-order conditions for an interior PCE are given by,

$$\begin{aligned}\frac{\theta + x_2}{2\theta + x_1 + x_2} &= \frac{c}{\alpha}(2\theta + x_1 + x_2);\\ \frac{\theta + x_1}{2\theta + x_1 + x_2} &= \frac{c}{\alpha}(2\theta + x_1 + x_2).\end{aligned}$$

Denote by x^C the effort of both agents at a PCE in the complete network case, and π^C its associated payoff. An inclusive PCE exists if and only if $\theta < \frac{\alpha}{4c}$, with symmetric equilibrium efforts given by $x_1^C = x_2^C = -\theta + \frac{\alpha}{4c}$. Note that when $\bar{\theta}$ becomes too high, i.e. $\theta \geq \frac{\alpha}{4c}$, there is a unique PCE where $x_1^C = x_2^C = 0$.

(ii) Suppose now that agents 1 and 2 have different perceptions of the environment. Specifically agent 2 does not perceive agent 1 as a competitor but agent 1 does, so that $\mathcal{N}_1 = \{2\}$ and $\mathcal{N}_2 = \emptyset$. The first-order conditions for an interior PCE are given by

$$\frac{\theta + x_2}{\bar{\theta} + x_1 + x_2} = \frac{c}{\alpha_1}(\bar{\theta} + x_1 + x_2); \tag{E.1}$$

$$\frac{\theta}{\bar{\theta} + x_2} = \frac{c}{\alpha_2}(\bar{\theta} + x_1 + x_2). \tag{E.2}$$

Whereas the equation for agent 1 is unchanged –to the exception of the conjecture α_1 in place of α– notice the impact of having $\mathcal{N}_2 = \emptyset$ on the first-order condition of agent 2. In addition, while agent 1 perfectly understands that $\bar{\theta}$ is decomposed as $\theta_1 + \theta_2 = 2\theta$, agent 2 only perceives $\bar{\theta} - \theta_2$ without understanding how this remainder is to be further decomposed. Indeed, agent 2 ignores how many agents there may be around, and she therefore remains only with an aggregative view of $\bar{\theta}$. Note that, given the confirmed conjecture requirement of

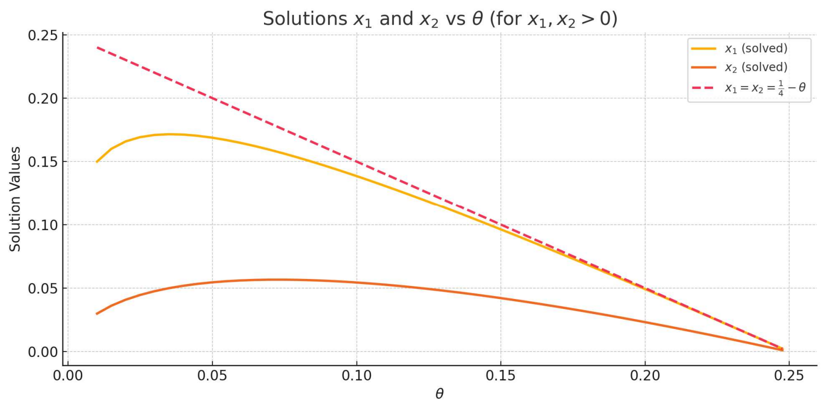


Figure E3: Inclusive perception-consistent equilibrium with $\theta_1 = \theta_2 = \theta$, $\bar{\theta} = 2\theta$, and $c/\alpha = 1$

a PCE, we have that $\alpha_i^*(\mathbf{x}^*) = \frac{\bar{\theta}+x_i^*+X^*_{\mathcal{N}_i}}{\theta+X^*}\alpha$. At an inclusive PCE $\mathbf{x}^*$, the confirmed-conjecture requirement transforms the first-order conditions into the following equality,

$$\frac{\bar{\theta} - \theta + X^*_{\mathcal{N}_i}}{(\bar{\theta} + x_i^* + X^*_{\mathcal{N}_i})(\bar{\theta} + X^*)}\alpha = c$$

We can show the following claim:

Claim. If effort profile $\mathbf{x}^*$ is an inclusive PCE, then :

- $x^C > x_1^* > x_2^*$.
- $\pi_1^* > \pi^C > \pi_2^*$.

Figure E3 illustrates Claim 1 for the case where $\frac{c}{\alpha} = 1$. For instance, pick the parameters constellation, $\theta = 0.1$, and $\alpha = c = 1$. Then $x^C = 0.15$, $x_1^* = 0.137$ and $x_2^* = 0.054$. In terms of equilibrium payoffs, we obtain that $\pi^C = 0.35$, while $\pi_1^* = 0.47$ and $\pi_2^* = 0.34$. Hence, due to her misperception of the environment, agent 2's equilibrium PCE effort is dramatically lower than agent 1's –the effort of agent 1 is more than twice agent 2's. Given the homogeneity in objective payoff functions, the gap in objective payoffs that results is only due to the position of agents in the perception network.

