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WP 2016 - Nr 43

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November 20, 2016

Abstract

We study absolute qualified majority rules in a setting with more than two alternatives. We show that given two qualified majority rules, if transitivity is desired for the societal outcome and if the thresholds of one of these rules are at least as high as the other's for any pair of alternatives, then at each preference profile the rule with higher thresholds results in a coarser social ranking. Hence all absolute qualified majority rules can be expressed as specific coarsenings of the simple majority rule.

Keywords: simple majority rule, qualified majority rules
JEL Codes: D7

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1 Introduction

Studies of majoritarianism date back to the seminal characterization of the simple majority rule for two alternatives by May (1952), where a majority winner is defined as an alternative that receives higher support with respect to its rival.[1] This pertains to so-called *relative* majoritarianism. When there is a threshold of excess support for an alternative to win, we have *relative qualified majority rules* considered by Fishburn (1973) and Saari (1990) and characterized by Llamazares (2006) and Houy (2007).

On the other hand, under *absolute (qualified) majority rules*, the winner is an alternative that receives the support of a given (qualified) majority. Absolute and relative majoritarianism diverge when indifferences in individual preferences are allowed. Fishburn (1973) delivers an early characterization of absolute majority rules while Austen-Smith and Banks (1999), Yi (2005), and Aşan and Sanver (2006) provide alternative characterizations. Sanver (2009) provides a unified assessment of characterizations for relative and absolute qualified majority rules, centered on the axioms of anonymity (equal treatment of individuals) and neutrality (equal treatment of alternatives).

Non-neutral versions of qualified majority rules are employed in legislative decision-making processes across the world. A major issue related to the application of these rules concerns the choice of the threshold and different thresholds are observed in use. For instance, in the United Nations Security Council, at least 9 members, *i.e.*, 60 percent, of 15 members must vote in favor to pass a draft resolution while in the Council of the European Union (CEU) a threshold of 55% is used.[2] Leech and Machover (2003) study the effect of the threshold on the voting power of each member, the blocking power of each member, the sensitivity of the rule, the ability of the collectivity to act, and the mean majority deficit, in the context of CEU.

This paper is about the effects of the thresholds when neutral absolute

[1] Aşan and Sanver (2002), Woeginger (2003), Miroiu (2004), and Woeginger (2005) provide alternative characterizations in societies with variable population.

[2] More precisely, as agreed in the Treaty of Lisbon on 13 December 2007 to be effective definitively from 1 November 2014, a qualified majority is reached when at least 55% of all member states, who comprise at least 65% of EU citizens, vote in favor of a proposal. Previously, some sort of a weighted majority voting system was in use. See http://www.consilium.europa.eu/en/council-eu/voting-system/qualified-majority.

qualified majority rules are used as social welfare functions in a setting with more than two alternatives. For the sake of better exposition of our major result, consider first the case of two alternatives and two absolute qualified majority rules. When the rule with smaller threshold regards the alternatives indifferent, the rule with higher threshold complies. When one of the alternatives wins under the rule with higher threshold, the rule with smaller threshold complies. For some of those profiles where one of the alternatives wins under the rule with smaller threshold, we will have indifference under the rule with higher threshold, as the excess support may not be sufficient. So the rule with a higher threshold leads to coarser comparisons in the sense that it produces indifferences more frequently. With more than two alternatives, on the other hand, it is well known that (qualified) majority rules can lead to cycles. Theorem 1 demonstrates that, with more than two alternatives, the coarsening property is inherited by the transitive closures of the absolute qualified majority rules, *i.e.*, the rule with at least as high thresholds will result in a coarser social ranking at each preference profile. It follows from this result that all absolute qualified majority rules can be expressed as specific coarsenings of the simple majority rule.

Section 2 is devoted to basic notation, definitions, and preliminary observations. Section 3 presents our main result.

2 Preliminaries

Let $N = \{2, \dots, n\}$ be the set of individuals, or the society, where $n \in \mathbb{N}$ and A is the set of alternatives the society confronts, with $\#A = m \geq 3$. Each individual $i \in N$ is endowed with a *preference* $P_i \in \mathcal{W}(A)$ where $\mathcal{W}(A)$ denotes the set of all weak orders, *i.e.*, complete and transitive binary relations over A. The set of all complete binary relations over A is denoted by $\Theta(A)$, hence we have $\mathcal{W}(A) \subset \Theta(A)$. Furthermore, let T^* stand for the strict counterpart for any $T \in \Theta(A)$, whereas $\tilde{T}$ stand for the indifference counterpart.[3] The preferences of all individuals in N, or the *preference profile*, is denoted by $P_N \in \mathcal{W}(A)^N$. Given any $T \in \Theta(A)$ and any $B \subseteq A$, the restriction of T to B is denoted by $T|_B \in \Theta(B)$. So for any $x, y \in B$,

