Implementation of the Lindahl Correspondance via Simple Indirect Mechanisms

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Abstract

Our paper proposes an original angle to study the free-rider problem in the provision of public goods when the regulator has no information about agents' preferences. For a given outcome - specifically a Lindahl allocation - we ask what assumptions have to be imposed on *simple* mechanisms (in a precisely defined sense) that have the ability to Nash-implement it. Our answer lies in two main results: i) transfers necessarily belongs to a class of mechanims that are linear in individual contributions to the public good, ii) there exists a subset of this class that fully implement Lindahl allocations. This subset encompasses, but does not reduce to, Walker (1981).

JEL codes: H41, C72, D62, D82.

Keywords: Lindahl allocations, mechanism design.

1   Introduction

How could a public authority cope with free-riding temptations in the provision of public goods when relevant pieces of information about contributors are not available? A substantial literature that addresses this archetypical regulation problem has accumulated over the past decades. The theoretical literature is made of two blocks.

The first one consists of a collection of papers, each offering a *mechanism* with the ability - once in place - to remove the original discrepancy between individual and collective rationalities. In other words a mechanism modifies the rules of the game so that resulting non-cooperative equilibria are also Pareto optimal allocations. It answers the question: *given a theory of agents' behaviors along with a particular distribution of the information, can we exhibit a mechanism that remove the free-rider problem?*

The second stream starts with the notion of a *social choice function* (SCF). It addresses the following question: *for the prevailing information structure and the appropriate concepts of equilibria, what kind of SCF could ever be implemented - or decentralized?* One typically obtains general possibility or impossibility results, but do not get a particular explicit mechanism. For instance, a well-known result states that if a SCF is implementable in Nash equilibrium, then it is *monotonic* (Maskin, 1999). Overall, this second stream provides an axiomatic picture on SCF that are implementable or not.

The present paper proposes a conceptual shift. We attack the problem from a third angle, which is a mixture of the two approaches described above. For a given outcome - specifically a *Lindahl correspondence* - we ask: *what properties have to be imposed on the mechanisms that have the ability to Nash implement the given outcome?* In other words, we axiomatize the means rather than the targets. We restrict attention to *simple mechanisms*, by which we mean functions that depend only of the profile of contributions to the public good. It is showed that simple mechanisms for attaining Lindahl allocations must satisfy a system of differential equations, that we call *LDS* (Lindahl Differential System). From LDS, two theorems are derived: *i*) Theorem *LT* establishes the existence of a unique class of simple mechanisms that solve *LDS;* and it also shows that, in this class, transfers to agent i are linear functions of his contribution, *ii*) Theorem *LLT* singles out a subclass of mechanisms for Lindahl allocation that encompasses, but does not reduce to, Walker (1981).

This paper is organized as follows. Section 2 presents the standard public good framework and its Lindahl equilibria. Section 3 introduces simple Nash-mechanisms for implementation of Lindahl allocations, and it axiomatizes this class of mechanisms. Section 4 concludes.

2 A public good economy and its Lindahl equilibria

There are $n \geq 2$ agents. Each agent is generically labelled i $(i = 1, ..., n)$, and is endowed with an exogenous income $y_i > 0$. He has two decision variables, a contribution $g_i \geq 0$ to a public good and his consumption $c_i \geq 0$ of a private good. There is no initial endowment of the public good G, which results from individual contributions *via* a production technology $G = \mathcal{G}(g_1, ..., g_n) \geq 0$. Each agent i has a preference relation defined over bundles (c_i, G), which admits a numerical representation $U^i(c_i, G; \theta_i)$, a continuous and increasing function of

its arguments c_i and G. Agent i's preference depends on a parameter - or type - denoted θ_i , taken in an open set $\Sigma_i \subseteq \mathbb{R}$ of the real numbers, $\theta_i \in \Sigma_i$. We assume that the Hessian matrix of $U^i(.,.;\theta_i)$ is definite negative, therefore this function is concave with respect to (c_i, G).

Let $\mathbf{y} = (y_1, ..., y_n)$ be the vector of exogenous incomes, $\boldsymbol{g} = (g_1, ..., g_n)$ the vector of contributions, $\mathbf{U} = (U^1, ..., U^n)$ the vector of utility functions and $\boldsymbol{\theta} = (\theta_1, ..., \theta_n)$ the vector of types (vectors are in bold characters). A public good economy[1] is a quadruplet

$$e \equiv (\mathbf{y}, \mathcal{G}, \mathbf{U}, \boldsymbol{\theta}) \in \Omega \ ,$$

for some set Ω of possible economies. In this paper we focus on a *pure public good* economy, where $\mathcal{G}(\boldsymbol{g}) = \sum_{i=1}^n g_i$ is the simple aggregation of individual contributions. A *feasible allocation* is a vector $(c_1, ..., c_n , G) \in R_+^{n+1}$ such that $\sum_{i=1}^n c_i + G \leq \sum_{i=1}^n y_i$. The set of all feasible allocations is a function of the vector of incomes and is denoted $\mathcal{F}(\mathbf{y})$.

