Polarization through the endogenous network

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Abstract

Processes of polarization have been documented in several applications. Nevertheless most of the theories built so far show how herding behavior and convergence of opinions tend to be a regularity in several contexts. In this paper we develop a model where agents correct their heterogeneous initial opinions averaging the opinions of their neighbors. The key contribution is to let the network take place endogenously. While the most known results are derived assuming the network to be strongly connected, we show how this component depends on the initial distribution of opinions. To do so, we characterize the process letting naive learning be a best reply function for agents. This allows to study the incentives in linking choices on the primitive process. Results show that, if opinions are not distributed uniformly, there always exist conditions on the strength of the social influence to prevent the network to be connected. This causes polarization both in the transition and in the long run.

Keywords: Homophily, Naive Learning, Polarization, Speed of Convergence

1 Introduction

Understanding the processes of opinion formation is a matter of interest for several reasons. In a society different social norms can coexist in contexts that share lot of cultural and environmental features. Or similarly, political opinions tend to be concentrated around two different poles. It has been documented also that in the latter specific case, looking at the process dynamically we observe a reinforcement, and therefore opinions become more polarized over time.[1]

Our approach is to explain this issue through strategic network formation. The idea is that individuals are on one hand influenced by other opinions, but the choice of social connections, i.e. sources of opinions, is not arbitrary. Thus whether a society ends up having convergent opinions would strongly depend by the way individuals interact with each other. Clearly if a society is split and there is no interaction between the several groups, the evolution of social norm within subgroups is independent from the others, and therefore is likely to exhibit disagreement.

We assume that evolution of opinions follows a process where agents are boundedly rational, and then focus on structural conditions on the nature of social relations. Thus we study a process known as *naive learning*, and we contribute to the existing stream of literature endowing agents with additional rationality with respect to the fully myopic scenario, letting them form the network optimally. Our approach could be described through the formulation of the DeGroot learning dynamic (see DeGroot (1974)), which simply states that at each point in time agents have an opinion equal to the average of opinions in their neighborhood in the previous period. Formally denoting with x_i the opinion of an agent i, and with μ_i the average of the opinions of i's neighborhood we have

$$x_{i,s} = \mu_{i,s-1}$$

This process has been studied in several papers, of which we mention among others Hegselmann and Krause (2002), DeMarzo et al. (2003) and Golub and Jackson (2010). In particular the latter shows that in order to have convergence some conditions on the structure of the adjacency matrix are needed and, specifically, it has to be irreducible and aperiodic. The former translates in a network being strongly connected[2], while the latter is easily satisfied assuming agents posing positive weight on their opinions.[3]

On top of these results, we let agents weight positively their own opinions, focusing therefore on conditions for which the network will be endogenously connected or disconnected.

[1]See Pew Research Center, June, 2014, "Political Polarization in the American Public" for a study on public in US, Andris et al. (2015) for evidence from the US House of Representatives, and how polarization is in this case a dynamic process.

[2]See Brualdi and Ryser (1991) for a formal proof.

[3]The condition requires that the largest common divisor of the path length is 1. See Jackson et al. (2008) for a discussion of this issue.

Thus the process that is going to be examined in this paper is

$$x_{i,s} = f\mu_i + (1 - f)t_{i,s} \quad (1)$$

where f is a weight that we call flexibility, and we let $t_{i,s} = x_{i,s-1}$. Through that lot of emphasis on the first period, where agents have no connections and are endowed exogenously with an opinion. Hence we have $t_{i,1} = t_{i,0}$, since $x_{i,0} = t_{i,0}$. Thus we let agents weight their initial opinions. When the process is initialized the network is not yet formed, and then afterwards we want it to update according to the evolution of opinions. Formally the process embeds a strong inertia, because agents update the network according to the opinion exhibited in the previous period and the opinion of new neighbors. The two are clearly correlated. Nevertheless, if the original DeGroot process is taken into account, there will be no evolution of the network, and therefore we need this modification. Moreover, this is the core of naive learning, because agents reinforce their bias over time. In other words, they are not able to exploit all the information that is available.

We initialize the problem endowing agents with an opinion, and depending on that they will form the network optimally. To achieve that we identify a payoff structure, through reverse engineering, such that the averaging of opinions described by equation 1 is a best reply function of the game. This allows for an analysis of strategic network formation, with opinions evolving accordingly.

Through this small change in the structure of the problem we are able to translate the conditions on the connectedness of the network assumed in existing works, into conditions on the ex-ante spectrum of opinions. This is a matter of interest because on top of this process the literature may advance inquiring the possible sources of interference that could either amplify or weaken the polarization process.

In term of results we first identify the possible equilibria of the network formation stage. Given a distribution the process boils down to a unique equilibrium characterized by agents ordered on the opinion space, forming connections with adjacent agents. Therefore the network in equilibrium exhibits *homophily*, which is in line with broad evidence on social networks, and more specifically with the literature on political opinions also, as documented by Gentzkow and Shapiro (2010), among others.

Hence we derive conditions on the initial distribution of opinions, sampled from an arbitrary distribution, such that in equilibrium the network will be either connected or disconnected. In particular we detect conditions on how profitable are connections, and how sensible are agents to others' opinions so that, when the distribution of agents is such that there is a mass of opinions that is relatively less represented than in its neighborhoods, the network will be disconnected. This leads to persistent divergence of opinion into a society. We believe it is an important contribution, especially considering that opinions take initially place under the influence of external factors. This could be the case in political opinions when considering media and politicians. Additional considerations could be drawn letting these external entities act strategically with respect to the network. That is another fundamental question that we leave for future work.

2 Literature Review

The literature has tried to explain disagreement using several techniques, clearly showing different possible outcomes. We want to contribute to the understanding of this issue using a network approach. This is not new, since research in networks has focused substantially on these issues. We could broadly define two approaches within this field, the *bayesian learning* and *naive learning*.

