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**Abstract**

Two countries produce goods and are penalized by the common pol-

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lution they generate. Each country maximizes an inter-temporal utility criterion, taking account of the pollution stock to which both contribute. The dynamic is in continuous time with possible sudden switches to less polluting technologies. The set of Nash equilibria, for which solutions also remain in the set of constraints, is the intersection of two manifolds in a certain state space. At the Nash equilibrium, the choices of the two countries are interdependent: different productivity levels after switching lead the more productive country to hasten and the less productive to delay the switch. In the absence of cooperation, efforts by one country to pollute less motivate the other to pollute more, or encourage the country that will be cleaner or less productive country after switching to delay its transition.

**Key words**: Pollution, dynamic game, Nash, viability theory.

**AMS classification**: 49M30, 49K30, 91B06, 91B5

# 1 Introduction

The economics of the commons (Hardin, 1968, 1994, 2008) with two non-cooperating countries sharing a common pollution may lead to Nash equilibria. The complexity of the dynamic usually makes it hard to find an analytic solution, and this holds even more so when trajectories have to remain within constraints and when the continuous dynamic can be reset by impulses on state variables.

Taking the production function linear in input and the input as a control variable, Boucekkine, Krawczyck, and Vallée (2011) proved the existence of a Nash equilibrium, where each player chooses the technology without regard for the other's choice. Here, we consider a Cobb-Douglas production function and the input as a state variable governed by a differential equation. This precludes finding analytic solutions, but we can deal with non-linearities and with the switching times from the more to the less polluting regime. Our solution is numerical and the procedure can be used whenever looking for Nash equilibria between two rivals.

Our solution to the problem of the commons innovates by exploiting the properties of *capture-viability kernels*. The capture-viability kernel of a closed set $\mathcal{K}$ with closed target $\Omega$ under a set-valued dynamic $F$ is the set of all initial states from which there exists at least one solution remaining in $\mathcal{K}$ and reaching $\Omega$ at a given time horizon, possibly infinite. Such initial states are called *viable*. Our innovation is to show that a state is a viable Nash equilibrium if and only if it is located at the intersection of certain boundaries of two capture-viability kernels. Fulton (1997), Sarkar, Gupta, and Pal (1998), and Agata (2010) proposed geometric solutions of Cournot oligopoly, but not in the framework of optimal control.

The solution by capture-viability kernels yields all valuation functions without solving the first-order Pontryagin necessary conditions for each set of initial conditions, without even solving any differential equation (Bonneuil and Boucekkine, 2014, 2016). State constraints are inherently taken into account, while they must be treated one by one in Pontryagin's or Hamilton-Jacobi-Bellman's methods, with the additional requirement to verify sufficient conditions, in the sense of Arrow's theorem (Kamien and Schwartz, 1991).

In dynamic programming, functional relationships often have to be assumed in order to derive the optimal feedback rules. This kind of assumption is not necessary here, which opens the way to searching for possible a posteriori relationships between controls and state variables at Nash equilibria, as well as a posteriori dependence between the two players.

We show that different productivity levels after the switch to a cleaner technology gives the more productive country a lead in the valuation function: this country hastens, and the less productive country delays, making the switch.

We reveal the non-corner solutions of the game: at Nash equilibrium, the players are not assumed to be selfish as in Boucekkine, Krawczyck, and Vallée (2011). Our consideration of non-linearities, through the Cobb-Douglas production functions, enriches the problem and leads to a more differentiated set of equilibria. The problem has infinite time horizon support. We solve it by finite time horizon approximation, which Bonneuil (2012) proved to be convergent. Because we deal with non-linearities, our solution, based on Bonneuil (2006)'s algorithm, is numerical.

After presenting the problem of common pollution (section 2) and viability theory (section 3.1), we explain how to infer Nash equilibria from capture-viability kernels (section 3.2). Numerical solving is achieved with the viability algorithm (section 4.1), which we use to consider three exemplary cases (section 4.2): same response from each country in terms of productivity and reducing pollution to the introduction of a cleaner technology; same productivity but different levels of pollution reduction, and different productivity levels with similar effect on pollution. We determine the Nash equilibria (section 4.3).

## 2 The problem

Two non-cooperating countries called 1 and 2 produce goods, generating a pollution equally detrimental to both countries. They do not trade in goods. Each country $j = 1, 2$ has consumption $C^{(j)}$ and capital $K^{(j)}$, with common capital share $\nu$ in the production function $A^{(j)}(K^{(j)})^{\nu}$. It uses a technology with progress level $A^{(j)}$ and marginal contribution to pollution $\alpha^{(j)}$. It has the possibility, at a date $t^{(j)}$ that it chooses, to switch from a technology characterized by $A^{(j)} =$

$A_1^{(j)}$ and $\alpha^{(j)} = \alpha_1^{(j)}$ to a less polluting technology characterized by $A_2^{(j)}$ and marginal contribution to pollution $\alpha_2^{(j)}$. Pollution comes from both countries and is proportional to production.

Country $j = 1, 2$ solves the program:

$$\max_{C^{(j)}, t^{(j)}} \int_0^{\infty} U(C^{(j)}(t), P(t)) e^{-\rho t}\, dt, \tag{1}$$

where $U$ is the utility function and $\rho$ the discount rate, under the continuous-time dynamic:

$$\begin{cases} K^{(j)\prime}(t) &=& A^{(j)}(t)(K^{(j)}(t))^{\nu} - C^{(j)}(t) - \delta_K K^{(j)}(t) \\ P'(t) &=& \sum_{i=1}^{2} \alpha^{(i)}(t) A^{(i)}(t)(K^{(i)}(t))^{\nu} - \delta_P P(t) \\ A^{(j)\prime}(t) &=& 0 \\ \alpha^{(j)\prime}(t) &=& 0 \\ A^{(j)}(0) &=& A_1^{(j)},\ \alpha^{(j)}(0) = \alpha_1^{(j)},\ K^{(j)}(0) = K_0^{(j)}, P(0) = P_0, \quad j = 1, 2, \end{cases} \tag{2}$$

and the impulse (or discrete-time part) at $t^{(j)}$, $j = 1, 2$:

$$\begin{cases} K^{(j)+} &=& K^{(j)-} \\ P^{+} &=& P^{-} \\ A^{(j)+} &=& A^{(j)-} + A_2^{(j)} - A_1^{(j)} \\ \alpha^{(j)+} &=& \alpha^{(j)-} + \alpha_2^{(j)} - \alpha_1^{(j)},\ j = 1, 2. \end{cases} \tag{3}$$