Let us now give a key insight of the perceived competition forces at play. It is important to observe that agent 1's perception of agent 2 doesn't imply that agent 1 knows that her competitor misperceives the environment, or that the environment ends at agent 2. Yet, the implicit transmission of information at a PCE shows an endogenous reflection of the perceived network on equilibrium actions. Agent 1 correctly anticipates agent 2's actions, which, in turn, implicitly transmits information about what agent 2 perceives as well − recall that $\mathcal{N}_2 = \emptyset$. At equilibrium, agent 1 correctly infer the value of competition so that $\alpha_1(\mathbf{x}^*) = \frac{\overline{\theta}+x_1^*+X^*_{\mathcal{N}_1}}{\overline{\theta}+X^*}\alpha = \alpha$. However, agent 2 misperceives the value of competition, with an equilibrium conjecture $\alpha_2(\mathbf{x}^*) = 0.647 < \alpha$. Agent 1 takes advantage of her position in the perceived network and receives a *perceived competition rent*. Agent 2 lags behind due to her informational disadvantage and, importantly, cannot benchmark the optimality of her action on other competitors. ◇

E.2 Exclusionary PCE

Let us revisit the application used above in section E.1 where,

$$\pi_i(x_i, \mathbf{x}_{-i}, \alpha) = \frac{\theta_i + x_i}{\overline{\theta} + x_i + \mathbf{x}_{-i}}\alpha - cx_i.$$

Let us impose that $\overline{\theta} = 0$ and $\theta_i = 0$ for each $i \in N$. We then obtain have a standard Tullock model, violating Assumption 2.1. Note that for $x_i > 0$, we have $\alpha_i(\mathbf{x}) = \frac{x_i+X_{\mathcal{N}_i}}{x_i+\mathbf{x}_{-i}}\alpha$. For $x_i = 0$, Assumption 3.1 holds with $\alpha_i(0, \mathbf{x}_{-i}) = \frac{X_{\mathcal{N}_i}}{\mathbf{x}_{-i}}\alpha$.

Therefore, a profile $\mathbf{x}^*$ is a PCE in the sense of Definition 2 if:

* for all $i \in N_+(\mathbf{x}^*)$,

$$c = \frac{X^*_{\mathcal{N}_i}}{(x_i^* + X^*_{\mathcal{N}_i})^2}\alpha_i(\mathbf{x}^*) = \frac{X^*_{\mathcal{N}_i}}{(x_i^* + X^*_{\mathcal{N}_i})X^*}\alpha, \tag{E.3}$$

* for all $i \in N_0(\mathbf{x}^*)$,

$$c \geq \frac{X^*_{\mathcal{N}_i}}{(\eta + X^*_{\mathcal{N}_i})^2}\alpha_i(0, \mathbf{x}^*_{-i}) = \frac{(X^*_{\mathcal{N}_i})^2}{(\eta + X^*_{\mathcal{N}_i})^2X^*}\alpha, \text{ for all } \eta > 0, \tag{E.4}$$

Note that the latter terms on the right-hand sides of Equations (E.3) and (E.4) are obtained by, respectively, replacing $\alpha_i(\mathbf{x}^*)$ and $\alpha_i(0, \mathbf{x}^*_{-i})$ by their confirmed-conjecture values.

For inactive agents $i \in N_0(\mathbf{x}^*)$, if $X^*_{\mathcal{N}_i} > 0$, the above condition is equivalent to $c \geq \frac{\alpha}{X^*}$, while it is automatically satisfied if $X^*_{\mathcal{N}_i} = 0$. We refer the reader to Theorem 3 for a precise statement in a more general case.

E.3 An illustration of the Marginal Feedback Condition.

Let us revisit our application where,

$$\pi_i(x_i, \mathbf{x}_{-i}, \alpha) = \frac{\theta_i + x_i}{\bar{\theta} + x_i + \mathbf{x}_{-i}} - cx_i.$$

Let us impose that $\bar{\theta} > \theta > 0$ and that $\theta_i \equiv \theta$ for each $i \in N$. First, separability holds, with $G(x) = \theta + x$, and $H(X, \alpha) = \frac{\alpha}{\bar{\theta}+X} - c$.[12] Next, we show that the marginal feedback condition holds. We have,

$$a(y, X^*) = \frac{\overline{\theta} + y}{\overline{\theta} + X^*}\alpha, \text{ and } \Phi_{\mathbf{x}^*}(y) = \frac{\partial H}{\partial X}(y, a(y, X^*)) = -\frac{a(y, X^*)}{(\overline{\theta} + y)^2} = -\frac{\alpha}{(\overline{\theta} + y)(\overline{\theta} + X^*)}.$$

Observe that $\Phi_{\mathbf{x}^*}(y)$ is strictly increasing in y, as required. Moreover,

$$G(x)\Phi_{\mathbf{x}^*}(y) = \frac{-\alpha}{\overline{\theta} + X^*}\frac{\theta + x}{\overline{\theta} + y}, \text{ while } G(x + \Delta)\Phi_{\mathbf{x}^*}(y + \Delta) = \frac{-\alpha}{\overline{\theta} + X^*}\frac{\theta + x + \Delta}{\overline{\theta} + y + \Delta}.$$

Next, since $\theta + x < \overline{\theta} + y$, we have

$$\frac{\theta + x}{\overline{\theta} + y} < \frac{\theta + x + \Delta}{\overline{\theta} + y + \Delta} \tag{E.5}$$

We finally obtain that $G(x)\Phi_{\mathbf{x}^*}(y) > G(x+\Delta)\Phi_{\mathbf{x}^*}(y+\Delta)$. The marginal feedback condition holds.