The logic of a *Lindahl equilibrium* can be presented as follows. Imagine that each agent can choose his own public good G_i, for which he has to pay an individualised price τ_i (or cost share). His problem is to choose the pair (c_i, G_i) so as to maximize $U^i(c_i, G_i; \theta_i)$ subject to the individual budget constraint:

$$c_i + \tau_i G_i = y_i \ . \tag{1}$$

Optimal interior decisions for agent i solve the first order condition[2]:

$$\frac{U_2^i(y_i - \tau_i G^i, G^i; \theta_i)}{U_1^i(y_i - \tau_i G^i, G^i; \theta_i)} \equiv MRS^i(y_i - \tau_i G^i, G^i; \theta_i) = \tau_i, \quad \forall i. \tag{2}$$

Those equations implicitly define each individual demand for the public good:

$$G_i = L^i(y_i, \tau_i; \theta_i), \quad \forall i.$$

A *Lindahl equilibrium* for an economy $e \in \Omega$ is defined as a vector $(\tau_1^*, ..., \tau_n^*)$ of prices and a level of public good G^* such that:

$$\sum_{i=1}^n \tau_i^* = 1 \ , \tag{3}$$

$$G^* = L^i(y_i, \tau_i^*; \theta_i), \quad \forall i. \tag{4}$$

For future reference, let the set of *interior Lindahl allocations* - or *Lindahl correspondence* - be defined as:

$$L(e) = \{(\{c_i^*\}_{i=1}^n , G^*) \subseteq \mathcal{F}(\mathbf{y}) \text{ such that (1) - (4) are satisfied}\}. \tag{5}$$

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[1] There are many pedagogic presentations of economic environments with public goods. For instance Oakland (1987). For a textbook introduction, see Laffont (1988), Cornes and Sandler (1996, part III) or, more recently, Hindriks and Myles (2013), Chapter 6.

[2] Since the Hessian of U^i is definite negative, one can check that the second order condition for a maximum is also satisfied.

Combining (1),(2), (3) and (4), one can deduce:

$$\sum_i c_i^* + G^* = \sum_i y_i \ , \tag{6}$$

and:

$$\sum_{i=1}^n MRS^i \left(c_i^*, G^*; \theta_i\right) = 1 \ , \tag{7}$$

which is the well-known Bowen-Lindahl-Samuelson condition (BLS, see for instance Laffont, 1988) for Pareto Optimality, *i.e.* Lindahl equilibria lead to Pareto optimal allocations. Actually, they have been considered as the natural extension of the notion of a competitive equilibrium for economies with private goods when public goods are involved (see the surveys offered by Milleron, 1972, Section III, and Roberts, 1974). An important consequence is that, at Lindahl allocations, adapted versions of the *First and Second Welfare Theorems* can be established for economies with public goods (see Foley, 1970, Milleron, 1972). In addition, Lindahl allocations respect various requirements, in terms of fair sharing of the cost of the public good (Buchholz and Peters, 2007), that explain the particular attention they have attracted over decades. In particular:

i) They are in the *core* of the economy (Foley, 1970, Section 6), so agents, alone or in groups, presumably have no rational reasons to object the Lindahl outcome.

ii) They are *individually rational* in following the sense:

$$U^i \left(y_i - \tau_i^* G^*, G^*; \theta_i\right) \geq U^i \left(y_i, 0; \theta_i\right), \forall i,$$

meaning that each participant is at least as well-off at the Lindahl allocation as he would have been had the public good not been provided[3]. This is a particular case of a more general fairness property - *voluntariness* - established by Silvestre (1984)[4].

iii) they are *envy-free* when all agents are endowed with the same exogenous income: no agent would like to interchange his Lindahl bundle with the Lindahl bundle of another agent (Sato, 1987),

iv) When the public good is a normal good, at a Lindahl price system, richer agents contribute more: due to the income effect, the larger the income, the larger the demand for the public good; but since at the equilibrium everybody consumes the same amount, rich agents face a higher price.

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[3] This definition of individual rationality should not be confused with another possible definition, which would require that a move from a Nash voluntary contribution equilibrium to a Lindahl equilibrium would make everyone better-off. This is a different issue, addressed in Shitovitz and Spiegel (1998). In general, not all agents prefer the Lindahl equilibrium over the Nash outcome.

