SCORING RULES AND IMPLEMENTATION IN ITERATIVELY UNDOMINATED STRATEGIES

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Abstract. We characterize voting procedures according to the solution that they implement when voters cast ballots strategically, applying iteratively undominated strategies. In elections with three candidates, the Borda Rule is the unique positional scoring rule that satisfies *unanimity* (U) (i.e., elects a candidate whenever it is unanimously preferred) and is *majoritarian after eliminating a worst candidate* (MEW)(i.e., if there is a unanimously disliked candidate, the majority-preferred among the other two is elected). In the larger class of direct mechanism scoring rules, Approval Voting is characterized by a single axiom – it is *majoritarian after eliminating a Pareto dominated candidate* (MEPD)(i.e., if there is a Pareto-dominated candidate, the majority-preferred among the other two is elected). However, it fails a desirable monotonicity property: a candidate that is elected for some preference profile, may lose the election once she gains further in popularity. In contrast, the Borda Rule is the unique direct mechanism scoring rule that satisfies U, MEW and *monotonicity* (MON). Finally, there exists no direct mechanism scoring rule that satisfies both MEPD and MON or Condorcet consistency (CON).

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1. Introduction

Voting procedures allow individual voters to cast ballots that are aggregated to arrive at a collective choice from a set of available alternatives. To compare voting procedures, we ask which alternatives may arise as the outcome of an election when voters cast their ballots strategically, potentially misrepresenting their preferences. Then, for any solution concept that describes voters' behaviour in voting games, we can map voters' preferences to possible election outcomes and thus arrive at a social choice correspondence said to be implemented by the voting procedure.

Ideally, our voting procedure should implement a normatively appealing social choice correspondence under mild assumptions restricting voters' behaviour. Arguably the mildest such restriction is to assume that voters play undominated strategies. Unfortunately, for all finite voting procedures,[1] we face the following

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[1] A finite voting procedure allows each voter to choose from a finite set of admissible ballots, as envisioned by both Gibbard [1973] and Satterthwaite [1975].

impossibility result: with at least three alternatives, any social choice function[2] that can be implemented in undominated strategies is either dictatorial or rules out the election of some candidate a priori. In its original formulation, the result is due to Gibbard [1973] and Satterthwaite [1975] who are concerned with implementation in dominant strategies; Jackson [1992] shows that if we consider finite voting procedures,[3] implementation in dominant strategies is equivalent to implementation in undominated strategies – any social choice function that can be implemented in dominant strategies can be implemented in undominated strategies and vice versa.

In this paper, we will focus on the case of three alternatives where these negative results first arise. Moreover, in light of these results, we content ourselves with implementing social choice correspondences and move to a stronger solution concept, considering implementation in *iteratively* undominated strategies.

Here we are able to derive three main characterisation results. First, in the class of positional scoring rules (including among others Plurality-, Antiplurality- and the Borda-Rule), the Borda Rule is the unique voting procedure implementing a social choice correspondence that satisfies *unanimity* (**U**) (i.e., uniquely selects an alternative whenever it is unanimously preferred) and is *majoritarian after eliminating a worst alternative* (**MEW**) (i.e., if there is a unanimously disliked alternative, the majority-preferred alternative among the other two is uniquely selected).

Second, in the larger class of direct mechanism scoring rules (including e.g. all positional scoring rules as well as Approval Voting), Approval Voting is characterized by a single axiom – it is the unique voting procedure that is *majoritarian after eliminating a Pareto-dominated alternative* (**MEPD**) (i.e., if there is a Pareto-dominated alternative, the majority-preferred alternative among the other two is uniquely selected).

Third, in the class of direct mechanism scoring rules, the Borda Rule is the unique voting procedure implementing a social choice correspondence that satisfies **U**, **MEW** and *monotonicity* (**MON**)(i.e., an alternative that is uniquely selected for some preference profile should still be uniquely selected for a preference profile where every voter ranks this alternative weakly higher).

Three recent papers most closely related to our results are [Dhillon and Lockwood, 2004], [Buenrostro et al., 2013] and [Núñez and Courtin, 2013] who all identify conditions for preferences profiles under which particular scoring rules yield a unique solution in iteratively undominated strategies. Dhillon and Lockwood [2004] consider the Plurality Rule with an arbitrary number of alternatives and provide sufficient and necessary conditions. Buenrostro et al. [2013] consider so called general scoring rules – a set that overlaps with the set of direct mechanism scoring rules that we consider – and provide sufficient conditions. Núñez and Courtin [2013] consider Approval Voting and provide sufficient-and-necessary conditions.

The use of iteratively undominated strategies as solution concept has a long tradition in the theory of voting where it was introduced by Farquharson [1969] under the name of *sophisticated voting*. It is particularly well suited to model strategic behaviour in elections where the number of voters is large relative to the number of available alternatives, as under these conditions voters typically find themselves in

[2]i.e. a single valued social choice correspondence

[3]Jackson's equivalence result is even more general in that he considers bounded mechanisms.

a position where they are not pivotal. As a result, it is easy to sustain any strategy as a best response so that the alternative solution concept of rationalizability has no bite. Similarly, under many intuitive voting procedures, if all voters vote 'in favour' of some arbitrary alternative, it should be elected and an individual deviation should be of no effect. But then, all alternatives are implemented in (some) Nash-equilibrium.

To restrict the set of alternatives implemented in Nash-equilibrium, we could consider refinements, such as undominated [Palfrey and Srivastava, 1991] or trembling-hand perfect equilibrium. However, these refinements leave a second problem of (pure strategy) Nash-implementation unaddressed. To illustrate this, consider two voters who both prefer a over b over c and who have to choose an alternative using the Antiplurality Rule where each voter votes against one alternative and the alternative with the least number of votes is chosen. Then, subject to specifying a tiebreaking procedure, it is easy to see that in any Nash-equilibrium one voter will vote for b while the other votes for c, so that the commonly preferred alternative a is chosen. However, both voters face a coordination problem in that it is unclear who should vote for b and who should vote for c. Hence, while a is the unique outcome in any Nash-equilibrium, it remains doubtful whether miscoordination may not in the end help b or c to arrive at tie with a and hence be potentially chosen.[4]

Many authors have studied the implementation in iteratively undominated strategies. If in each iteration only strictly dominated strategies are removed, Börgers [1995] shows that only dictatorial social choice functions can be implemented, unless we restrict the set of possible preference profiles to exclude cases where voters preferences are identical. If weakly dominated strategies are removed as well (as we will assume throughout this paper), Moulin [1979] shows that there exist voting procedures that implement anonymous and Pareto efficient social choice functions. Abreu and Matsushima [1994] show that any social choice function may be implemented, when voters can be fined for what are identified as misrepresentations of preferences. For that, they require a large strategy space where each voter reports not just her own preferences and the preferences of some 'neighbour', but also in total K preference profiles, i.e. tuple of all voters' preferences, where K has to be chosen arbitrarily large in order to allow fines to become arbitrarily small. While this allows them to derive a remarkably permissive implementation result, the sheer size of the strategy space (as well as the introduction of fines) rules out the use of their mechanism for elections with many voters.

In order to restrict attention to voting procedures that can be readily applied in practice, we limit our analysis to rules where the size of the strategy space is no larger than the number of possible preference relations that a voter may hold; that is, we consider voting procedures that can be interpreted as direct mechanisms. Moreover, we will consider scoring rules, for which Myerson [1995] provides an axiomatisation based on *reinforcement* and *overwhelming majority*: Consider a voting procedure where each voter has access to the same set of strategies, i.e. can cast the same admissible ballots, and where the set of such strategies is independent

[4]To rule out examples such as this, we could consider mixed-strategy equilibria and demand that any outcome sustained by such an equilibrium, is contained in the set of alternatives chosen by the social choice correspondence. However, such an analysis requires that voters' preferences over *lotteries* of alternatives are common knowledge, which constitutes an additional, strong, assumption.

of the number of voters participating in the election. Reinforcement then demands that if ballots are evaluated for two separate districts and in each district the same alternative is elected, then in a joint district, this alternative should be elected as well. Overwhelming majority demands that if some group of voters, or rather the ballots that they cast, are replicated sufficiently often, the election outcome in the general election has to agree with the outcome of an election where only ballots of the overwhelmingly large, replicated group are considered.[5]

Together with the requirement that a voting procedure be neutral (with respect to a relabelling of alternatives) and anonymous (with respect to a relabelling of voters), these axioms uniquely characterize scoring rules in the class of *all* voting procedures. Hence, unless one is willing to give up on any of these desirable properties, restricting our attention to scoring rules comes at no further loss of generality.

The paper is organised as follows. Section 2 defines voting games and their solution by iterative elimination of dominated strategies. Section 3 defines normative criteria for social choice correspondences. Section 4 characterizes scoring rules with respect to the social choice correspondences that they implement. Section 5 concludes.

2. Technicalities

2.1. **Candidates and voters.** Throughout this paper, we consider a set of three candidates (or alternatives) $A = \{a, b, c\}$ and a finite set of voters I with generic element i. Each voter's preferences are assumed to be given by a strict linear order $\succ_i$ on A. In consequence, there are six distinct sets of voters, characterized by their preferences that we denote $I_{xyz} = \{i \in I | x, y, z \in A, x \succ_i y \succ_i z\}$ and whose generic element we refer to as i_{xyz}. A preference profile is denoted as $\succ_I = (\succ_i)_{i\in I}$.

2.2. **Scoring rules.** Scoring rules allow each voter i to cast a ballot $v_i = (v_i^a, v_i^b, v_i^c)$ from the same set of admissible ballots $V \subset \mathbb{R}^3$. We assume that ballots are *neutral* with respect to a relabelling of candidates; formally, for any admissible ballot $v_i = (k, l, m) \in V$, each permutation of v_i is also an admissible ballot. A ballot is called an abstention if it takes the form $v_i = (k, k, k)$.

Using Cartesian products, we define $V^0 = \prod_{i\in I} V$ and $V_{-i}^0 = \prod_{j\neq i} V$ and denote generic elements as v and v_{-i}. We refer to $v \in V^0$ as a ballot profile and denote the associated score of some candidate x as $|v^x| = \sum v_i^x$. For an opposing ballot profile $v_{-i} \in V_{-i}^0$ we define $|v_{-i}^x| = \sum v_{j\neq i}^x$.

A candidate wins the election if her score is higher than any other candidate's score. To deal with ties, we rely on the report of a tiebreaker, labelled t, who has to chose a strict linear order $\rhd$ on A, where $\triangleright$ denotes the set of such orders.[6] Then, for given v and $\rhd$, candidate x wins the election whenever she has a weakly higher

[5]For a detailed description of both axioms, see [Myerson, 1995].

[6]If one objects to the introduction of an additional player, another option would be to break ties by a multiplayer version of "matching pennies": ask each voter to report a number $t_i \in \{0, 1, .., 5\}$, set $t = \sum t_i$ mod 6 and let each of the 6 possible outcomes $t = \{0, 1, ..., 5\}$ correspond to one of the 6 possible linear orders $\rhd \in \triangleright$. For our purposes, the two approaches are essentially equivalent, as the tiebreaker will be assumed to be indifferent, so that neither the tiebreaker's set of possible reports, nor the voters' set of possible reports t_i can be reduced using elimination of weakly dominated strategies.

score than all other candidates and, in case of a tie, is ranked first according to $\triangleright$. Formally, x wins if and only if

$$\forall y \neq x : |v^x| \geq |v^y| \text{ and } |v^x| = |v^y| \implies x \triangleright y.$$

Note that for any reported ballot profile v and a report by the tiebreaker $\triangleright$, there exists a unique winner. If we would refrain from breaking ties in a deterministic manner, outcomes would either be set-valued or take the form of a lottery over alternatives. To analyse voting games induced by a scoring rule, we would then have to amend voters preferences, for example to include preferences over sets of candidates[7] or by specifying von Neumann - Morgenstern utility functions. Instead we opt for deterministic tiebreaking which allows us to base our analysis exclusively on ordinal preferences over candidates.

We will consider scoring rules that can be interpreted as direct mechanisms, i.e. rules where the size of voters' strategy space is bounded by the number of voters' types. A scoring rule as described above is a *direct mechanism scoring rule* if, after the removal of abstentions,[8] we have $|V| \leq 6$. For *positional scoring rules*, V is taken to be the set of permutation of $(1, s, 0)$, where $s \in [0, 1]$ is a fixed parameter that characterizes the rule. The most notable positional scoring rules are the *Plurality Rule*, corresponding to $s = 0$, the *Antiplurality Rule* ($s = 1$) and the *Borda Rule* ($s = \frac{1}{2}$).

Other direct mechanism scoring rules, are rules that allow voters to either vote for one candidate or split their vote between two – we refer to such rules as *vote-splitting scoring rules*. Formally, for a vote-splitting scoring rule, V consists of all permutations of $(s, s, 0)$ and $(1 - s, 0, 0)$, $s \in [0, 1]$. If $s = \frac{1}{3}$, voters have a fixed budget of points that they can award to one candidate or split between two. If $s \neq \frac{1}{3}$, splitting is either rewarded or punished by changes in the budget. The most notable such rule is *Approval Voting*, where $s = \frac{1}{2}$. Note that $s = 1$ is equivalent to the Antiplurality rule, while $s = 0$ corresponds to the Plurality Rule. Hence, both Approval Voting and the Borda Rule can be thought of as 'half-way' between the Plurality and Antiplurality Rule. Our first result will show that positional and vote-splitting scoring rules are essentially the only direct mechanism scoring rules.

In a slight abuse of notation, we will at times identify a scoring rule and the set of admissible ballots and denote both by V.

2.3. **Voting games.** Together, the set of candidates, voters' preferences, a scoring rule and a tiebreaker - assumed to be indifferent between candidates - give rise to a complete information voting game $\Gamma(\succ_I, V^0)$ with a set of players $I \cup \{t\}$. In each game $\Gamma(\succ_I, V^0)$, a strategy profile $(v, \triangleright) \in V^0 \times \triangleright$ determines a unique outcome $g(v, \triangleright) \in A$.

We will also consider restricted games $\Gamma(\succ_I, V')$, where each voter's strategies are restricted to some set $V'_i \subseteq V$ and the space of ballot profiles is denoted $V' = \prod_{i \in I} V'_i$.

[7]Equivalently to the approach followed here, we could refrain from breaking ties and extend each preference relation $\succ_i$ to pairs of subsets of A, by defining for all $A', A'' \subset A$

$$A' \succ_i A'' \quad :\iff \quad A' \neq A'' \text{ and } \forall x \in A' \backslash A'', y \in A'' : x \succ_i y \text{ and } \forall x \in A', y \in A'' \backslash A' : x \succ_i y.$$

[8]Since abstentions represent dominated strategies, removing them will not affect our analysis.

Accordingly, the space of opponents' ballot profiles is denoted $V'_{-i} \prod_{j \neq i} V'_j$. Where all voters $i \in I_{xyz}$ have the same (restricted) strategy set, we denote it $V'_{xyz} = V'_i$.

2.4. **Iteratively Undominated Strategies.** In particular, we will focus on restricted games where weakly dominated strategies have been removed.

Definition 1. A strategy $v_i \in V'_i$ is *weakly dominated* in $\Gamma(\succ_I, V')$ if there exists $w_i \in V'_i$ such that for all $v_{-i} \in V'_{-i}$, $\triangleright \in \triangleright$

$$g(w_i, v_{-i}, \triangleright) \succ_i g(v_i, v_{-i}, \triangleright) \text{ or } g(w_i, v_{-i}, \triangleright) = g(v_i, v_{-i}, \triangleright)$$

with $g(w_i, v_{-i}, \triangleright) \succ_i g(v_i, v_{-i}, \triangleright)$ for at least one $v_{-i} \in V'_{-i}$ and $\triangleright \in \triangleright$.

Strategies $\triangleright \in \triangleright$ are never dominated, as the tiebreaker is assumed to be indifferent between all outcomes $g(v, \triangleright) \in A$. Hence, in iteratively removing dominated strategies, we can focus on voters $i \in I$. First, we define the set of undominated strategies as $V_i^1 = V \backslash \{v_i \in V | v_i \text{ is weakly dominated in } \Gamma(\succ_I, V^0)\}$. We will make use of the following useful fact.

Fact 1. In approval voting games, the set of undominated strategies V_i^1 for a voter of type i_{xyz} consists of all ballots $v_i \in V$ such that $v_i^x = 1/2$ and $v_i^z = 0$ [Brams and Fishburn, 1978]. For positional scoring rule voting games, i_{xyz}'s undominated strategies are all ballots $v_i \in V$, such that $v_i^x \geq s$ and $v_i^z \leq s$ (see Proposition 1 in [Buenrostro et al., 2013]).

Next, we move to the iterative elimination of dominated strategies and define

$$V_i^{m+1} = V_i^m \backslash \{v_i \in V_i^m | v_i \text{ is weakly dominated in } \Gamma(\succ_I, V^m)\}, \text{ for } m \in \mathbb{N}.$$

Clearly, $V_i^{m+1} \neq \varnothing$, as it is impossible for all strategies in V_i^m to be dominated by one another.[9] Also, as V is finite, there exists some $\overline{m}$, such that no further restrictions are possible; $V^m = V^{\overline{m}}$, for all $m \geq \overline{m}$. This leads us to the following solution of a voting game.

Definition 2. For a voting game $\Gamma(\succ_I, V^0)$ we define its *solution in iteratively undominated strategies* as the set of possible outcomes after iteratively eliminating all weakly dominated strategies, and denote it as

$$S(\succ_I, V) = \{x \in A | \exists v \in V^{\overline{m}} : \forall y \in A : |v^x| \geq |v^y|\}.$$

We say that V implements the *social choice correspondence* $S(\cdot, V)$ that maps preference profiles onto subsets of A.

2.5. **Order independence and elimination of duplicate strategies.** In solving games via an iterative elimination of dominated strategies, we followed Moulin [1979] in that we eliminated *all* weakly dominated strategies when moving from V^m to V^{m+1}.[10] This raises the question, whether a different order of elimination, where only some individuals' dominated strategies are eliminated at each step, might yield a different solution.

Fortunately, Marx and Swinkels [1997] assure us that this is not the case. More precisely, their Theorem 1 ensures that once we reach a restricted game $\Gamma(\succ_I, V')$ such that no further strategy can be eliminated based on weak dominance, $\Gamma(\succ_I, V')$

[9] Recall that $\succ_i$ is a strict linear order.