To see a case in which the marginal feedback condition fails, consider a strengthening of the ability for agents to get rents, by imposing that $\theta > \gamma\bar{\theta}$ for $\gamma > 0$, large. Incentives to compete vanish, and Equation (E.5) no longer holds. The marginal feedback condition is violated.

E.4 Ordinal Centrality does not imply Closedness

Consider again the example introduced in Section E.1 where,

$$\pi_i(x_i, \mathbf{x}_{-i}, \alpha) = \frac{\theta_i + x_i}{\bar{\theta} + x_i + \mathbf{x}_{-i}}\alpha - cx_i.$$

Assuming that $\theta_i = \theta_j = \theta$ for all $i, j \in N$, it is easy to see that the ordinal centrality result, Equation (13), holds. Indeed, recall Equation (7) that defined equilibrium actions:

$$\mathbf{G}_{\mathbf{x}_+^*}\mathbf{x}_+^* = \frac{c\left(X^* + \overline{\theta}\right)}{\alpha - c\left(X^* + \overline{\theta}\right)}\mathbf{x}_+^* + \frac{\alpha}{\alpha - c\left(X^* + \overline{\theta}\right)}\boldsymbol{\theta} + \frac{c\overline{\theta}\left(\overline{\theta} + X^*\right) - \alpha\overline{\theta}}{\alpha - c\left(X^* + \overline{\theta}\right)}\mathbf{1},$$

[12] This is up to a constant term, i.e. $h(x_i, X, \alpha) = G(x_i)H(X, \alpha) + \theta c$.

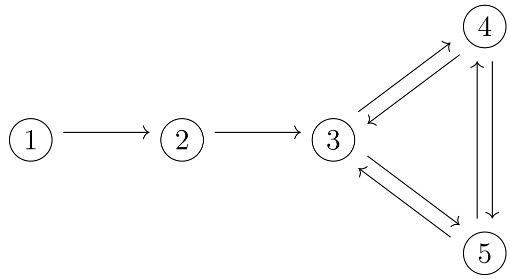


Figure E4: Ordinal Centrality without closedness

Thanks to the assumption that $\theta_i = \theta_j$, the last two terms on the right-hand side are common to all agents. Since the left-hand side is equivalent to $\mathbf{x}_{\mathcal{N}_i}$, it is clear that $X^*_{\mathcal{N}_i} > X^*_{\mathcal{N}_j}$ implies that $x^*_i > x^*_j$. Regarding closedness of the set of active agents, notice that we have the following:

* for all $i \in N_+(\mathbf{x}^*)$,

$$\frac{c}{\alpha}(\bar{\theta} + X^*) = \frac{\bar{\theta} - \theta + X^*_{\mathcal{N}_i}}{(\bar{\theta} + x^*_i + X^*_{\mathcal{N}_i})};$$

* for all $i \in N_0(\mathbf{x}^*)$,

$$\frac{c}{\alpha}(\bar{\theta} + X^*) \geq \frac{\bar{\theta} - \theta + X^*_{\mathcal{N}_i}}{(\bar{\theta} + X^*_{\mathcal{N}_i})}.$$

Hence, it might be the case that $X^*_{\mathcal{N}_i} > 0$ and $x^*_i = 0$, without violating these conditions. Note also that the proof for closedness in Proposition 2 also relies on the assumption that $G(0) = 0$, whereas here $G(0) = \theta$. To see an example where ordinal centrality holds, while closedness fails, let $N = \{1, .., 5\}$, and consider the perception network $\mathbf{G}$ shown in Figure E4. For simplicity, let $\bar{\theta} = \theta$ –assuming that θ is arbitrarily close to $\bar{\theta}$ would suffice. It can be shown that there exists a PCE profile $\mathbf{x}^* = (0, x^*_2, x^*_3, x^*_4, x^*_5)$, with $x^*_2 = 0.0197$, $x^*_j = \bar{x} = 0.1394$ for $j = 3, 4, 5$. In this case, the first-order condition of agent 1 gives $x^*_2/(\bar{\theta} + X^*)(\bar{\theta} + x^*_2) < 1$, hence $x^*_1 = 0$. It can also be checked that Equation (7) holds for the set of active agents.