The former assumes fully rational agents that are thus able to exploit at best the information in their possess. In this context herding behavior is a natural result, and it is hard to escape from convergence of opinions. Therefore we should depart from full rationality. In this context researchers have been focusing on minimal conditions on bounded rationality such that non-convergence could be achieved as a result. One example of that is given by Yildiz et al. (2013) which assumes the presence of *stubborn agents* which do not pay attention to the information available, but simply stick to their own opinion. Our approach differ from this stream of literature since we assume a boundedly rational process, and add rationality on top of it.

Indeed this paper is closely related to the literature on *naive learning*, which began with the paper by DeGroot (1974). More recently this framework has been investigated in several other works that are a fortiori related to this paper.

DeMarzo et al. (2003) brings explicitly the network into the process of updating, and assumes an environment with *persuasion bias* and *uni-dimensional opinions*. The former means that agents hear the news, and then are influenced according to an exogenously given *listening matrix*, without accounting for repetition of information. This pushes toward convergence since the sources of information are over-counted because of the social structure brought in by the network. Uni-dimensional opinions instead means that opinions can be summarized by a single opinion. Although this is a result in that paper, we use here opinions that are uni-dimensional and can thus be summarized on an interval of a line.

Golub and Jackson (2010) exploiting some of the results in the paper summarized above, derive general conditions on the structure of the adjacency matrix such that consensus may occur. One of the main results, among others, is that the network should be strongly connected and its adjacency matrix be aperiodic, which means that the highest common divisor of the cycles length must be one. This is easily satisfied allowing for self cycles, with agent then taking into account their own opinion. Thus we keep this assumption but we study conditions on the distribution of opinions such that the network will be endogenously disconnected, which would then prevent consensus to happen. To the best of our knowledge, no other works attempts to do so.

With a completely different approach, Krause (2000) and Hegselmann and Krause (2002) do not analyze directly the network, but assume that the hearing matrix, which is in fact a stochastic matrix that summarizes a Markov process, is such that agents exhibit *bounded confidence*. This is implemented through an exogenous rule for the opinions to be taken into account only if they are similar enough. Under this circumstances, if this distance parameter is too relevant, consensus is not achieved. In our paper such a rule is not exogenous but is a consequence of the endogenous network, although follows from the payoff structure. The results are richer because they take into account relevant parameters to analyze different scenarios, and are derived for arbitrary distributions of opinions.

Another closely related paper to ours is Melguizo (2015) where the evolutions of opinions is based on *salience of attributes*, meaning the difference in behavior between individuals sharing a characteristic or lacking it. Agents are endowed with a vector of characteristics, and if there is a unique most salient characteristic the agents will assign, dynamically, growing weight to those who share the same trait. Since characteristics are binary groups can at most be two, and that is the case in which disagreement occurs. While this can be reasonable in several contexts, we believe the modeling explicitly the strategic network formation allows for a better understanding of the drivers of such results, and to derive richer results, too.

Most of the analysis of the model is here focused on the first period of the dynamic process, since this will determine the full evolution of opinions in the society. Nevertheless we contribute also to the existing literature on the speed of convergence to consensus. In this stream of literature we find Golub and Jackson (2012), among others. They study a process with agents belonging to a finite set of groups, and assume that interactions are more frequent toward same-type agents. In such a context convergence will always be reached, and thus it is shown how the stronger the same-group interactions (*homophily*), the slower the convergence to consensus. Here we show how, on top of this result, the speed of convergence is non linear due to the endogeneity of the network. In particular we identify thresholds on the diameter of the network such that the process speeds up dramatically and converge.

Thus more homophilous networks tend to exhibit slower speed of convergence, but because of the endogenous network they will show also a more prominent non-linearity in the latest stages.

For different reasons, this paper is also related on the literature of network games with endogenous networks. On of the first attempts in this context is provided by Galeotti and Goyal (2010), and followed by Kinateder and Merlino (2014). These papers examine games of public goods and thus they differ substantially from the game proposed here, which formally is a game of complements. A similar model to the one in this paper is analyzed in Bolletta (2015). The main difference is that here we study directed networks, following therefore a fully non-cooperative approach. In that paper instead the model is solved under *pairwise Nash stability*, and the undirected network allows to focus on the agents less prone to form links. Under a perspective of optimal policy design that aim to foster interactions, the analysis identifies individuals more susceptible to incentives.

The rest of the paper is organized as follows. Next section describes the model, analyzed in steps assuming exogenous network first in section 3.1, then letting it be endogenous in section 3.2. The latter contains the main results of the paper. Thus we analyze the full dynamic process and speed of convergence in section 4. Discussion in section 5 concludes. Proofs are in Appendix.

3 The model

3.1 Network as given

Consider the following one-shot game between two players.[4] Each player i is characterized by an *opinion* (or *type*) $t_i \in [0,1]$ and by a *flexibility* $f \in [0,1]$. The action of each player i is $x_i \in \mathbb{R}$, and the payoff for agent i is, given the action x_j of the other player,

$$\pi_i(x_i, x_j) = V - f(x_i - x_j)^2 - (1-f)(x_i - t_i)^2 \quad , \tag{2}$$

V is the value from the interaction, and then there are some costs for adaptation and coordination. V here could represent the idea that other agents have information that is useful, and we assume that it is linear and symmetric across agents.[5] The parameter f weights these two components, and that is why we have called it flexibility. One can already

[4] The two players version of this game is similar to Bisin et al. (2006).

[5] This is a simplification. However introducing an extra layer of heterogeneity in this dimension would only complicate the nature of the results, without delivering interesting insights.

see that agents with low flexibility weigh more adaptation. High flexibility agents weigh more the coordination term.