[4] Voluntariness requires that no agent is better-off after a reduction of his contribution that is accompanied by a reduction of the public good in the same proportion. See Sylvestre (1984, Section 3).

4

Despite those appealing allocative properties, Lindahl equilibria face well-known problems when it comes to implementation. Agents are assumed to behave as price-takers, even though they are unique agents on the demand side of their respective market for the public good or, are assumed to reveal truthfully their quantity demanded of the public good at each given price. This constitutes an important obstacle for the implementation of a Lindahl equilibrium[5]. Economists in the last decades have looked for alternative ways for extracting truthful information, using *Implementation Theory* and the concept of a *game form*.

3   A public good game form

Denote $\mathbf{G}^{-i}$ the $(n-1)$-dimensional vector made of contributions other than g_i, and let $g_{-i} = \sum_{h \neq i} g_h$ stand for the sum of the other agents' contributions.

Suppose an incentive scheme is put in place where each agent i pays or makes a transfer:
$$T^i \left(g_i, \mathbf{G}^{-i}\right).$$

Those transfers (a subsidy when $T^i < 0$, a tax when $T^i > 0$), can be used to define a *public good game form (PGGF)*. Let $\boldsymbol{T} = \left(T^1, ..., T^n\right)$ stand for the vector of transfers put in place. A PGGF consists of: *i)* n action compact sets $S^h \subset \mathbb{R}, \forall h = 1, ..., n$, where an action is a contribution to the public good, *ii)* a set of feasible outcomes $(c_1, ..., c_n, G) \in \mathcal{F}(\mathbf{y})$, *iii)* an outcome function $\phi : \Pi_{i=1}^n S^i \times \mathbb{R}_+ \longrightarrow \mathcal{F}(\mathbf{y})$, given more precisely by:

$$\phi\left(\boldsymbol{g}, \boldsymbol{y}\right) \equiv \left(y_1 - g_1 + T^1\left(g_1, \mathbf{G}^{-1}\right), ..., y_n - g_n + T^n\left(g_n, \mathbf{G}^{-n}\right), \sum_i^n g_i\right).$$

In a game form, contrary to a standard game, payoff functions are not exogenously given. They are endogenously determined so that a pre-specified outcome of interest emerges as an equilibrium of strategic choices from the agents. In this paper, our *institution design* - or *mechanism design* - problem is to characterize necessary conditions for an outcome function $\phi\left(\boldsymbol{g}, \boldsymbol{y}\right)$, or equivalently a game form, to admit Nash equilibria that are also a Lindahl allocations.

A few mechanisms have been proposed in order to Nash-implement a Lindahl allocation (Hurwicz, 1979, Walker, 1981, Tian, 1989, Nakamura, 1989, Vega-Redondo, 1989, Kim, 1993, de Trenqualye, 1994, Li, Nakamura and Tian, 1995, Chen 2002, Van Hessen, 2013, Rouillon, 2013). Not all those mechanisms belong to the class of mechanisms that can be associated to the PGGFs and the family of transfers $T^i\left(g_i, \mathbf{G}^{-i}\right)$ we are using in this paper. For instance Kim (1993), de Trenqualye (1994), Chen (2002), Van Hessen (2013) and Rouillon (2013)

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[5]See Samuelson (1954). For this reason, Lindahl equilibria are also called Lindahl *pseudo-equilibria*.

rest of more sophisticated strategy spaces, for they require that agents report their prediction about the other agents' contributions, or about the level of public good that will be provided. Put differently, in order to encompass those mechanisms we would have to start with extended strategy spaces and a more complex class of transfers. We defend our simpler framework for its realism. Transfers whose amounts are partly conditioned to forecasts about decisions, despite their conceptual interest, have been criticized because they do not appear to be natural or they are lacking simplicity. See the survey in Moore (1992, in particular Section X of part I). This concern for simple mechanisms appears in various forms in the theoretical and experimental literatures. Recent examples of the former are Baskar, Sen and Vohra (1995), Segal (2007); and for the latter see Bracht, Figuières and Ratto (2008), Iyengar and Kamenica (2010), or Midler, Figuières and Willinger (2015).

Throughout the paper, we impose some regularity on transfers:

Assumption 1 ($\mathbf{C}^1$). *Each $T^i\left(g_i, \mathbf{G}^{-i}\right)$ is of class C^1.*

In the sequel, transfers that satisfy $\mathbf{C}^1$ are said to be *regular*. Assumption $\mathbf{C}^1$ ensures that the outcome function ϕ is also C^1, so that small changes in an agent's choice does not result in large changes in the ensuing allocation. Insights gained in such games are robust when agents are slightly uncertain about rivals' strategies[6]. For more details, see Postlewaite and Wettstein (1989).