F Examples illustrating Proposition 5

Example F1. Let $N = \{1, ..., 16\}$, $\mathbf{G}$ be defined as in Definition 1, with $|M_1| = 4$, $|M_2| = 5$, and $|M_3| = 7$ and $M_1 \succ M_2 \succ M_3$. In order to obtain a positive perceived competition rent, community M_1 needs to have a sufficient number of links towards M_2. Yet, the final result is also conditional on the links from M_2 to M_3.

(i) Agents in M_1 get higher payoffs than those in M_2 if

$$\frac{\lambda_{M_1}}{|M_3| - |M_1|} > 1 \iff \lambda_{M_1} > 3$$

(ii) Agents in M_1 get higher payoffs than those in M_3 if,

$$\left(\frac{\lambda_{M_1}}{|M_3| - |M_1|}\right)\left(\frac{\lambda_{M_2}}{|M_3| - |M_2|}\right) > 1 \iff \lambda_{M_1}\lambda_{M_2} > 6 \tag{F.1}$$

Note that agents in M_1 can obtain a lower payoff than those in M_2, while still getting a higher payoff than agents in M_3. Hence, payoffs are not necessarily strictly ranked according to the ranking $\succ$, across communities. Of course, if the inequality in Equation F.1 is reversed, agents in the root of the PCE get higher payoffs than those in the highest ranked community. In that case, the density of M_3–its inner-community perception–, is stronger than the direct knowledge advantage that agents in M_1 have–their outer-community perception. $\diamond$

Example F2. Let $\mathbf{G} \in \mathcal{G}$ be a regular communities graph with 3 communities such that $M_1 \succ M_2 \succ M_3$, and $|M_1| = 3$, $|M_2| = 5$, and $|M_3| = 4$. Next, $\gamma_{M_1} = 1$, $\gamma_{M_2} = 2$, and $\gamma_{M_3} = 3$. Finally, $\lambda_{M_1} = 1$, $\lambda_{M_2} = 1$, and $\lambda_{M_3} = 0$. Consider the PCE $\mathbf{x}^*$ with root M_3. Clearly, $\bar{\pi}_{M_1} = \frac{1}{2}\bar{\pi}_{M_2}$ and $= \bar{\pi}_{M_2} = \bar{\pi}_{M_3}$. Hence, for M_1, being active at all PCEs doesn't imply that its payoff are greater than those in the active but dominated communities. Notice that there are two possible levers to increase community M_1's payoff: an increase in inner-community perception–as given by γ_{M_1}, or an increase in outer-community perception–as given by λ_{M_1}. In this instance, other things equal, both have the same impact: an increase by 1 of either of these numbers equalizes payoffs across communities. Instead, suppose that both λ_{M_1} and λ_{M_2} increase by one. Then, M_1 benefits from the increase in outer-perception of M_2. The payoff of M_1 and M_2 are equalized and agents in M_3 obtain the lowest payoff. $\diamond$

Example F3. *Driving forces of PCEs with Tullock*

• Consider Figure 3(a). We already know that there are two communities: $M_1 = \{1, 2, 3\}$ and $M_2 = \{4, 5, 6\}$, with $M_1 \succ M_2$. Hence there is an equilibrium with root M_1, where only agents $1, 2$, and 3 are active. Since $\rho(\mathbf{G}_{M_2}) = \rho(\mathbf{G}_{M_1})$, there is no equilibrium with root M_2. As a result, there is a unique equilibrium, an exclusionary PCE with $x_1^* = x_2^* = x_3^* = \frac{2\alpha}{9c}$ and $x_4^* = x_5^* = x_6^* = 0$.

• Consider again Figure 3(b). The strongly connected component M_1 is relatively less dense compared to Figure3(a) and, thus, its spectral radius is smaller. The partial order is unchanged: $M_1 \succ M_2$. Hence, there is again an exclusionary equilibrium $\mathbf{x}^*$, with root $M_1 = \{1, 2, 3\}$. However, equilibrium actions are different: $x_1^* = x_2^* = \frac{1}{(1+\sqrt{2})^2}\frac{\alpha}{c}$, $x_3^* = \frac{\sqrt{2}}{(1+\sqrt{2})^2}\frac{\alpha}{c}$, and

$x_4^* = x_5^* = x_6^* = 0$. Since $\rho(\mathbf{G}_{M_2}) = 2 > \sqrt{2} = \rho(\mathbf{G}_{M_1})$, there is also a PCE $\mathbf{y}^*$ with root M_2, where all agents are active, with the following action levels: $y_1^* = \frac{\alpha}{9c}$, $y_2^* = \frac{7\alpha}{45c}$, $y_3^* = \frac{2\alpha}{15c}$, and $y_i^* = \frac{4\alpha}{45c}$ for $i \in \{4,5,6\}$.

• Finally, consider the perception network displayed in Figure 4 with $M_1 = \{1,2,3\}$, $M_2 = \{4,5\}$, and $M_3 = \{6,7,8,9\}$, with $M_1 \succ M_2 \succ M_3$. Each community's subnetwork is complete, so that $\rho(\mathbf{G}_{M_2}) = 1 < \rho(\mathbf{G}_{M_1}) = 2 < \rho(\mathbf{G}_{M_3}) = 3$.