[10] Other authors in the context of voting theory, most notably Farquharson [1969], have used the same solution concept under the name of 'sophisticated voting'.

will be equivalent to $\Gamma(\succ_I, V^{\overline{m}})$ up to the elimination of duplicate strategies and the renaming of strategies. In particular, the set of possible outcomes of both games will be the same.

This is because, in our voting games, the elimination of dominated strategies satisfies what Marx and Swinkels [1997] call 'transference of decisionmaker indifference': whenever a voter i, for a given opposing strategy profile, is indifferent between outcomes $g(v_i, v_{-i}, \rhd)$ and $g(v_i', v_{-i}, \rhd)$, then so is every other player. This is of course satisfied, as i will only be indifferent if both outcomes coincide.[11]

Moreover, whether in the process of iterative elimination, we chooses at some point to eliminate a single (of multiple) duplicate strategies, will be of no effect; the game $\Gamma(\succ_I, V')$ that we reach eventually will be equivalent to $\Gamma(\succ_I, V^{\overline{m}})$ up to the elimination of duplicate strategies and the renaming of strategies.

To see this, suppose that in the game $\Gamma(\succ_I, V^m)$ there are two duplicate but undominated strategies $v_i, \tilde{v}_i \in V_i^m$, of which we choose to eliminate only $\tilde{v}_i$ when moving to the next restricted game. If $\tilde{v}_i$ could at some step be instrumental in eliminating another strategy v_j based on weak dominance, the remaining duplicate v_i will suffice to eliminate v_j. If v_i was eliminated based on weak dominance before it becomes instrumental in eliminating v_j, $\tilde{v}_i$ would have been eliminated as well.

3. Axioms

We want to compare and characterize scoring rules according to the social choice correspondences that they implement. In particular, we ask for which preference profiles the induced voting games have a unique solution in iteratively undominated strategies – and which outcomes are selected in that case. A minimal and prominent requirement is unanimity.

Definition 3. A scoring rule V is said to satisfy *Unanimity* (**U**), if for any preference profile $\succ_I$ such that $I = I_{abc} \cup I_{acb}$, we have $S(\succ_I, V) = \{a\}$.

Where there is no universal agreement, we have to weigh some voters' preferences against others', in order to choose an alternative. In the case of two alternatives, fairness and efficiency force us to accept simple majority as the guiding principle,[12] but when the number of alternatives grows, it is unclear how this principle should be adjusted.

However, if one of three alternatives is unanimously agreed to be the worst, we are essentially in a situation with just two relevant alternatives, so that a simple majority should suffice to determine the optimal alternative. We can formalize this idea as follows.

Definition 4. Consider an arbitrary preference profile $\succ_I$ such that $I = I_{abc} \cup I_{bac}$. A scoring rule V is said to be *Majoritarian after Eliminating a Worst Alternative* (**MEW**), if $|I_{abc}| > |I_{bac}|$ implies $S(\succ_I, V) = \{a\}$.

[11]Indifference of the tiebreaker does not transfer to indifference of other voters. However, this is unproblematic, as the principle of 'transference of decisionmaker indifference' is only required to hold for players whose strategies are eliminated (see Definition 2 in [Marx and Swinkels, 1997]).

[12]See May [1952] who provides an axiomatisation of the Majority Rule. His symmetry axioms can be seen as an embodiment of fairness, while the *positive responsiveness* axiom may be seen as a requirement of efficiency.

$$\begin{tikzcd}
 & & \mathbf{U} \\
\mathbf{CON} \arrow[r, Rightarrow] & \mathbf{MEPD} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & \\
 & & \mathbf{MEW}
\end{tikzcd}$$

FIGURE 1. Logical relations between intra-profile axioms

A similar situation arises, when one of three alternatives is unanimously agreed to be worse than some other alternative. Again, one might think that the former, Pareto dominated, alternative should be disregarded and the decision between the remaining two alternatives should be made by simple majority.

Definition 5. Consider an arbitrary preference profile $\succ_I$ such that $I = I_{abc} \cup I_{acb} \cup I_{bac}$. A scoring rule V is said to be *Majoritarian after Eliminating a Pareto Dominated Alternative* (**MEPD**), if for $|I_{abc}| + |I_{acb}| > |I_{bac}|$, we have $S(\succ_I, V) = \{a\}$, while for $|I_{abc}| + |I_{acb}| < |I_{bac}|$, we have $S(\succ_I, V) = \{b\}$.

The formal definition reveals what might be a controversial property of **MEPD**: some alternative b might be chosen by the social choice correspondence $S(\cdot, V)$ based on its majority support over another alternative a, even though it may only be a that, according to **MEPD**, forces us to eliminate c, based on Pareto dominance.

In defence of **MEPD**, observe that it unifies both preceding axioms, i.e. **MEPD** implies both **MEW** and **U**. Moreover, it is implied by another, well known requirement, formulated by the Marquis de Condorcet, according to which an alternative should be chosen whenever it is supported by a majority against any other alternative.[13]

Definition 6. Consider an arbitrary preference profile $\succ_I$. A scoring rule V is said to be *Condorcet consistent* (**CON**), if $S(\succ_I, V) = \{a\}$ whenever

$$|I_{abc}|+|I_{acb}|+|I_{bac}| > |I_{bca}|+|I_{cab}|+|I_{cba}| \quad \text{and} \quad |I_{abc}|+|I_{acb}|+|I_{cab}| > |I_{cba}|+|I_{bac}|+|I_{bca}|.$$

To see that **CON** implies **MEPD**, observe that whenever b is supported by a majority against a, and a Pareto dominates c, b will also be supported by a majority against c and should therefore be chosen according to **CON**.

Figure 1 presents the logical relations between the axioms described so far. Note that they are all intra-profile axioms, i.e. they all concern the behaviour of a social choice correspondence within given preference profiles. The next axiom concerns its behaviour across profiles. For that, consider an arbitrary profile $\succ_I = (\succ_i)_{i \in I}$. Another profile $\succ'_I = (\succ'_i)_{i \in I}$ is said to be an a-monotonic transformation of $\succ_I$, iff

$$\forall i \in I: \quad a \succ_i b, c \Longrightarrow a \succ'_i b, c \quad \text{and} \quad b \succ_i c \Longleftrightarrow b \succ'_i c,$$

i.e. such that a is more popular under $\succ'_I$, while the ordering of b and c remains unchanged.[14] Then, if a is the unique solution under $\succ_I$, it should remain so under $\succ'_I$.

[13]Note that Condorcet famously pointed out that such an alternative may not exist when pairwise majority comparisons yield a cycle, cf. de Condorcet [1785] p. lxi.

[14]The requirement that the ranking between b and c remains unchanged, makes the following notion of monotonicity weaker than Maskin-monotonicity, which is required for Nash-implementation.

Definition 7. A scoring rule V is said to satisfy *Monotonicity* (**MON**), if for any preference profile $\succ_I$ and an a-monotonic transformation $\succ'_I$ we have

$$S(\succ_I, V) = \{a\} \implies S(\succ'_I, V) = \{a\}.$$

Monotonicity is particularly important where candidates are engaged in electoral competition, i.e. where they can choose a policy platform and thereby affect their position in voters' rankings of candidates. A violation of monotonicity could create perverse incentives for candidates – a candidate may then increase her chance of election by adjusting her platform with the *only* effect to hurt some group within the electorate, moving her down in that groups' rankings of candidates (while leaving everyone's ranking of the other candidates unchanged). Then, under a violation of monotonicity, it could be that the candidate is uniquely selected by only after the change in platform, i.e. after she has lost in popularity.

4. Results

Our first result maps out the class of scoring rules under consideration, by showing that positional and vote-splitting scoring rules are essentially the only direct mechanism scoring rules; the only other scoring rules are slight variations of the Plurality and Antiplurality Rule. For that, we normalize ballots in a way that exchanges some strategies for duplicate counterparts.

Theorem 1. *Consider a direct mechanism scoring rule V. Then up to the elimination of abstentions and a normalization of ballots, one of the following four cases applies. The set of admissible ballots V consists of*

(1) *all permutations of* $(1, s, 0)$, $s \in [0, 1]$.

(2) *all permutations of* $(s, s, 0)$ *and* $(1 - s, 0, 0)$, $s \in [0, 1]$.

(3) *all permutations of* $(1, 0, 0)$ *and* $(s, 0, 0)$, $s \in [0, 1]$.

(4) *all permutations of* $(1, 1, 0)$ *and* $(s, s, 0)$, $s \in [0, 1]$.

The intuition behind Theorem 1 is straightforward. Suppose V contains an admissible ballot b with three distinct entries. Since V is neutral with respect to a relabelling of candidates, the corresponding 6 permutations of b are also included in, and exhaust, V. Normalizing then yields case (1). If V contains a ballot b with two identical entries, it also contains all 3 of its permutations. This leaves room for another ballot b' which can have only 3 permutations itself, i.e. must contain two identical entries as well. Normalizing b and b', as well as their permutations yields one of the cases (2)-(4). A slightly more formal proof is found in the Appendix.

Within the class of direct mechanism scoring rules, we will show that the Borda Rule, and the social choice correspondence implemented by it, occupy a particularly prominent position. For that, the next two results establish sufficient-and-necessary conditions on preference profiles for the associated Borda Rule voting games to have a unique solution in iteratively undominated strategies.

Theorem 2. *Consider a Borda Rule voting game $\Gamma(\succ_I, V^0)$. A candidate $x \in A$, is the unique solution, i.e. $S(\succ_I, V) = \{x\}$, if we can label candidates so that one of the* *following three conditions is satisfied:*

$$
\begin{aligned}
&(1) && |I_{xyz}| > |I_{xzy}| + |I_{yxz}| + 2|I_{yzx}| + 2|I_{zxy}| + 2|I_{zyx}|, \\
&(2)\ \textit{or} && |I_{xyz}| > |I_{xzy}| + |I_{yxz}| + 2|I_{yzx}| + 2|I_{zxy}| + 2|I_{zyx}| - 1 \quad \textit{and} \quad |I_{zxy}| > |I_{yxz}|, \\
&(3)\ \textit{or} && |I_{xyz}| > |I_{xzy}| + |I_{yxz}| + 2|I_{yzx}| + 2|I_{zxy}| + 2|I_{zyx}| - 2 \quad \textit{and} \quad |I_{xzy}| > 0.
\end{aligned}
$$

The proof for all three cases proceeds as follows. We first show that either y or z can be ruled out as an element of $S(\succ_I, V)$, as after a few rounds of eliminating dominated strategies we have $|v^x| > |v^y|$ or $|v^x| > |v^z|$. Then, the election is effectively over x and one remaining alternative candidate, and x wins, as it is supported by a majority. We present the proof for case (1) here, and relegate cases (2) and (3) to the Appendix.

Assume (1) holds. After eliminating dominated strategies, we know by Fact 1 that

$$\min_{v \in V^1} |v^x| - |v^z| = 1/2|I_{xyz}| - 1/2|I_{xzy}| - 1/2|I_{yxz}| - |I_{yzx}| - |I_{zxy}| - |I_{zyx}| > 0,$$

so that z is ruled out as an outcome. But then, in the game $\Gamma(\succ_I, V^1)$, for any voter i who prefers x over y, $v_i = (v_i^x, v_i^y, v_i^z) = (1, 0, \frac{1}{2})$ is a best reply for every opposing strategy profile $(v_{-i}, \rhd) \in V_{-i}^1 \times \rhd$ as it maximizes the impact that i has on $|v^x| - |v^y|$. If another ballot $\tilde{v}_i \neq v_i$ is also a best reply against every $(v_{-i}, \rhd)$, then $\tilde{v}_i$ is a duplicate strategy. If on the other hand $\tilde{v}_i$ is a worse reply than v_i against some $(v_{-i}, \rhd)$, it is dominated and hence eliminated as we move to V^2.

To determine the possible outcomes in $\Gamma(\succ_I, V^2)$, we can assume that all $i \in I_{xyz} \cup I_{xzy} \cup I_{zxy}$ cast ballot $v_i = (1, 0, 1/2)$ - any other remaining strategy in V_i^2 would be a duplicate strategy and produce the same outcome. But then, x is the unique outcome after two rounds of eliminating dominated strategies, as by condition (1)

$$|v^x| \geq |I_{xyz}| + |I_{xzy}| + |I_{zxy}| > |I_{yxz}| + |I_{yzx}| + |I_{zyx}| \geq |v^y|.$$

This completes the proof for case (1); the proof for cases (2) and (3) is found in the appendix. The next Theorem shows that the conditions of Theorem 2 are also necessary.

Theorem 3. *Consider a Borda Rule voting game $\Gamma(\succ_I, V^0)$. No candidate can be excluded as a winner, i.e. $S(\succ_I, V) = A$, if for any labelling of candidates $x, y, z \in A$ the following three conditions are satisfied:*

$$
\begin{aligned}
&(1) && |I_{xyz}| \leq |I_{xzy}| + |I_{yxz}| + 2|I_{yzx}| + 2|I_{zxy}| + 2|I_{zyx}| \\
&(2)\ \textit{and} && |I_{xyz}| = |I_{xzy}| + |I_{yxz}| + 2|I_{yzx}| + 2|I_{zxy}| + 2|I_{zyx}| \implies |I_{zxy}| \leq |I_{yxz}| \\
&(3)\ \textit{and} && |I_{xyz}| \geq |I_{xzy}| + |I_{yxz}| + 2|I_{yzx}| + 2|I_{zxy}| + 2|I_{zyx}| - 1 \implies |I_{xzy}| = 0.
\end{aligned}
$$

Intuitively, under the conditions of Theorem 3, each group of voters I_{xyz} is small relative to the other groups, bringing us close to a balanced profile where each group is of the same size. For such a balanced profile, it is clear that no outcome can be ruled out.

The proof rests on a Lemma, which shows that if each I_{xyz} is small enough, no strategies beyond the initially dominated ones are eliminated in the process of iterated elimination.

Lemma 1. *Suppose that for any labelling of candidates $x, y, z \in A$ we have*

$$|I_{xyz}| < |I_{xzy}| + |I_{yxz}| + 2|I_{yzx}| + 2|I_{zxy}| + 2|I_{zyx}| - 2.$$

Then for the Borda Rule voting game $\Gamma(\succ_I, V^0)$*, the elimination of dominated strategies stops after one round so that* $\Gamma(\succ_I, V^1) = \Gamma(\succ_I, V^{\overline{m}})$*. Moreover,* $S(\succ_I, V) = A$*.*

The proof of Theorem 3 then delineates the remaining cases where some I_{xyz} may be larger than assumed in Lemma 1 so that some initially undominated strategies are eliminated, yet the process of elimination stops before any outcome can be ruled out. Both the proof of Lemma 1 and remaining proof of Theorem 3 require a large number of case distinctions and are relegated to the appendix.

Corollary 1. *The Borda Rule satisfies both* **U** *and* **MEW***.*

Proof. Assume that $I = I_{abc} \cup I_{acb}$. Without loss of generality, we can assume $|I_{abc}| \geq |I_{acb}|$. By Theorem 2, a is the unique solution as $|I_{abc}| > |I_{acb}| - \mathbb{1}_{\{|I_{acb}|>0\}} = |I_{bac}| + |I_{acb}| + 2|I_{bca}| + 2|I_{cab}| + 2|I_{cba}| - \mathbb{1}_{\{|I_{acb}|>0\}}$.

Assume on the other hand that $I = I_{abc} \cup I_{bac}$ and $|I_{abc}| > |I_{bac}|$. Then by Theorem 2, a is the unique solution, as $|I_{abc}| > |I_{bac}| = |I_{bac}| + |I_{acb}| + 2|I_{bca}| + 2|I_{cab}| + 2|I_{cba}|$. □

The above corollary overlaps with results in Buenrostro et al. [2013] who provide sufficient conditions for scoring rule voting games to be dominance solvable, i.e. have a unique solution in iteratively undominated strategies. The corollary extends beyond their Theorem 1 and Theorem 2, in that it includes the case $I = I_{abc} \cup I_{acb}$, $|I_{abc}| = |I_{acb}|$, i.e. we show that a unanimously preferred candidate a is the unique solution even if the electorate is split in half. What might be more remarkable though, is the exceptional position among positional scoring rules that Corollary 1 grants to the Borda Rule:

Theorem 4. *The Borda Rule is the unique positional scoring rule that satisfies* **U** *and* **MEW***. In particular, positional scoring rules with* $s < \frac{1}{2}$ *violate* **U***, while positional scoring rules with* $s > \frac{1}{2}$ *violate* **MEW***.*

The proof can be found in the appendix. To understand the intuition behind Theorem 4, assume that $s > 1/2$ and everyone agrees that c is the worst alternative. Furthermore, if the groups I_{abc} and I_{bac} are roughly of the same size, it is possible that a and b receive roughly the same score so that a single voter is pivotal. In such a situation, awarding a score of s to the least preferred alternative c – and a score of zero to the second best alternative a or b – may be undominated, or even a unique best response, as it tips the election in favour of the most preferred alternative. Yet, if awarding a score of s to c cannot be ruled out based on weak dominance, c may win with an average score of $s > 1/2$ while a and b are tied with an average score of about $1/2$.

Similarly, assume that that $s < 1/2$ and that alternative a is unanimously preferred. If the electorate is split in half between the groups I_{abc} and I_{acb} and every voter supports their second best alternative by awarding it a score of one, a receives an average score of at most $s < 1/2$ while b and c will be tied with an average score of $1/2$. An individual who deviates and supports a would then hand the election to their least preferred candidate. Hence, for each voter, supporting their second best alternative is undominated – as long as everyone else may still support their second best alternative. But then supporting the second best alternative can never be eliminated based on weak dominance which establishes both b and c as element of the solution $S(\succ_I, V)$.

In light of Theorem 4, it is natural to ask whether there exist other direct mechanism scoring rules, beyond the Borda Rule that simultaneously satisfy unanimity and are majoritarian after eliminating a worst candidate. The most prominent direct mechanism scoring rule not covered by Theorem 4 is Approval Voting, for which Núñez and Courtin [2013] provide necessary-and-sufficient conditions for the associated voting games to be dominance solvable, i.e. to have a unique solution in iteratively undominated strategies. In fact, we find that Approval Voting satisfies even the stronger axiom of being majoritarian after eliminating a Pareto dominated candidate – and that it is the only direct mechanism scoring rule that satisfies it.

Theorem 5. *Approval Voting is the unique direct mechanism scoring rule that satisfies* **MEPD**. *In particular, vote-splitting scoring rules with* $s < \frac{1}{2}$ *and scoring rules where* V *consists of all permutations of* $(1,0,0)$ *and* $(s,0,0)$ *violate* **U**, *while vote-splitting scoring rules with* $s > \frac{1}{2}$ *and scoring rules where* V *consists of all permutations of* $(1,1,0)$ *and* $(s,s,0)$ *violate* **MEW**.