Let $\{\widehat{g}_i\}_{i=1}^n$ be a Nash equilibrium of the PGGF. At such an equilibrium, taking as given the equilibrium profile of others' contributions, agent i contribution g_i maximizes

$$U^i\left(y_i - g_i + T^i\left(g_i, \widehat{\mathbf{G}}^{-i}\right), g_i + \widehat{g}_{-i}; \theta_i\right) .$$

If $T^i_{g_i}$ is the partial derivative of T^i with respect to g_i, the first order condition for an interior optimal decision is:

$$MRS^i\left(\widehat{c}_i, \widehat{G}; \theta_i\right) = 1 - T^i_{g_i} , \tag{8}$$

for each agent i.

Define $N^T\left(e\right)$ as the set of interior Nash equilibria[7] of the PGGF defined by the profile of transfers T. Those particular allocations, $\left(\{\widehat{c}_i\}_{i=1}^n, \widehat{G}\right) \in \mathcal{F}\left(\mathbf{y}\right)$

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[6]Actually, it is sufficient that ϕ be continuous to guarantee this result.

[7]Our analysis applies to PGGF that admit Nash equilibria in pure strategies. Sufficient conditions for the existence of such Nash equilibria are easy to find out, but they are also more restrictive than one would like. They would ask that the composite function:

$$V^i\left(g_i, G^{-i}; \theta_i\right) \equiv U^i\left(y_i - g_i + T^i\left(g_i, G^{-i}\right), g_i + g_{-i}; \theta_i\right)$$

be continuous and quasi-concave with respect to g_i, $\forall G^{-i}$. In general this has to be a joint property of U^i and T^i, unless more structure is imposed. For instance, a particular case of interest that meets these conditions is

$$V^i\left(g_i, G^{-i}; \theta_i\right) = y_i - g_i + T^i\left(g_i, G^{-i}\right) + \theta_i v^i\left(g_i + g_{-i}\right),$$

where $v^i\left(g_i + g_{-i}\right)$ is a C^2 function, and in addition $\frac{d^2}{(dG)^2}v^i \leq 0$ and $\frac{\partial^2}{(\partial g_i)^2}T^i \leq 0$.

necessarily satisfy (8). Ideally, one would like to identify the class of transfers that fully implements the Lindahl correspondence:

$$\mathcal{T}^L \equiv \left\{ T = \left\{ T^i \left(g_i, \mathbf{G}^{-i} \right) \right\}_{i=1}^n \text{ such that } N^T \left(e \right) = L \left(e \right) \ , \ \forall e \in \Omega \right\}.$$

A less ambitious requirement would be to achieve Pareto optimality. For that purpose, the condition to be imposed on transfers is easy to identify. Summing-up the equations (8) over the agents:

$$\sum_{i=1}^n MRS^i \left(\widehat{c}_i, \widehat{G}; \theta_i \right) = n - \sum_{i=1}^n T_{g_i}^i. \tag{9}$$

Compare the above equation (9) with the BLS equation (7), and a requirement immediately follows: if an interior Nash equilibrium of the PGGF corresponds to a particular interior Pareto optimal allocation $\left(\{\widehat{c}_i\}_{i=1}^n \ , \widehat{G} \right)$, transfers necessarily meet the following requirement:

Axiom 1 (Transfers for Pareto Optimal Allocations (TPOA)). *For a given Pareto optimal allocation $\left(\widehat{g}_i, \widehat{\mathbf{G}}^{-i} \right)$, corresponding transfers are such that*

$$\sum_{i=1}^n T_{g_i}^i \left(\widehat{g}_i, \widehat{\mathbf{G}}^{-i} \right) = n - 1. \tag{10}$$

If in addition transfers are required to be balanced, at *and* off equilibrium, then:

Axiom 2 (Budget Balance (BB)).

$$\sum_{i=1}^n T^i \left(g_i, \mathbf{G}^{-i} \right) = 0, \tag{11}$$

for any arbitrary $\left(g_i, \mathbf{G}^{-i} \right)$.

Not all the existing mechanisms are balanced in the strong sense required by the axiom **BB**[8]. For instance Kim (1993), de Trenqualye (1994), Chen (2002), Van Hessen (2013) and Rouillon (2013), are balanced only when agents play the Nash equilibrium. Our main reason for imposing **BB** is pragmatic. Although one may hope that agents' behavior conforms with what the theory predicts, one may also expect that few agents will make out-of-equilibrium decisions. As a

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[8] A weaker form requires that transfers be weakly balanced, in the sense:

$$\sum_{i=1}^n T^i \left(g_i, \mathbf{G}^{-i} \right) \leq 0, \tag{12}$$

for any arbitrary $\left(g_i, \mathbf{G}^{-i} \right)$.

result, there may remain a deficit or a surplus of money. A deficit is problematic because it means that the funding comes at the expense of other sectors in the economy (or future agents / generations). And a surplus is problematic as well. Because it could be redistributed to all agents and create a Pareto improvement, but at the same time this redistribution could undermine the incentive properties of the mechanism under consideration.

