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Abstract

When data exhibit high volatility and jumps, which are common features in most high frequency financial time series, forecasting becomes even more challenging. Using high frequency exchange rate data, we show that wavelets, which are robust to high volatility and jumps, are useful forecasters in high frequency settings when high volatility is a dominant feature that affects estimation zones, forecasting zones or both. The results indicate that decomposing the time series into homogeneous components that can then be used in time series forecast models is critical. Different components become more useful than others for different data features associated with a volatility regime. We cover a wide range of linear and nonlinear time series models for forecasting high frequency exchange rate return series. Our results indicate that when data display nonstandard features with high volatility, nonlinear models outperform linear alternatives. However, when data are in low volatility ranges for both estimations and forecasts, simple linear autoregressive models prevail, although considerable denoising of the data via wavelets is required.

JEL Classification: C12, C22
Keywords: Wavelets, Forecasting, High Frequency Data, Nonlinear Models, Maximum Overlap Discrete Wavelet Transformation.

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1 Introduction

A wavelet multiresolution analysis can be used to decompose a time series into different timescale components, and a model can be fitted to each component to improve the forecast accuracy of the series as a whole. When data exhibit high volatility and jumps, which are common features in most high frequency financial time series, forecasting becomes more challenging. Wavelets, which are capable of separating most heterogeneous data generation processes (DGP) into their homogeneous components, have the potential for use in forecasting high frequency financial series. Multiresolution analysis (MRA) wavelets can be used to decompose data series into high and low frequencies components. A low frequency (or smooth) component represents general trends of the data by eliminating the noisy part of high frequency (details) via a MRA. If the data display standard features, then this smooth component alone can be used to forecast the series because it contains more homogeneous information than the whole data. However, if data exhibit nonstandard features, then high frequency components (details) can be more useful than smooth components and the original data.

Using high frequency exchange rate data with such features, we partition our data into low and high frequency zones, and show that wavelets are useful for forecasting all portions of our data.[1] The low and high frequency components of the data are used for estimation and forecasting interchangeably. By using these different estimations and forecast zones, we also posit that the wavelets component that represents the most useful forecaster may vary with the data features in the estimation and forecasts zones. Smooth components are usually but not always the most useful forecasters. Strikingly, when both estimation forecasts are performed, highly volatile data environment detail components become more efficient over the smooth components. For the remaining cases, if low volatility prevails in either the estimation and forecasting zones or both, then smooth components dominate.

For forecasting using wavelet components, we employ a vast range of linear and nonlinear models. The results indicate that nonlinear models should be preferred in all cases except when both estimation and forecasting have low volatility exposure, for which a simple linear model outperforms numerous nonlinear ones . However, the wavelet component used in the model requires several denoising iterations via a MRA analysis.

Whether forecasts are targeted for short or long horizons appears to matter only in the choice of multiresolution level. In long forecast horizons, additional denoising is advisable before using smooth components.

The remainder of this paper is organized as follows. Section 2 outlines how wavelets can be used for to improve forecasting via multiresolution analyses. Section 3 displays the linear and nonlinear models used in forecasting. Section 4 explains the data and shows how the data are divided into different estimation and forecasting volatility zones in relation to the pseudo-out-of-sample forecasting zones. Section 5 illustrates the results. Section 6 presents the conclusions.

2 Forecasting with wavelets via multiresolution analyses

In principle, wavelet analyses can be realized at all arbitrary time scales. However, if only key features of the data are in question, then this analysis may not be necessary. Thus, the discrete wavelet transformation (DWT) is more efficient and parsimonious than the continuous wavelet transformation. When forecasting in an environment with nonstandard data features as emphasized above, splitting data into homogeneous features is critical, and one component may overrule the other under different volatility characteristics.

Let $w(\lambda, t)$ be the change in the average of the time series $\mathbf{x}$, where t is the time and λ refers to the scale associated with the average. The DWT is a subsampling of $w(\lambda, t)$ with only dyadic

[1] Although the potential usefulness of wavelets in forecasting has been acknowledged, few wavelet applications are used for forecasting in economics. Arino and Vidakovic (1995)) focuses on car sales forecasts; Wong et al. (2003) provide an application to exchange rates; Conejo et al. (2005) and Tan et al. (2010) forecast electricity prices; Fernandez (2007) focuses on forecasting shipments of US manufactured items; Rua (2011) introduces the wavelet approach for factor-augmented forecasting; and Zhang et al. (2017) consider forecasting with wavelets using high frequency exchange rate data.

scales, i.e., λ is of the form 2^{j-1}, $j = 1, 2, 3, \ldots$, and within a given dyadic scale, 2^{j-1}, t's are separated by multiples of 2^j.

Let $\mathbf{x}$ be a dyadic length vector ($N = 2^J$) of observations. The length N vector of the discrete wavelet coefficients $\mathbf{w}$ is obtained by

$$\mathbf{w} = \mathcal{W}\mathbf{x},$$

where $\mathcal{W}$ is an $N \times N$ real-valued orthonormal matrix (based on the wavelet type) that defines the DWT, which satisfies $\mathcal{W}^T\mathcal{W} = I_N$ ($n \times n$ identity matrix).[2] The nth wavelet coefficient w_n is associated with a particular scale and a particular set of times. The vector of wavelet coefficients may be organized into $J + 1$ vectors,

$$\mathbf{w} = [\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_J, \mathbf{v}_J]^T,$$

where $\mathbf{w}_j$ is a length $N/2^j$ vector of wavelet coefficients associated with changes on a scale of length $\lambda_j = 2^{j-1}$ and $\mathbf{v}_J$ is a length $N/2^J$ vector of scaling coefficients associated with averages on a scale of length $2^J = 2\lambda_J$.