We use the conventional utility function (Boucekkine et al., 2011):

$$U^{(j)}(C, P) = \ln(C) - \beta^{(j)} P, \tag{4}$$

where $\beta^{(j)}$ is a country-specific parameter, $j = 1, 2$. The controls are $t^{(1)}$, $t^{(2)}$, $C^{(1)}$, and $C^{(2)}$, and the state variables $K^{(1)}$, $K^{(2)}$, $P$, $A^{(1)}$, $A^{(2)}$, $\alpha^{(1)}$, and $\alpha^{(2)}$.

System {2, 3} constitutes a differential system in continuous-discrete time, also called hybrid dynamic (Bensoussan and Menaldi, 1997) under the constraints

$$\mathcal{K} := \{K^{(1)} \geq K^{(1)}_{\min}, K^{(2)} \geq K^{(2)}_{\min}, P \geq 0, A^{(1)} \geq A^{(1)}_{\min}, A^{(2)} \geq A^{(2)}_{\min}, \alpha^{(1)} \geq \alpha^{(1)}_{\min}, \alpha^{(2)} \geq \alpha^{(2)}_{\min}\} = \mathbb{R}^{+7}. \tag{5}$$

Nash equilibria are obtained when country 1 maximizes its inter-temporal utility, that is, solves (1) with $j = 1$, considering that country 2 does the same, that is, solves (1) with $j = 2$.

# 3 Method: Capture-Viability, Optimization, Algorithm

## 3.1 Impulse Dynamical Systems and Capture-Viability

An impulse differential inclusion $(F, R)$ consists in a continuous-time set-valued map $F : X \to X$:

$$x' \in F(x) = \{f(x, u, v), u \in V_u(x), v \in V_v(x)\}, \tag{6}$$

where $f$ is a function such that $F$ satisfies the regularity properties (3.1) stated below and $V_u(x)$ and $V_v(x)$ are closed sets, and a discrete-time reset map $R : X \to X$, giving a successor state

$$x^{i+1} \in R(x^i) \tag{7}$$

to the state $x^i$ at control impulse time $t_i$.

For $\mathcal{K} \subset \mathbb{R}^m$ a closed set, the problem of capture-viability is to delineate all states $x_0 \in \mathcal{K}$ from which there exists at least one trajectory under the dynamic $(F, R)$ remaining in $\mathcal{K}$ until a given time horizon $T$ and hitting a closed subset $\Omega \subset \mathcal{K}$ in finite time. Such states are said to be *viable* in $\mathcal{K}$ under $(F, R)$ with target $\Omega$. Aubin (1991) states that, when the Marchaud assumptions:

**Hypothesis 3.1.**

$$\begin{array}{ll} (i) & F \textit{ is upper semi-continuous with non empty compact, convex values,} \\ (ii) & \exists\, c \in \mathbb{R} \textit{ such that} \sup_{y \in F(x)} \parallel y \parallel < c(\parallel x \parallel + 1), \\ (iii) & R \textit{ is upper semi-continuous with compact domain and compact values} \end{array} \tag{8}$$

hold true, then there exists a maximal set of viable states —called *capture-viability kernel*— containing all sets of viable states. The hybrid capture-viability kernel of $\mathcal{K}$ under the hybrid dynamic $(F, R)$ is denoted $\mathrm{Viab}_{(F,R)}(\mathcal{K}, \Omega)$.

$S_F(x_0, v(\cdot))$ is the set of absolutely continuous solutions to (6) starting from $x_0$, and for any subset $X$ of $\mathbb{R}^m$, $S_F(X, v(\cdot)) := \bigcup_{x_0 \in X} S_F(x_0, v(\cdot))$. The scalar product is denoted by $< \, , \, >$.

**Definition 3.2.** *Consider set-valued maps $F : \mathbb{R}^m \mapsto \mathbb{R}^m$ and $R : \mathbb{R}^m \mapsto \mathbb{R}^m$ satisfying Assumptions 3.1.*

- *For a continuous-time system described by the differential inclusion $x' \in$*

*$F(x)$, $\mathcal{K}$ is a viability domain with target $\Omega$ under $F$ if and only if*

$$\forall x \in \mathcal{K} \setminus \Omega, \ \ \forall y \in NP_{\mathcal{K}}(x), \quad \exists y \in F(x), < y, y > \leq 0\,, \tag{9}$$

*where $NP_{\mathcal{K}}(x)$ is the normal cone to $\mathcal{K}$ in $x$. If $\mathcal{K}$ is not a viability domain with target, there exists a largest closed capture-viability domain contained in $\mathcal{K}$ denoted by* $\mathrm{Capt}_F(\mathcal{K}, \Omega)$ *and called the* capture-viability kernel of $\mathcal{K}$ with target $\Omega$. *It is the largest closed set of initial conditions in $\mathcal{K}$ from which there exists at least one trajectory viable in $\mathcal{K}$ with target $\Omega$.*

- *For an impulse system $(F, R)$, $\mathcal{K}$ is an impulse capture-viability domain with target $\Omega$ under $(F, R)$ if and only if it is a viability domain with target $(\Omega \cup R^{-1}(\mathcal{K} \cup \Omega))$ under $F$, namely*

$$\forall x \in \mathcal{K} \setminus (\Omega \cup R^{-1}(\mathcal{K} \cup \Omega)), \ \ \forall y \in NP_{\mathcal{K}}(x), \quad \exists y \in F(x), < y, y > \leq 0. \tag{10}$$

*If $\mathcal{K}$ is not an impulse capture-viability domain with target, there exists a largest closed impulse capture-viability domain contained in $\mathcal{K}$, called* impulse capture-viability kernel of $\mathcal{K}$ with target $\Omega$ *and denoted* $\mathrm{Capt}_{(F,R)}(\mathcal{K}, \Omega)$. *It is the set of initial conditions in $\mathcal{K}$ from which there exists at least one trajectory of $(F, R)$ viable in $\mathcal{K}$ with target $\Omega$.*

Quincampoix and Veliov (1998) proved the first point, Saint-Pierre (2002) the second point.