The fact that the Borda Rule, while satisfying **U** and **MEW**, fails to satisfy **MEPD**, follows from Theorem 3. For example, consider a preference profile $\succ_I$ where $I = I_{abc} \cup I_{acb} \cup I_{bac}$ and $|I_{abc}| = |I_{acb}| = |I_{bac}| = n \geq 2$. Then by Theorem 3 $S(\succ_I, V) = A$, while **MEPD** requires a to be the unique solution. All other positional scoring rules violate either **U** or **MEW** and hence also **MEPD**, see Theorem 4.

In order to show that Approval Voting satisfies **MEPD** consider a preference profile where a Pareto dominates c, i.e. such that $I = I_{abc} \cup I_{acb} \cup I_{bac}$. Then after eliminating dominated strategies, no voter awards a higher score to c than to a (see Fact 1), so that for any, $v \in V^1$, the score of a is weakly larger than the score of c.

Moreover, if there exists a voter $i \in I_{abc}$, she will vote either $(1/2, 1/2, 0)$ or $(1/2, 0, 0)$, thereby ensuring that $|v^a| > |v^c|$ and ruling out outcome c after one round of elimination. In the next step, each voter will award a score $s = 1/2$ to her preferred among the remaining candidates a and b and a score of zero to the other candidate. Then, the candidate supported by a majority is the only remaining outcome after two rounds of elimination of dominated strategies.

If on the other hand $|I_{abc}| = 0$, so that $I = I_{acb} \cup I_{bac}$, we have to consider two cases. First consider $|I_{acb}| > |I_{bac}|$, where a is preferred by a majority over b. Then, for any $v \in V^1$, we have $|v^a| \geq \frac{|I_{acb}|}{2} > \frac{|I_{bac}|}{2} = |v^b|$ so that b is ruled out as an outcome. In the next step, each voter will support a among the two remaining candidates, so that a is the only remaining outcome after two rounds of elimination.

Finally, if $I = I_{acb} \cup I_{bac}$ and $|I_{bac}| > |I_{acb}|$, we know that for any $v \in V^1$, $|v^b| = \frac{|I_{bac}|}{2} > \frac{|I_{acb}|}{2} \geq |v^c|$, such that c is ruled out as an outcome. In the next step, every voter will support either a or b over the other, so that the majority candidate b is the only remaining outcome after two rounds of elimination of dominated strategies.

It remains to show that no other direct mechanism scoring rule satisfies **MEPD**. For that, the reader is referred to the Appendix.

We are now left with only two direct mechanism scoring rules that satisfy **U** and **MEW**, namely the Borda Rule and Approval Voting where only the latter satisfies the even stronger axiom **MEPD**. However, Approval Voting fails monotonicity, as can be seen in the following example.

Example 1. Consider a preference profile $\succ_I$ where $I = I_{abc} \cup I_{bac} \cup I_{cab}$ and

$$|I_{abc}| = 2, \qquad |I_{bac}| = 4, \qquad |I_{cab}| = 3.$$

After eliminating dominated strategies, it is clear that b will have a score of at least $\frac{|I_{bac}|}{2} = 2$, while the score of c is equal to $\frac{|I_{cab}|}{2} < 2$ (see Fact 1). This reduces the game further, to an election between a and b, which a wins with a score of $|v^a| = \frac{|I_{abc}|+|I_{cab}|}{2} = \frac{5}{2} > 2 = \frac{|I_{bac}|}{2} = |v^b|$. Hence a is the unique solution of $\Gamma(\succ_I, V^0)$.

But, if a increases in popularity, so that we now have $\succ'_I$ with $I = I'_{abc} \cup I'_{bac} \cup I'_{cab}$ and $|I'_{abc}| = |I'_{bac}| = |I'_{cab}| = 3$, candidate c is not sure to lose against b so that the game cannot be reduced to an election between a and b. No other candidate is sure to lose either, so that by results in Núñez and Courtin [2013], we know that $\Gamma(\succ'_I, V)$ is not dominance solvable, i.e. has no unique solution in iteratively undominated strategies.[15]

In contrast to Approval Voting, the Borda Rule satisfies monotonicity:

Theorem 6. *The Borda Rule is the unique direct mechanism scoring rule that satisfies* **U**, **MEW** *and* **MON**.

Proof. In light of Theorem 4 and 5 as well as Example 1, it remains to show that the Borda Rule satisfies monotonicity. For that, assume that some candidate, say a, is the unique solution in $\Gamma(\succ_I, V^0)$. Then we know from Theorem 2 and 3 that, up to relabelling of candidates b and c, one of the three conditions are satisfied

(1) $\quad |I_{abc}| > |I_{acb}| + |I_{bac}| + 2|I_{bca}| + 2|I_{cab}| + 2|I_{cba}|,$

(2) or $\quad |I_{abc}| > |I_{acb}| + |I_{bac}| + 2|I_{bca}| + 2|I_{cab}| + 2|I_{cba}| - 1 \quad$ and $\quad |I_{cab}| > |I_{bac}|,$

(3) or $\quad |I_{abc}| > |I_{acb}| + |I_{bac}| + 2|I_{bca}| + 2|I_{cab}| + 2|I_{cba}| - 2 \quad$ and $\quad |I_{acb}| > 0.$

Note that as we move to an a-monotonic transformation of $\succ_I$, this

- weakly increase $|I_{abc}|$,
- weakly decrease $|I_{acb}| + |I_{bac}| + 2|I_{bca}| + 2|I_{cab}| + 2|I_{cba}|$,
- and weakly relaxes the inequality $|I_{acb}| > 0$.

Hence, if initially conditions (1) or condition (3) were satisfied, they continue to hold, so that a is still the unique solution. If initially only condition (2) was satisfied, the inequality $|I_{cab}| > |I_{bac}|$ could cease to hold when moving to an a-monotonic transformation of $\succ_I$

- as $|I_{bac}|$ increase (some i moves from I_{bac} to I_{abc}),
- or $|I_{cab}|$ shrinks (some i moves from I_{cab} to I_{acb}).

However, then in both cases (1) will be satisfied, as $|I_{acb}|+|I_{bac}|+2|I_{bca}|+2|I_{cab}|+2|I_{cba}|$ is decreased by one. In either case, a remains the unique solution. □

We conclude this section with two impossibility results.

Corollary 2. *No social choice correspondence that satisfies both* **MEPD** *and* **MON** *can be implemented by a direct mechanism scoring rule.*

Corollary 3. *No social choice correspondence that satisfies* **CON** *can be implemented by a direct mechanism scoring rule.*

[15] For an Approval Voting game to have a unique solution, there has to be an alternative that is ranked first more often than some other alternative is ranked first or second, see Núñez and Courtin [2013].

The first impossibility result is an immediate implication of Theorem 5 and Example 1. The second impossibility follows from Theorem 5 and a result by Peress [2008] who shows that even when a strict Condorcet winner exists, Approval Voting allows for undominated Nash-equilibria where some other alternative is elected – such equilibrium strategies are never eliminated in the process of iterative elimination of dominated strategies.

5. Conluding Remarks

While the analysis of social choice correspondences that can be implemented in iteratively undominated strategies has occupied the minds of many social choice theorists, a complete characterization has remained elusive.

This paper hopes to contribute to such a characterization by a change in perspective. Instead of considering all mechanisms, we begin by concentrating on a limited, yet comparatively large class of voting procedures that includes prominent and intuitive rules. For that class, we are able to characterize voting procedures using a small number of intuitive axioms that are based on simple majority and monotonicity. In particular, Approval Voting and the Borda Rule stand out as optimal voting procedures with respect to our axioms.

For a class of more general mechanisms, our results raise a number of questions. Is Approval Voting still the unique scoring rule that is majoritarian after eliminating a Pareto dominated alternative (**MEPD**), once we drop the direct mechanism restriction? Does there exist a scoring rule or a more general (bounded) mechanism that not only satisfies **MEPD** but is also monotonic? Such a new mechanism could then be seen an improvement over both Approval Voting and the Borda Rule in conducting elections involving three candidates. For elections involving more than three candidates, one may ask whether our axioms, **MEPD** and *Majoritarian after Eliminating a Worst alternative* (**MEW**), can be extended so as to yield analogous characterisations of Approval Voting and the Borda Rule.

We hope that questions such as these will stimulate future research.

Appendix

Proof of Theorem 1. Consider a ballot $b = (k, l, m) \in V$ and assume w.l.o.g. that $k \geq l \geq m$. Since V is assumed to be neutral, it also includes all permutations of b.

If $k > l > m$, the 6 permutations exhaust V; normalizing all ballots in V by replacing k by $k' = \frac{k-m}{k-m} = 1$, l by $l' = \frac{l-m}{k-m} \in [0,1]$ and m by $m' = \frac{m-m}{k-m} = 0$ yields case (1).

If two entries of b coincide, V contains 3 permutations of b. If those are the only elements of V, we can normalize ballots such that $k' = 1$ and $m' = 0$ which again yields case (1). If V contains another non-abstention ballot $b' = (p, q, r)$, then two of its three entries must coincide - if all were distinct, V would contain not only all permutations of b but also of b', violating $|V| \leq 6$.

W.l.o.g assume $p \geq q \geq r$. If $k = l > m$ and $p > q = r$, normalizing each permutation of b by subtracting m and each permutation of b' by subtracting r before dividing each ballot by $k-m+p-r$ yields $k' = l' = \frac{k-m}{k-m+p-r}$, $m' = 0$, $p' = \frac{p-r}{k-m+p-r}$ and $q' = r' = 0$, which corresponds to case (2).

If $k > l = m$ and $p > q = r$, assume w.l.o.g. that $k - m \geq p - r$. Normalizing each permutation of b by subtracting m and each permutation of b' by subtracting r

before dividing each ballot by $k-m$ yields $k'=1$, $l'=m'=0$, $p'=\frac{p-r}{k-m}\le 1$ and $q'=r'=0$, which corresponds to case (3).

If $k=l>m$ and $p=q>r$, assume w.l.o.g. that $k-m\ge p-r$. Normalizing each permutation of b by subtracting m and each permutation of b' by subtracting r before dividing each ballot by $k-m$ yields $k'=l'=1$, $m'=0$, $p'=q'=\frac{p-r}{k-m}\le 1$ and $r'=0$, which corresponds to case (4). □

Proof of Theorem 2. In light of the arguments presented in Section 4, the two remaining cases are (2) and (3). Assume condition (2) holds, so that

$$\min_{v\in V^1}|v^x|-|v^z| = 1/2|I_{xyz}|-1/2|I_{xzy}|-1/2|I_{yxz}|-|I_{yzx}|-|I_{zxy}|-|I_{zyx}|\ge 0.$$

Then, for any i_{xyz} in $\Gamma(\succ_I,V^1)$, ballot $v_i=(v_i^x,v_i^y,v_i^z)=(1,\frac{1}{2},0)$ is a weakly better reply than $\tilde{v}_i=(\frac{1}{2},1,0)$ against any $v_{-i}\in V^1_{-i}$:

(i) if for $\tilde{v}=(\tilde{v}_i,v_{-i})$, $|\tilde{v}^y|>|\tilde{v}^x|\ge|\tilde{v}^z|$, then for $v=(v_i,v_{-i})$, we have $|v^y|\ge|v^x|>|v^z|$,
(ii) if for $\tilde{v}=(\tilde{v}_i,v_{-i})$, $|\tilde{v}^x|\ge|\tilde{v}^y|,|\tilde{v}^z|$, then for $v=(v_i,v_{-i})$, we have $|v^x|>|v^y|,|v^z|$.

Hence, $\tilde{v}_i$ is either dominated by $v_i=(1,\frac{1}{2},0)$, or it is a duplicate strategy. Eliminating $\tilde{v}_i$ and moving to the restricted game, $\Gamma(\succ_I,V')$, where $V'_{xyz}=V^1_{xyz}\setminus\{(\frac{1}{2},1,0)\}=\{(1,0,\frac{1}{2}),(1,\frac{1}{2},0)\}$ and $V'_j=V^1_j$ for all $j\notin I_{xyz}$ we find that

$$\begin{aligned}\min_{v\in V'}|v^x|-|v^y| &= 1/2|I_{xyz}|+1/2|I_{xzy}|-|I_{yxz}|-|I_{yzx}|-1/2|I_{zxy}|-|I_{zyx}|\\ &= \underbrace{1/2|I_{xyz}|-1/2|I_{xzy}|-1/2|I_{yxz}|-|I_{yzx}|-|I_{zxy}|-|I_{zyx}|}_{\ge 0}\underbrace{-1/2|I_{yxz}|+1/2|I_{zxy}|}_{>0}+|I_{xzy}|>0,\end{aligned}$$

which rules out y as an outcome of $\Gamma(\succ_I,V')$. But then, in the game $\Gamma(\succ_I,V')$, for any voter i who prefers x over z, $v_i=(1,1/2,0)$ is a best reply as it maximizes i's impact on $|v^x|-|v^z|$. Eliminating dominated or duplicate strategies and moving to $\Gamma(\succ_I,V'')$, where $V''_i=\{(1,1/2,0)\}$ for all $i\in I_{xyz}\cup I_{xzy}\cup I_{yxz}$ and $V''_j=V'_j=V^1_j$ for all $j\notin I_{xyz}\cup I_{xzy}\cup I_{yxz}$, we find that for all $v\in V''$

$$\begin{aligned}|v^x| \ge |I_{xyz}|+|I_{xzy}|+|I_{yxz}| &> 2|I_{xzy}|+2|I_{yxz}|+2|I_{yzx}|+2|I_{zxy}|+2|I_{zyx}|-1\\ &\ge |I_{yzx}|+|I_{zxy}|+|I_{zyx}|\ge|v^z|,\end{aligned}$$

where the strict inequality follows from directly from condition (2), while the next weak inequality follows from the fact that $|I_{zxy}|>0$ by condition (2). Hence, x is the unique outcome after iteratively eliminating dominated strategies from $\Gamma(\succ_I,V^0)$.

Finally, assume condition (3) holds, so that

$$\min_{v\in V^1}|v^x|-|v^z| = 1/2|I_{xyz}|-1/2|I_{xzy}|-1/2|I_{yxz}|-|I_{yzx}|-|I_{zxy}|-|I_{zyx}|\ge -1/2.$$

Then, for any i_{xzy} in $\Gamma(\succ_I,V^1)$, ballot $v_i=(v_i^x,v_i^y,v_i^z)=(1,0,1/2)$ is a weakly better reply than $\tilde{v}_i=(1/2,0,1)$ against any $v_{-i}\in V^1_{-i}$:

(i) if for $\tilde{v}=(\tilde{v}_i,v_{-i})$, $|\tilde{v}^x|\ge|\tilde{v}^y|$, then for $v=(v_i,v_{-i})$, we have $|v^x|>|v^y|,|v^z|$,
(ii) if for $\tilde{v}=(\tilde{v}_i,v_{-i})$, $|\tilde{v}^y|>|\tilde{v}^x|,|\tilde{v}^z|$, so that $v=(v_i,v_{-i})$ can only yield a weakly better outcome for i_{xzy},
(iii) if for $\tilde{v}=(\tilde{v}_i,v_{-i})$, $|\tilde{v}^z|\ge|\tilde{v}^y|>|\tilde{v}^x|$, then $|\tilde{v}^z|=|\tilde{v}^x|+1/2$ and$|\tilde{v}^z|=|\tilde{v}^y|$. But then $2(|\tilde{v}^x|+|\tilde{v}^y|+|\tilde{v}^z|)=2(3|\tilde{v}^x|+1)$. However, as each voter awards score that sum to $\frac{3}{2}$, $2(|\tilde{v}^x|+|\tilde{v}^y|+|\tilde{v}^z|)$ would have to be divisible by three - a contradiction.

Hence, $\tilde{v}_i$ is either dominated by $v_i = (1, 0, 1/2)$, or it is duplicate. Eliminating $\tilde{v}_i$ and moving to the restricted game, $\Gamma(\succ_I, V')$, where $V'_{xzy} = V^1_{xzy} \setminus \{(\frac{1}{2}, 0, 1)\} = \{(1, 0, \frac{1}{2}), (1, \frac{1}{2}, 0)\}$ and $V'_j = V^1_j$ for all $j \notin I_{xyz}$, condition (3) yields

$$\begin{aligned}\min_{v \in V'} |v^x| - |v^z| &= 1/2|I_{xyz}| + 1/2|I_{xzy}| - 1/2|I_{yxz}| - |I_{yzx}| - |I_{zxy}| - |I_{zyx}| \\ &= \underbrace{1/2|I_{xyz}| - 1/2|I_{xzy}| - 1/2|I_{yxz}| - |I_{yzx}| - |I_{zxy}| - |I_{zyx}| + 1}_{>0} \underbrace{-1 + |I_{xzy}|}_{\geq 0} > 0,\end{aligned}$$

which rules out z as an outcome of $\Gamma(\succ_I, V')$. But then, in the game $\Gamma(\succ_I, V')$, for any voter i who prefers x over y, $v_i = (1, 0, 1/2)$ is a best reply as it maximizes i's impact on $|v^x| - |v^y|$. Eliminating dominated or duplicate strategies and moving to $\Gamma(\succ_I, V'')$, where $V''_i = \{(1, 0, 1/2)\}$ for all $i \in I_{xyz} \cup I_{xzy} \cup I_{zxy}$ and $V''_i = V'_i = V^1_i$ for all $i \notin I_{xyz} \cup I_{xzy} \cup I_{zxy}$, we find that for all $v \in V''$

$$\begin{aligned}|v^x| \geq |I_{xyz}| + |I_{xzy}| + |I_{zxy}| &> \underbrace{2|I_{xzy}|}_{\geq 2} + |I_{yxz}| + 2|I_{yzx}| + 3|I_{zxy}| + 2|I_{zyx}| - 2 \\ &\geq |I_{yxz}| + |I_{yzx}| + |I_{zyx}| \geq |v^z|\end{aligned}$$

by condition (3). Hence, x is the unique outcome after iteratively eliminating dominated strategies from $\Gamma(\succ_I, V^0)$. □

Proof of Lemma 1. Suppose that for any labelling of candidates, we have

$$(\star) \qquad |I_{xyz}| < |I_{xzy}| + |I_{yxz}| + 2|I_{yzx}| + 2|I_{zxy}| + 2|I_{zyx}| - 2.$$

Now consider a voter of type i_{abc}. We will show that after one round of elimination, no strategy $v_i = (v^a_i, v^b_i, v^c_i)$ in $V^1_i = \{(1, 1/2, 0), (1, 0, 1/2), (1/2, 1, 0)\}$ is dominated in the game $\Gamma(\succ_I, V^1)$ and that each outcome $a, b, c \in A$ is possible.