Since M_1 is $\succeq$-maximal, there is an exclusionary PCE with root M_1. Using Theorem 3, the equilibrium actions are given by:

$$x_i^* = \frac{2\alpha}{9c} \text{for } i \in \{1,2,3\},\ x_j^* = 0 \text{ for } j \in \{4,5,6,7,8,9\}.$$

Since $M_1 \succ M_2$ and $\rho(\mathbf{G}_{M_2}) < \rho(\mathbf{G}_{M_1})$, there is no equilibrium with root M_2.
Finally, Since $\rho(\mathbf{G}_{M_3}) > \max\{\rho(\mathbf{G}_{M_1}), \rho(\mathbf{G}_{M_2})\}$, there also exists an inclusive PCE y^* with root M_3 such that the set of active individuals is $\bar{M}_3 = N$. This equilibrium is given by:

$$y_i^* = \frac{3\alpha}{56c} \text{for } i \in \{1,4,5\},\ y_j^* = \frac{9\alpha}{112c} \text{ for } j \in \{2,3\}, \text{ and } \ y_k^* = \frac{3\alpha}{28c}, \text{ for } k \geq 6.$$

◇

G Welfare and Policy Interventions

In most network models, the planner may ask which network maximizes total welfare. When the game is with strategic complementarities, the complete network or a nested-split graph is often the optimal network (König et al., 2014; Belhaj et al., 2016; Chen et al., 2022). Interestingly, this is not true here and giving more information to agents by adding links is not necessarily good news for agents' welfare.[13]

G.1 Aggregate Equilibrium Payoffs and Efforts

Here, we discuss some key facts about aggregate payoffs (welfare) and efforts. For this, we keep the assumption on the homogeneity of agents up to their positions in the perception network—that is, for each agent i, $h_i(x_i, X) = h(x_i, X) = x_i f(X)$.

Pick any perception-consistent equilibrium $\mathbf{x}^*$ with root M. Then, the payoff for any agent $i \in N_+(\mathbf{x}^*)$ is given by $\pi_i(\mathbf{x}^*) = x_i^* f(X^*)$. It is easy to see that $(\rho(\mathbf{G}_M) + 1) f(X^*) =$

[13] Remember that our game is neither with strategic complementarities nor with strategic substitutes.

$-X^*f'(X^*)$. The aggregate utility is then given by:

$$\sum_i \pi_i(\mathbf{x}^*) = X^* f(X^*) = \frac{-(X^*)^2 f'(X^*)}{\rho(\mathbf{G}_M)+1}. \tag{G.1}$$

By Theorem 3, the equilibrium utility of any $i \in N_+(\mathbf{x}^*)$ is proportional to her equilibrium effort and is therefore proportional to her eigenvector centrality in the sub-network of active agents. However, this only informs us about the relative utility of active agents at equilibrium, but it does not tell us anything about the aggregate utility itself. For the two applications we looked at, we can nevertheless say more. For the Tullock contest function, we have:

$$\sum_i \pi_i(x_i^*) = \frac{\alpha}{\rho(\mathbf{G}_M)+1}. \tag{G.2}$$

Equation (G.2) displays aggregate utility as an explicit function of $\rho(\mathbf{G}_M)$, the spectral radius of the part of the perception network associated with the equilibrium –with root M. Clearly, aggregate utility decreases with an increase in density. In the general case, as can be seen in equation (G.1), we cannot make such a statement because X^* implicitly depends on $\rho(\mathbf{G}_M)$.

Can we say anything about the relationship between the density of the perception network and the aggregate effort, X^*, at the unique stable PCE $\mathbf{x}^*$? By Theorem 3, we have:

$$\rho(\mathbf{G}) + 1 = \frac{-Xf'(X^*)}{f(X^*)}.$$

Then, if f is concave, the aggregate effort at the stable PCE increases with $\rho(\mathbf{G})$. This condition is sufficient, but not necessary. Indeed, for the Tullock model, this statement holds, even though f is convex. An immediate corollary is that the *aggregate effort* at a PCE is always smaller than at the Nash equilibrium. It is worth noting that, at the individual level, there is no such relation between density of the perception network and equilibrium efforts. For instance, in the perception network of Figure 4, while agents 1 to 5 exert less effort at the stable PCE than at the Nash equilibrium, it is the opposite for agents 6 to 9. In Figure 1(b), only agent 2 exerts more effort at the stable PCE than at the Nash equilibrium.[14] The effort of a given agent at a PCE can be either larger or smaller than the Nash effort, depending of their eigenvector centrality. Importantly, note that more central agents can be located anywhere in the hierarchy of communities.