[3] So we have xT^*y if and only if xTy holds but yTx does not. Also, $x\tilde{T}y$ if and only if we have both xTy and yTx.

we have $xTy \iff xT|_By$. We call a mapping $f : \mathcal{W}(A)^N \to \Theta(A)$ an *aggregation rule*. An aggregation rule whose range is transitive is called a *social welfare function* (SWF). Hence, a social welfare function is a mapping $f : \mathcal{W}(A)^N \to \mathcal{W}(A)$ that assigns each preference profile a social ranking. Given any aggregation rule f, we write $f^*(\cdot)$ for the strict counterpart of $f(\cdot)$ and $\tilde{f}(\cdot)$ for its indifference counterpart.

Definition 1 *Given any $Q, T \in \Theta(A)$, we say that Q is a* coarsening *of T iff for any $x, y \in A$ we have*

(i) $xT^*y \implies xQy$ and

(ii) $x\tilde{T}y \implies x\tilde{Q}y$.

We denote the set of all coarsenings of T by $\gamma(T)$. Note that $\gamma(T) \neq \emptyset$ as $T \in \gamma(T)$ for any $T \in \Theta(A)$. Furthermore, for the full indifference R_o over A with xR_oy for all $x, y \in A$, we have $R_o \in \gamma(T)$ for any T.

A set $C \in 2^A\backslash\{\emptyset\}$ is called a *cycle* with respect to $T \in \Theta(A)$ if and only if it can be written as $C = \{x_1, \ldots, x_{\#C}\}$ such that $x_iTx_{i+1}$for all $i \in \{1, \ldots, \#C - 1\}$ and $x_{\#C}Tx_1$. Note that the definition allows two degeneracies, namely full indifference over C and C being a singleton. Every $T \in \Theta(A)$ induces a unique ordered partition of cycles $\pi(T) = \{C_1, \ldots, C_k\}$ for some $k \in \{1, \ldots, m\}$ such that xT^*y for all $x \in C_i$ and for all $y \in C_j$ whenever $1 \leq i < j \leq k$.[4] When T is a weak order, for any $i \in \{1, \ldots, k\}$, C_i is degenerate, *i.e.*, C_i is either a singleton or a full indifference.

Given any $T \in \Theta(A)$, the *transitive closure of* T is denoted by $\tau(T) \in \mathcal{W}(A)$. The strict counterpart of $\tau(T)$ is denoted by $\tau^*(T)$ and the indifference counterpart by $\tilde{\tau}(T)$. The transitive closure of an intransitive binary relation transforms the relation into a weak order by replacing non-degenerate cycles with full indifferences. Formally, given $T \in \Theta(A)$ and $\pi(T) = \{C_1, \ldots, C_k\}$, the transitive closure $\tau(T)$ is such that

(i) $x\tau(T)y$ for all $x, y \in C_i$ with $i \in \{1, \ldots, k\}$ and

(ii) $x\tau^*(T)y$ whenever $x \in C_i$ and $y \in C_j$ with $1 \leq i < j \leq k$.

[4]See Proposition 1.3.3. in Laslier (1997) where this statement is proven for $T \in \Theta(A)$ with xT^*y or yT^*x for all $x, y \in A$.

Note that, $\forall T \in \Theta(A)$, we have $\tau(T) \in \gamma(T)$ and if $T \in \mathcal{W}(A)$, then $\tau(T) = T$.

Given any two ordered partitions $\bar{A} = \{A_i\}_{i\in\{1,\ldots,k\}}$ and $\bar{A}' = \{A'_i\}_{i\in\{1,\ldots,k'\}}$ of A with $1 \le k' \le k \le m$, we say that $\bar{A}'$ is an *order-preserving coarsening* of $\bar{A}$ if and only if there exists a function $f : \{1,\ldots,k\} \to \{1,\ldots,k'\}$ such that for all $i, j \in \{1,\ldots,k\}$ we have $A_i \subseteq A'_{f(i)}$ and $i < j \implies f(i) \le f(j)$.

So given any ordered partition $\bar{A}$, we can construct another partition that is an order-preserving coarsening, by joining any two elements A_i and A_j of $\bar{A}$ provided that their union includes all elements of $\bar{A}$ which lie between A_i and A_j. The following lemma shows that any coarsening R' of a weak order R induces a partition $\pi(R')$ that is an order-preserving coarsening of the partition $\pi(R)$.