Claim 1. *Neither* $(1, 0, 1/2)$ *nor* $(1, 1/2, 0)$ *is dominated by* $(1/2, 1, 0)$*. Moreover,* $(1, 0, 1/2)$ *is not dominated by* $(1, 1/2, 0)$ *and both* a *and* b *are possible outcomes.*

Proof. We will proof the claim by constructing an opposing strategy profile for which (i) $v_i = (1, 0, 1/2)$ and $v_i = (1, 1/2, 0)$ yield outcome a while $v_i = (1/2, 1, 0)$ yields b and (ii) another opposing profile for which $v_i = (1, 0, 1/2)$ yields a while $v_i = (1, 1/2, 0)$ yields b. To find such profiles, observe that

$$\max_{v \in V^1} |v^a| - |v^b| = |I_{abc}| + |I_{acb}| + 1/2|I_{bac}| - 1/2|I_{bca}| + |I_{cab}| + 1/2|I_{cba}| \geq 1$$

as otherwise $(\star)$ would be violated for $x = b$, $y = c$ and $z = a$. Similarly,

$$\min_{v \in V^1} |v^a| - |v^b| = -1/2|I_{abc}| + 1/2|I_{acb}| - |I_{bac}| - |I_{bca}| - 1/2|I_{cab}| - |I_{cba}| \leq -1$$

as otherwise $(\star)$ would be violated for $x = a$, $y = c$ and $z = b$. Adjusting opponents' strategies one by one, we can generate a profile v_{-i} such that $|v^a_{-i}| \approx |v^b_{-i}|$. Holding $|v^c_{-i}|$ as small as possible in the process, leads us to the following 5 case distinctions.

Case 1.1 We know that by $(\star)$,

$$\underbrace{2|I_{abc}| + 2|I_{acb}| + |I_{bac}| - |I_{bca}| + 2|I_{cab}| + |I_{cba}|}_{=:n} \geq 2.$$

Suppose

$$\underbrace{|I_{abc}| + |I_{acb}| + |I_{bac}| - |I_{bca}| + 2|I_{cab}| + |I_{cba}|}_{=n - |I_{abc}| - |I_{acb}|} < 0.$$

Construct v_{-i} as follows:

- $n-1 < |I_{abc}| + |I_{acb}| - 1$ of $I_{abc} \backslash \{i\} \cup I_{acb}$ chose $v_j = (1, 1/2, 0)$,
- all remaining $j \in I_{abc} \backslash \{i\} \cup I_{acb}$ chose $v_j = (1, 0, 1/2)$,
- all $j \in I_{bac}$ chose $v_j = (1, 1/2, 0)$,
- all $j \in I_{bca}$ chose $v_j = (1/2, 1, 0)$,
- all $j \in I_{cab}$ chose $v_j = (1, 0, 1/2)$,
- all $j \in I_{cba}$ chose $v_j = (1/2, 0, 1)$.

Then,

$$\begin{aligned}|v^a_{-i}| - |v^b_{-i}| &= (1/2\,n - 1/2) + (|I_{abc}| - 1 + |I_{acb}| - n + 1) + 1/2|I_{bac}| - 1/2|I_{bca}| + |I_{cab}| + 1/2|I_{cba}| \\ &= -1/2\,n + |I_{abc}| + |I_{acb}| + 1/2|I_{bac}| - 1/2|I_{bca}| + |I_{cab}| + 1/2|I_{cba}| - 1/2 = -1/2\end{aligned}$$

and

$$\begin{aligned}|v^a_{-i}| - |v^c_{-i}| &= n - 1 + 1/2(|I_{abc}| - 1 + |I_{acb}| - n + 1) + |I_{bac}| + 1/2|I_{bca}| + 1/2|I_{cab}| - 1/2|I_{cba}| \\ &= 1/2\,n + 1/2|I_{abc}| + 1/2|I_{acb}| + |I_{bac}| + 1/2|I_{bca}| + |I_{cab}| - 1/2|I_{cba}| - 1 \\ &\geq 3/2|I_{abc}| - 1 \geq 1/2.\end{aligned}$$

Hence for $a \triangleright b$, $v_i = (1, 0, 1/2), (1, 1/2, 0)$ yield a while $v_i = (1/2, 1, 0)$ yields b. Moreover, if $b \triangleright a$, $v_i = (1, 0, 1/2)$ yields a while $v_i = (1, 1/2, 0)$ yields b.

Case 1.2 Suppose

$$\underbrace{|I_{abc}| + |I_{acb}| + |I_{bac}| - |I_{bca}| + 2|I_{cab}| + |I_{cba}|}_{=:n} \geq 0$$

but

$$\underbrace{|I_{abc}| + |I_{acb}| + |I_{bac}| - |I_{bca}| + 2|I_{cab}| - 2|I_{cba}|}_{=n-3|I_{cba}|} < 0.$$

Construct v_{-i} as follows:

- all $j \in I_{abc} \backslash \{i\} \cup I_{acb} \cup I_{bac}$ chose $v_j = (1, 1/2, 0)$,
- all $j \in I_{bca}$ chose $v_j = (1/2, 1, 0)$,
- all $j \in I_{cab}$ chose $v_j = (1, 0, 1/2)$,
- $\lfloor \frac{n}{3} \rfloor < |I_{cba}|$ of I_{cba} chose $v_j = (0, 1, 1/2)$,
- all remaining $j \in I_{cba}$ chose $v_j = (1/2, 0, 1)$.

Then,

$$\begin{aligned}|v^a_{-i}| - |v^b_{-i}| &= \underbrace{1/2|I_{abc}| - 1/2 + 1/2|I_{acb}| + 1/2|I_{bac}| - 1/2|I_{bca}| + |I_{cab}| + 1/2|I_{cba}|}_{=\frac{n}{2}-\frac{1}{2}} - \frac{3}{2}\left\lfloor \frac{n}{3} \right\rfloor \\ &= \begin{cases} -1/2 & \text{if } n \bmod 3 = 0 \\ 0 & \text{if } n \bmod 3 = 1 \\ 1/2 & \text{if } n \bmod 3 = 2 \end{cases}\end{aligned}$$

and

$$|v^a_{-i}| - |v^c_{-i}| = |I_{abc}| - 1 + |I_{acb}| + |I_{bac}| + 1/2|I_{bca}| + 1/2|I_{cab}| - 1/2|I_{cba}| \geq 0,$$

as otherwise, $(\star)$ would be violated for $x = c$, $y = b$ and $z = a$. Hence for $v_i = (1, 0, 1/2)$, a is elected independent of $\triangleright$. If $|v^a_{-i}| - |v^b_{-i}| \in \{-1/2, 0\}$, and $a \triangleright b$, then $v_i = (1, 1/2, 0)$ yields outcome a while $v_i = (1/2, 1, 0)$ yields b. If $|v^a_{-i}| - |v^b_{-i}| \in \{0, 1/2\}$, and $b \triangleright a$, then $v_i = (1, 1/2, 0)$ yields a while $v_i = (1/2, 1, 0)$ yields b.

If $|v^a_{-i}| - |v^b_{-i}| = -1/2$ and $b \rhd a$, then $v_i = (1, 0, 1/2)$ yields a while $v_i = (1, 1/2, 0)$ yields b. If $|v^a_{-i}| - |v^b_{-i}| \in \{0, 1/2\}$, observe that $\lfloor \frac{n}{3} \rfloor < |I_{cba}|$, so that there is some $j \in I_{cba}$ who chooses $v_j = (1/2, 0, 1)$. A switch by j to $\tilde{v}_j = (0, 1/2, 1)$ yields $|\tilde{v}^a_{-i}| - |\tilde{v}^b_{-i}| \in \{-1, -1/2\}$ and $|\tilde{v}^a_{-i}| - |\tilde{v}^c_{-i}| \geq -1/2$. If $|\tilde{v}^a_{-i}| - |\tilde{v}^b_{-i}| = -1$ and $a \rhd b, c$, then $v_i = (1, 0, 1/2)$ yields a while $v_i = (1, 1/2, 0)$ yields b. If $|\tilde{v}^a_{-i}| - |\tilde{v}^b_{-i}| = -1/2$ and $b \rhd a \rhd c$, then $(1, 0, 1/2)$ yields a while $(1, 1/2, 0)$ yields b.

Case 1.3 Suppose

$$\underbrace{|I_{abc}| + |I_{acb}| + |I_{bac}| - |I_{bca}| + 2|I_{cab}| - 2|I_{cba}|}_{=:n} \geq 0$$

but

$$\underbrace{-|I_{abc}| + |I_{acb}| - |I_{bac}| - |I_{bca}| + 2|I_{cab}| - 2|I_{cba}|}_{=n-2|I_{abc}|-2|I_{bac}|} < 0.$$

Construct v_{-i} as follows:

- $\lfloor \frac{n}{2} \rfloor \leq |I_{abc}| - 1 + |I_{bac}|$ of $I_{abc}\backslash\{i\} \cup I_{bac}$ chose $v_j = (1/2, 1, 0)$,
- all remaining $j \in I_{abc}\backslash\{i\} \cup I_{bac}$ chose $v_j = (1, 1/2, 0)$.
- all $j \in I_{acb}$ chose $v_j = (1, 1/2, 0)$,
- all $j \in I_{bca}$ chose $v_j = (1/2, 1, 0)$,
- all $j \in I_{cab}$ chose $v_j = (1, 0, 1/2)$,
- all $j \in I_{cba}$ chose $v_j = (0, 1, 1/2)$.

Then,

$$\begin{aligned} |v^a_{-i}| - |v^b_{-i}| &= -\left\lfloor \frac{n}{2} \right\rfloor + \underbrace{1/2(|I_{abc}| - 1 + |I_{bac}|) + 1/2|I_{acb}| - 1/2|I_{bca}| + |I_{cab}| - |I_{cba}|}_{\frac{n}{2} - \frac{1}{2}} \\ &= \begin{cases} -1/2 & \text{if } n \bmod 2 = 0, \\ 0 & \text{if } n \bmod 2 = 1. \end{cases} \end{aligned}$$

First, consider the case $n \bmod 2 = 0$, so that $|v^a_{-i}| - |v^b_{-i}| = -1/2$. Towards a contradiction, assume $|v^a_{-i}| < |v^c_{-i}|$. Then $|v^b_{-i}| \leq |v^c_{-i}|$ and $3|v^c_{-i}| > |v^a_{-i}| + |v^b_{-i}| + |v^c_{-i}| = \frac{3}{2}(|I| - 1)$. But $3|v^c_{-i}| \leq \frac{3}{2}(|I| - 1)$ as no $j \in I\backslash\{i\}$ awards more than $v^c_j = 1/2$. Hence, $|v^a_{-i}| \geq |v^c_{-i}|$. Then, for $a \rhd b$, $v_i = (1, 0, 1/2), (1, 1/2, 0)$ yield a while $v_i = (1/2, 1, 0)$ yields b. Moreover, for $b \rhd a$, $v_i = (1, 0, 1/2)$ yields a while $v_i = (1, 1/2, 0)$ yields b.

Next, consider the case $n \bmod 2 = 1$, so that $|v^a_{-i}| = |v^b_{-i}|$. Towards a contradiction, assume $|v^a_{-i}| \leq |v^c_{-i}|$. Then $3|v^c_{-i}| \geq |v^a_{-i}| + |v^b_{-i}| + |v^c_{-i}| = \frac{3}{2}(|I| - 1)$. Moreover $3|v^c_{-i}| \leq \frac{3}{2}(|I| - 1)$ as no $j \in I\backslash\{i\}$ awards more than $v^c_j = 1/2$. Hence, $|v^c_{-i}| = \frac{1}{2}(|I| - 1)$ which requires $I\backslash\{i\} = I_{cab} \cup I_{cba}$. Then $(\star)$ requires

$$|I_{cab}| + 1 \leq |I_{cba}| + \underbrace{2|I_{abc}| - 2}_{=0} \qquad \text{and} \qquad |I_{cba}| \leq |I_{cab}| + \underbrace{2|I_{abc}| - 2}_{=0}$$

- a contradiction. Instead we conclude that $|v^a_{-i}| - |v^c_{-i}| \geq \frac{1}{2}$. Then, for $a \rhd b$, $v_i = (1, 0, 1/2), (1, 1/2, 0)$ yield a while $v_i = (1/2, 1, 0)$ yields b. To see that $v_i = (1, 0, 1/2)$ can be a better reply than $(1, 1/2, 0)$, consider first the case that $\lfloor \frac{n}{2} \rfloor < |I_{abc}| - 1 + |I_{bac}|$. Then there is some $j \in I_{abc}\backslash\{i\} \cup I_{bac}$ who chooses $v_j = (1, 1/2, 0)$. A switch by j to $\tilde{v}_j = (1/2, 1, 0)$ yields $|\tilde{v}^a_{-i}| - |\tilde{v}^b_{-i}| = -1$ and $|\tilde{v}^a_{-i}| - |\tilde{v}^c_{-i}| \geq 0$, so that for $a \rhd b$, $v_i = (1, 0, 1/2)$

yields a while $(1, 1/2, 0)$ yields. If instead $\lfloor \frac{n}{2} \rfloor = |I_{abc}| - 1 + |I_{bac}|$, then, as n is odd,

$$2|I_{abc}| - 2 + 2|I_{bac}| = 2\lfloor \frac{n}{2} \rfloor = n - 1 = |I_{abc}| + |I_{acb}| + |I_{bac}| - |I_{bca}| + 2|I_{cab}| - 2|I_{cba}| - 1$$
$$\iff |I_{abc}| + |I_{bac}| = |I_{acb}| - |I_{bca}| + 2|I_{cab}| - 2|I_{cba}| + 1.$$

If $I_{bac} \cup I_{bca} \cup I_{cab} = \varnothing$, this would yield $|I_{acb}| = |I_{abc}| + 2|I_{cba}| - 1$, contradicting $(\star)$. Hence, there is some $j \in I_{bac} \cup I_{bca} \cup I_{cab}$. A switch by either $j \in I_{bac} \cup I_{bca}$ from $v_j = (1/2, 1, 0)$ to $\tilde{v}_j = (0, 1, 1/2)$ or by $j \in I_{bac}$ from $v_j = (1, 0, 1/2)$ to $\tilde{v}_j = (1/2, 0, 1)$ yields $|\tilde{v}^a_{-i}| - |\tilde{v}^b_{-i}| = -1/2$ and $|\tilde{v}^a_{-i}| - |\tilde{v}^c_{-i}| \geq -1/2$, so that for $b \rhd a \rhd c$, $v_i = (1, 0, 1/2)$ yields a, while $v_i = (1, 1/2, 0)$ yields b.

Case 1.4 Suppose

$$\underbrace{-|I_{abc}| + |I_{acb}| - |I_{bac}| - |I_{bca}| + 2|I_{cab}| - 2|I_{cba}|}_{=:n} \geq 0$$

but

$$\underbrace{-|I_{abc}| + |I_{acb}| - |I_{bac}| - |I_{bca}| - |I_{cab}| - 2|I_{cba}|}_{=n-3|I_{cab}|} < 0.$$

Construct v_{-i} as follows:

- all $j \in I_{abc} \backslash \{i\}$ chose $v_j = (1/2, 1, 0)$.
- all $j \in I_{acb}$ chose $v_j = (1, 1/2, 0)$,
- all $j \in I_{bac} \cup I_{bca}$ chose $v_j = (1/2, 1, 0)$,
- $\lceil \frac{n}{3} \rceil \leq |I_{cab}|$ of I_{cab} chose $v_j = (0, 1/2, 1)$
- all remaining $j \in I_{cab}$ chose $v_j = (1, 0, 1/2)$,
- all $j \in I_{cba}$ chose $v_j = (0, 1, 1/2)$.

Then,

$$|v^a_{-i}| - |v^b_{-i}| = \underbrace{-1/2|I_{abc}| + 1/2 - 1/2|I_{bac}| + 1/2|I_{acb}| - 1/2|I_{bca}| + |I_{cab}| - |I_{cba}|}_{=\frac{n}{2}+\frac{1}{2}} - \frac{3}{2}\left\lceil \frac{n}{3} \right\rceil$$
$$= \begin{cases} 1/2 & \text{if } n \bmod 3 = 0 \\ -1/2 & \text{if } n \bmod 3 = 1 \\ 0 & \text{if } n \bmod 3 = 2 \end{cases}$$

and

$$|v^a_{-i}| - |v^c_{-i}| = |v^b_{-i}| - |v^c_{-i}| + |v^a_{-i}| - |v^b_{-i}|$$
$$= \underbrace{|I_{abc}| - 1 + |I_{bac}| + 1/2|I_{acb}| + |I_{bca}| - 1/2|I_{cab}| + 1/2|I_{cba}|}_{=:k} + |v^a_{-i}| - |v^b_{-i}|.$$

Since $\mathbb{1}_{\{|I_{abc}|>0\}}$, $(\star)$ yields $1/2|I_{cab}| \leq 1/2|I_{cba}| + 1/2|I_{acb}| + |I_{abc}| + |I_{bca}| + |I_{bac}| - \frac{3}{2}$, so that $k \geq 1/2$. Hence, $|v^a_{-i}| - |v^c_{-i}| > |v^a_{-i}| - |v^b_{-i}| \geq 0$, so that for $v_i = (1, 0, 1/2)$, a is elected independent of $\rhd$. If $|v^a_{-i}| - |v^b_{-i}| \in \{-1/2, 0\}$, and $a \rhd b$, then $v_i = (1, 1/2, 0)$ yields outcome a while $v_i = (1/2, 1, 0)$ yields b. If $|v^a_{-i}| - |v^b_{-i}| \in \{0, 1/2\}$, and $b \rhd a$, then $v_i = (1, 1/2, 0)$ yields a while $v_i = (1/2, 1, 0)$ yields$b$.