With transfers of the form $T^i\left(g_i, \mathbf{G}^{-i}\right)$, identifying conditions (8) with (2), (3) and (4), if a Nash equilibrium of the PGGF is also a Lindahl equilibrium, necessarily, $\forall i = 1, ..., n, \forall e \in \Omega$:

$$\frac{U_2^i\left(y_i - g_i + T^i\left(g_i, \mathbf{G}^{-i}\right), g_i + g_{-i}; \theta_i\right)}{U_1^i\left(y_i - g_i + T^i\left(g_i, \mathbf{G}^{-i}\right), g_i + g_{-i}; \theta_i\right)} = 1 - T_{g_i}^i = \tau_i \ ,$$

$$g_i - T^i\left(g_i, \mathbf{G}^{-i}\right) = \tau_i G \ ,$$

$$\sum_{i=1}^n \left(\frac{g_i}{G} - \frac{T^i\left(g_i, \mathbf{G}^{-i}\right)}{G}\right) = \sum_{i=1}^n \tau_i = 1.$$

Or, getting rid of the individualized prices:

$$\frac{U_2^i\left(y_i - \left(1 - T_{g_i}^i\right) * G, G; \theta_i\right)}{U_1^i\left(y_i - \left(1 - T_{g_i}^i\right) * G, G; \theta_i\right)} = 1 - T_{g_i}^i \ , \tag{13}$$

$$g_i - T^i\left(g_i, \mathbf{G}^{-i}\right) = \left(1 - T_{g_i}^i\right) * G \ , \tag{14}$$

$$G - \sum_{i=1}^n T^i\left(g_i, \mathbf{G}^{-i}\right) = G \ . \tag{15}$$

Equation (13) is useless from an implementation perspective, because it requires information about utility functions that is not known to the mechanism designer. However, it indicates explicitly that the mechanism relies on a behavioral assumption, that rationality is of the usual "utility-maximizer" kind. Equation (15) is automatically satisfied provided that transfers are budget-balanced.

The second equation, or the set of second equations since there is one equation (14) for each agent, is crucial. It reveals a structure about the kind of transfers we are looking for. They must solve (14) for any Lindahl allocation in $L(e)$. Since the institution designer does not know the agents' preferences (in particular the vector $\boldsymbol{\theta}$ of parameters), the profile of sought transfers must be built without resorting to such pieces of information, *i.e.* transfers must solved this set of equations $\forall e \in \Omega$. The set of public good economies we consider encompasses cases where the Lindahl allocations vary continuously with the vector $\boldsymbol{\theta}$. Therefore transfers must satisfy (14) not only at a particular point $\left(g_i, \mathbf{G}^{-i}\right)$ but on a manifold of Lindahl allocations $\mathcal{M}$, *i.e.* $\forall\left(g_i, \mathbf{G}^{-i}\right) \in \mathcal{M}$. This means (14) cannot be considered as a set of equations; it must be viewed as a system of non autonomous differential equations, to which a natural initial condition can be specified. Consider a particular Lindahl allocation $\left(\{c_i^*\}_{i=1}^n, G^*\right) \in L(e)$. Denote:

$$T^{i*} = T^i\left(g_i^*, \mathbf{G}^{*-i}\right),$$

the numerical value of agent i's transfer when computed at this Lindahl allocation. Then the set of equations (14) can be rewritten:

Definition 1 (Lindahl Differential System (LDS)).

$$\begin{cases} T^i_{g_i}\left(g_i, \mathbf{G}^{-i}\right) = \frac{T^i\left(g_i, \mathbf{G}^{-i}\right)}{G} + 1 - \frac{g_i}{G} \ , \\ T^i\left(g_i^*, \mathbf{G}^{*-i}\right) = T^{i*} \ , \end{cases} \qquad \forall i, \forall \left(g_i, \mathbf{G}^{-i}\right) \in \mathcal{M}. \qquad (16)$$

For a manifold of Lindahl allocations, **LDS** singles out a particular system of non autonomous differential equations whose solutions $\left\{T^i\right\}_{i=1}^n$ are transfers with the ability to induce the considered Lindahl allocations.