Now consider a directed network with n agents, where the neighbors of node i are given by the set d_i, with cardinality $k_i = |d_i|$. We let the network be directed for two main reasons. In terms of interpretation we believe that in opinion formation, where the structure of social interactions has been often represented through a "listening matrix" is more consistent with what we observe. Moreover there is a technical reason, that is we solve the model through a fully non-cooperative approach.[6] The same game as above is played on the network but now an agent must choose the same action as before, taking into account all her neighbors' choices. The payoff structure considering the network thus become:[7]

$$\begin{aligned} \pi_i(x_i, \mathbf{x}_j) &= \sum_{j \in d_i} \left(V - f(x_i - x_j)^2 - (1-f)(x_i - t_i)^2 \right) \\ &= k_i \left(V - (1-f)(x_i - t_i)^2 \right) - f \sum_{j \in d_i} (x_i - x_j)^2 \quad . \end{aligned} \tag{3}$$

where the unique best response for agent i is

$$x_i^*(\mathbf{x}_j) = f\mu_i + (1-f)t_i \quad , \tag{4}$$

where we have called $\mu_i \equiv \frac{\sum_{j \in d_i} x_j}{k_i}$. This is equation 1, already described in the introduction. At this stage we analyze one period only, and in a later section we discuss dynamics of the model.

The timing of the on period game is as follows.

Definition **1.** *TIMING:*

- *Opinions t are exogenously assigned to agents*
- *Agents form the network*
- *Opinions are updated into x, according to the network and best replies*

[6]See Bolletta (2015) for the study of a similar model under undirected network. There it is shown that it is still feasible to perform the analysis, although some extra conditions are required in terms of rationality to refine multiplicity of equilibria in the network formation stage. In that paper farsightedness is used as a refinement. We preferred here to stand to a more realistic assumption on individual's rationality, which in addition allows us to solve the model through the concept of Nash equilibrium.

[7]Note that although we focus on the following payoff structure, what really matters for the results that follow is the best reply scheme. Indeed one could redirect the analysis that follows to a payoff structure consistent with other parametric forms already used in the literature, such as Calvó-Armengol et al. (2009) among others.

From the timing we introduce the equilibrium concept. Formally we see this as a sequential game, that we can therefore solve by backward induction. In particular we focus now in the solution of the system of best replies, when the network is given, and therefore μ_i is uniquely defined for all i. Then in the next section we move to the previous stage letting agents form the network, to finally move to the very initial stage on the distribution of opinions to characterize the possible equilibria.

To proceed, let us introduce some more notation to rewrite the system of best replies in matrix form. To do so we call:

- F the diagonal matrix of all flexibilities, so that $F \equiv \begin{pmatrix} f_1 & & 0 \\ & \ddots & \\ 0 & & f_n \end{pmatrix}$;
- $\mathbf{t}$ the vector of all types;
- and D the adjusted adjacency matrix such that $D_{ij} \equiv \begin{cases} \frac{1}{k_i} & \text{if } j \in d_i, \\ 0 & \text{otherwise.} \end{cases}$

Note that we let f be homogeneous, although the model could account for heterogeneity in this dimension. Moreover we underline once again that the entries into the matrix D have to be determined in equilibrium. Then a compact way to write (4) is

$$(I - FD)\mathbf{x} = (I - F)\mathbf{t} \quad . \tag{5}$$

Lemma 1. *Equation (5) has a unique solution* $\mathbf{x} \in [0, 1]^n$.

Proof. See Appendix. □

This simple result states that for a given network there is a unique Nash equilibrium of the game. This is crucial because we can now move forward and go study the network formation. Before doing that we focus on the payoff structure and derive a formula for the *payoff in equilibrium*, which dramatically helps us in the analysis of agents' strategies. Thus let us move back to the Nash equilibrium. From (3) and (4), the payoff in equilibrium is[8]

$$\pi_i \quad = k_i \left(V - f(1-f)(\mu_i - t_i)^2 - f\sigma_i^2\right) \tag{6}$$

where we have called $\sigma_i^2 \equiv \frac{\sum_{j \in d_i}(x_j - \mu_i)^2}{k_i}$ the variance of the actions of i's neighbors. Given that the payoff structure is quadratic, it is not surprising that second moments of the distribution of behaviors appear in the analysis. Nevertheless, this is a result that is not highlighted from

[8]See Appendix for the complete derivation

previous works, although it is particularly meaningful. Interestingly, we see how fully flexible agents ($f = 1$) have preferences only for homogeneous groups, and they would not care about which opinion the group exhibits. Therefore they will form the group with agents that share the most similar opinions. For agents that instead are not prone to change their opinion ($f = 0$), others' opinion are completely irrelevant, and therefore connections are formed at cost 0. For intermediate values of flexibility ($f = 1/2$), agents both care about having homogeneous and similar opinion groups. Next Remark formally states some comparative statics on f on the payoffs.

Remark **1.** *The payoff of an agent in equilibrium depends:*

- *quadratically on μ_i, with a maximum when $\mu_i = t_i$ – this effect is the most detrimental when $f = \frac{1}{2}$;*
- *linearly on σ_i^2 – this effect is the most detrimental when $f \to 1$.*

So, the payoff seems always maximum when $f = 0$, so that $x_i = t_i$ (but it could be a problem to find neighbors when the network is endogenous).
The first order effect (i.e. fixing others' best responses) of increasing flexibility up to $\frac{1}{2}$ decreases welfare, but when $f > \frac{1}{2}$ an larger f could increase welfare if the actions of neighbors have low variance.

3.2 Endogenous network

The concept of equilibrium here is sub-game perfect Nash equilibrium. Since the network is directed the solution is fully non-cooperative, and it is worked out in two stages, by backward induction. In fact the agent will first choose their neighbors, and then will update their behaviors according to equation 4. In the previous section we established uniqueness of equilibrium for a given network and we can therefore focus now on the network formation stage.

Before moving to the result, we propose here a definition that allows us to partially characterize the equilibrium.

Definition **2.** *An equilibrium is **ordered** if each agent in a network g has an interval of neighbors $\{i - a_i, i + b_i\}$ such that $a_{i+1} \leq a_i + 1$ and $b_i \geq b_{i+1} + 1$.*

The above definition can be described as follows. We call an equilibrium to be ordered if agents are matched only with their closest neighbors. Clearly every agent will have different bounds on their neighborhood, and we could characterize those only departing from a given distribution of opinions. The following result formally states that this configuration is the unique that can arise in equilibrium.