If $|v^a_{-i}| - |v^b_{-i}| = -1/2$ and $b \rhd a$, then $v_i = (1, 0, 1/2)$ yields a while $v_i = (1, 1/2, 0)$ yields b. If $|v^a_{-i}| - |v^b_{-i}| = 1/2$, then $n \bmod 3 = 0$ and hence $\lceil \frac{n}{3} \rceil = \frac{n}{3} < |I_{cab}|$, so that there is some $j \in I_{cab}$ who chooses $v_j = (1, 0, 1/2)$. A switch by j to $\tilde{v}_j = (0, 1/2, 1)$ yields

$|\tilde{v}^a_{-i}| - |\tilde{v}^b_{-i}| = -1$ and $|\tilde{v}^a_{-i}| - |\tilde{v}^c_{-i}| \geq -1/2$. Hence, for $a \rhd b$, $v_i = (1, 0, 1/2)$ yields a while $v_i = (1, 1/2, 0)$ yields b.

If $|\tilde{v}^a_{-i}| - |\tilde{v}^b_{-i}| = 0$ then $n \bmod 3 = 2$. If in addition $|I_{bac}| + |I_{bca}| = 0$, then by $(\star)$

$$\underbrace{-|I_{abc}| + |I_{acb}| - |I_{bac}| - |I_{bca}| - |I_{cab}| - 2|I_{cba}|}_{=n-3|I_{cab}|} = -|I_{abc}| + |I_{acb}| - 2|I_{bac}| - 2|I_{bca}| - |I_{cab}| - 2|I_{cba}| \leq -2.$$

Hence $\lceil \frac{n}{3} \rceil = \frac{n}{3} + \frac{1}{3} < \frac{n}{3} + \frac{2}{3} \leq |I_{cab}|$, so that there is some $j \in I_{cab}$ who chooses $v_j = (1, 0, 1/2)$. A switch by j to $\tilde{v}_j = (1/2, 0, 1)$ yields $|\tilde{v}^a_{-i}| - |\tilde{v}^b_{-i}| = -1/2$ and $|\tilde{v}^a_{-i}| - |\tilde{v}^c_{-i}| \geq -1/2$. Thus, for $b \rhd a \rhd c$, $v_i = (1, 0, 1/2)$ yields a while $v_i = (1, 1/2, 0)$ yields b. If instead $|I_{bac}| + |I_{bca}| > 0$, let some $j \in I_{bac} \cup I_{bca}$ switch from $v_j = (1/2, 1, 0)$ to $\tilde{v}_j = (0, 1, 1/2)$. Then $|\tilde{v}^a_{-i}| - |\tilde{v}^b_{-i}| = -1/2$ and $|\tilde{v}^a_{-i}| - |\tilde{v}^c_{-i}| \geq -1/2$. Hence, for $b \rhd a \rhd c$, $v_i = (1, 0, 1/2)$ yields a while $v_i = (1, 1/2, 0)$ yields b.

Case 1.5 Suppose

$$\underbrace{-|I_{abc}| + |I_{acb}| - |I_{bac}| - |I_{bca}| - |I_{cab}| - 2|I_{cba}|}_{=:n} \geq 0.$$

We know that

$$\underbrace{-|I_{abc}| + |I_{acb}| - 2|I_{bac}| - 2|I_{bca}| - |I_{cab}| - 2|I_{cba}|}_{=n-|I_{bac}|-|I_{bca}|} \leq -2,$$

as otherwise $(\star)$ would be violated for $x = a$, $y = c$ and $z = b$. Construct v_{-i} as follows:

- all $j \in I_{abc} \backslash \{i\}$ chose $v_j = (1/2, 1, 0)$.
- all $j \in I_{acb}$ chose $v_j = (1, 1/2, 0)$,
- all $j \in I_{cab}$ chose $v_j = (0, 1/2, 1)$,
- all $j \in I_{cba}$ chose $v_j = (0, 1, 1/2)$,
- $n + 2 \leq |I_{bac}| + |I_{bca}|$ of $I_{bac} \cup I_{bca}$ chose $v_j = (0, 1, 1/2)$,
- all remaining $j \in I_{bac} \cup I_{bca}$ chose $v_j = (1/2, 1, 0)$.

Then,

$$|v^a_{-i}| - |v^b_{-i}| = -1/2|I_{abc}| - 1/2 + 1/2 + 1/2|I_{acb}| - 1/2|I_{cab}| - |I_{cba}| - 1/2|I_{bac}| - 1/2|I_{bca}| - 1/2\,(n+2) = -1/2$$

and

$$\begin{aligned} |v^a_{-i}| - |v^c_{-i}| &= 1/2|I_{abc}| - 1/2 + |I_{acb}| - |I_{cab}| - 1/2|I_{cba}| + 1/2|I_{bac}| + 1/2|I_{bca}| - (n+2) \\ &= 3/2|I_{abc}| + 3/2|I_{bac}| + 3/2|I_{bca}| + 3/2|I_{cba}| - 3/2 \geq 0. \end{aligned}$$

Hence for $a \rhd b$, $v_i = (1, 0, 1/2), (1, 1/2, 0)$ yields a while $v_i = (1/2, 1, 0)$ yields b. Moreover, for $b \rhd a$, $v_i = (1, 0, 1/2)$ yields a while $v_i = (1, 1/2, 0)$ yields b. *Claim 1* $\diamond$

Claim 2. *Neither* $(1/2, 1, 0)$ *nor* $(1, 1/2, 0)$ *is dominated by* $(1, 0, 1/2)$. *Moreover,* $(1/2, 1, 0)$ *is not dominated by* $(1, 1/2, 0)$ *and both b and c are possible outcomes.*

Proof. We will proof the claim by constructing an opposing strategy profile for which (i) $v_i = (1/2, 1, 0)$ and $v_i = (1, 1/2, 0)$ yield outcome b while $v_i = (1, 0, 1/2)$ yields c and (ii) another opposing profile for which $v_i = (1/2, 1, 0)$ yields b while $v_i = (1, 1/2, 0)$ yields c. To find such profiles, observe that

$$\max_{v \in V^1} |v^b| - |v^c| = |I_{abc}| + 1/2|I_{acb}| + |I_{bac}| + |I_{bca}| - 1/2|I_{cab}| + 1/2|I_{cba}| \geq 3/2$$

as otherwise $(\star)$ would be violated for $x = c$, $y = a$ and $z = b$. Similarly,

$$\min_{v \in V^1} |v^b| - |v^c| = -1/2|I_{abc}| - |I_{acb}| + 1/2|I_{bac}| - 1/2|I_{bca}| - |I_{cab}| - |I_{cba}| \leq -1$$

as otherwise ($\star$) would be violated for $x = b$, $y = a$ and $z = c$. Adjusting opponents' strategies one by one, we can generate a profile v_{-i} such that $|v_{-i}^b| \approx |v_{-i}^c|$. Holding $|v_{-i}^a|$ as small as possible in the process, leads us to the following 5 case distinctions.

Case 2.1 We know that by ($\star$),

$$\underbrace{2|I_{abc}| + |I_{acb}| + 2|I_{bac}| + 2|I_{bca}| - |I_{cab}| + |I_{cba}|}_{=:n} \geq 3.$$

Suppose

$$\underbrace{2|I_{abc}| + |I_{acb}| + |I_{bac}| + |I_{bca}| - |I_{cab}| + |I_{cba}|}_{=n-|I_{bac}|-|I_{bca}|} \leq 0.$$

Construct v_{-i} as follows:

- all $j \in I_{abc} \backslash \{i\}$ chose $v_j = (1/2, 1, 0)$,
- all $j \in I_{acb}$ chose $v_j = (1, 1/2, 0)$,
- $n - 1 < |I_{bac}| + |I_{bca}|$ of $I_{bac} \cup I_{bca}$ chose $v_j = (0, 1, 1/2)$,
- all remaining $j \in I_{bac} \cup I_{bca}$ chose $v_j = (1/2, 1, 0)$,
- all $j \in I_{cab}$ chose $v_j = (0, 1/2, 1)$,
- all $j \in I_{cba}$ chose $v_j = (0, 1, 1/2)$.

Then,

$$|v_{-i}^b| - |v_{-i}^c| = |I_{abc}| - 1 + 1/2|I_{acb}| + |I_{bac}| + |I_{bca}| - 1/2(n-1) - 1/2|I_{cab}| + 1/2|I_{cba}| = -1/2$$

and

$$\begin{aligned} |v_{-i}^b| - |v_{-i}^a| &= 1/2|I_{abc}| - 1/2 - 1/2|I_{acb}| + 1/2|I_{bac}| + 1/2|I_{bca}| + 1/2(n-1) + 1/2|I_{cab}| + |I_{cba}| \\ &= 3/2|I_{abc}| + 3/2|I_{bac}| + 3/2|I_{bca}| + 3/2|I_{cba}| - 3/2 \geq 6, \end{aligned}$$

since by assumption for Case 2.1 we have $|I_{bac}| + |I_{bca}| \geq 4$. Then for $b \rhd c$, $v_i = (1/2, 1, 0), (1, 1/2, 0)$, yield b whereas $v_i = (1, 0, 1/2)$ yields c. Moreover, for $c \rhd b$, $v_i = (1/2, 1, 0)$ yields b while $v_i = (1, 1/2, 0)$ yields c.

Case 2.2 Suppose

$$\underbrace{2|I_{abc}| + |I_{acb}| + |I_{bac}| + |I_{bca}| - |I_{cab}| + |I_{cba}|}_{=:n} > 0$$

but

$$\underbrace{2|I_{abc}| - 2|I_{acb}| + |I_{bac}| + |I_{bca}| - |I_{cab}| + |I_{cba}|}_{=n-3|I_{acb}|} \leq 0.$$

Construct v_{-i} as follows:

- all $j \in I_{abc} \backslash \{i\}$ chose $v_j = (1/2, 1, 0)$,
- $\lfloor \frac{n-1}{3} \rfloor < |I_{acb}|$ of I_{acb} chose $v_j = (1/2, 0, 1)$,
- all remaining $j \in I_{acb}$ chose $v_j = (1, 1/2, 0)$,
- all $j \in I_{bac} \cup I_{bca}$ chose $v_j = (0, 1, 1/2)$,
- all $j \in I_{cab}$ chose $v_j = (0, 1/2, 1)$,
- all $j \in I_{cba}$ chose $v_j = (0, 1, 1/2)$.

Then,

$$|v^b_{-i}| - |v^c_{-i}| = \underbrace{|I_{abc}| - 1 + {}^1\!/_2|I_{acb}| + {}^1\!/_2|I_{bac}| + {}^1\!/_2|I_{bca}| - {}^1\!/_2|I_{cab}| + {}^1\!/_2|I_{cba}|}_{\frac{n}{2}-1} - \frac{3}{2}\left\lfloor \frac{n-1}{3} \right\rfloor$$

$$= \begin{cases} -{}^1\!/_2 & \text{if } n-1 \bmod 3 = 0 \\ 0 & \text{if } n-1 \bmod 3 = 1 \\ {}^1\!/_2 & \text{if } n-1 \bmod 3 = 2 \end{cases}$$

and

$$|v^b_{-i}| - |v^a_{-i}| = {}^1\!/_2|I_{abc}| - {}^1\!/_2 - {}^1\!/_2|I_{acb}| + |I_{bac}| + |I_{bca}| + {}^1\!/_2|I_{cab}| + |I_{cba}| \geq {}^1\!/_2,$$

as otherwise, $(\star)$ would be violated for $x = a$, $y = c$ and $z = b$. If $|v^b_{-i}| - |v^c_{-i}| = -{}^1\!/_2$, and $b \rhd c, a$, then $v_i = (1, {}^1\!/_2, 0), ({}^1\!/_2, 1, 0)$ yields b while $v_i = (1, 0, {}^1\!/_2)$ yields c. Moreover, for $c \rhd b, a$, $v_i = ({}^1\!/_2, 1, 0)$ yields b while $v_i = (1, {}^1\!/_2, 0)$ yields c.

Next, consider $|v^b_{-i}| - |v^c_{-i}| = 0$. Again, for $b \rhd a$, then $v_i = (1, {}^1\!/_2, 0), ({}^1\!/_2, 1, 0)$ yields b while $v_i = (1, 0, {}^1\!/_2)$ yields c. For a profile where $v_i = ({}^1\!/_2, 1, 0)$ is a better reply than $(1, {}^1\!/_2, 0)$, let some $j \in I_{acb}$ switch from $v_j = (1, {}^1\!/_2, 0)$ to $\tilde{v}_j = (1, 0, {}^1\!/_2)$. Then $|\tilde{v}^b_{-i}| - |\tilde{v}^c_{-i}| = -1$ and $|\tilde{v}^b_{-i}| - |\tilde{v}^a_{-i}| \geq 0$. Hence, for $b \rhd c \rhd a$, $v_i = ({}^1\!/_2, 1, 0)$ yields b while $v_i = (1, {}^1\!/_2, 0)$ yields c.

Finally, consider v_{-i} where $|v^b_{-i}| - |v^c_{-i}| = {}^1\!/_2$ or rather the neighbouring profile $\tilde{v}_{-i}$ where some $j \in I_{cab}$ has switched from $v_j = (1, {}^1\!/_2, 0)$ to $\tilde{v}_j = (1, 0, {}^1\!/_2)$. Then $|\tilde{v}^b_{-i}| - |\tilde{v}^c_{-i}| = -{}^1\!/_2$ and $|\tilde{v}^b_{-i}| - |\tilde{v}^a_{-i}| \geq 0$. Hence, for $b \rhd c \rhd a$, a or b is elected for $v_i = (1, {}^1\!/_2, 0), ({}^1\!/_2, 1, 0)$ while $v_i = (1, 0, {}^1\!/_2)$ yields c. For a profile where $v_i = ({}^1\!/_2, 1, 0)$ is a better reply than $(1, {}^1\!/_2, 0)$, consider $\hat{v}_{-i}$, which differs from v_{-i} in that some $j \in I_{acb}$ switches from $v_j = (1, {}^1\!/_2, 0)$ to $\hat{v}_j = ({}^1\!/_2, 0, 1)$. Then $|\hat{v}^b_{-i}| - |\hat{v}^c_{-i}| = -1$, $|\hat{v}^b_{-i}| - |\hat{v}^a_{-i}| \geq {}^1\!/_2$ so that for $b \rhd c$, $v_i = ({}^1\!/_2, 1, 0)$ yields b while $v_i = (1, {}^1\!/_2, 0)$ yields c.

Case 2.3 Suppose

$$\underbrace{2|I_{abc}| - 2|I_{acb}| + |I_{bac}| + |I_{bca}| - |I_{cab}| + |I_{cba}|}_{=:n} > 0$$

but

$$\underbrace{2|I_{abc}| - 2|I_{acb}| + |I_{bac}| - |I_{bca}| - |I_{cab}| - |I_{cba}|}_{=n-2|I_{bca}|-2|I_{cba}|} \leq 0.$$

Construct v_{-i} as follows:

- all $j \in I_{abc} \backslash \{i\}$ chose $v_j = ({}^1\!/_2, 1, 0)$
- all $j \in I_{acb}$ chose $v_j = ({}^1\!/_2, 0, 1)$,
- all $j \in I_{bac}$ chose $v_j = (0, 1, {}^1\!/_2)$,
- $\lfloor \frac{n}{2} \rfloor \leq |I_{bca}| + |I_{cba}|$ of $I_{bca} \cup I_{cba}$ chose $v_j = (0, {}^1\!/_2, 1)$,
- all remaining $j \in I_{bca} \cup I_{cba}$ chose $v_j = (0, 1, {}^1\!/_2)$.
- all $j \in I_{cab}$ chose $v_j = (0, {}^1\!/_2, 1)$,

Then,

$$|v^b_{-i}| - |v^c_{-i}| = \underbrace{|I_{abc}| - 1 - |I_{acb}| + 1/2|I_{bac}| + 1/2|I_{bca}| - 1/2|I_{cab}| + 1/2|I_{cba}|}_{\frac{n}{2}-1/2} - \left\lfloor \frac{n}{2} \right\rfloor$$

$$= \begin{cases} -1/2 & \text{if } n \bmod 2 = 0, \\ 0 & \text{if } n \bmod 2 = 1, \end{cases}$$

and

$$\begin{aligned} |v^b_{-i}| - |v^a_{-i}| &= 1/2|I_{abc}| - 1/2 - 1/2|I_{acb}| + |I_{bac}| + |I_{bca}| + 1/2|I_{cab}| + |I_{cba}| - \frac{1}{2}\left\lfloor \frac{n}{2} \right\rfloor \\ &\geq 1/2|I_{abc}| - 1/2 - 1/2|I_{acb}| + |I_{bac}| + |I_{bca}| + 1/2|I_{cab}| + |I_{cba}| - \frac{n}{4} \\ &= \underbrace{3/4|I_{bac}| + 3/4|I_{bca}| + 3/4|I_{cab}| + 3/4|I_{cba}|}_{\geq 3/4,\text{ by } (\star)} - 1/2 > 0. \end{aligned}$$

For $b \rhd c \rhd a$, both $v_i = (1, 1/2, 0), (1/2, 1, 0)$ yields b while $v_i = (1, 0, 1/2)$ yields c. Moreover, if $|v^b_{-i}| - |v^c_{-i}| = -1/2$ and $c \rhd a, b$ then $v_i = (1/2, 1, 0)$ yields b while $v_i = (1, 1/2, 0)$ yields c.

If on the other hand $|v^b_{-i}| - |v^c_{-i}| = 0$ then $\lfloor \frac{n}{2} \rfloor = \frac{n-1}{2} < |I_{bca}| + |I_{cba}|$, so that there is some $j \in I_{bca} \cup I_{cba}$ who chooses $v_j = (0, 1, 1/2)$. Letting her switch to $\tilde{v}_j = (0, 1/2, 1)$ gives $|\tilde{v}^b_{-i}| - |\tilde{v}^c_{-i}| = -1$ and $|\tilde{v}^b_{-i}| - |\tilde{v}^a_{-i}| \geq 0$, so that for $b \rhd c \rhd a$, $v_i = (1/2, 1, 0)$ yields b while $v_i = (1, 1/2, 0)$ yields c.

Case 2.4 Suppose

$$\underbrace{2|I_{abc}| - 2|I_{acb}| + |I_{bac}| - |I_{bca}| - |I_{cab}| - |I_{cba}|}_{=:n} > 0$$

but

$$\underbrace{-|I_{abc}| - 2|I_{acb}| + |I_{bac}| - |I_{bca}| - |I_{cab}| - |I_{cba}|}_{=n-3|I_{abc}|} \leq 0.$$

Construct v_{-i} as follows:

- $\lceil \frac{n}{3} \rceil - 1 \leq |I_{abc}| - 1$ of $I_{abc} \setminus \{i\}$ chose $v_j = (1, 0, 1/2)$
- all remaining $j \in I_{abc} \setminus \{i\}$ chose $v_j = (1/2, 1, 0)$
- all $j \in I_{acb}$ chose $v_j = (1/2, 0, 1)$,
- all $j \in I_{bac}$ chose $v_j = (0, 1, 1/2)$,
- all $j \in I_{bca} \cup I_{cab} \cup I_{cba}$ chose $v_j = (0, 1/2, 1)$.