Lemma 1 $\gamma(R) = \{Q \in \mathcal{W}(A) : \pi(Q)$ *is an order-preserving coarsening of* $\pi(R)\}$, $\forall R \in \mathcal{W}(A)$.

Proof. Fix some $R \in \mathcal{W}(A)$ and denote

$$\{Q \in \mathcal{W}(A) : \pi(Q) \text{ is an order-preserving coarsening of } \pi(R)\}$$

by $\Delta(R)$. To show $\gamma(R) = \Delta(R)$ we establish $\gamma(R) \subseteq \Delta(R)$ and $\Delta(R) \subseteq \gamma(R)$. For the former, take any $R' \in \gamma(R)$. Let $\pi(R) = \{A_1,\ldots,A_k\}$ and $\pi(R') = \{A'_1,\ldots,A'_{k'}\}$ for some $k, k' \in \{1,\ldots,m\}$. First note that R' is a weak order. Now, take any $A_i \in \pi(R)$. In case $A_i = \{x\}$ for some $x \in A$, as $\pi(R')$ partitions A, we have $x \in A'_{i'}$ hence $A_i \subseteq A'_{i'}$, for some $i' = \{1,\ldots,k'\}$. Now consider the case $\#A \ge 2$ and take any distinct $x, y \in A_i$. We have $x\tilde{R}y$, hence $x\tilde{R}'y$ as $R' \in \gamma(R)$, which implies $x, y \in A'_{i'}$ for some $i' \in \{1,\ldots,k'\}$. Thus $A_i \subseteq A'_{i'}$ for some $i' \in \{1,\ldots,k'\}$. Now, construct $f : \{1,\ldots,k\} \to \{1,\ldots,k'\}$, which, for each $i \in \{1,\ldots,k\}$, $f(i) = i'$ such that $A_i \subseteq A'_{i'}$. By construction, $A_i \subseteq A'_{f(i)}$, for all $i \in \{1,\ldots,k\}$. We need to show that $i < j \implies f(i) \le f(j)$. Suppose for a contradiction that there exists $i, j \in \{1,\ldots,k\}$ with $i < j$ such that $f(j) > f(i)$. Hence for all $x \in A_{f(i)}$ and $y \in A_{f(j)}$, we have yR'^*x. However, xR^*y as $x \in A_i$ and $y \in A_j$ with $i < j$, which is a contradiction because $R' \in \gamma(R)$ implies $xR'y$. Thus we conclude that $\gamma(R) \subseteq \Delta(R)$.

To show $\Delta(R) \subseteq \gamma(R)$, take any $R' \in \Delta(R)$ and let $f : \{1,\ldots,k\} \to \{1,\ldots,k'\}$ be the function that ensures that $\pi(R')$ is an order-preserving coarsening of $\pi(R)$. Take any $x, y \in A$ such that $x\tilde{R}y$. Hence $x, y \in A_i$ for some $i \in \{1, \ldots, k\}$. As $x, y \in A_{f(i)}$ for some $f(i) \in \{1, \ldots, k'\}$, we have $x\tilde{R}'y$ as well. Next, take any $x, y \in A$ such that xR^*y. Then there exist $i, j \in \{1, \ldots, k\}$ with $i < j$ such that $x \in A_i$ and $y \in A_j$. If $f(i) = f(j)$, we have $x\tilde{R}'y$. If $f(i) \neq f(j)$, we must have $f(i) < f(j)$ and hence xR'^*y. So we have $xR'y$ as well, which means $R' \in \gamma(R)$. Hence we established that $\Delta(R) \subseteq \gamma(R)$, which completes the proof. ■

3 Result

Let $\hat{A}$ denote the set of all subsets of A with cardinality 2 and let $n(x, y; P_N) = \#\{i \in N : xP_i^*y\}$ be the number of individuals who prefer x to y at P_N. The smallest integer greater than $n/2$ is denoted by $n^* = \lfloor \frac{n}{2} + 1 \rfloor$. For any collection $\mathbf{q} = \{q_{\{x,y\}}\}_{\{x,y\} \in \hat{A}}$ of pairwise thresholds $q_{\{x,y\}} \in \{n^*, \ldots, n\}$, we define the aggregation rule $f_{\mathbf{q}} : \mathcal{W}(A)^N \to \Theta(A)$ such that $\forall x, y \in A$ and $\forall P_N \in \mathcal{W}(A)^N$, we have

$$x f^*_{\mathbf{q}}(P_N) y \iff n(x, y; P_N) \geq q_{\{x,y\}}.$$

The *absolute* $\mathbf{q}-$*majority rule* is a social welfare function that is constructed by the transitive closures of the social rankings under $f_{\mathbf{q}}$ at each $P_N \in \mathcal{W}(A)^N$, and is denoted by $\tau(f_{\mathbf{q}}(\cdot))$.