## 3.2 Viable Nash equilibria by capture-viability kernels

The problem {2, 3} is a problem of capture-viability with impulse. It can be written as a differential inclusion with impulse:

$$\left\{\begin{array}{rcl} x'(t) & \in & F(x(t)) := \{\big(A^{(1)}(t)(K^{(1)}(t))^{\nu} - C^{(1)}(t) - \delta_K K^{(1)}(t), \\ & & A^{(2)}(t)(K^{(2)}(t))^{\nu} - C^{(2)}(t) - \delta_K K^{(2)}(t), \\ & & \sum_{i=1}^{2} \alpha^{(i)}(t)A^{(i)}(t)(K^{(i)}(t))^{\nu} - \delta_P P(t), 0, 0, 0, 0\big) \\ & & \quad \mid C^{(1)} \in V^{(1)}, C^{(2)} \in V^{(2)}, t^{(1)} \geq 0, t^{(2)} \geq 0\} \\ x^{+} & = & R(x^{-}) := \{{K^{(1)}}^{-}, {K^{(2)}}^{-}, P^{-}, {A^{(1)}}^{-} + A_2^{(1)} - A_1^{(1)}, {A^{(2)}}^{-}, \\ & & {\alpha^{(1)}}^{-} + \alpha_2^{(1)} - \alpha_1^{(1)}, {\alpha^{(2)}}^{-}\} \quad \text{at } t^{(1)}, \\ x^{+} & = & R(x^{-}) := \{{K^{(1)}}^{-}, {K^{(2)}}^{-}, P^{-}, {A^{(1)}}^{-}, {A^{(2)}}^{-} + A_2^{(2)} - A_1^{(2)}, \\ & & {\alpha^{(1)}}^{-}, {\alpha^{(2)}}^{-} + \alpha_2^{(2)} - \alpha_1^{(2)}\} \quad \text{at } t^{(2)}, \end{array}\right. \tag{11}$$

where $x := (K^{(1)}, K^{(2)}, P, A^{(1)}, A^{(2)}, \alpha^{(1)}, \alpha^{(2)})$, and where $V^{(j)} \subset \mathbb{R}^{+}$, $j = 1, 2$, are closed sets, under the state constraints defined by $\mathcal{K}$ (defined in (5)).

Moreover, Bonneuil (2012) showed that $x_0 \in \mathbb{R}^n$ is a solution to the optimization problem

$$\left\{\begin{array}{l} \max_{u \in V_u} \int_0^{\infty} L(x(t))\, dt \\ x'(t) \in F(x(t)) := \{f(x(t), u(t)) \ \mid \ u(t) \in V_u\} \\ x^{+} = R(x^{-}) \quad \text{at } t^{(1)}, \cdots, t^{(n)} \\ \forall t,\ x(t) \in \mathcal{K} \\ x(0) = x_0, \end{array}\right. \tag{12}$$

where $t^{(1)}, \cdots, t^{(n)}$ are discrete times, if and only if it is located on the upper boundary in the direction of high $y$ of the capture-viability kernel $\text{Capt}_{(F,R,-L)}(\mathcal{K}\times$

$\mathbb{R}, \mathcal{K} \times \{0\})$ for the augmented dynamic

$$\left\{\begin{array}{rcl} x'(t) & \in & F(x(t)) \\ x^{+} & = & R(x^{-}) \quad \text{at } t^{(1)}, \cdots, t^{(n)} \\ y'(t) & = & -L(x(t)) \\ \forall t,\ x(t) & \in & \mathcal{K} \\ x(0) & = & x_0 \\ y(0) & = & y_0. \end{array}\right. \tag{13}$$

Bonneuil (2012) also showed that the infinite time problem ($T = \infty$) is conveniently approximated by $T < \infty$ large enough, which is made possible by the discount term $\exp(-\rho t)$. The numerical results below correspond to $T = 30$ and $\rho = 5\%$. We present the method for the approximation of (1) in finite time horizon $T$. The criterion becomes

$$\max_{C^{(j)}, t^{(j)}} \int_0^T U(C^{(j)}(t), P(t)) e^{-\rho t}\, dt. \tag{14}$$

By increasing the time horizon $T$ to infinity, the solution of problem (14) converges to problem (1).

**Proposition 3.3.** *The valuation function in the infinitesimal horizon control problem:*

$$\max_{C^{(j)}, t^{(j)}} \int_0^\infty U(C^{(j)}(t), P(t)) e^{-\rho t}\, dt, \quad j = 1, 2, \tag{15}$$

*with* $(C^{(1)}(0), C^{(2)}(0), P(0)) = (C_0^{(1)}, C_0^{(2)}, P_0)$*, is related to the capture-viability kernel*

$$\text{Capt}^{(\infty)}_{(F,R,-L)}(K \times \mathbb{R}^{+}, K \times \{0\}) := \bigcup_{T \geq 0} \text{Capt}^{(T)}_{(F,R,-L)}(K \times \mathbb{R}^{+}, K \times \{0\}) \tag{16}$$

*by:*

$$\overset{\infty \sup}{V}(x) = \sup_{(x,y)\in \mathrm{Capt}_{(F,R,-L)}(K\times I\!R^+, K\times\{0\})} y \tag{17}$$

*Proof:* in Bonneuil (2012).