PROPOSITION **2.** *An ordered equilibrium exists and it is unique.*

Proof. See Appendix. □

To get an intuition on this result, let us recall equation 6. In particular we could broadly interpret the payoff deriving from a set of links as follows. First agents want that the average behavior in their neighborhood is close enough to their initial opinion. That is agents are *homophilous.* Therefore agents with similar opinions will tend to match together. This already pushes strongly towards uniqueness of the equilibrium. In addition to that we should consider the second term of equation 6, which shows how agents have preference over the diversity of opinions inside their neighborhood. Therefore we can say that agents will tend to match with their most similar agents, and the reference groups are well defined since the variance term ensures sharp bounds on it.

Given the previous result we can now determine the conditions on the initial distribution of opinions such that the network will be disconnected. In particular we will have that the spread of opinions is greater than the initial distribution of opinions if the network is disconnected, while lower if instead the network is connected.

The model allows to define a triple $\langle V, f, T\rangle$ which will map into a network configuration and a distribution of opinions $\langle G, X\rangle$. Thus it is a matter of interest to understand under which condition the distribution of ex-post opinions is more polarized than the distribution of initial opinions. Simply defining a Gini type of measure, we can compare the vector $\mathbf{x}$ with the vector $\mathbf{t}$. As it turns out, the weighted adjacency matrix is all matters here. Such a result is therefore comparable with all the literature on DeGroot processes. We need to find condition on aperiodicity, irreducibility and other Markov Chain results for the adjacency matrix.

To test for the polarization into the population of interest, we should first define a Gini type coefficient for our specific context. To do so, let us introduce a vector

$$\bar{\xi}' = (-n+1, -n+3, ..., n-1, n+1)$$

Thus, if we calculate $\bar{\xi}'\bar{x}$ with $\bar{x}$ being the vector of actions in equilibrium, we have a measure of dispersion of actions. Clearly this will not be between 0 and 1 as the typical Gini index, since we would need to weight it, but we are rather interested in the difference in dispersion between equilibrium behaviors $\bar{x}$ and initial opinions $\bar{t}$. We want then to analyse the sign of our measure of polarization $P(g,t) = \bar{\xi}'(\bar{x} - \bar{t})$. If positive we would have polarization, if negative we would have centralization, and if 0 then we would have that the endogenous network would not matter at all. Interestingly, if we use equation 5, we could simplify this measure, as shown by the following simple algebra.

$$\begin{aligned} P(g,t) &= \bar{\xi}'(\bar{x} - \bar{t}) \\ &= \bar{\xi}'[I - \frac{f}{1-f}(I - fD)]\bar{x} \\ &= \frac{f}{1-f}\bar{\xi}'[D - I]\bar{x} \end{aligned} \tag{7}$$

From this we see clearly how the sign depends only on D, while it is independent of f. In particular the equation above is going to be positive if D is disconnected, negative otherwise. This is to put emphasis once more on the relevance of the network formation stage in the process analyzed here. Provided that, we move now to the main result of the paper, which provide sufficient conditions on the distribution of opinions such that the network is going to be disconnected in equilibrium.

Proposition 3. *If $F(t)$ is not uniform, there is a $\bar{V}$ such that for every $V \in (0, \bar{V})$ there exists a $\bar{f}$ such that any $f \in (\bar{f}, 1)$ the network is not empty and exhibits multiple components.*

Proof. See Appendix. □

From the above result we learn that any distribution where there is a mass of opinions which is more represented leads the network to be disconnected. In particular there is a couple of values V, f that summarizes the sufficient condition for the network to be disconnected.

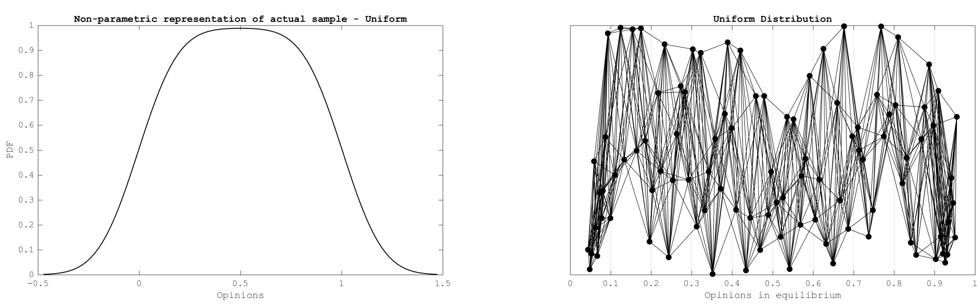
The result hold for any distribution but the uniform, since we can always identifies some values low enough such that for the less representative group it is not profitable to form connections with the larger group. The intuition is that if not uniform there must be an interval in $[0, 1]$ such that the mass of agents is higher than in another point.

Interestingly in order to have multiple components we need agents to weight enough social influence from neighbors. This is not immediately intuitive, since on one hand a higher f pushes towards conformism of opinions which makes the network easier to be connected. Nevertheless this implies also that in the first place agents prefer to have homogeneous neighborhood, and that is why in equilibrium they are likely to take place locally, with segregation arising.

In the next example we show simulations of the model for several distributions, showing how the uniform would be strongly connected, while other distributions, namely a bimodal and a normal, favor the arousal of more groups not inter-connected.

Example 1. We show in this numerical example what happens with given initial distributions of opinions. Through simulations of the model we show how the system, under some fixed set of parameters, reacts differently if we let sample the initial opinions from different distributions. In each of the figures it is shown on the left the non parametric approximation of the distribution of opinions generated to initialize the model. Then the figures on the right show the equilibrium network arisen after we let the agents form their links optimally. Moreover it is shown on the horizontal axis the value for each node of the equilibrium behaviors.