Then,

$$|v^b_{-i}| - |v^c_{-i}| = \underbrace{|I_{abc}| - 1 - |I_{acb}| + 1/2|I_{bac}| - 1/2|I_{bca}| - 1/2|I_{cab}| - 1/2|I_{cba}|}_{\frac{n}{2}-1} - \frac{3}{2}\left\lceil \frac{n}{3} \right\rceil + \frac{3}{2}$$

$$= \begin{cases} 1/2 & \text{if } n \bmod 3 = 0, \\ -1/2 & \text{if } n \bmod 3 = 1, \\ 0 & \text{if } n \bmod 3 = 2. \end{cases}$$

and

$$|v^a_{-i}| - |v^c_{-i}| = 1/2|I_{abc}| - 1/2 - 1/2|I_{acb}| - 1/2|I_{bac}| - |I_{bca}| - |I_{cab}| - |I_{cba}| \leq -3/2,$$

as otherwise $(\star)$ would be violated for $x = a$, $y = b$, $z = c$. For $v_i = (1/2, 1, 0)$, b is elected independently of $\rhd$. If $|v^b_{-i}| - |v^c_{-i}| \in \{-1/2, 0\}$ and $b \rhd c$, then $v_i = (1, 1/2, 0)$ yields b while $v_i = (1, 0, 1/2)$ yields c. If $|v^b_{-i}| - |v^c_{-i}| \in \{0, 1/2\}$ and $c \rhd b$, then $v_i = (1, 1/2, 0)$ yields b while $v_i = (1, 0, 1/2)$ yields c.

Moreover, if $|v^b_{-i}| - |v^c_{-i}| = -1/2$ and $c \rhd b$, then $v_i = (1/2, 1, 0)$ yields b while $v_i = (1, 1/2, 0)$ yields c. Next, if $|v^b_{-i}| - |v^c_{-i}| = 0$, then $n \bmod 3 = 2$ and hence $\lceil \frac{n}{3} \rceil - 1 = \frac{n+1}{3} - 1$. Towards a contradiction, assume that $\frac{n+1}{3} - 1 = |I_{abc}| - 1$. Then

$$3|I_{abc}| - 1 = n \leq 2|I_{abc}| - 2|I_{acb}| + |I_{bac}| - |I_{bca}| \implies |I_{bac}| \geq |I_{bca}| + |I_{abc}| + 2|I_{acb}| - 1,$$

which violates $(\star)$. Thus, we know that there exist either some $j \in I_{abc} \backslash \{i\}$ who votes $v_j = (1/2, 1, 0)$. Letting $j \in I_{abc} \backslash \{i\}$ switch to $\tilde{v}_j = (1, 1/2, 0)$ gives $|\tilde{v}^b_{-i}| - |\tilde{v}^c_{-i}| = -1/2$ and $|\tilde{v}^a_{-i}| - |\tilde{v}^c_{-i}| \leq 1$. Then for $c \rhd a, b$, $v_i = (1/2, 1, 0)$ yields b while $v_i = (1, 1/2, 0)$ yields c.

Finally, if $|v^b_{-i}| - |v^c_{-i}| = 1/2$ then $n \bmod 3 = 0$ and hence $\lceil \frac{n}{3} \rceil = \frac{n}{3}$. Towards a contradiction, assume that $\frac{n}{3} - 1 = |I_{abc}| - 1$. Then

$$3|I_{abc}| = n \leq 2|I_{abc}| - 2|I_{acb}| + |I_{bac}| - |I_{bca}| \implies |I_{bac}| \geq |I_{bca}| + |I_{abc}| + 2|I_{acb}|,$$

which violates $(\star)$. Thus, we know that there exist either some $j \in I_{abc} \backslash \{i\}$ who votes $v_j = (1/2, 1, 0)$. Letting $j \in I_{abc} \backslash \{i\}$ switch to $\tilde{v}_j = (1, 0, 1/2)$ gives $|\tilde{v}^b_{-i}| - |\tilde{v}^c_{-i}| = -1$ and $|\tilde{v}^a_{-i}| - |\tilde{v}^c_{-i}| \leq -3/2$. Then for $b \rhd c$, $v_i = (1/2, 1, 0)$ yields b while $v_i = (1, 1/2, 0)$ yields c.

Case 2.5 Suppose

$$\underbrace{-|I_{abc}| - 2|I_{acb}| + |I_{bac}| - |I_{bca}| - |I_{cab}| - |I_{cba}|}_{=:n} > 0.$$

We know that

$$\underbrace{-|I_{abc}| - 2|I_{acb}| + |I_{bac}| - |I_{bca}| - 2|I_{cab}| - 2|I_{cba}|}_{=n-|I_{cab}|-|I_{cba}|} \leq -2.$$

Construct v_{-i} as follows:

- all $j \in I_{abc} \backslash \{i\}$ chose $v_j = (1, 0, 1/2)$
- all $j \in I_{acb}$ chose $v_j = (1/2, 0, 1)$,
- all $j \in I_{bac}$ chose $v_j = (0, 1, 1/2)$,
- all $j \in I_{bca}$ chose $v_j = (0, 1/2, 1)$,
- $n + 2$ of $I_{cab} \cup I_{cba}$ chose $v_j = (1/2, 0, 1)$,
- all remaining $j \in I_{cab} \cup I_{cba}$ chose $v_j = (0, 1/2, 1)$.

Then,

$$|v^b_{-i}| - |v^c_{-i}| = -1/2|I_{abc}| + 1/2 - |I_{acb}| + 1/2|I_{bac}| - 1/2|I_{bca}| - 1/2|I_{cab}| - 1/2|I_{cba}| - 1/2(n + 2) = -1/2$$

and

$$\begin{aligned}|v^a_{-i}| - |v^c_{-i}| &= 1/2|I_{abc}| - 1/2 - 1/2|I_{acb}| - 1/2|I_{bac}| - |I_{bca}| - |I_{cab}| - |I_{cba}| + 1/2(n + 2) \\ &= -3/2|I_{acb}| - 3/2|I_{bca}| - 3/2|I_{cab}| - 3/2|I_{cba}| + 1/2 \leq -4,\end{aligned}$$

since by assumption for case 2.5, $|I_{cab}| - |I_{cba}| \geq 3$. Then for $b \rhd c$, $v_i = (1/2, 1, 0), (1, 1/2, 0)$ yields b while $v_i = (1, 0, 1/2)$ yields c. Moreover, if $c \rhd b$, $v_i = (1/2, 1, 0)$ yields b and while $v_i = (1, 1/2, 0)$ yields c. *Claim 2* $\diamond$

Together, Claim 1 and 2 show that each outcome is possible in $\Gamma(\succ_I, V^1)$ and that for $i \in I_{abc}$, $V^2_i = V^1_i$. In the same way, i.e. just by relabelling candidates in Claim

1 and 2, we find that for any $j \in I_{xyz}$ $V_j^2 = V_j^1$. Then by induction $V^m = V^1$, for all $m \geq 1$. This completes the proof. $\square$

Proof of Theorem 3. For $x, y, z \in A$, define $|O_{xyz}| := |I_{xzy}| + |I_{yxz}| + 2|I_{yzx}| + 2|I_{zxy}| + 2|I_{zyx}|$ and to fix labels, assume w.l.o.g. that $|I_{abc}| - |O_{abc}| \geq |I_{xyz}| - |O_{xyz}|$ for all $x, y, z \in A$. We will show that each election outcome is possible under some ballot profile, where each voter i chooses a strategy v_i that is undominated. To guide our construction, we make use of the following fact.

Claim 1. *Consider a ballot profile v such that $|v^x| = |v^y| = |v^y|$ and some voter $i \in I_{xyz}$ such that $v_i^x = 1$. Then v_i is undominated in any Game $\Gamma(\succ_I, V')$ where $v \in V' \subset V^1$.*

Proof. Consider the case $v_i = (v_i^x, v_i^y, v_i^z) = (1, 1/2, 0)$. If $x \rhd y, z$, then i's most preferred outcome x is realized. On the other hand, a switch to $\tilde{v}_i = (1/2, 1, 0)$ would yield outcome y and a switch to $\tilde{v}_i = (1, 0, 1/2)$ would yield z. Hence $v_i = (1, 1/2, 0)$ is undominated.

If $v_i = (v_i^x, v_i^y, v_i^z) = (1, 0, 1/2)$ and $x \rhd y, z$ outcome x is realized, while a switch to $\tilde{v}_i = (1/2, 1, 0)$ or $\tilde{v}_i = (1, 1/2, 0)$ would yield y. Hence $v_i = (1, 0, 1/2)$ is undominated. *Claim 1* $\diamond$

Case 1: $|I_{abc}| = |O_{abc}| = |I_{abc}| + |I_{bac}| + 2|I_{bca}| + 2|I_{cab}| + 2|I_{cba}|$

By the assumptions of Theorem 3 we have $|I_{cab}| \leq |I_{bac}|$ and $|I_{acb}| = 0$. If $|I_{bca}| + |I_{cab}| + |I_{cba}| = 0$, so that $|I_{abc}| = |I_{bac}|$, consider a ballot profile v where all i_{abc} chose $v_i = (1, 0, 1/2) \in V_i^1$ while all i_{bac} chose $v_i = (0, 1, 1/2) \in V_i^1$. Then $|v^a| = |v^b| = |v^c|$, so that by claim 1 each v_i is undominated and hence no outcome can be ruled out via iterated elimination of dominated strategies.

Next, consider $|I_{bca}| + |I_{cab}| + |I_{cba}| > 0$. To show that no outcome can be eliminated, we will construct two strategy profiles where a, b and c are possible outcomes (depending on $\rhd$) and show that no individual strategy used in the construction can be eliminated based on weak domination.

Profile $v \in V^1$:

- each $i \in I_{abc}$ chooses $v_i = (1, 0, 1/2)$,
- each $i \in I_{bac}$ chooses $v_i = (0, 1, 1/2)$,
- each $i \in I_{bca} \cup I_{zxy} \cup I_{zyx}$ chooses $v_i = (0, 1/2, 1)$.

Then,

$$|v^a| - |v^c| = 1/2|I_{abc}| \underbrace{-1/2|I_{bac}| - |I_{bca}| - |I_{cab}| - |I_{cba}|}_{-1/2|I_{abc}|} = 0,$$

while

$$\begin{aligned} |v^a| - |v^b| &= |I_{abc}| - |I_{bac}| - 1/2|I_{bca}| - 1/2|I_{cab}| - 1/2|I_{cba}| \\ &= 3/2|I_{bca}| + 3/2|I_{cab}| + 3/2|I_{cba}| \geq 3/2, \end{aligned}$$

so that both a and c are possible outcomes, depending on $\rhd$. If $c \rhd a$, then c is elected while any unilateral deviation to some $\tilde{v}_i \in V_i^1$ by some $i \in I_{bca} \cup I_{cab} \cup I_{cba}$ would yield outcome a. Hence, for $i \in I_{bca} \cup I_{cab} \cup I_{cba}$, $(0, 1/2, 1)$ is the unique best response and thus undominated in any game $\Gamma(\succ_I, V^n)$ where $v \in V^n \subset V^1$.

Profile $v' \in V^1$:

- let $|I_{bac}| - |I_{cab}|$ of I_{abc} chose $v_i' = (1, 0, 1/2)$

- let the remaining i_{abc} choose $v'_i = (1, 1/2, 0)$
- let each $i \in I_{cab}$ choose $v'_i = (0, 1/2, 1)$,
- let each $i \in I_{bac} \cup I_{bca} \cup I_{cba}$ chooses $v'_i = (0, 1, 1/2)$.

Then,

$$\begin{aligned} |v^a| - |v^b| &= |I_{bac}| - |I_{cab}| + 1/2(|I_{abc}| - |I_{bac}| + |I_{cab}|) - 1/2|I_{cab}| - |I_{bac}| - |I_{bca}| - |I_{cba}| \\ &= 1/2|I_{abc}| \underbrace{-1/2|I_{bac}| - |I_{bca}| - |I_{cab}| - |I_{cba}|}_{=1/2|I_{abc}|} = 0, \end{aligned}$$

while

$$\begin{aligned} |v^a| - |v^c| &= 1/2(|I_{bac}| - |I_{cab}|) + (|I_{abc}| - |I_{bac}| + |I_{cab}|) - |I_{cab}| - 1/2|I_{bac}| - 1/2|I_{bca}| - 1/2|I_{cba}| \\ &= |I_{abc}| - |I_{bac}| - 1/2|I_{bca}| - 1/2|I_{cab}| - 1/2|I_{cba}| \\ &= 3/2|I_{bca}| + 3/2|I_{cab}| + 3/2|I_{cba}| \geq 3/2, \end{aligned}$$

so that both a and b are possible outcomes, depending on $\rhd$. If $b \rhd a$, then b is elected while any unilateral deviation to some $\tilde{v}_i \in V_i^1$ by some $i \in I_{bac} \cup I_{bca} \cup I_{cab}$ would yield outcome a. Hence, for $i \in I_{bac} \cup I_{bca} \cup I_{cab}$, $(0, 1, 1/2)$ is the unique best response and thus undominated in any game $\Gamma(\succ_I, V^n)$ where $v' \in V^n \subset V^1$.

It remains to check that for i_{abc}, $(1, 0, 1/2)$ and $(1, 1/2, 0)$ are undominated in any game $\Gamma(\succ_I, V^n)$ where $v, v' \in V^n \subset V^1$.

For that, consider again profile v where $v_i = (1, 0, 1/2)$ and assume that $c \rhd b, a$, so that c is elected. A switch by i to $(1/2, 1, 0)$ would also yield c, as we would now have $|v^a| = |v^c|$ and $|v^a| \geq |v^b|$. On the other hand, a switch to $(1, 1/2, 0)$ would yield a, as we would now have $|v^a| > |v^c|$ and $|v^a| > |v^b|$. Hence, for i_{abc}, $(1, 1/2, 0)$ is the unique best response and thus undominated in any game $\Gamma(\succ_I, V^n)$ where $v, v' \in V^n \subset V^1$.

Similarly, consider profile v' where some i_{abc} chooses $v'_i = (1, 1/2, 0)$ and assume that $b \rhd a, c$, so that b is elected. A switch by i to $(1/2, 1, 0)$ would yield b, as we would now have $|v^b| > |v^a|$ and $|v^a| \geq |v^c|$. On the other hand, a switch to $(1, 0, 1/2)$ would yield a, as we would now have $|v^a| > |v^b|$ and $|v^a| > |v^c|$. Hence, for i_{abc}, $(1, 0, 1/2)$ is the unique best response and thus undominated in any game $\Gamma(\succ_I, V^n)$ where $v, v' \in V^n \subset V^1$.

Case 2: $|I_{abc}| = |O_{abc}| - 1 = |I_{abc}| + |I_{bac}| + 2|I_{bca}| + 2|I_{cab}| + 2|I_{cba}| - 1$

By the assumptions of Theorem 3, we have $|I_{acb}| = 0$. Moreover, we know that $|I_{bca}| + |I_{cab}| + |I_{cba}| > 0$ as otherwise $I = I_{abc} \cup I_{bac}$ and $|I_{abc}| = |I_{bac}| - 1$; this would imply $|I_{bac}| > |I_{abc}| = |I_{bca}| + |I_{abc}| + 2|I_{acb}| + 2|I_{cba}| + 2|I_{cab}|$ and hence violate the assumptions of Theorem 3.

First, assume that $|I_{bca}| = 1$ and $|I_{bac}| + |I_{cab}| + |I_{cba}| = 0$ so that $|I_{abc}| = 1$. Let $i \in I_{abc}$ choose $v_i = (1, 0, 1/2)$ and $j \in I_{bac}$ choose $v_j = (0, 1, 1/2)$. Then, $|v^a| = |v^b| = |v^c| = 1$ so that by claim 1 each v_i is undominated and hence no outcome can be ruled out via iterated elimination of dominated strategies.

Next, assume that either $|I_{bca}| \neq 1$ or $|I_{bac}| + |I_{cab}| + |I_{cba}| > 0$. We construct a ballot profile v as follows:

- some $j \in I_{abc}$ chooses $v_j = (1, 1/2, 0)$
- $|I_{bac}| + |I_{bca}| + |I_{cab}| + |I_{cba}| - 1$ of I_{abc} choose $v_j = (1, 0, 1/2)$
- $|I_{bca}| + |I_{cab}| + |I_{cba}| - 1$ of I_{abc} choose $v_j = (1/2, 1, 0)$
- all $j \in I_{bac}$ choose $v_j = (0, 1, 1/2)$,
- all $j \in I_{bca} \cup I_{cab} \cup I_{cba}$ choose $v_j = (0, 1/2, 1)$.

Then,

$$|v^a| = |v^b| = |v^c| = |I_{bac}| + 3/2|I_{bca}| + 3/2|I_{cab}| + 3/2|I_{cba}| - 1/2,$$

and any candidate may win, depending on $\triangleright$.

To see that each strategy used in the construction of v is undominated in $\Gamma(\succ_I, V^n)$ where $v \in V^n \subset V^1$, consider $i \in I_{abc}$ who chooses $v_i = (1, 1/2, 0)$. By claim 1, v_i is undominated. Moreover, if $c \triangleright a, b$, then outcome c is realized. Only a switch to $\tilde{v}_i = (1/2, 1, 0)$ would yield b, while a switch to $\tilde{v}_i = (1, 0, 1/2)$ would yield c as well. Hence, $v_i = (1/2, 1, 0)$ is undominated.

If there is some $i \in I_{abc}$ who votes $v_i = (1, 0, 1/2)$ (i.e. if $|I_{bac}|+|I_{bca}|+|I_{cab}|+|I_{cba}|-1 > 0$), then $v_i = (1, 0, 1/2)$ is undominated by claim 1. Similarly, for each $j \in I_{bac} \cup I_{cab} \cup I_{cba}$, strategy v_j is undominated by claim 1.