On this basis, we now innovate by finding all the Nash equilibria of {1, 2, 3}. We do this by solving the two optimization programs (14) jointly, with valuation functions $y^{(1)}$ and $y^{(2)}$. The idea is first to identify viable states in $\mathcal{K} \times [0,T] \times \mathbb{R}^2$ with target $\mathcal{K} \times \{T, 0, 0\}$ under the dynamic {2, 3} augmented with the three differential equations:

$$\begin{cases} t' &= 1 \\ y^{(j)\prime}(t) &:= -U(C^{(j)}(t), P(t))e^{-\rho t},\ j = 1, 2, \end{cases} \tag{18}$$

with initial conditions

$$x(0) = x_0,\ t(0) = 0;\ y^{(1)}(0) = y_0^{(1)}, y^{(2)}(0) = y_0^{(2)}. \tag{19}$$

This is achieved by Bonneuil (2006)'s viability algorithm applied to identify viable states belonging to $\mathrm{Capt}_{\{2,3,18\}}(\mathcal{K} \times \mathbb{R}^3, \mathcal{K} \times \{T, 0, 0\})$ of the augmented dynamic. This is performed in the ten-dimensional state space ($K^{(1)}$, $K^{(2)}$, $P$, $A^{(1)}$, $A^{(2)}$, $\alpha^{(1)}$, $\alpha^{(2)}$, $t$, $y^{(1)}$, $y^{(2)}$) with the four controls $t^{(1)}$, $t^{(2)}$, $C^{(1)}$, $C^{(2)}$. However, the initial conditions of $t$, $A^{(j)}$, and $\alpha^{(j)}$ are fixed, so there remain five dimensions in which to search for viable states.

The boundary in the direction of high $y^{(1)}$ of $\mathrm{Capt}_{\{2,3,18\}}(\mathcal{K}\times\mathbb{R}^3, \mathcal{K}\times\{T, 0, 0\})$. gathers all initial conditions from which there exists a trajectory satisfying all state constraints and such that $y^{(1)}(0)$ is maximal, considering the variables taken by country 2 as given. The same is true for the boundary in the direction of high $y^{(2)}$, such that $y^{(2)}(0)$ is maximal, considering the values of the variables taken by country 1 as given. The set of Nash viable equilibria, where each country maximizes its valuation function taking into account that the other country also maximizes its valuation function, is then exactly the intersection of these two boundaries (necessary and sufficient conditions, by construction).

# 4 Numerical set-up

## 4.1 Viability Algorithm

The set of constraints $\mathcal{K}$ and the target $\Omega$ are defined by inequalities:

$$\mathcal{K} \cup \Omega =: \{\xi(z) \leq 0\}. \tag{20}$$

A viable state satisfies:

$$\text{Inf}_{z(.)\in S(z)}\text{Sup}_{t\in[0,T]}\xi(z(t)) \leq 0 \tag{21}$$

Bonneuil (2006) uses stochastic optimization to solve (21). The state $z$ is viable if and only if at least one solution $z(.)$ with $z(0) = z$ exists. Instead of trying $z$ at random, a search for a viable $z$ is done by traveling in the state space with a probability of decreasing the cost $\text{Inf}_{z(.)\in S(z)}\text{Sup}_{t\in[0,T]}\xi(z(t))$ over $z$, until this cost ever becomes non positive. This probability increases with the total number of trials. The search for $z$ is proceeded by stochastic optimization, too.

With the addition of the auxiliary variables $y^{(1)}$, $y^{(2)}$, and time $t$ now considered as a state variable in the augmented dynamic, the search for a viable optimum for population 1 consists in first finding a viable state $(x, y_0^{(1)}, y_0^{(2)})$, then increasing $y_0^{(1)}$ step by step at $x$ and $y_0^{(2)}$ fixed until obtaining a non viable state. Refinement yields $y_0^{(1)}$ on the boundary of the capture-viability kernel in the direction of high $y^{(1)}$ of the viability kernel associated with the augmented dynamic (Bonneuil, 2012). There is no need to compute the whole capture-viability kernel, which is very time-consuming. The same applies when seeking the viable optimum for population 2.

### 4.2 Three representative cases

We consider three important cases, so as to understand the effect of productivity and nuisance differentials:

- case 1: the introduction of cleaner energy decreases the marginal contribution to pollution and the productivity of capital equally in both countries: $A_1^{(1)} = A_1^{(2)} = 1.5$, $A_2^{(1)} = A_2^{(2)} = 1.1$, $\alpha_1^{(1)} = \alpha_1^{(2)} = 0.05$, $\alpha_2^{(1)} = \alpha_2^{(2)} = 0.03$;
- case 2: the introduction of cleaner energy decreases the productivity of capital in both countries equally; it also decreases the marginal contribution to pollution in both countries, but in country 2 more than in country 1: $A_1^{(1)} = A_1^{(2)} = 1.5$, $A_2^{(1)} = 1.3$, $A_2^{(2)} = 1.0$, $\alpha_1^{(1)} = \alpha_1^{(2)} = 0.05$, $\alpha_2^{(1)} = \alpha_2^{(2)} = 0.03$;

- case 3: the introduction of cleaner energy decreases the productivity of capital in both countries, but in country 2 more than in country 1; it also decreases the marginal contribution to pollution equally in both countries: $A_1^{(1)} = A_1^{(2)} = 1.5$, $A_2^{(1)} = A_2^{(2)} = 1.1$, $\alpha_1^{(1)} = \alpha_1^{(2)} = 0.05$, $\alpha_2^{(1)} = 0.03$, $\alpha_2^{(2)} = 0.01$.