Parameters are chosen to be, throughout all the simulations, $n = 100, V = 0.001, f = 0.7$. Thus we generate respectively a uniform, bimodal and normal distribution. As we can see, while in the uniform the network is strongly connected, in the other cases several components arise.

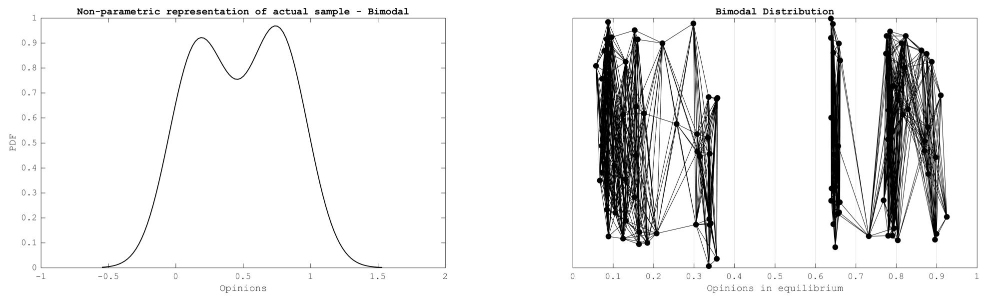


(a) (b)

Figure 1: The network is strongly connected.

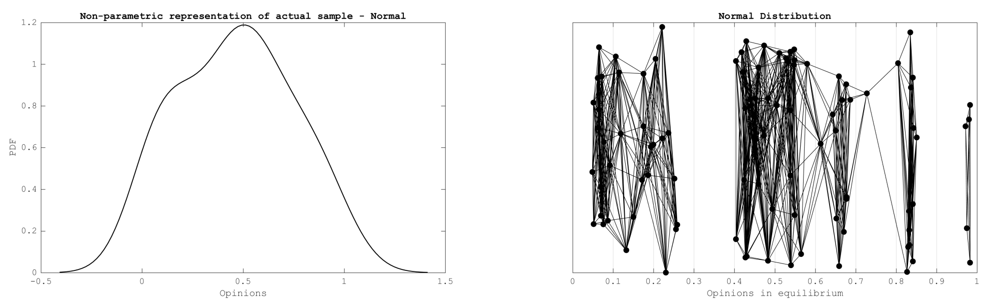
4 Dynamic setting

In this section we extend the result shown before to a dynamic setting. In particular we could simply assume that at any point in time an agent's opinion correspond to her ex-post opinion, determined in equilibrium in the previous period. Formally we have then $t_{i,s} = x_{i,s-1}$, where $s \in \{0, 1, .., S\}$ denotes discrete time and with $t_{i,0} = t_i$. Clearly we want the process to be myopic. In fact agents at every step s will only maximize utility at that given point in time.



(a) (b)

Figure 2: Two components of similar size arise.



(a) (b)

Figure 3: Three components arise, with a big central group, and smaller groups on the tails.

In this way we are able to initialize the model with a distribution of times and study both the long-run behavior of the society and the step-by-step evolution of the process. In particular we are able also to see deeper how the result obtained before relate to the well known results found in DeMarzo et al. (2003) and Golub and Jackson (2010). This papers show how if the network is strongly connected and weighted adjacency matrix is primitive and aperiodic (as it turns out the latter implies the former), consensus is always reached.

Our model could be in fact compared to the model by Krause (2000) and Hegselmann and Krause (2002), where he shows that if agents give weight only to those who have similar opinions, convergence of opinions will be ensured only within each component, even if the network is strongly connected. Somehow in these models connections do not matter, as it matter instead the agents to which it is given weight in the process of opinions updating. The rule to determine the closest neighbors to which pay attention is imposed exogenously, while again our model generate that in equilibrium through the endogenous formation of the network.

Remark 2. *Convergence takes place within every component of the network that is endogenously determined in the first time step.*

The result follows from the analysis in Hegselmann and Krause (2002), with the only difference brought in by the endogenous network, and therefore we omit a formal proof. Nevertheless, we have that if the network is disconnected at the first time step it will remain disconnected, and viceversa. Consensus will take place only partially within every component that arouse from the network.

Accommodating our model to a dynamic version we see that all that matters for convergence is the formation of the network in the first period. Then one or more components arise accordingly, and over time we observe a contraction of opinions toward a unique one, in each component that has arouse. At the same time, the network will become more and more connected over time, converging to the complete one.

The endogenous network shapes the adjacency matrix into blocks, and then over time there is an increase in density of positive entries because the actions are closer and closer. As a result we get a reinforcement over the process of convergence. We can surely say that an increase in V and in f would push both over a faster convergence. This is intuitive, but considering the flexibility, we know also that it makes easier to break up the network.

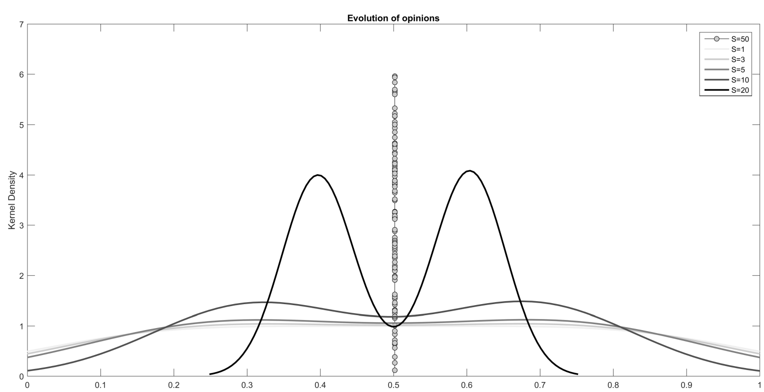


Figure 4: Non parametric approximation of the distribution of opinions over time. Parameters are chosen to be $f = 0.5$, $V = 0.005$, $N = 100$, and t uniformly distributed. Curves represent different steps in time denoted by S. At $S = 50$ we represented the network after convergence, with random coordinates on the vertical axis, and opinions in the horizontal one, which converged to $x = 0.5017$.