Finally, assume that $|I_{bca}| > 0$ so that there is some $j \in I_{bca}$ who chooses $v_j = (0, 1/2, 1)$. Then either $|I_{bca}| > 1$, or $|I_{bca}| = 1$ and $|I_{bac}| + |I_{cab}| + |I_{cba}| > 0$, so that in either case $|I_{bac}|+|I_{bca}|+|I_{cab}|+|I_{cba}|-1 > 0$ and $v_i = (1, 0, 1/2)$ is undominated for $i \in I_{abc}$. Then, letting voter $i \in I_{abc}$ who chooses $v_i = (1, 1/2, 0)$ switch to $\tilde{v}_i = (1, 0, 1/2)$ yields $|\tilde{v}^a| = |v^a|$, $|\tilde{v}^b| = |v^b| - 1/2$ and $|\tilde{v}^c| = |v^c| + 1/2$, so that $j \in I_{bca}$'s second most preferred candidate c wins. A switch by j to $(0, 1, 1/2)$ would again equalize candidates' scores and render j's least preferred candidate a a possible outcome. A switch to $(1/2, 1, 0)$ would even yield a independent of $\triangleright$. Hence, $v_j = (0, 1/2, 1)$ is undominated.

Case 3: $|I_{abc}| = |O_{abc}| - 2 = |I_{abc}| + |I_{bac}| + 2|I_{bca}| + 2|I_{cab}| + 2|I_{cba}| - 2$

Assume first that $|I_{acb}| + |I_{bca}| + |I_{cab}| + |I_{cba}| \geq 2$ and $|I_{bac}| + |I_{bca}| + |I_{cab}| + |I_{cba}| \geq 2$. We construct a ballot profile v for which each candidate is a possible outcome as follows:

- 2 of I_{abc} chooses $v_j = (1, 1/2, 0)$
- $|I_{acb}| + |I_{bca}| + |I_{cab}| + |I_{cba}| - 2$ of I_{abc} choose $v_j = (1/2, 1, 0)$
- $|I_{bac}| + |I_{bca}| + |I_{cab}| + |I_{cba}| - 2$ of I_{abc} choose $v_j = (1, 0, 1/2)$
- all $j \in I_{bac}$ choose $v_j = (0, 1, 1/2)$,
- all $j \in I_{bca} \cup I_{cab} \cup I_{cba}$ choose $v_j = (0, 1/2, 1)$.

Then,

$$|v^a| = |v^b| = |v^c| = |I_{bac}| + 3/2|I_{bca}| + 3/2|I_{cab}| + 3/2|I_{cba}| - 1,$$

and any candidate may win, depending on $\triangleright$.

In light of claim 1, we only need to check the undominatedness of strategies $v_i = (1/2, 1, 0)$, $i \in I_{abc}$, and $v_i = (0, 1/2, 1)$, $i \in I_{bca}$. For that, note that if $c \triangleright a, b$, then outcome c is realized. A switch by some $i \in I_{abc}$ with $v_i = (1, 1/2, 0)$ to $\tilde{v}_i = (1/2, 1, 0)$ would yield b, while a switch to $\tilde{v}_i = (1, 0, 1/2)$ would yield c as well. Hence, $v_i = (1/2, 1, 0)$ is undominated.

If there is some $i \in I_{bca}$ who votes $v_i = (0, 1/2, 1)$, let her switch to $\tilde{v}_i = (0, 1, 1/2)$. In addition, let some $j \in I_{abc}$ switch from $v_j = (1, 1/2, 0)$ to $\tilde{v}_j = (1, 0, 1/2)$. Then $|\tilde{v}^a| = |\tilde{v}^b| = |\tilde{v}^c|$ and by claim 1, both $\tilde{v}_i$ and $\tilde{v}_j$ are undominated in any game $\Gamma(\succ_i, V')$ where $\tilde{v} \in V'$. Moreover, for ballot profile $\tilde{v}$, if $a \triangleright b, c$, then i_{bca}'s least preferred candidate a is elected. Only a switch to $v_i = (0, 1/2, 1)$ can prevent this and yields c. Hence $v_i = (0, 1/2, 1)$ is undominated.

Now, assume that $|I_{acb}| + |I_{bca}| + |I_{cab}| + |I_{cba}| < 2$ or $|I_{bac}| + |I_{bca}| + |I_{cab}| + |I_{cba}| < 2$. This can be split up further as follows:

(1) $|I_{bca}| + |I_{cab}| + |I_{cba}| = 0$ and $|I_{acb}| < 2$
(2) $|I_{bca}| + |I_{cab}| + |I_{cba}| = 0$ and $|I_{bac}| < 2$

(3) $|I_{bca}| + |I_{cab}| + |I_{cba}| = 1$ and $|I_{acb}| = 0$ and $|I_{bac}| = 0$
(4) $|I_{bca}| + |I_{cab}| + |I_{cba}| = 1$ and $|I_{acb}| = 0$ and $|I_{bac}| > 0$
(5) $|I_{bca}| + |I_{cab}| + |I_{cba}| = 1$ and $|I_{acb}| > 0$ and $|I_{bac}| = 0$

Consider (1): Since $|I_{bca}| + |I_{cab}| + |I_{cba}| = 0$, it follows

$$|I_{abc}| = |I_{acb}| + |I_{bac}| - 2 \leq |I_{acb}| + \big(|I_{abc}| + 2|I_{acb}| - 2\big) - 2 = |I_{abc}| + 3|I_{acb}| - 4$$

which implies $|I_{acb}| \geq 2$ – a contradiction.

Consider (2): Since $|I_{bca}| + |I_{cab}| + |I_{cba}| = 0$, it follows

$$|I_{abc}| = |I_{acb}| + |I_{bac}| - 2 \leq \underbrace{\big(|I_{abc}| + 2|I_{bac}| - 2\big)}_{\geq |I_{acb}|} + |I_{bac}| - 2 = |I_{abc}| + 3|I_{bac}| - 4$$

which implies $|I_{bac}| \geq 2$ – a contradiction.

Consider (3): Then $|I_{abc}| = |I_{acb}| + |I_{bac}| + 2(|I_{bca}| + |I_{cab}| + |I_{cba}|) - 2 = 0$ so that I consists of a single voter $i \in I_{bac} \cup I_{cab} \cup I_{cba}$ – a contradiction to the assumptions of Theorem 3.

Consider (4): Then $|I_{abc}| = |I_{bac}| > 0$. Moreover $|I_{bca}| = 0$ as otherwise $|I_{bac}| - |O_{bac}| = |I_{bac}| - |I_{bca}| - |I_{abc}| = -1 > -2 = |I_{abc}| - |I_{bac}| - 2|I_{bca}| = |I_{abc}| - |O_{bca}|$. Construct ballot profile v as follows.

- some $j \in I_{abc}$ chooses $v_j = (1, 1/2, 0)$
- remaining $j \in I_{abc}$ choose $v_j = (1, 0, 1/2)$
- all $j \in I_{bac}$ choose $v_j = (0, 1, 1/2)$
- $j \in I_{cab} \cup I_{cba}$ chooses $v_j = (1/2, 0, 1)$

Then $|v^a| = |v^b| = |v^c| = |I_{abc}| + 1/2 = |I_{bac}| + 1/2$ and each strategy v_j is undominated by Claim 1.

Consider (5): Then $|I_{abc}| = |I_{acb}| > 0$. Moreover $|I_{cab}| = 0$ as otherwise $|I_{acb}| - |O_{acb}| = |I_{acb}| - |I_{abc}| - |I_{cab}| = -1 > -2 = |I_{abc}| - |I_{bac}| - 2|I_{bca}| = |I_{abc}| - |O_{bca}|$. We will construct tree strategy profiles v, $\tilde{v}$ and $\hat{v}$ and show that each strategy used in the construction is undominated in any game $\Gamma(\succ_I, V')$ where $v, \tilde{v}, \hat{v} \in V'$. First construct ballot profile v as follows.

- some $j \in I_{abc}$ chooses $v_j = (1, 1/2, 0)$
- all remaining $j \in I_{abc}$ choose $v_j = (1/2, 1, 0)$
- all $j \in I_{acb}$ choose $v_j = (1, 1/2, 0)$
- $j \in I_{bca} \cup I_{cba}$ chooses $v_j = (0, 1, 1/2)$

Then $|v^a| = 3/2|I_{abc}| + 1/2$, $|v^b| = 3/2|I_{abc}| + 1/2$ and $|v^c| = 1/2$ and if $b \triangleright a$, b is chosen. A voter $j \in I_{abc} \cup I_{acb}$ who votes $v_j = (1, 1/2, 0)$ could change the outcome to a by switching to $(1, 0, 1/2)$, but not by switching to any other strategy. Hence for $j \in I_{abc} \cup I_{acb}$, $(1, 0, 1/2)$ is undominated. If voter $j \in I_{bca} \cup I_{cba}$ would switch to any other strategy, the outcome would also be a, so that for her $(0, 1, 1/2)$ is established to be undominated.

Next construct ballot profile $\tilde{v}$ as follows.

- all $j \in I_{abc}$ choose $\tilde{v}_j = (1, 0, 1/2)$
- some $j \in I_{acb}$ chooses $\tilde{v}_j = (1, 0, 1/2)$
- all remaining $j \in I_{acb}$ choose $\tilde{v}_j = (1/2, 0, 1)$
- $j \in I_{bca} \cup I_{cba}$ chooses $\tilde{v}_j = (0, 1/2, 1)$

Then $|\tilde{v}^a| = 3/2|I_{abc}| + 1/2$, $|\tilde{v}^b| = 1/2$ and $|\tilde{v}^c| = 3/2|I_{abc}| + 1/2$ and if $c \rhd a$, c is chosen. A voter $j \in I_{abc} \cup I_{acb}$ who votes $\tilde{v}_j = (1, 0, 1/2)$ could change the outcome to a by switching to $(1, 1/2, 0)$, but not by switching to any other strategy. Hence for $j \in I_{abc} \cup I_{acb}$, $(1, 1/2, 0)$ is undominated. If voter $j \in I_{bca} \cup I_{cba}$ would switch to any other strategy, the outcome would also be a, so that for her $(0, 1/2, 1)$ is established to be undominated.

Finally construct ballot profile $\hat{v}$ as follows.

- one $j \in I_{abc}$ chooses $\hat{v}_j = (1, 1/2, 0)$
- all remaining $j \in I_{abc}$ choose $\hat{v}_j = (1/2, 1, 0)$
- all $j \in I_{acb}$ choose $\hat{v}_j = (1/2, 0, 1)$
- $j \in I_{bca} \cup I_{cba}$ chooses $\hat{v}_j = (0, 1, 1/2)$

Then $|\hat{v}^a| = |\hat{v}^b| = |\hat{v}^c| = |I_{abc}| + 1/2$, so that for $c \rhd a, b$, outcome c is realized. A switch by $j \in I_{abc}$ to $(1, 0, 1/2)$ would also yield c, but a switch to $(1/2, 1, 0)$ yields b. Hence, for $j \in I_{abc}$, $(1/2, 1, 0)$ is undominated. A construction symmetric to $\hat{v}$ shows that for $j \in I_{abc}$, $(1/2, 0, 1)$ is undominated, which completes the proof for Case 3.

Case 4: $|I_{abc}| < |O_{abc}| - 2$

Then, $|I_{xyz}| < |O_{xyz}| - 2$ for all $x, y, z \in A$ and Lemma 1 completes the proof.

□

Proof of Theorem 4. We first consider positional scoring rules with $s < \frac{1}{2}$ and show that for any fixed s, there exist preference profiles with $I = I_{abc} \cup I_{acb}$, where the induced voting game fails to elect a after iterated elimination of dominated strategies.

Assume that $|I_{abc}| = |I_{acb}| = n$ with $n > \frac{2-2s}{1-2s} \geq 2$. We will show that the ballot profile v, given by $v_{i_{abc}} = (s, 1, 0)$ and $v_{i_{acb}} = (s, 0, 1)$ respectively, survives the iterative elimination of dominated strategies.

Consider $\Gamma(\succ_I, V^1)$ and assume that all voters $i \in I_{abc}$ chose $v_i = (s, 1, 0)$ while voters $i \in I_{acb}$ chose $v_i = (s, 0, 1)$. Then $|v^b| = |v^c| = n$ while $|v^b| - |v^a| = |v^c| - |v^a| = n - 2ns = n(1 - 2s) > 2 - 2s > 1$. Thus, the winner is either b or c, depending on $\rhd$. If i_{abc} would switch to a different strategy, $(1, s, 0), (1, 0, s) \in V^1_{i_{abc}}$ that awards fewer points to candidate b, c would win the election independent of $\rhd$. Hence, neither $(1, s, 0)$ nor $(1, 0, s)$ dominate $(s, 1, 0)$ for voter i_{abc}, so that $v_{i_{abc}} = (s, 1, 0) \in V^2_{i_{abc}}$.

A symmetric argument applies to i_{acb} for whom $v_i = (s, 0, 1) \in V^2_{acb}$. But then, we can again consider the ballot profile v in $\Gamma(\succ_I, V^2)$ and show that neither strategy is dominated and eliminated as we move to V^3. By induction it follows that the two strategies are never eliminated.

Moreover, we have already seen that for strategy profile v, candidate a does not win the election which concludes the proof for the case $s < \frac{1}{2}$.

Next, we consider the case of Antiplurality, i.e. $s = 1$. Assume that all voters agree on the ranking $a \succ_i b \succ_i c$, so that $V^1_i = \{(1, 1, 0), (1, 0, 1)\}$. If in $\Gamma(\succ_I, V^1)$ all voters $j \neq i$ chose $v_j = (1, 1, 0)$, then i can ensure the election of a by casting the ballot $v_i = (1, 0, 1)$, whereas $v'_i = (1, 1, 0)$ would lead to the election of b whenever $b \rhd a$. Hence, $(1, 0, 1)$ is not dominated. Similarly, if all $j \neq i$ cast ballot $v_j = (1, 0, 1)$ and the tiebreaker chooses $b \rhd a$, i's unique best reply is $v_i = (1, 1, 0)$. But then, voters' strategy sets cannot be narrowed down any further than $V^1_i = \{(1, 1, 0), (1, 0, 1)\}$, so that a is not the unique solution in iteratively undominated strategies.

Last, consider the case $s \in (\frac{1}{2}, 1)$. Assume that $I = I_{abc} \cup I_{bac}$ and $|I_{abc}| = n+1 > n = |I_{bac}|$ with $n > \frac{2}{(2s-1)(1-s)} > 2$, so that in particular $2sn - n > 2$ and

$$(\star\star) \qquad s > \frac{n+2}{2n} \qquad \text{and} \qquad n > \frac{n+2}{2s}.$$

We will show that in the process of iterative elimination, strategies $(1,s,0), (1,0,s) \in V^1_{abc}$ and $(s,1,0), (0,1,s) \in V^1_{bac}$ are never weakly dominated and hence not eliminated. But then, b remains a possible outcome throughout the sequence of restricted games: if all i_{abc} vote $(1,s,0)$ while all i_{bac} vote $(0,1,s)$, candidates scores are

$$|v^a| = n+1, \qquad |v^b| = n + s(n+1), \qquad |v^c| = sn.$$

As $s > \frac{1}{2}$ and $n > 1$, candidate b then wins the election.

First, let us remind ourselves that the sets of undominated strategies are

$$V^1_{abc} = \{(1,s,0),(1,0,s),(s,1,0)\} \text{ and } V^1_{bac} = \{(s,1,0),(0,1,s),(1,s,0)\}.$$

To show that $\{(1,s,0),(1,0,s)\} \subseteq V^{m+1}_{abc} \subseteq V^m_{abc}$ and $\{(s,1,0),(0,1,s)\} \subseteq V^{m+1} \subseteq V^m_{bac}$ for all $m \geq 1$ we consider 6 cases.

Case 1: *For $i \in I_{abc}$, $(1,s,0)$ can be a better reply than $(1,0,s)$ in $\Gamma_s(\succ_I, V^m)$.* Consider the situation of $i \in I_{abc}$ who faces an opposing strategy profile where

- $n - x$ voters $j \in I_{abc}$ vote $v_j = (1,s,0)$,
- x voters $j \in I_{abc}$ vote $v_j = (1,0,s)$,
- all n voters $j \in I_{bac}$ vote $v_j = (0,1,s)$,
- $x = \lceil \frac{n}{2s} - \frac{1}{2} \rceil$,
- $c \rhd b$.

This profile is well defined, as

$$x = \left\lceil \frac{n}{2s} - \frac{1}{2} \right\rceil < \frac{n}{2s} + \frac{1}{2} < \frac{n}{2s} + \frac{1}{2s} < n+1.$$

If i chooses $v_i = (1,s,0)$, the associated candidates' scores are $|v^a| = n+1$, $|v^b| = s(n-x+1)+n$ and $|v^c| = s(n+x)$. Then, b wins as its score is larger than c's

$$|v^b| - |v^c| = n + s - 2sx > n + s - 2s\left(\frac{n}{2s} + \frac{1}{2}\right) = 0,$$

while c's score is larger than a's:

$$|v^a| - |v^c| = n + 1 - sn - sx \leq n + 1 - sn - s\left(\frac{n}{2s} - \frac{1}{2}\right) = \underbrace{\frac{n+1+s}{2}}_{<sn,\text{ see } (\star\star)} - sn < 0.$$

If on the other hand i chooses $v_i = (1,0,s)$, b's score is at most as high as c's, so that b never wins (ties are broken in favour of c):

$$|v^b| - |v^c| = n - s - 2sx \leq n - s - 2s\left(\frac{n}{2s} - \frac{1}{2}\right) = 0.$$

Instead, c would win as its score has increased an hence is still larger than a's.

Case 2: *For $i \in I_{abc}$, $(1,s,0)$ can be a better reply than $(s,1,0)$ in $\Gamma(\succ_I, V^m)$.* (This case is only relevant if $(s,1,0) \in V^m_{abc}$). Consider the situation of $i \in I_{abc}$ who faces an opposing strategy profile where

- n voters $j \in I_{abc}$ vote $v_j = (1,s,0)$,
- n voters $j \in I_{bac}$ vote $v_j = (s,1,0)$.

Ballot $v_i = (1,s,0)$ would elect a, whereas $(s,1,0)$ would elect b.

Together, case 1 and 2 imply that $(1,s,0) \in V^{m+1}_{abc}$. Next, we show that $(1,0,s) \in V^{m+1}_{abc}$.

Case 3: *For $i \in I_{abc}$, $(1,0,s)$ can be the unique best reply in $\Gamma(\succ_I, V^m)$:*
Consider the situation of $i \in I_{abc}$ who faces an opposing strategy profile where

- n voters $j \in I_{abc}$ vote $v_j = (1,s,0)$,
- 1 voter $j \in I_{bac}$ vote $v_j = (0,1,s)$,
- $n-1$ voters $j \in I_{bac}$ vote $v_j = (s,1,0)$.