We draw initial states $(K_0^{(1)}, K_0^{(2)}, P_0, y_0^{(1)}, y_0^{(2)})$ at random, test their viability status by the viability algorithm, reject those tested not viable, until obtaining 300 viable states. For each of them, a one-dimensional optimization yields the highest value of $y_0^{(1)}$ (at $y_0^{(2)}$ fixed). These points belong to the upper boundary in the direction of high $y^{(1)}$ of the capture viability kernel $\mathrm{Capt}_{\{2,3,18\}}(\mathcal{K} \times \mathbb{R}^3, \mathcal{K} \times \{T, 0, 0\})$. We approximate this manifold by penalized least squares on the basis of the regular grid $\{0, 0.1, \cdots, 1.0\}^3$ of the cube $(K_0^{(1)}, K_0^{(2)}, P_0)$ scaled to $[0, 1]^3$.

Similarly, we obtain 300 points $(K_0^{(1)}, K_0^{(2)}, P_0, y_0^{(1)}, y_0^{(2)})$ viable in $\mathcal{K} \times \mathbb{R}^2$ and for which the valuation function $y_0^{(2)}$ is maximal. These points likewise delineate a manifold numerically approximated by penalized least squares.

The smoothed manifold built from the 300 scattered points obtained on each upper boundary must be truncated so as to retain only the states for which $(K_0^{(1)}, K_0^{(2)}, P_0)$ is viable for {2, 3}. To do this, we smooth the viability kernel of problem {2, 3} by penalized least squares, successively $P_0$ as a function of $K_0^{(1)}$ and $K_0^{(2)}$, then $K_0^{(1)}$ as a function of $K_0^{(2)}$ and $P$, and finally $K_0^{(2)}$ as a function of $K_0^{(1)}$ and $P_0$. The computation of one viable state $x$ with its two optima $y_0^{(1)}$ and

$y_0^{(2)}$ requires 11 computing hours on a Dell Precision M6600, or 137 computing days for the $2 \times 300$ points.

### 4.3 Nash equilibria

From the two truncated smoothed manifolds representing the two upper boundaries of $\text{Capt}_{\{2,3,18\}}(\mathcal{K} \times \mathbb{R}^3, \mathcal{K} \times \{T, 0, 0\})$ in the direction of either $y^{(1)}$ or $y^{(2)}$, we compute the intersection of the two surfaces of viable optima. This intersection is the locus of viable Nash equilibria. Figure 1 shows an example of a section at given $P_0$. The switching times $t_1$ and $t_2$, as well as the mean values of consumption $\overline{C^{(j)}}_{t<t_j}$ and $\overline{C^{(j)}}_{t \geq t_j}$, $j = 1, 2$, associated with these Nash equilibria likewise are obtained by penalized least squares from the values associated with the computed viable optima.

The valuation functions at *viable Nash equilibria* and their associated switching times and mean consumption values all depend only on the initial conditions $K_0^{(1)}$, $K_0^{(2)}$, and $P_0$ through the maximization (14). Therefore, in order to characterize Nash equilibria, we cannot take switching times or consumption levels as explanatory variables in regressions. These control variables are endogenous, and are entirely determined by the initial conditions $K_0^{(1)}$, $K_0^{(2)}$, and $P_0$. That is why we characterize the viable Nash equilibria in regressing the valuation function and its controls in a system of seemingly unrelated regressions having only these initial conditions as explanatory variables. The three systems, one for each case,

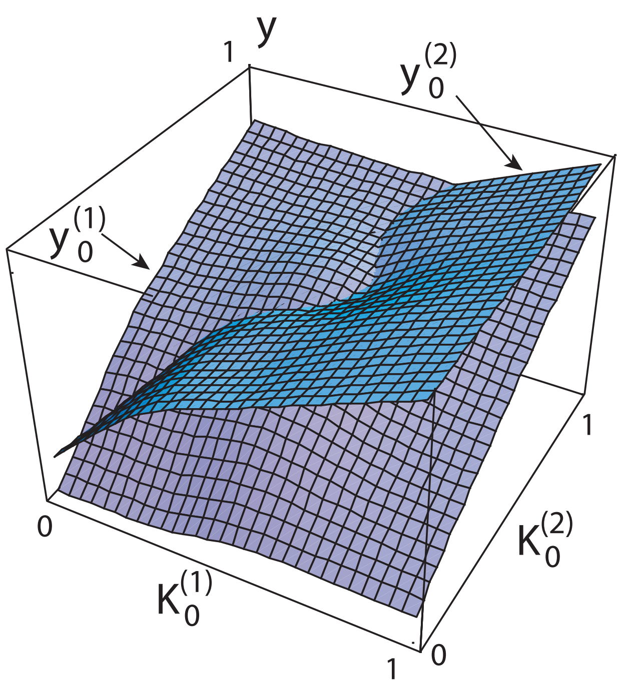


Figure 1: Section at $P_0 = 0.1$ (for $P_0$ normalized between 0 and 1) of the Nash equilibria as intersection of the boundaries in the direction of high $y^{(j)}$, $j = 1, 2$, of the two capture-viability kernels. Coordinates scaled between 0 (minimal computed value) and 1 (maximal computed value). Parameter values: $\rho = 0.05$, $\nu = 0.5$, $\delta_K = 0.1$, $\delta_P = 0.1$.

are embedded in a single model, to allow testable comparisons (with $\tau$ denoting transposition):

$$Y = \sum_{i=1}^{3} B^\tau X 1_{\text{case } i} \tag{22}$$

where $Y^\tau = (y, t_1, t_2, \overline{C^{(1)}}_{t<t_1}, \overline{C^{(2)}}_{t<t_2}, \overline{C^{(1)}}_{t\geq t_1}, \overline{C^{(2)}}_{t\geq t_2})$ is the vector made of the valuation function at the Nash equilibrium, the associated switching times, and the mean consumption values. The explanatory variables are $X^\tau = (1, K_0^{(1)}, K_0^{(2)}, P_0)$, $B$ is a matrix of coefficients, and $1_{\text{case } i}$ is the indicator of case $i = 1, 2, 3$.