We now analyze three different policies. We first consider a policy that only adds one link in the network (thus, giving more information about competitors to a pair of agents) and

[14]The equilibrium effort at the Nash equilibrium is equal to $\frac{8}{81}\frac{\alpha}{c}$ in Figure 2, while it is equal to $\frac{5}{36}\frac{\alpha}{c}$ in Figure (1b).

examine whether it affects the effort of the agents in the economy. Next, we study the standard key-player policy, the aim of which is to determine the agents who should be removed from the network to maximize total activity. Finally, we consider a social mixing policy that examines how merging two separate networks affects the total activity in the (merged) network.

G.2 Adding Links

We now consider the policy implications of our model. We start with the simplest intervention: *Given a network and its unique stable perception-consistent equilibrium, what would happen if we add a link between two agents?*

Consider networks with a unique dominant component. We only focus on *stable* perception-consistent equilibria. We examine whether adding a link from individual i to individual j has an impact on individual efforts. If we do not make additional assumptions on the payoff structure, adding links does not have a clear impact on the aggregate equilibrium effort.[15] Hence we only obtain results on relative efforts in the general model:

Proposition G6. *Pick a network* $(N, \mathbf{G})$ *with* $\mathbf{x}^*$ *being the asymptotically stable perception-consistent equilibrium.*[16] *Suppose that* $i, j \in N_+(\mathbf{x}^*)$*, and* $g_{ij} = 0$*. Let* $\widehat{\mathbf{G}}$ *be the network obtained from* $\mathbf{G}$ *by adding a link from* i *to* j*. Then,* $\widehat{\mathbf{G}}$ *admits an asymptotically stable perception-consistent equilibrium* $\widehat{\mathbf{x}}^*$ *that has the following properties:*

(i) $N_+(\widehat{\mathbf{x}}^*) \subseteq N_+(\mathbf{x}^*)$,

(ii) for any $k \in N$*, we have* $\frac{\widehat{x}_i^*}{x_i^*} > \frac{\widehat{x}_k^*}{x_k^*}$.

Proof of Proposition G6. Let M be the root of $\mathbf{x}^*$, meaning that $N_+(\mathbf{x}^*) = \bar{M}$, and $\rho := \rho(\mathbf{G}_{\bar{M}}) = \rho(\mathbf{G})$. The network $\hat{\mathbf{G}}$ also has a unique dominant component, $\hat{M}$. Either $\hat{M} = M$ (if, for instance there is no path from j to i), or $\hat{M}$ is a community which did not exist in $\mathbf{G}$. When it is the case, we have $i, j \in \hat{M}$, $\hat{M} \subset \bar{M}$ and $\rho(\mathbf{G}_{\hat{M}}) \geq \rho$. Hence there is a unique stable equilibrium $\hat{\mathbf{x}}$ (with root $\hat{M}$) in $\hat{\mathbf{G}}$, $N_+(\hat{\mathbf{x}}) \subset N_+(\mathbf{x})$ and $\hat{\rho} := \rho(\hat{\mathbf{G}}_{\hat{M}}) = \rho(\hat{\mathbf{G}})$.

We now prove that *(ii)* holds. Note that $\hat{\mathbf{x}}^* \neq \mathbf{x}^*$: suppose by contradiction that $\mathbf{x}^* = \hat{\mathbf{x}}^*$. Let $k \neq i$ with $k \in N^+(\mathbf{x}^*)$. Then $\rho x_k^* = (\mathbf{G}\mathbf{x}^*)_k = (\hat{\mathbf{G}}\mathbf{x}^*)_k = \hat{\rho}\hat{x}_v^*$, implying that $\rho = \hat{\rho}$. Thus $\rho x_i^* = (\mathbf{G}\mathbf{x}^*)_i = (\hat{\mathbf{G}}\mathbf{x}^*)_i - x_j^* = \rho x_i^* - x_j^*$, a contradiction. Consider the following subsets of

[15] Observe that, in the general case, X^* is increasing with $\rho(\mathbf{G})$ if $f(X)(-f'(X) - Xf''(X)) + X(f'(X))^2 > 0$. A sufficient condition is that $Xf'(X)$ is non-increasing. However, it is not necessary. For example, in the linear Tullock model, $Xf'(X) = -\alpha/X$. Nevertheless, $f(X)(-f'(X) - Xf''(X)) + X(f'(X))^2) = c\alpha/X^2 > 0$

[16] Recall that our analysis of the stable PCE holds under the assumption of unique dominating component, see Online Appendix B.3 for details.

agents:

$$K_+ := \left\{ k \in \hat{M} : \frac{\hat{x}_k^*}{x_k^*} \geq \frac{\hat{x}_l^*}{x_l^*} \ \forall l \in \hat{M} \right\}, \ \ K_- := \left\{ k \in \hat{M} : \frac{\hat{x}_k^*}{x_k^*} \leq \frac{\hat{x}_l^*}{x_l^*} \ l \in \hat{M} \right\}.$$