Ballot $v_i = (1,0,s)$ would then elect a, as $|v^a| - |v^b| = 1-s > 0$. Should i choose $(1,s,0)$, b would be elected as we would have $|v^a| - |v^b| = 1-2s < 0$. Ballot $v_i = (s,1,0)$ would only further increase b's lead over a.

Case 4: *For $i \in I_{bac}$, $(0,1,s)$ can be the unique best reply in $\Gamma(\succ_I, V^m)$*: Consider the situation of $i \in I_{bac}$ who faces an opposing strategy profile where

- $n+1$ voters $j \in I_{abc}$ vote $v_j = (1,s,0)$,
- $n-1$ voters $j \in I_{bac}$ vote $v_j = (s,1,0)$.

Ballot $v_i = (0,1,s)$ would then elect b, as $|v^a| - |v^b| = 1-2s < 0$. Should i choose $(s,1,0)$, a would be elected, as we would have $|v^a| - |v^b| = 1-s > 0$. Ballot $v_i = (1,s,0)$ would only further increase a's lead over b.

From case 4, we learn that $(0,1,s) \in V^{m+1}_{bac}$. The last two cases establish that $(s,1,0) \in V^{m+1}_{bac}$, which concludes the proof.

Case 5: *For $i \in I_{bac}$, $(s,1,0)$ can be a better reply than $(1,s,0)$ in $\Gamma(\succ_I, V^m)$:*
(This case is only relevant if $(1,s,0) \in V^m_{i_{bac}}$). Consider the situation of $i \in I_{bac}$ who faces an opposing strategy profile where

- $n+1$ voters $j \in I_{abc}$ vote $v_j = (1,s,0)$,
- x voters $j \in I_{bac}$ vote $v_j = (1,s,0)$,
- 1 voters $j \in I_{bac}$ vote $v_j = (0,1,s)$,
- $n-2-x$ voters $j \in I_{bac}$ vote $v_j = (s,1,0)$,
- $x = \left\lceil \frac{4s-3}{2-2s} \right\rceil \geq 0$,
- $a \rhd b$.

This profile is well defined, as $x < n-2$:
$$x = \left\lceil \frac{4s-3}{2-2s} \right\rceil < \frac{4s-3}{2-2s} + 1 = \frac{2s-1}{2-2s} < \frac{2s-1}{1-s} = \frac{1}{1-s} - 2 < n-2.$$
If i chooses $v_i = (s,1,0)$, b wins the election as its score is larger than $|v^c| = s$ and larger than a's score $|v^a|$:
$$|v^a| - |v^b| = 1-2s+(2-2s)x < 1-2s+(2-2s)\left(\frac{4s-3}{2-2s}+1\right) = 1-2s+4s-3+2-2s = 0.$$
If on the other hand, i chooses $v_i = (1,s,0)$, b's score is weakly less than a's:
$$|v^a| - |v^b| = 3-4s+(2-2s)x \geq 3-4s+(2-2s)\left(\frac{4s-3}{2-2s}\right) = 0.$$
As ties are broken in favour of a, b would lose the election.

Case 6: *For $i \in I_{bac}$, $(s,1,0)$ can be a better reply than $(0,1,s)$ in $\Gamma(\succ_I, V^m)$:*
Consider the situation of $i \in I_{bac}$ who faces an opposing strategy profile where

- $n+1$ voters $j \in I_{abc}$ vote $v_j = (1,0,s)$,
- $n-x-1$ voters $j \in I_{bac}$ vote $v_j = (s,1,0)$,

- x voters $j \in I_{bac}$ vote $v_j = (0, 1, s)$,
- $x = \lceil \frac{n+1}{2s} - \frac{3}{2} \rceil$,
- $c \triangleright a$.

This profile is well defined, since

$$x = \left\lceil \frac{n+1}{2s} - \frac{3}{2} \right\rceil < \frac{n+1}{2s} - \frac{1}{2} = \frac{n+2}{2s} - \frac{1+s}{2s} \overset{(\star\star)}{<} n - \frac{1+s}{2s} < n$$

and

$$x \geq \frac{\overbrace{n+1}^{>3}}{\underbrace{2s}_{<2}} - \frac{3}{2} > 0.$$

If i chooses $v_i = (s, 1, 0)$, the associated candidates' scores are $|v^a| = n + 1 + s(n - x)$, $|v^b| = n$ and $|v^c| = s(n + 1 + x)$. Hence, a is elected with a higher score than b and c:

$$|v^a| - |v^c| = n + 1 - s - 2sx > n + 1 - s - 2s\left(\frac{n+1}{2s} - \frac{1}{2}\right) = 0.$$

If on the other hand, i chooses $v_i = (0, 1, s)$, a's score is weakly less than c's:

$$|v^a| - |v^c| = n + 1 - 3s - 2sx \leq n + 1 - 3s - 2s\left(\frac{(n+1)}{2s} - \frac{3}{2}\right) = 0.$$

As ties are broken in favour of c, and $|v^c| \geq |v^a| \geq n + 1 > n = |v^b|$, c would be elected. □

Proof of Theorem 5. Let us first analyse scoring rules where V consist of all permutations of $(1, 1, 0)$ and $(s, s, 0)$. For that, consider a preference profile where all voters share the same preferences, $a \succ_i b \succ_i c$.

Claim 1. *For all i, if V_i^m includes at least one of the two ballots $(1, 0, 1)$ or $(s, 0, s)$ as well at least one of the two ballots $(1, 1, 0)$ or $(s, s, 0)$ then after eliminating strategies that are dominated in the game $\Gamma(\succ_I, V^m)$, V_i^{m+1} will contain at least one of the two ballots $(1, 0, 1)$ or $(s, 0, s)$ as well at least one of the two ballots $(1, 1, 0)$ or $(s, s, 0)$.*

Proof. In the game $\Gamma(\succ_I, V^m)$, consider the ballot profile v where all voters choose either $(1, 0, 1)$ or $(s, 0, s)$ so that $|v^a| = |v^c| > |v^b|$ and c is elected if $c \triangleright a \triangleright b$. If an individual voter i switches to $\tilde{v}_i = (1, 1, 0)$ or $\tilde{v}_i = (s, s, 0)$, the outcome is a. If instead she would switch to $(0, 1, 1)$ or $(0, s, s)$ (provided that these are still included in V_i^m), the outcome would be c as well. Hence, at least one of the ballots $(1, 1, 0)$, $(s, s, 0)$ is undominated and included in V_i^{m+1}.

Analogously, consider the ballot profile v where all voters choose either $(1, 1, 0)$ or $(s, s, 0)$ so that $|v^a| = |v^b| > |v^c|$ and b is elected if $b \triangleright a \triangleright c$. If an individual voter i switches to $\tilde{v}_i = (1, 0, 1)$ or $\tilde{v}_i = (s, 0, s)$, the outcome is a. If instead she would switch to $(0, 1, 1)$ or $(0, s, s)$ (provided that these are still included in V_i^m), the outcome would be b as well. Hence, at least one of the ballots $(1, 0, 1)$, $(s, 0, s)$ is undominated and included in V_i^{m+1}. *Claim 1* ◇

Since initially they are included in the set of admissible ballots, at least on of $(1, 1, 0)$ and $(s, s, 0)$ survives the process of iterative elimination of dominated strategies. Then, in the game $\Gamma(\succ_I, V^{\overline{m}})$, if all voters choose either $(1, 1, 0)$ or $(s, s, 0)$, b is a possible outcome and hence included in $S(\succ_I, V)$. Thus, such a scoring rule violates **MEW** (as well as **U**).

Next, let us analyse scoring rules where V consist of all permutations of $(1,0,0)$ and $(s,0,0)$. If $s=1$, the rule is the Plurality rule, for which we know by Theorem 4 that it violates **U**. If $s<1$, consider a preference profile such that $I = I_{abc} \cup I_{acb}$ and $|I_{abc}| = |I_{acb}| > 1$.

Claim 2. *If V^m_{abc} includes $(0,1,0)$ while V^m_{acb} includes $(0,0,1)$, then both strategies are undominated in the game $\Gamma(\succ_I, V^m)$ and V^{m+1}_{abc} includes $(0,1,0)$ while V^{m+1}_{acb} includes $(0,0,1)$.*

Proof. In the game $\Gamma(\succ_I, V^m)$, consider the ballot profile v where all voters $i \in I_{abc}$ choose $(0,1,0)$ while all $i \in I_{acb}$ choose $(0,0,1)$, so that $|v^b| = |v^c| > |v^a|+1$. For $b \rhd c$, the outcome is b. If an individual voter $i \in I_{abc}$ switches to $(1,0,0)$, $(0,0,1)$, $(s,0,0)$, $(0,s,0)$ or $(0,0,s)$ the outcome is c, as it has the highest score. Hence, $v_i = (0,1,0)$ is undominated and included in V^{m+1}_{abc}. By a symmetric argument, $(0,0,1)$ is included in V^{m+1}_{acb}. *Claim 2* ◇

By induction, we know that $(0,1,0) \in V^{\overline{m}}_{abc}$ and $(0,0,1) \in V^{\overline{m}}_{acb}$. Then, in the game $\Gamma(\succ_I, V^{\overline{m}})$, all voters $i \in I_{abc}$ choose $(0,1,0)$ while all $i \in I_{acb}$ choose $(0,0,1)$, the outcome is either b or c. Thus, the scoring rule violates **U**.

Now, let us consider vote-splitting scoring rules, i.e. scoring rules where V consists of all permutations of $(s,s,0)$ and $(1-s,0,0)$. We want to show that such a rule violates unanimity if $s < 1/2$. For that, consider a profile such that $I = I_{abc} \cup I_{acb}$ and $|I_{abc}| = |I_{acb}| > 1$.

Claim 3. *If V^m_{abc} includes $(0,1-s,0)$ while V^m_{acb} includes $(0,0,1-s)$, then both strategies are undominated in the game $\Gamma(\succ_I, V^m)$ and V^{m+1}_{abc} includes $(0,1-s,0)$ while V^{m+1}_{acb} includes $(0,0,1)$.*

Proof. In the game $\Gamma(\succ_I, V^m)$, consider the ballot profile v where all voters $i \in I_{abc}$ choose $(0,1-s,0)$ while all $i \in I_{acb}$ choose $(0,0,1-s)$, so that $|v^b| = |v^c| > 1$ while $|v^a| = 0$. For $b \rhd c$, the outcome is b. If an individual voter $i \in I_{abc}$ switches to $(1-s,0,0)$, $(0,0,1-s)$, $(s,s,0)$, $(0,s,s)$ or $(s,0,s)$ the outcome is c, as it has the highest score. Hence, $v_i = (0,1-s,0)$ is undominated and included in V^{m+1}_{abc}. By a symmetric argument, $(0,0,1-s)$ is included in V^{m+1}_{acb}. *Claim 3* ◇

By induction, we know that $(0,1-s,0) \in V^{\overline{m}}_{abc}$ and $(0,0,1-s) \in V^{\overline{m}}_{acb}$. Then, in the game $\Gamma(\succ_I, V^{\overline{m}})$, all voters $i \in I_{abc}$ choose $(0,1-s,0)$ while all $i \in I_{acb}$ choose $(0,0,1-s)$, the outcome is either b or c. Thus, the scoring rule violates **U**.

Finally, we want to show that a vote-splitting scoring rule violates **MEW** if $s \in (1/2, 1)$ ($s=1$ corresponds to the Antiplurality Rule, for which we know from Theorem 4 that it violates **MEW**). For that, consider a profile such that $I = I_{abc} \cup I_{bac}$, $|I_{abc}| = n+1$ and $|I_{bac}| = n > \frac{1}{(1-s)(2s-1)}$. We will show that strategies $(s,s,0), (s,0,s), (1-s,0,0) \in V^m_{i_{abc}}$ and $(0,s,s), (0,1-s,0) \in V^m_{i_{bac}}$ are undominated in $\Gamma(\succ_I, V^m)$ and hence included in $V^{m+1}_{i_{abc}}$ and $V^{m+1}_{i_{bac}}$ respectively.

(i) *For $i \in I_{abc}$, $(s,s,0)$ is undominated in $\Gamma(\succ_I, V^m)$.*

Consider the the ballot profile v where

- i votes $v_i = (s,s,0)$
- one $j \in I_{abc}$ votes $v_j = (s,0,s)$
- remaining $n-1$ of I_{abc} vote $v_j = (1-s,0,0)$
- all n of I_{bac} vote $v_j = (0,s,s)$

Then $|v^b| = |v^c| = s(n+1)$ and $|v^a| = 2s + (1-s)(n-1)$, so that

$$|v^a| - |v^b| = -sn + 3s + n - 1 - sn - s = (1-n)(2s-1) < 0$$

and b is elected for $b \rhd c$. A switch by i to any other ballot $\tilde{v}_i \in V$ would never raise the score of a and would either reduce the score of b or increase the score of c, thereby changing the outcome to c. Hence $(s,s,0)$ is undominated.

(ii) *For $i \in I_{abc}$, $(s,0,s)$ is undominated.*
Consider the the ballot profile v where

- i votes $v_i = (s,0,s)$
- n of I_{abc} vote $v_j = (1-s,0,0)$
- $n - \left\lfloor \frac{s}{2s-1} \right\rfloor$ of I_{bac} vote $v_j = (0,1-s,0)$
- $\left\lfloor \frac{s}{2s-1} \right\rfloor$ of I_{bac} vote $v_j = (0,s,s)$

Then

$$|v^a| - |v^b| = s + (1-2s)\left\lfloor \frac{s}{2s-1} \right\rfloor \geq s + (1-2s)\frac{s}{2s-1} = 0$$

and

$$|v^a| - |v^b| = s + (1-2s)\left\lfloor \frac{s}{2s-1} \right\rfloor < s + (1-2s)\left(\frac{s}{2s-1} - 1\right) = 2s - 1.$$

Moreover, $|v^a| - |v^c| = n(1-s) - s\left\lfloor \frac{s}{2s-1} \right\rfloor > \frac{1}{2s-1} - \frac{s^2}{2s-1} > 0$ so that and a is elected for $a \rhd b$. A switch by i to ballot $(1-s,0,0)$ would change the score difference $|v^a| - |v^b|$ by $-s + (1-s) = 1-2s$ so that b overtakes a. As any other ballot would change the difference $|v^a| - |v^b|$ even more in b's favour, we conclude that $(s,0,s)$ is undominated.

(iii) *For $i \in I_{abc}$, $(1-s,0,0)$ is not dominated by $(s,0,s)$ or $(0,0,1-s)$.*
Consider the ballot profile v where all $j \in I_{abc}$ vote $v_j = (1-s,0,0)$ while all $j \in I_{bac}$ vote $v_j = (0,s,s)$. Then $|v^b| = |v^c| = sn$ which is larger than $|v^a| = (1-s)(1+n)$as n is large. Then for $b \rhd c$, b is elected while a switch by i to $(s,0,s)$ or $(0,0,1-s)$ would yield c as outcome.

(iv) *For $i \in I_{abc}$, $(1-s,0,0)$ is not dominated by $(s,s,0)$, $(0,s,s)$, $(0,1-s,0$ or $(0,0,1-s)$.*
Consider the ballot profile v where all $j \in I_{abc}$ vote $v_j = (1-s,0,0)$ while all $j \in I_{bac}$ vote $v_j = (0,1-s,0)$. Then $|v^a| - |v^b| = 1-s$ and $|v^c| = 0$ and a is elected. A switch by i to $(s,s,0)$ or $(0,0,s)$ would yield $|v^a| = |v^b|$, so that for $b \rhd a$, a would no longer be elected. Any other ballot would change the difference $|v^a| - |v^b|$ even more in b's favour, ruling out a as well.

(v) *For $i \in I_{bac}$, $(0,s,s)$ is undominated.*
Consider the ballot profile v where one $j \in I_{abc}$ votes $v_j = (s,s,0)$ while n of I_{abc} vote $v_j = (1-s,0,0)$ and all $j \in I_{bac}$ votes $v_j = (0,1-s,0)$. Then $|v^a| = |v^b|$ and $|v^c| = 0$ so that for $a \rhd b$, b is elected. Then, for some i_bac, only a switch to $(0,s,s)$ would increase the difference $|v^b| - |v^a|$ and hence yield outcome b.

(vi) *For $i \in I_{bac}$, $(0,1-s,0)$ is undominated by $(s,s,0)$ (only relevant if $(s,s,0) \in V_i^m$).*
If $(s,s,0) \in V_i^m$, consider the ballot profile v where every voter votes $(s,s,0)$. Then if $a \rhd b$, candidate a is elected. A switch be i to $(0,1-s,0)$ yields outcome b.

(vii) *For $i \in I_{bac}$, $(0,1-s,0)$ is undominated by $(1-s,0,0)$, $(1-s,0,0)$, $(0,s,s)$ and $(s,0,s)$.*

Consider the the ballot profile v where

- one $j \in I_{abc}$ votes $(s, 0, s)$,
- n of I_{abc} vote $(1 - s, 0, 0)$,
- $\left\lceil \frac{s}{1-s} \right\rceil > 1$ of I_{bac} vote $(0, 1 - s, 0)$,
- $n - \left\lceil \frac{s}{1-s} \right\rceil$ of I_{bac} vote $(0, s,)$.

Then

$$|v^b| - |v^c| = (1 - s)\left\lceil \frac{s}{1 - s} \right\rceil - s \in [0, 1 - s)$$

and

$$|v^a| - |v^c| = n(1 - s) - \left(n - \left\lceil \frac{s}{1 - s} \right\rceil\right) s = n(1 - 2s) + s\left\lceil \frac{s}{1 - s} \right\rceil < -\frac{1}{1 - s} + \frac{s^2}{1 - s} < 0$$

so the b is elected for $b \rhd c$. If i_{bac} switches from $(0, 1 - s, 0)$ to either $(1 - s, 0, 0)$, $(1 - s, 0, 0)$, $(0, s, s)$ or $(s, 0, s)$, she would reduce the payoff difference $|v^b| - |v^c|$ by at least $1 - s$, so that c's score would be higher than the score of b, ruling out b as an outcome.

Together, (i)-(vii) establish that for each $i \in I_{abc}$, $(s, s, 0) \in V_i^{\overline{m}}$ while for each $i \in I_{bac}$, $(0, s, s) \in V_i^{\overline{m}}$. But then b remains a possible outcome in the game $\Gamma(\succ_I, V^{\overline{m}})$, violating **MEW** which requires that a is the only remaining outcome after the iterative elimination of dominated strategies has run its course.

□

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