Actually, we can be much more precise about regular transfers that respect **LDS**:

Theorem (Linear Transfers (LT)). *There exists a unique class of regular transfers that abides by **LDS**. In this class, transfers take the form:*

$$T^i\left(g_i, \mathbf{G}^{-i}\right) = A^i\left(\mathbf{G}^{-i}\right) * g_i + \left[A^i\left(\mathbf{G}^{-i}\right) - 1\right] * g_{-i} \ , \qquad (17)$$

for some function $A^i\left(\mathbf{G}^{-i}\right)$.

Proof. Fix an arbitrary vector of admissible contributions $\left(g_i^*, \mathbf{G}^{*-i}\right)$ to serve as an initial condition. If one holds $\mathbf{G}^{*-i}$ constant then **LDS** is an ordinary non autonomous differential equation. Since each $T^i\left(., \mathbf{G}^{-i}\right)$ is C^1 (Assumption $\mathbf{C}^1$), the partial derivative $T^i_{g_i}$ is continuous over its domain of definition, S^h, which is compact. Therefore $T^i_{g_i}$ has an upper bound, meaning that $T^i\left(., \mathbf{G}^{-i}\right)$ is Lipschitz continuous with respect to the first argument, for all admissible $\mathbf{G}^{-i}$. Then by application of the Cauchy-Lipschitz Theorem there exists a unique (local) solution to equation **LDS**. And because of Assumption $\mathbf{C}^1$ this solution varies continuously with $\mathbf{G}^{-i}$ (still in application of the Cauchy-Lipschitz Theorem). This proves the first statement of the theorem (uniqueness). As for the second statement, it is easy to check that (17) indeed solves equation **LDS**, so it gives the unique solution we are looking for. ∎

Expression (17) is the first main finding of this paper. Its statement substantially reduces the possibilities for transfers formulae. It defines a class of transfers that encompasses but is not limited to $\mathcal{T}^{L9}$. A natural follow-up question is whether the **TPOA** property is guaranteed for transfers (17). The answer is affirmative provided the condition **BB** is also imposed.

Corollary 1. *If a profile of transfers $\left\{T^i\left(g_i, \mathbf{G}^{-i}\right)\right\}_{i=1}^n$ respects $\mathbf{C}^1$, **BB** and **LDS** at a particular allocation $\left(\{\widehat{g}_i\}_{i=1}^n, \widehat{\mathbf{G}}\right)$, then it also respects **TPOA** at $\left(\widehat{g}_i, \widehat{\mathbf{G}}^{-i}\right)$.*

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[9]More on this below.

Proof. According to Theorem 3, summing transfers that respect **LDS** at $\left(\widehat{g}_i, \widehat{\mathbf{G}}^{-i}\right)$ one finds:

$$\sum_{i=1}^{n} T^i \left(\widehat{g}_i, \widehat{\mathbf{G}}^{-i}\right) = \sum_{i=1}^{n} A^i \left(\widehat{\mathbf{G}}^{-i}\right) * \widehat{g}_i + \sum_{i=1}^{n} \left[A^i \left(\widehat{\mathbf{G}}^{-i}\right) - 1\right] * \left(\widehat{\mathbf{G}} - \widehat{g}_i\right) ,$$

$$= \widehat{\mathbf{G}} * \sum_{i=1}^{n} A^i \left(\widehat{\mathbf{G}}^{-i}\right) - (n-1) * \widehat{\mathbf{G}} = 0 ,$$

where the last equality is obtained by virtue of **BB**. This last expression simplifies to:

$$\sum_{i=1}^{n} A^i \left(\widehat{\mathbf{G}}^{-i}\right) = n - 1 .$$

Because transfers are of the form (17), observe that $A^i \left(\widehat{\mathbf{G}}^{-i}\right) = T_{g_i}^i \left(\widehat{g}_i, \widehat{\mathbf{G}}^{-i}\right)$. Therefore:

$$\sum_{i=1}^{n} A^i \left(\widehat{\mathbf{G}}^{-i}\right) = n - 1 \iff \sum_{i=1}^{n} T_{g_i}^i \left(\widehat{g}_i, \widehat{\mathbf{G}}^{-i}\right) = n - 1 ,$$

which is the **TPOA** axiom. ∎

The class of such transfers therefore fully implements the Pareto correspondence. Let us single out this class of transfers:

$$\mathcal{T}^P \equiv \left\{T = \left\{T^i \left(g_i, \mathbf{G}^{-i}\right)\right\}_{i=1}^{n} \text{ such that } \mathbf{C}^1, \ \mathbf{BB} \text{ and } \mathbf{LDS} \text{ are respected}\right\} .$$