We actually prove a stronger property, namely that $i \in K_+$. Note that if $k \neq i$ and $k \in K_+$ then $\frac{\hat{x}_k^*}{x_k^*} = \frac{\sum_{w \in \mathcal{N}_k} \hat{x}_w^*}{\sum_{w \in \mathcal{N}_k} x_w^*}$. Hence $w \in K_+$ for all $w \in \mathcal{N}_k$. By a recursive argument this implies that, if k is connected to w through a path then $w \in K_+$. The same property also holds for K_-. As a consequence $i \in K_+ \cup K_-$. If this were not the case there would exist two nodes $k_+ \neq i$ and $k_- \neq i$ such that $k_+ \in K_+$ and $k_- \in K_-$, which would imply that elements of M belong to both K_+ and K_-, a contradiction.
Suppose first that we are in the case where $\hat{\rho} > \rho$, and let $k \neq i$. Suppose that $k \in K_+$. Then

$$\frac{1}{\hat{\rho}} = \frac{\hat{x}_k^*}{(\hat{\mathbf{G}}\hat{\mathbf{x}}^*)_k} = \frac{\hat{x}_k^*}{\sum_{w \in \mathcal{N}_k} \hat{x}_w^*} \geq \frac{x_k^*}{\sum_{w \in \mathcal{N}_k} x_w^*} = \frac{x_k^*}{(\mathbf{G}\mathbf{x}^*)_k} = \frac{1}{\rho},$$

a contradiction. Hence $K_+ = \{i\}$.
Suppose now that $\hat{\rho} = \rho$. Showing that $i \in K_+$ is equivalent to showing that $i \notin K_-$. Suppose by contradiction that $i \in K_-$. Then

$$\frac{1}{\rho} = \frac{\hat{x}_i^*}{(\hat{\mathbf{G}}\hat{\mathbf{x}}^*)_i} = \frac{\hat{x}_i^*}{\sum_{w \in \mathcal{N}_i} \hat{x}_w^* + \hat{x}_j^*} < \frac{\hat{x}_i^*}{\sum_{w \in \mathcal{N}_i} \hat{x}_w^*} \leq \frac{x_i^*}{\sum_{w \in \mathcal{N}_i} x_w^*} = \frac{x_i^*}{X_{\mathcal{N}_i}^*} = \frac{1}{\rho},$$

where the strict inequality follows from the fact that $j \in N_+(\hat{\mathbf{x}})$ (see above). This is a contradiction. Thus $i \in K_+$. □

Since both players i and j initially belong to the set of active agents, there is no reason why adding a link between them should induce a positive effort from an initially inactive agent. Indeed, the spectral radius of the subgraph of inactive agents remains the same while the spectral radius of the set of agents who were initially active can only increase. In other words, $N_+(\widehat{\mathbf{x}}^*) \subseteq N_+(\mathbf{x}^*)$. This is part (i). Additionally, agent i becomes relatively more central compared to other agents, when adding a link from i to j. This implies that the relative increase in effort is maximal for agent i. This is captured by part (ii) of the proposition. Note that, if $f(\cdot)$ is such that X^* increases with the spectral radius of the graph (see subsection G.1 for a discussion), then part (ii) of Proposition G6 automatically implies that $\hat{x}_i^* > x_i^*$.

G.3 Key Players

Another possible intervention involves removing one agent from the economy. This is known as the *key-player* policy (Zenou, 2016) and is particularly relevant in the crime application (Ballester et al., 2006, 2010) but also in the conflict application (König et al., 2017), because governments want to target these individuals (the key players) in order to reduce total activity X (total crime or total conflict).

In the general version of the model, there is no clear relationship between network density (captured by the spectral radius $\rho(\mathbf{G})$) and total equilibrium effort X^* (see Section G.1), and it is therefore difficult to obtain a general result. However, a specific case such as $h_i(x_i, X) = h(x_i, X) = x_i f(X)$ for each agent i allows us to derive some interesting results—which then apply to the Tullock application.

Proposition G7. *Consider the (linear) Tullock contest game. Let* $\mathbf{x}^*$ *be the (unique) asymptotically stable equilibrium of the simple network* $(N, \mathbf{G})$ *and* $\widehat{\mathbf{x}}^*$ *the (unique) asymptotically stable equilibrium of the simple network* $(N\backslash\{i\}, \mathbf{G}_{N\backslash\{i\}})$*. Then,* $\widehat{X}^* \leq X^*$*.*

Proof of Proposition G7. We have $X^* = \frac{\alpha\rho(\mathbf{G})}{c[1+\rho(\mathbf{G})]}$ and $\widehat{X}^* \leq \frac{\alpha\rho(\mathbf{G}_{N\backslash\{i\}})}{c[1+\rho(\mathbf{G}_{N\backslash\{i\}})]}$. By standard results, $\rho(\mathbf{G}) \geq \rho(\mathbf{G}_{N\backslash\{i\}})$. Hence $\widehat{X}^* \leq X^*$. □

Proposition G7 shows that, for instance in the Tullock model, removing an agent will never increase total equilibrium effort. Indeed, the largest eigenvalue either stays the same or is reduced; and the latter decreases total effort. However, the distribution of efforts may be greatly altered, as shown in the following example.