Figures 4 and 5 show the results of a simulation focusing respectively on the distribution of opinions over time, and on the speed of convergence vìs a vìs with the average degree.

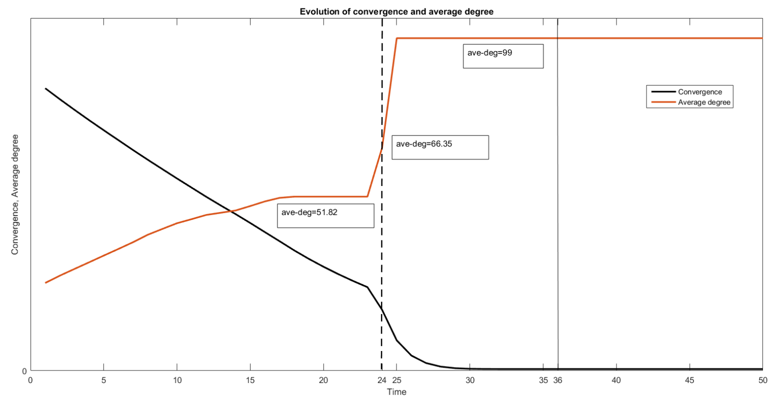


Figure 5: Speed of convergence and average degree. Time is on the horizontal axis, and distance between vector of opinions and expected convergence point, given by 0.5. Scale omitted since what really matters is the slope of this curve. Convergence happens after 36 periods, and there is a significant increase in speed when the network start converging to the complete one, which happens suddenly in period 24. This figure refers to the same simulation explained in Figure 1.

Parameters are chosen to be $S = 50$ periods, with $n = 100$ agents and t uniformly distribute. From Proposition 3, we know that the network is surely connected, if not considered the banal case where it is empty.

5 Discussion

In this paper we showed how through the endogenous network tend to be disconnected. In particular this happens when forming connections is not too profitable, and agents are particularly sensible to social influence from their neighborhoods. This conditions lead opinions in the long run to diverge, and thus consensus is never reached.

In several real world examples the choice from the agents of their neighbors is non-negligible. From this results we learn therefore that considering a network formation process has dramatic results on several dimensions, namely the network configuration, equilibrium behaviors, long run opinions and speed of convergence.

We wanted to contribute in this sense to the existing literature on opinion formation processes, because this could really open new research questions. In particular it comes natural the question regarding possible nuisances that may alter the distribution of opinions at any given point in time. This could explain cyclic behavior of political parties, strategic provision of information from media and evolution of social norms. These are relevant questions well explored in the literature, but we believe that this work can consistently contribute having a richer understanding of them.

APPENDIX

Payoff in equilibrium

From (3) and (4), the payoff in equilibrium is

$$\begin{aligned}
\pi_i &= k_i\left(V - f^2(1-f)(\mu_i - t_i)^2\right) - f\sum_{j\in d_i}(x_i^* - x_j)^2 \\
&= k_i\left(V - f(1-f)(\mu_i - t_i)^2\right) - f\sum_{j\in d_i}\left[(x_i^* - x_j)^2 + (1-f)^2(\mu_i - t_i)^2\right] \\
&= k_i\left(V - f(1-f)(\mu_i - t_i)^2\right) - f\sum_{j\in d_i}(x_j - \mu_i)^2 \\
&= k_i\left(V - f(1-f)(\mu_i - t_i)^2 - f\sigma_i^2\right) \quad ,
\end{aligned} \tag{8}$$

where we have called $\sigma_i^2 \equiv \frac{\sum_{j\in d_i}(x_j - \mu_i)^2}{k}$ the variance of the actions of i's neighbors.

5.1 Proof of Lemma 1

Proof: There is clearly a unique solution to the unconstrained equation (5), because $(I - F)$ and $(I - FD)$ are always full rank matrices.

Now suppose that the maximum element of $\mathbf{x}$, call it x_m is such that $x_m > 1$. Then, by (4), it is a convex combination of $\frac{\sum_{j\in d_m} x_j}{\ell_m}$ and t_m. But since $t_m < 1$, it must be that $\frac{\sum_{j\in d_m} x_j}{\ell_m} > x_m$, which contradicts the initial assumption.

In the same way it is impossible that the minimum element of $\mathbf{x}$ is less than 0. □

5.2 Payoff differential

Recall the payoff in equilibrium

$$\pi_i = k_i(V - f(1-f)(\mu_i - t_i)^2 - f\sigma_i^2) \tag{9}$$

From which we can define the additional payoff deriving from the addition of a link with an agent j

$$\pi_{i\to j} = (k_i + 1)(V - f(1-f)\left(\frac{\mu_i k_i + x_j}{k_i + 1} - t_i\right)^2 - f\sigma_{i\to j}^2) \tag{10}$$

Given equation 10, we can now define the payoff differential, simply given by the difference $\Delta\pi_{i\to j} = (\pi_{i\to j} - \pi_i)$

$$\Delta\pi_{i\to j} = V - f(1-f)\left((k_i+1)\theta' - k_i\theta\right) - f\left((k_i+1)\sigma^2_{i\to j} - k_i\sigma^2_i\right) \tag{11}$$

where $\theta' = \left(\frac{\mu_i k_i + x_j}{k_i+1} - t_i\right)^2$ and $\theta = (\mu_i - t_i)^2$. Let us now focus on this difference. Thus

$$\begin{aligned}
&(k_i+1)\theta' - k_i\theta = \\
&= (k_i+1)\left(\frac{\mu_i k_i + x_j}{k_i+1} - t_i\right)^2 - k_i(\mu_i - t_i)^2 \\
&= (k_i+1)\left((\mu_i - t_i) - \frac{1}{k_i+1}(\mu_i - x_j)\right)^2 - k_i(\mu_i - t_i)^2 \\
&= (\mu_i - t_i)^2 + \frac{1}{k_i+1}(\mu_i - x_j)^2 - 2(\mu_i - t_i)(\mu_i - x_j) \\
&= (t_i - x_j)^2 - \frac{k_i}{k_i+1}(\mu_i - x_j)^2
\end{aligned} \tag{12}$$