The valuation function at *viable Nash equilibria* $y$, capital values $K_0^{(1)}$ and $K_0^{(2)}$, pollution level $P_0$, and consumption levels are re-scaled between 0 and 1 within their ranges of variation. Significant coefficients at 5% are indicated by a star; standard deviations appear in parentheses below the coefficients. We present the three cases successively, although they are estimated jointly in (22).

- Case 1 (after technological switch, both countries are of equal productivity and contribute equally to pollution)

$$\begin{cases}
E(y) &= \underset{(0.01)}{0.67^*} + \underset{(0.02)}{0.20^*}\, K_0^{(1)} + \underset{(0.02)}{0.20^*}\, K_0^{(2)} - \underset{(0.01)}{0.10^*}\, P_0 \\
E(t_1) &= \underset{(0.21)}{13.23^*} + \underset{(0.45)}{5.70^*}\, K_0^{(1)} + \underset{(0.45)}{0.95^*}\, K_0^{(2)} + \underset{(0.29)}{1.23^*}\, P_0 \\
E(t_2) &= \underset{(0.17)}{12.87^*} + \underset{(0.36)}{1.01^*}\, K_0^{(1)} + \underset{(0.36)}{5.81^*}\, K_0^{(2)} + \underset{(0.23)}{1.71^*}\, P_0 \\
E(\overline{C^{(1)}}_{t<t_1}) &= \underset{(0.019)}{0.16^*} + \underset{(0.041)}{0.12^*}\, K_0^{(1)} + \underset{(0.04)}{0.38^*}\, K_0^{(2)} + \underset{(0.026)}{0.013}\, P_0 \\
E(\overline{C^{(2)}}_{t<t_2}) &= \underset{(0.02)}{0.11^*} + \underset{(0.04)}{0.41^*}\, K_0^{(1)} + \underset{(0.04)}{0.14^*}\, K_0^{(2)} - \underset{(0.022)}{0.010}\, P_0 \\
E(\overline{C^{(1)}}_{t\geq t_1}) &= \underset{(0.02)}{0.30^*} - \underset{(0.04)}{0.22^*}\, K_0^{(1)} + \underset{(0.04)}{0.34^*}\, K_0^{(2)} - \underset{(0.025)}{0.001}\, P_0 \\
E(\overline{C^{(2)}}_{t\geq t_2}) &= \underset{(0.02)}{0.25^*} + \underset{(0.04)}{0.35^*}\, K_0^{(1)} - \underset{(0.04)}{0.19^*}\, K_0^{(2)} - \underset{(0.024)}{0.008}\, P_0,
\end{cases} \tag{23}$$

where $E$ denotes expectancy. As expected, because the two countries have the same parameters, the coefficients of $K_0^{(1)}$ and $K_0^{(2)}$ are not significantly different from each other. The higher the pollution level at the start, the lower the maximal valuation function. The higher the initial capital value, the higher the consumption level in the same country before the switching and the lower the consumption level thereafter: people consume more when their production capacity is higher. Expecting that the other country's inhabitants are doing the same, they increase consumption, while the higher the other country's initial capital, the more they delay their transition to the less polluting technology.

The higher the initial capital value, the more each country consumes before switching and the longer it delays switching, while reducing its consumption after switching. Combining the equations for consumption and timing, the mean consumption over the entire period $[0, T]$ increases with the initial capital values of both countries.

*The Nash equilibria then reflect a race to consume more when the other participant is also expected to consume more.* Logically, a longer time to switching is associated with more pollution, that is, with a higher initial pollution level.

- Case 2 (after technological switch, Country 2's productivity is less than

Country 1's)

$$
\left\{
\begin{array}{rcl}
E(y) & = & \underset{(0.01)}{0.70} + \underset{(0.02)}{0.23^*}\, K_0^{(1)} + \underset{(0.005)}{0.030^*}\, K_0^{(2)} - \underset{(0.01)}{0.10^*}\, P_0 \\
E(t_1) & = & \underset{(0.98)}{23.71^*} - \underset{(2.18)}{23.89^*}\, K_0^{(1)} - \underset{(0.47)}{1.97^*}\, K_0^{(2)} - \underset{(0.45)}{5.23^*}\, P_0 \\
E(t_2) & = & \underset{(0.33)}{16.51^*} + \underset{(0.73)}{12.51^*}\, K_0^{(1)} - \underset{(0.16)}{9.96^*}\, K_0^{(2)} - \underset{(0.15)}{7.08^*}\, P_0 \\
E(\overline{C^{(1)}}_{t<t_1}) & = & \underset{(0.01)}{0.18^*} + \underset{(0.01)}{0.56^*}\, K_0^{(1)} - \underset{(0.001)}{0.031^*}\, K_0^{(2)} - \underset{(0.001)}{0.030^*}\, P_0 \\
E(\overline{C^{(2)}}_{t<t_2}) & = & \underset{(0.01)}{0.24^*} - \underset{(0.01)}{0.05^*}\, K_0^{(1)} + \underset{(0.01)}{0.46^*}\, K_0^{(2)} - \underset{(0.01)}{0.04^*}\, P_0 \\
E(\overline{C^{(1)}}_{t\geq t_1}) & = & \underset{(0.01)}{0.28^*} + \underset{(0.01)}{0.21^*}\, K_0^{(1)} + \underset{(0.01)}{0.07^*}\, K_0^{(2)} - \underset{(0.001)}{0.012^*}\, P_0 \\
E(\overline{C^{(2)}}_{t\geq t_2}) & = & \underset{(0.01)}{0.32^*} - \underset{(0.01)}{0.04^*}\, K_0^{(1)} + \underset{(0.01)}{0.11^*}\, K_0^{(2)} - \underset{(0.01)}{0.08^*}\, P_0.
\end{array}
\right. \tag{24}
$$

The asymmetry of productivity after the transition has modified the directions of association in switching times and initial capital values. The lower productivity of Country 2 makes Country 1 leader of the game: the valuation function depends mainly on consumption in Country 1, then on its initial capital endowment $K_0^{(1)}$ (coefficient 0.23 in the expression of $E(y)$), because Country 2's capital grows more slowly. Pollution then comes mainly from Country 1, whose capital grows faster after switching.