Example G4. Key players and the spread of efforts across neighborhoods
Consider the Tullock application and the perception network displayed in Figure 3(a) (Example 3). We have shown that there is a unique stable perception-consistent equilibrium where the only active agents belong to $\succeq$-maximal community $M_1 = \{1, 2, 3\}$ with $x_1^* = x_2^* = x_3^* = \frac{2\alpha}{9c}$ and thus the total effort is $X^* = \frac{2\alpha}{3c}$. Let us now remove agent 1. It is easily verified that the unique (stable) equilibrium is now an inclusive PCE $\mathbf{x}^{[-1]*}$ where $N_+(\mathbf{x}^*) = \{2, 3, 4, 5, 6\}$, even though the total effort remains the same at $\frac{2\alpha}{3c}$. Indeed, by removing agent 1, the spectral radius of $M_1 = \{1, 2, 3\}$ decreases from 2 to 1 and becomes strictly smaller than the (unchanged) spectral radius of $M_2 = \{4, 5, 6\}$, which is equal to 2. As a result, the only stable PCE is an inclusive equilibrium. Removing an agent can thus have the *counter-productive effect* of making inactive agents active. In the standard key-player policy (Zenou, 2016), this is not possible since total effort always decreases as *all* agents reduce their individual effort. ◇

G.4 Social Mixing

We conclude this section with a brief look at the issue of *social mixing*. To this end, we depart slightly from our initial model in which there was one (simple) network. Suppose, instead, that we start with two disconnected (simple) networks $(N^1, \mathbf{G}^1)$ and $(N^2, \mathbf{G}^2)$, each of which has a unique stable PCE. As above, we obtain results only for specific cases. We consider here social mixing in the Tullock contest function model.[17] Think of social mixing as starting with two fully segregated neighborhoods, each endowed with their own resources α^1 and α^2. The key question for the planner is whether merging these two neighborhoods (social mixing) into a connected network $(N, \mathbf{G})$, with $N = N^1 + N^2$, $\alpha = \alpha^1 + \alpha^2$, leads to an increase in total activity and resources.

Proposition G8. *Consider the (linear) Tullock contest game. Let $(N^1, \mathbf{G^1})$ and $(N^2, \mathbf{G}^2)$ be two simple networks endowed with resources equal to α_1 and α_2, respectively. Let $\mathbf{x}^{1*}$ (resp. $\mathbf{x}^{2*}$) be the unique stable PCE of $(N^1, \mathbf{G^1})$ (resp. $(N^2, \mathbf{G}^2)$), with root M_1 (resp. M_2).Let also $(N, \mathbf{G})$ be the network obtained from $(N^1, \mathbf{G^1})$ and $(N^2, \mathbf{G}^2)$ in which $N = N^1 \cup N^2$, $\alpha = \alpha^1 + \alpha^2$, with $g_{ij} = 1$ and $g_{k\ell} = 1$ for some $(i, \ell) \in M_1$, $(j, k) \in M_2$. Then, there is a unique stable PCE $\mathbf{x}^*$ of $(N, \mathbf{G})$ satisfying $\rho(\mathbf{x}^*) = \rho(\mathbf{G})$, and $X^* > X^{1*} + X^{2*}$.*

Proof of Proposition G8. We have

$$X^1 = \frac{\alpha^1}{c}\frac{\rho(\mathbf{G}^1)}{\rho(\mathbf{G}^1)+1}; \;\; X^2 = \frac{\alpha^2}{c}\frac{\rho(\mathbf{G}^2)}{\rho(\mathbf{G}^2)+1}; \;\; X = \frac{\alpha^1+\alpha^2}{c}\frac{\rho(\mathbf{G})}{\rho(\mathbf{G})+1}$$

We have $\rho(\mathbf{G}) = \rho(M_1 \cup M_1) > \max\left\{\rho(\mathbf{G}^1), \rho(\mathbf{G}^2)\right\}$. Hence

$$X^1 + X^2 = \frac{\alpha^1}{c}\frac{\rho(\mathbf{G}^1)}{\rho(\mathbf{G}^1)+1} + \frac{\alpha^2}{c}\frac{\rho(\mathbf{G}^2)}{\rho(\mathbf{G}^2)+1}\frac{\alpha^1+\alpha^2}{c} < \frac{\rho(\mathbf{G})}{\rho(\mathbf{G})+1} = X.$$

□

Proposition G8 shows that the total effort in any new stable PCE of the connected network $(N, \mathbf{G})$ is higher than the sum of total efforts in each disconnected neighborhood. Hence, linking the two neighborhoods is beneficial to aggregate effort. On the other hand, the distribution of resources between agents in N^1 and N^2 is less clear. Indeed, distribution in the new equilibrium depends on the specific connections that are formed between the two groups. It is therefore possible to have some agents who are worse off following the mixing of the two neighborhoods.

[17] Similar results can be obtained for the Cournot model.