For any $\mathbf{q} = \{q_{\{x,y\}}\}_{\{x,y\} \in \hat{A}}$ and $\mathbf{r} = \{r_{\{x,y\}}\}_{\{x,y\} \in \hat{A}}$, we write $\mathbf{q} \geq \mathbf{r}$ if and only if for all $\{x, y\} \in \hat{A}$ we have $q_{\{x,y\}} \geq r_{\{x,y\}}$.

Theorem 1 *If* $\mathbf{q} \geq \mathbf{r}$, *then* $\tau(f_{\mathbf{q}}(P_N))$ *is a coarsening of* $\tau(f_{\mathbf{r}}(P_N))$ *for all* $P_N \in \mathcal{W}(A)$.

Proof. Let $\mathbf{q} \geq \mathbf{r}$. Take any $P_N \in \mathcal{W}(A)^N$ and $x, y \in A$. We prove the theorem by establishing

$(\star)$ $x\tilde{\tau}(f_{\mathbf{r}}(P_N))y \implies x\tilde{\tau}(f_{\mathbf{q}}(P_N))y$ and

$(\star\star)$ $x\tau^*(f_{\mathbf{r}}(P_N))y \implies x\tau(f_{\mathbf{q}}(P_N))y$.

To see $(\star)$, let $x\tilde{\tau}(f_{\mathbf{r}}(P_N))y$. We have either (i) $x\tilde{f}_{\mathbf{r}}(P_N)y$ or (ii) $x, y \in C \in \pi(f_{\mathbf{r}}(P_N))$ while C is non-degenerate and $xf^*_{\mathbf{r}}(P_N)y$ without loss of generality.

In case (i) we have $n(x,y;P_N) < r_{\{x,y\}}$, hence $n(x,y;P_N) < q_{\{x,y\}}$ as well. Thus, $x\tilde{f}_{\mathbf{q}}(P_N)y$, which implies $x\tilde{\tau}(f_{\mathbf{q}}(P_N))y$. In case ($ii$), we have either (†) $n(x,y;P_N) < q_{\{x,y\}}$, in which case we have $x\tilde{f}_{\mathbf{q}}(P_N)y$ hence $x\tilde{\tau}(f_{\mathbf{q}}(P_N))y$, or (‡) $n(x,y;P_N) \geq q_{\{x,y\}}$. Recall that there is a cycle and hence a subset of alternatives $\{z_1,\ldots,z_t\}$ such that $yf_{\mathbf{r}}(P_N)z_1\cdots z_tf_{\mathbf{r}}(P_N)x$. As $r_{\{a,b\}} > n/2$ for all $a,b \in A$, we also have $yf_{\mathbf{q}}(P_N)z_1\cdots z_tf_{\mathbf{q}}(P_N)x$. Hence in case (‡) too, transitive closure of $f_{\mathbf{q}}$ regards x and y indifferent, *i.e.*, $x\tilde{\tau}(f_{\mathbf{q}}(P_N))y$.

For ($\star\star$), let $x\tau^*(f_{\mathbf{r}}(P_N))y$. We have $n(x,y;P_N) \geq r_{\{x,y\}} > n/2$, hence $n(y,x;P_N) < q_{\{x,y\}}$, which implies that $xf_{\mathbf{q}}(P_N)y$. So we have $x\tau(f_{\mathbf{q}}(P_N))y$, which completes the proof. ■

Let $\tau(f_{\boldsymbol{\mu}}(\cdot))$ be the *simple majority rule* with $\mu_{\{x,y\}} = n^*$ for all $x,y \in A$. Theorem 1 has the following immediate corollary.

Corollary 1 *For any preference profile $P_N \in \mathcal{W}(A)^N$, the social ranking $\tau(f_{\mathbf{q}}(P_N))$ under any absolute $\mathbf{q}-$majority rule is a coarsening of the social ranking $\tau(f_{\boldsymbol{\mu}}(P_N))$ under the simple majority rule.*

Thus, setting the quota at simple majority ensures the most refined social ranking and switching to a qualified majority can only coarsen the social preference without reversing the order of any pair.[5]

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[5]We note the coherence of our result with Dasgupta and Maskin (2008), who, assuming a continuum of voters, establish the robustness of simple majority in the sense of being "more decisive" than any other pairwise independent SWF.

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