To compensate, the higher $K_0^{(1)}$ (coefficient -23.89 in the expression of $E(t_1)$), the earlier Country 1 switches. Meanwhile, Country 2 consumes with respect to its capital capacities allowed by its initial capital $K_0^{(2)}$ (coefficients 0.46 in $E(\overline{C^{(2)}}_{t<t_2})$ and 0.11 in $E(\overline{C^{(2)}}_{t\geq t_2})$), but, to keep up with Country 1, delays switching all the more as $K_0^{(1)}$ is high (coefficient 12.51 in $E(t_2)$), and continues using the more polluting technology as long as

possible. However, delaying because of country 1's higher productivity is possible comes at the price of reduced consumption (coefficients -0.05 and -0.04 in $E(\overline{C^{(2)}}_{t\geq t_2})$). *The differential in productivity encourages the more productive country to switch earlier and the less productive country to switch later than they would do in the absence of a differential.*

More pollution in any case triggers earlier technological switching (coefficients -5.23 in $E(t_1)$ and -7.08 in $E(t_2)$), and limits consumption (coefficients -0.030 in$E(\overline{C^{(1)}}_{t<t_1})$, -0.04 in $E(\overline{C^{(2)}}_{t<t_2})$, -0.012 in $E(\overline{C^{(1)}}_{t\geq t_1})$, -0.08 in $E(\overline{C^{(2)}}_{t\geq t_2})$).

The higher the initial capital value $K_0^{(1)}$, the more Country 1 consumes before and after switching (coefficients 0.56 in $E(\overline{C^{(1)}}_{t<t_1})$ and 0.21 in $E(\overline{C^{(1)}}_{t\geq t_1})$), but also the sooner the switch (coefficient -23.89 in $E(t_1)$), and the later for Country 2 (coefficient 12.51 in $E(t_2)$). Country 1, expecting Country 2 to switch to a less productive and less polluting technology, is better off consuming when Country 2 pollutes less. Conversely, Country 2, anticipating that Country 1 will use its advantage to hasten its switch, is better off postponing its own switching as long as $K_0^{(1)}$ is high.

As a result, combining the equations of consumption and timing, the mean consumption for each country over the entire period $[0, T]$ increases with initial capital then decreases: *"poor" countries consume less but for longer with the more polluting technology; while "rich" countries consume more* *(and pollute more) but for a shorter period of time.*

- Case 3 (after technological switch, Country 2 contributes less to total pollution than Country 1)

$$
\begin{cases}
E(y) & = \underset{(0.01)}{0.60^*} + \underset{(0.04)}{0.58^*}\, K_0^{(1)} + \underset{(0.03)}{0.04}\, K_0^{(2)} - \underset{(0.01)}{0.05^*}\, P_0 \\
E(t_1) & = \underset{(0.06)}{11.21^*} + \underset{(0.52)}{7.13^*}\, K_0^{(1)} + \underset{(0.37)}{9.25^*}\, K_0^{(2)} - \underset{(0.06)}{0.78^*}\, P_0 \\
E(t_2) & = \underset{(0.02)}{5.73^*} - \underset{(0.17)}{1.16^*}\, K_0^{(1)} - \underset{(0.12)}{3.99^*}\, K_0^{(2)} - \underset{(0.02)}{0.67^*}\, P_0 \\
E(\overline{C^{(1)}}_{t<t_1}) & = \underset{(0.01)}{0.15^*} + \underset{(0.01)}{0.67^*}\, K_0^{(1)} + \underset{(0.01)}{0.02^*}\, K_0^{(2)} - \underset{(0.01)}{0.03^*}\, P_0 \\
E(\overline{C^{(2)}}_{t<t_2}) & = -\underset{(0.01)}{0.03^*} + \underset{(0.07)}{0.58^*}\, K_0^{(1)} + \underset{(0.05)}{0.32^*}\, K_0^{(2)} - \underset{(0.01)}{0.13^*}\, P_0 \\
E(\overline{C^{(1)}}_{t\geq t_1}) & = \underset{(0.01)}{0.32^*} + \underset{(0.04)}{0.09^*}\, K_0^{(1)} - \underset{(0.03)}{0.14^*}\, K_0^{(2)} + \underset{(0.01)}{0.09^*}\, P_0 \\
E(\overline{C^{(2)}}_{t\geq t_2}) & = \underset{(0.01)}{0.14^*} + \underset{(0.01)}{0.10^*}\, K_0^{(1)} + \underset{(0.01)}{0.21^*}\, K_0^{(2)} - \underset{(0.001)}{0.013^*}\, P_0.
\end{cases}
\tag{25}
$$

The valuation again depends mainly (coefficient 0.58 in $E(\overline{C^{(2)}}_{t<t_2})$) on the country contributing most to pollution, which is also the country that waits longer to switch, the higher the initial capital values (coefficients 7.13 and 9.25 in $E(t_1)$). This is because Country 1 has to produce longer under the more polluting regime to attain the same cumulative discounted utility as Country 2, whose utility is less penalized by pollution. At the Nash equilibrium, to obtain the same utility, the country with the higher pollution penalty after the transition remains as long as possible in the more productive regime. This strategy is more polluting, and consequently detrimental to Country 2. This continues to the point that in Country 1, higher consumption means more pollution (coefficients 0.03 in $E(\overline{C^{(1)}}_{t<t_1})$ and 0.09 in $E(\overline{C^{(1)}}_{t\geq t_1})$).