Now let us focus on the term $\left((k_i+1)\sigma^2_{i\to j} - k_i\sigma^2_i\right)$

$$\begin{aligned}
&\left((k_i+1)\sigma^2_{i\to j} - k_i\sigma^2_i\right) = \\
&= \frac{k_i+1}{k_i+1}\left(\sum_h (x_h - \mu'_i)^2 + (x_j - \mu'_i)^2\right) - \sum_h (x_h - \mu_i)^2 \\
&= \left(\sum_h \left(x_h - \frac{k_i\mu_i + x_j}{k_i+1}\right)^2 + \left(x_j - \frac{k_i\mu_i + x_j}{k_i+1}\right)^2\right) - \sum_h (x_h - \mu_i)^2 \\
&= \sum_h \left(x_h - \frac{k_i\mu_i}{k_i+1} - \frac{x_j}{k_i+1}\right)^2 - \left(\frac{k_i}{k_i+1}(x_j - \mu_i)\right)^2 - \sum_h (x_h - \mu_i)^2 \\
&= \sum_h \left(x_h - \mu_i + \frac{\mu_i}{k_i+1} - \frac{x_j}{k_i+1}\right)^2 - \frac{k_i^2}{(k_i+1)^2}(x_j - \mu_i)^2 - \sum_h (x_h - \mu_i)^2 \\
&= \sum_h (x_h - \mu_i)^2 + \sum_h \left(\frac{1}{k_i+1}(\mu_i - x_j)^2\right) - 2\sum_h \left(\frac{1}{k_i+1}(x_h - \mu_i)(\mu_i - x_j)\right) + \\
&- \frac{k_i^2}{(k_i+1)^2}(x_j - \mu_i))^2 - \sum_h (x_h - \mu_i)^2 \\
&= \frac{k_i}{k_i+1}(1 - \frac{k_i}{k_i+1})(\mu_i - x_j)^2
\end{aligned} \tag{13}$$

Therefore combining the two previous results we get

$$\begin{aligned}\Delta\pi_{i\to j} &= V - f(1-f)\left((t_i - x_j)^2 - \frac{k_i}{k_i+1}(\mu_i - x_j)^2\right)\\ &- f\left(\frac{k_i}{k_i+1}(1-\frac{k_i}{k_i+1})(\mu_i - x_j)^2\right)\\ &= V - f(1-f)\,(t_i - x_j)^2 - f\frac{k_i}{k_i+1}\left(f - \frac{k_i}{k_i+1}\right)(\mu_i - x_j)^2\end{aligned} \tag{14}$$

5.3 Proof of Proposition 1

Proof. We proceed by steps, first addressing existence constructing the equilibrium. In particular we show that any equilibrium configuration must be ordered, as defined in 2, which then partially characterizes the equilibrium, too. Finally we show that it is also unique.

It is easy to check that if the equilibrium is ordered we have that

$$\begin{aligned} x_i > x_j &\leftrightarrow t_i > t_j\\ x_i < x_j &\leftrightarrow t_i < t_j\\ x_i = x_j &\leftrightarrow t_i = t_j\end{aligned} \tag{15}$$

Given that, we show that it cannot exist another equilibrium different from a ordered one, such that any of the above conditions hold true. Consider now the minimum x_i in the population, and call it x_0. Without loss of generality we let $x_0 \leq x_1 \leq ... \leq x_N$, and to prove the result basically we want to match every action with a type such that $t_i \mapsto x_i$ for all $i \in N$. Thus we claim that i is such that t_i is the minimum initial opinion in the population, calling it t_0. We can see this through the payoff in equilibrium, which we recall

$$k_i\left(V - f(1-f)(\mu_i - t_i)^2 - f\sigma_i^2\right)$$

Recalling the equation we see that the highest utility can be achieved minimizing the value of μ_i, given that we are considering t_0, and the variance term ensures sharp bounds on it. To see it rigorously we derive the payoff differential deriving from the addition of a link, and which is given by

$$\Delta\pi_{i\to j} = V - f(1-f)\,(t_i - x_j)^2 - f\frac{k_i}{k_i+1}\left(f - \frac{k_i}{k_i+1}\right)(\mu_i - x_j)^2 \tag{16}$$

Therefore links are added as long as the equation above is positive. Moreover any non-ordered scenario would let profitable deviation for t_0, and thus that would not be an equilibrium.

Note that this holds for any $f \in (0,1)$. Indeed if $f < \frac{k_i}{k_i+1}$ the second term on the right hand side would be positive. Nevertheless if $f < \frac{k_i}{k_i+1}$ then $f(1-f) > f\frac{k_i}{k_i+1}\left(f - \frac{k_i}{k_i+1}\right)$ is always true, and thus this does not alter the scheme we have built. If $f = 0$ all links would be added, and if instead $f = 1$ then $f > \frac{k_i}{k_i+1}$. This shows that in equilibrium the lowest t_i will exhibit the lowest x_i. Similarly this holds for x_N and t_N.

Now let us focus on the second lowest value of x, x_1. Following a similar argument, we claim that it should belong to the agent with the second lowest opinion in the population, t_1. This is slightest more tedious to prove, since agent i can form links both at her right and left. We can in fact reproduce the procedure we used to prove that $t_0 \mapsto x_0$, with the extra step set by the presence of agent t_0. Similarly with respect to agent t_0, agent t_1 would form links with agents $x_2 \leq x_3 \leq ... \leq x_{b_1}$, where b_1 is the optimal threshold as in the definition of ordered equilibrium. Moreover it is likely that agent t_1 would form a link with t_0. To see this we should see again payoff in equilibrium. The addition of such link would generate an increase in σ_1^2, but a decrease of the term $(\mu_1 - t_1)^2$. Clearly it can happen that such a link is not optimal, but in either case it would not contradict the ordered equilibrium scenario. Symmetrically this holds for t_{N-1}.