Note that $\mathcal{T}^P$ can still be large. Walker (1981) lies in the class of transfers (17). Indeed, setting:

$$A^i \left(\mathbf{G}^{-i}\right) = 1 - \left(\frac{1}{n} + g_{i+2} - g_{i+1}\right) .$$

yields Walker's mechanism and one can check that it indeed abides by **LDS**. Setting:

$$A^i \left(\mathbf{G}^{-i}\right) = 1 - \frac{1}{n} , \quad \forall i,$$

yields Falkinger's mechanism and also respects **LDS**, though it does not necessarily implements a Lindahl allocation. Further arguments are necessary to discriminate among the elements of $\mathcal{T}^P$. The set of Lindahl allocations is narrower. The initial condition in system **LDS** would obviously add precision about $A^i \left(\mathbf{G}^{-i}\right)$ in formula (17), in order to implement a particular Lindahl allocation. However such an information is typically not available in our implementation framework.

Despite this lack of information, a subset of $\mathcal{T}^P$ that fully implements Lindahl allocations can be found. Consider a Lindahl equilibrium, with a vector of individual prices $\tau^* = (\tau_1^*, ..., \tau_n^*)$ and a level of public good G^*. Under what

condition do transfers of the form indicated in Theorem LT (i) yield a vector of non-negative contributions $g^* = (g_1^*, ..., g_n^*)$ that sum up to G^* and such that the share of each individual contribution is $g_i^*/G^* = \tau_i^*$? And (ii) achieve (i) for any Lindahl allocation $(g_i^*, \mathbf{G}^{*-i}) \in \mathcal{M}$? Using (2) and (13), we are therefore looking for a set of functions $A^i(.)$ that solve the system:

$$\begin{cases} \tau_i^* = \frac{g_i^*}{G^*} = 1 - A^i(\mathbf{G}^{*-i}) \ , \quad \forall i, \ \forall (g_i^*, \mathbf{G}^{*-i}) \in \mathcal{M} \ , \\ \sum_i g_i^* = G^* \ . \end{cases} \tag{18}$$

Clearly if $A^i(.)$ is a constant function, this is impossible. Consider then the class of linear functions: $A^i(\mathbf{G}^{-i}) = \alpha + \sum_{h=1}^k \beta_h g_{i+h}$ where k can be any integer between 2 and $n-1$ and the subscripts $i+h$ are defined so that $i+h = i+h-n$ whenever $i+h > n$. Walker's transfer is a particular case where $k = 2$ and $\beta_2 = -\beta_1 = 1$.

Because at a Lindahl price system $\sum \tau_i^* = 1$, a first deduction drawn from (18) is:

$$n(1-\alpha) - \sum_{i=1}^n \sum_{h=1}^k \beta_h g_{i+h}^* = 1 \ , \quad \forall (g_i^*, \mathbf{G}^{*-i}) \in \mathcal{M} \ .$$

Therefore, since this equation is supposed to hold for all profiles $(g_i^*, \mathbf{G}^{*-i}) \in \mathcal{M}$:

$$\alpha = 1 - 1/n \ , \quad \sum_{h=1}^k \beta_h = 0 \ , \tag{19}$$

where the second equality obtains because:

$$\sum_{i=1}^n \sum_{h=1}^k \beta_h g_{i+h}^* = G^* \sum_{h=1}^k \beta_h \ .$$

This shows that, for the class of transfers where $A^i(\mathbf{G}^{-i}) = \alpha + \sum_{h=1}^k \beta_h g_{i+h}$, conditions (19) form necessary conditions to implement a Lindahl allocation.

Conditions (19) are also sufficient conditions, for we can establish that the system (18) now has a unique solution:

Lemma 1. *For a given profile of Lindahl prices $\{\tau_i^*\}_{i=1}^n$ and a given Lindahl level of public good G^*, the system (18) with functions $A^i(\mathbf{G}^{-i}) = (n-1)/n + \sum_{h=1}^k \beta_h g_{i+h}$, $\sum_{h=1}^k \beta_k = 0$, has a unique solution.*

Proof. If the determinant of the linear system (18) is different from 0 then a solution exists and it is unique. The matrix of the system is:

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & \cdots & 1 \\ 0 & -\beta_1 & -\beta_2 & -\beta_3 & \cdots & \cdots & -\beta_{n-1} \\ -\beta_{n-1} & 0 & -\beta_1 & -\beta_2 & \cdots & \cdots & -\beta_{n-2} \\ -\beta_{n-2} & -\beta_{n-1} & 0 & -\beta_1 & \cdots & \cdots & -\beta_{n-3} \\ -\beta_{n-3} & -\beta_{n-2} & -\beta_{n-1} & 0 & \cdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \cdots & \vdots \\ -\beta_2 & -\beta_3 & \cdots & \cdots & \cdots & 0 & -\beta_1 \end{bmatrix}$$