Meanwhile, expecting this behavior, Country 2's best strategy is to switch early so as to limit the pollution level, a behavior which is favorable to Country 1. More pollution entails less consumption (coefficients -0.13 in $E(\overline{C^{(2)}}_{t<t_2})$ and -0.013 in $E(\overline{C^{(2)}}_{t\geq t_2}))$. The Nash equilibrium is cynical in that the polluter is encouraged to pollute by the other player, who is cleaner. Anticipating this, Country 2 consumes (and pollutes) all the more, the higher the initial capital values Country 1 starts from. *Non cooperation has the effect that reducing one's contribution results in more total consumption and more total pollution,.* Country 2, which gains utility by polluting less after the switch, is better off hastening this transition when both initial capital values are high.

The mean consumption over the entire period $[0, T]$ increases for both countries, as in the symmetric case.

Numerically, consumption levels $C^{(1)}$ and $C^{(2)}$ of countries 1 and 2, which are controls of the problem $\{1, 2, 3\}$, do not look bang bang. To confirm, we ran the optimization procedure from identical 100 initial values of the state variables, by admitting only extreme values $\min C^{(j)}$ and $\max C^{(j)}$ for the controls $C^{(j)}$, $j = 1, 2$. For each of these 100 initial points, we find smaller optimal values $y^{(1)}$ and $y^{(2)}$ with bang-bang controls than with control variables $C^{(j)}$ admissible in $[\min C^{(j)}, \max C^{(j)}]$.

# 5 Conclusion

Taking Nash equilibria as located at the intersection of the two boundaries of the capture viability kernels in the directions of high $y^{(1)}$ and high $y^{(2)}$ allows us to compute Nash equilibria without resorting to simplifying endogenous assumptions on the controls, to take state constraints into account, to deal with non linearity, and to avoid the round-about method based on solving differential equations. This is an opportunity opened conjointly by the concept of capture-viability, by Bonneuil's (2012) theorem on the location of viable optima, and by Bonneuil's (2006) viability algorithm in large state dimension.

By examining three games: the symmetric case, the case where one country is less productive after switching to less polluting technology, and the case where one country contributes less to pollution after switching, we highlighted the interdependence of the two countries at the Nash equilibrium, again without having postulated any endogenous mechanism, as is often done. The model and the examined cases allow us to predict the conditions for technological switching: we showed that the absence of cooperation leads to undesirable consequences, where efforts to pollute less motivate the other country to pollute more, or encourage the country that will be cleaner or less productive after switching, to delay its transition.

# References


[1] d'Agata, A. (2010) Geometry of Cournot-Nash equilibrium with application to commons and anticommons. The Journal of Economic Education, 41(2), 169-176.

[2] Aubin, J.P.: Viability Theory, Boston: Birkhäuser 1991

[3] Bensoussan A, Menaldi JL. Hybrid Control and Dynamic Programming. Dynamics of Continuous, Discrete and Impulse Systems **3**, 395-442 (1997).

[4] Bonneuil, Noël (2006) 'Computing the viability kernel in large state dimension,' *Journal of Mathematical Analysis and Applications* 323(2), 1444-1454

[5] Bonneuil N. (2012) Maximum under continuous-discrete-time dynamic with target and viability constraints, *Optimization* 61(8), 901-913.

[6] Bonneuil, N. and Boucekkine, R. (2014). Viable Ramsey economies, *Canadian Journal of Economics, 47*(2), 421-444.

[7] Bonneuil, N. and Boucekkine, R. (2016) Optimal transition to renewable energy with threshold of irreversible pollution, *European Journal of Operational Research*, 248, 257-262.

[8] Boucekkine, R., Krawczyk, J. B., and Vallée, T. (2011) Environmental quality versus economic performance: a dynamic gam e approach. *Optimal Control, Applications and Methods* 32, 29-46.

[9] Fullon, M. (1997) A graphic analysis of the Cournot-Nash and Stackelberg models. *Journal of Economic Education* 28, 36-40.

[10] Hardin, G. (1968) The Tragedy of the Commons Science 162 (3859): 12431248.

[11] Hardin, G. (1994). The Tragedy of the Unmanaged Commons. Trends in Ecology & Evolution 9 (5): 199, 90097-3.

[12] Hardin, G. (2008). Tragedy of the Commons. In David R. Henderson. Concise Encyclopedia of Economics (2nd ed.). Indianapolis: Library of Economics and Liberty.

[13] Kamien, M.I. and Schwartz, (1991) Dynamic Optimization. N.L. Amsterdam: Elsevier.

[14] Karp, L. and Zhang J.: Taxes versus quantities for a stock pollutant with endogenous abatement costs and asymmetric information. Economic Theory **49**, 371-409 (2012).

[15] Makris, M.: Necessary conditions for infinite horizon discounted two-stage optimal control problems. Journal of Economic Dynamics and Control **25**,1935-1950 (2001).

[16] Ngo Van Long (2010). A Survey of dynamic games. Singapore: World Scientific Printers.

[17] Sarkar, J., Gupta, B., and Pal, D. (1998) A geoemtric solution of a Cournot oligopoly with nonidentical firms. *Journal of Economic Education* 29, 118-126.

[18] Tahvonen, O.: Fossil fuels, stock externalities, and backstop technology, Canadian Journal of Economics **30**, 855-874 (1997).

[19] Tahvonen, O., and Withagen C.: Optimality of irreversible pollution accumulation. Journal of Economic Dynamics and Control **20**,1775-1795 (1996).

[20] Tomiyama, K.: Two-stage optimal control problems and optimality conditions. Journal of Economic Dynamics and Control **9**, 317-337 (1985).

[21] Shaikh, M., and P. Caines: On the hybrid optimal control problem: Theory and algorithms. IEEE Transactions on Automatic Control **52**, 1587-1603 (2007)