Therefore we showed that in equilibrium $t_i \mapsto x_i$ for $i \in \{0, 1, N-1, N\}$. Iterating the reasoning, with the due differences because of the availability of agents at both sides, the result holds for all other agents.

The set of relations described by 15 is always true in equilibrium. We have shown that if the above set of condition holds the equilibrium must be ordered and that in equilibrium the ordered scenario is the only possible outcome. This thus addresses existence and the partial characterization.

To show that it is unique, we just have to consider that with a given distribution of opinions t, there is unique possible ordering scheme that we may call Z, that would hold. Assume it is not, this means that some agent i has formed an ordered neighborhood Z' with bounds differing from Z, because in all other cases it would not be ordered. Nevertheless this is not possible in equilibrium, since either there would exist profitable deviations by adding links from Z to Z', or severing links, or viceversa. As a small caveat, it can happen as an extremely specific case, that two or more agents share the same initial opinion. Therefore some other agents may be indifferent over any of these links. Under the standard hypothesis that agents would then randomize over such links, multiple equilibria may arise. Nevertheless if two or more agents share the same t, their position on the space of types is permutation free, thus we can always relabel agents to fit definition 2. Moreover the final configuration of the network is isomorphic, and the distribution of x will not be affected by that.

This conclues the proof. $\square$

5.4 Proof of Proposition 2

Proof. Let us begin with a definition.

DEFINITION **3.** *Call S the matrix with agents in columns ordered according to their type, and with rows all the possible set of links. Thus S is an $N \times \sum_{k=0}^{N} 2^k$. Moreover at any entry S_{ij} such that $i < j$ and j is such that $\sum_{k=0}^{j-2} 2^k$ the set of links is such that i is connected with all j.*

From this it heuristically follows the Lemma below.

LEMMA **4.** *If all $i \leq j$ choose a S_{ih} s.t. $h \leq \sum_{k=0}^{j-2} 2^k$, and all $i > j$ a $S_{ih'}$ s.t $h' > \sum_{k=0}^{j-2} 2^k$, then the network is* ***disconnected****.*

Now let us define a simple equation that shows the differential of payoff when comparing two different set of links. Note that a set of links characterizes a triple $\langle d_i, \mu_i, \sigma_i^2 \rangle$, which summarizes all the relevant information to the agents. Comparing every possible couple of set of links s, s', the set s is preferred to s' if and only if

$$V > \frac{f_i(1-f_i)(d_i(\mu_i - t_i)^2 - d_i'(\mu_i' - t_i)^2) + f_i(d_i\sigma_i^2 - d_i'\sigma_i^{2'})}{d_i - d_i'} \tag{17}$$

If agents draw their types from a uniform distribution $F^{uni}(t)$ we have that $dF^{uni}(t)/dt = 1$, for every t. It is therefore a 45° degree line if plotting $F(t)$ over t. Now take any other $F(t)$. We can compare such distributions studying the derivative of $F(t)$ around the 45° degree line.

Call t_1 and t_2 the levels of t such that $dF(t)/dt = dF^{uni}(t)/dt$. Finally call $\bar{t}$ the level of t such that $dF(t)/dt$ is minimal in $t \in (0, 1)$. Note that by construction $t_1 < \bar{t} < t_2$. Therefore we have to show that for every $t \in (t_1, \bar{t})$ there exists a set of links S_{ih} such that Lemma 4 is satisfied, i.e. S_{ih} such that $h \leq \sum_{k=0}^{j-2} 2^k$, and that set of links is going to be preferred to any other. By symmetry it would hold for agents between $t \in (t_2, \bar{t})$.

To show that assume that there exists an alternative set of links S_{ih}' such that $h > \sum_{k=0}^{j-2} 2^k$ that would lead any agent i with $t \in (t_1, \bar{t})$ better off, i.e. $U_i(S_{ih}) < U_i(S_{ih}')$. Recalling now equation 17 one can see that for an agent to be better off it must be that, with a little abuse of notation, $\langle d_i', \mu_i', \sigma_i^{2'} \rangle \succ_i \langle d_i, \mu_i, \sigma_i^2 \rangle$, meaning that she would prefer S_{ih}' to S_{ih}.

Now assume that $d_i = d_i'$, while $\sigma_i^2 < \sigma_i^{2\prime}$ by construction. It follows then that $|\mu_i - t_i| \lesseqgtr |\mu_i' - t_i|$ for all i with $t \in (t_1, \bar{t})$. Therefore the differential in variance leads always the agent worse off, and therefore everything depends by the relation between the new μ_i. In particular, given the configuration we are studying, if we take any agent with $t_i < \bar{t}$, and with $t_i > t_1$ (or conversely $t_i > \bar{t}, t_i < t_2$), $\mu_i' > \mu_i$ (or $\mu_i' < \mu_i$). Therefore agent i would experience a benefit in this case. Nevertheless, since the variance increases, it all depends on the weight f, and the local sampling of types t around the point where $dF(t)/dt$ in $t \in (0,1)$ is minimal. In other words when there is a "hole" in the distribution we need that agents weight enough the variance term in order to direct all their links to a single direction, rather than to both.

Moreover, V must be small enough. Clearly we are not interested in the trivial case where $V = 0$, and thus the network will be empty. This case is completely non informative, and thus we consider $V > 0$. To see that imagine V is really large. Then the optimal network would be the complete network. Therefore there must exist a threshold value $\bar{V}$ such that all said above holds true. Similarly, we must have $\bar{f}$ large enough. Imagine there are some agents with $f_i = \varepsilon$ with $\varepsilon > 0$ arbitrarily small. For those agents equation 17 holds true for any V small enough, leading these agents to connect the network. Therefore there is a threshold value of $\bar{f}$ such that, again, all said above holds true. □

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