Suppose, by way of contradiction, that $Det(M) = 0$. This means that the column vectors $C_1, ..., C_n$, of matrix A are linearly dependent. In this case we can exhibit real numbers $\alpha_i, i = 1, ..., n-1$, such that the last column vector C_n can be expressed as:

$$C_n = \alpha_1 C_1 + ... + \alpha_{n-1} C_{n-1}$$

For the first line of the matrix, this would mean:

$$\alpha_1 + ... + \alpha_{n-1} = 1, \tag{20}$$

For the second line, we would have:

$$\alpha_1 * 0 + \alpha_2 * (-\beta_1) + \alpha_3 * (-\beta_2) + ... + \alpha_{n-1} * \left(-\beta_{n-2}\right) = -\beta_{n-1} . \tag{21}$$

But we also know, for the chosen class of functions $A^i \left(\mathbf{G}^{-i}\right)$, that:

$$\beta_{n-1} = -\left(\beta_1 + \beta_2 + ... + \beta_{n-2}\right) , \tag{22}$$

and therefore, using (21) and (22):

$$\beta_1 + \beta_2 + ... + \beta_{n-2} = \alpha_2 \beta_1 + \alpha_3 \beta_2 + ... + \alpha_{n-1} \beta_{n-2} .$$

Repeating the logic with the third line, one obtains:

$$\alpha_1 * \left(-\beta_{n-1}\right) + \alpha_2 * 0 + \alpha_3 * (-\beta_1) + ... + \alpha_{n-1} * \left(-\beta_{n-3}\right) = -\beta_{n-2} .$$

And because:

$$\beta_{n-2} = -\left(\beta_1 + \beta_2 + ... + \beta_{n-3} + \beta_{n-1}\right) ,$$

necessarily:

$$\beta_1 + \beta_2 + ... + \beta_{n-3} + \beta_{n-1} = \alpha_1 \beta_{n-1} + \alpha_3 \beta_1 + ... + \alpha_{n-1} \beta_{n-3} .$$

This can be repeated for any line, and overall it is easy to check that there is only one possibility $\alpha_i = 1, \forall i = 1, .., n-1$, in contradiction with (20), hence the conclusion of the lemma. ∎

We now consider the reverse question, that is let one starts with this class of transfers and check if one arrives at a Lindahl equilibrium. From individual best responses (8), we have:

$$\begin{aligned} MRS^i &= 1 - T_{gi}^i = 1 - A^i \left(\mathbf{G}^{-i}\right) , \\ &= \frac{1}{n} - \sum_{h=1}^{k} \beta_h g_{i+h} , \quad i = 1, ..., n . \end{aligned} \tag{23}$$

The above best responses correspond indeed to a situation where individual Lindahl prices are:

$$\tau_i^* = \frac{1}{n} - \sum_{h=1}^{k} \beta_h g_{i+h} .$$

And summing-up all these expressions (23):

$$\sum_i MRS^i = \sum_i \tau_i^* = 1 - \sum_i \sum_{h=1}^{k} \beta_h g_{i+h} \ ,$$

and thus:

$$\sum_i MRS^i = 1, \tag{24}$$

because:

$$\sum_{i=1}^{n} \sum_{h=1}^{k} \beta_h g_{i+h}^* = G^* \sum_{h=1}^{k} \beta_h = 0 \ .$$

Lemma 1 along with (23) and (24) establish our second theorem, which generalizes Walker's mechanism:

Theorem (Linear Lindahl Transfers (LLT)). *Take an economy $e \in \Omega$ with at least three agents. There is a one-to-one correspondence between the Nash equilibrium generated by the class of transfers:*

$$T^i \left(g_i, \mathbf{G}^{-i} \right) \;\; = \;\; \left(\frac{n-1}{n} + \sum_{h=1}^{k} \beta_h g_{i+h} \right) * g_i \;\; + \;\; \left[\frac{1}{n} + \sum_{h=1}^{k} \beta_h g_{i+h} \right] * g_{-i} \ ,$$

$$\sum_{h=1}^{k} \beta_h \;\; = \;\; 0 \ ,$$

and the Lindahl allocation of e.

4 Summary

This paper focuses on the implementation of Lindahl allocations by means of simple transfers. Our contribution is to identify some structure in the set of such transfers. We establish two main results: *i*) Theorem LT states the conditions under which transfers are linear, *ii*) Theorem LLT identifies a whole class of linear transfers - which generalizes Walker's formula (1981) - that implements a Lindahl allocation.

A follow-up to the approach that we use would be to start with other families of transfers *a priori* given - for instance, transfers may also be conditioned on observable socio demographic categories - and then determine under what conditions it is possible to achieve a Lindahl allocation *via* such transfers.

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