Using the DWT, we can formulate an additive decomposition of $\mathbf{x}$ by reconstructing the wavelet coefficients at each scale independently. Let $\mathbf{D}_j = \mathcal{W}_j^T\mathbf{w}_j$ define the jth level *wavelet detail* associated with changes in $\mathbf{x}$ at the scale λ_j (for $j = 1, \ldots, J$). The wavelet coefficients $\mathbf{w}_j = \mathcal{W}_j\mathbf{x}$ represent the portion of the wavelet analysis (decomposition) attributable to scale λ_j, whereas $\mathcal{W}_j^T\mathbf{w}_j$ is the portion of the wavelet synthesis (reconstruction) attributable to scale λ_j. For an observation vector with length $N = 2^J$, the vector $\mathbf{D}_{J+1}$ is equal to the sample mean of the observations.

A MRA may now be defined via

$$x_t = D_{j+1,t} + D_{j,t} + D_{j-1,t} + ... + D_{1,t} \tag{1}$$

That is, each observation x_t is a linear combination of wavelet detail coefficients at time t. Let $\mathbf{S}_j = \sum_{k=j+1}^{J+1}\mathbf{D}_k$ define the jth level wavelet smooth component. Although the wavelet detail $\mathbf{D}_j$ is associated with variations at a particular scale, $\mathbf{S}_j$ is a cumulative sum of these variations and will be increasingly smooth as j increases. In fact, $\mathbf{x} - \mathbf{S}_j = \sum_{k=1}^{j}\mathbf{D}_k$ such that only lower scale details (high frequency features) from the original series remain. Hence, Equation (1) can be rewritten

$$x_t = S_{j,t} + D_{j,t} + D_{j-1,t} + ... + D_{1,t} \tag{2}$$

The terminology "detail" and "smooth" were used by Percival and Walden (2006) to describe additive decompositions from Fourier and wavelet transforms. The goal is to examine data at different resolutions with this representation. The smooth part is coarse, and we are examining local averages, i.e., low frequency trends and the sample mean. The detail is the deviation from the smooth part.

A variation of the DWT is called the maximum overlap DWT (MODWT). Similar to the DWT, the MODWT is a subsampling at the dyadic scale; however, compared with the DWT, the analysis involves all times t rather than multiples of 2^j. The retention of all possible times eliminates the alignment effects of DWT and leads to more efficient time series representations at multiple time scales.

For a given level j, to produce forecasts for x_t, we fit time series models to each component of x_t separately. Using Equation (2), a composite forecast for x_{t+h} can be reconstructed from the h-step ahead forecasts of individual components; i.e., $\hat{S}_{j,t+h}, \hat{D}_{j,t+h}, \hat{D}_{j,t+h}, ..., \hat{D}_{j,t+h}$. Their sum $\hat{C}_{j,t+h}$ refers to this composite forecast of x_{t+h}.

$$\hat{C}_{j,t+h} = \hat{S}_{j,t+h} + \hat{D}_{j,t+h} + \hat{D}_{j-1,t+h} + ... + \hat{D}_{1,t+h} \tag{3}$$

Alternatively, the forecasts of all individual components can be separately used to forecast x_{t+h} alone. Notice that j is not allowed to be smaller than 1. Otherwise, for $j = 0$, when no wavelet decomposition is applied to the data, $\hat{C}_{0,t+h} = \hat{S}_{0,t+h}$, which is identical to $\hat{x}_{t+h}$; i.e. the forecast of x_{t+h} is generated using the unfiltered data.

[2] DWT is an orthonormal transform, and orthonormality implies that $\mathbf{x} = \mathcal{W}^T\mathbf{w}$ and $||w||^2 = ||x||^2$.

3 Forecast Evaluation and Forecast Models

We use the root mean square forecast error (RMSFE) as the forecast evaluation criteria and calculate the following RMSFEs for different components of wavelets.

$$RMSFE = \sqrt{\sum_{l=1}^{\tau} \left(x_{t+h} - \hat{C}^l_{j,t+h}, \hat{S}^l_{j,t+h}, \hat{D}^l_{j,t+h}, \hat{x}^l_{j,t+h} \right)^2 / \tau} \tag{4}$$

where, $\hat{C}_{j,t+h}$, $\hat{S}_{j,t+h}$, and $\hat{D}_{j,t+h}$ refer to the forecast obtained from the composite, smooth or detail series at the jth level wavelet decomposition,respectively; and $\hat{x}_{j,t+h}$ refers to the forecasts of unfiltered data. τ denotes the number of times that the h-step ahead forecast error is calculated.

To produce each of these forecasts, we use several univariate time series models. The comparison models include linear autoregressive (AR) models, autoregressive moving-average (ARMA) models, logistic smooth transition autoregressive (LSTAR) models, self-exciting threshold autoregressive (SETAR) models, Markov-switching autoregressive (MS-AR) models, autoregressive neural network (ARNN) models, and AR augmented with GARCH volatility component (AR-GARCH) models.

Let y_t represent either x_t, or $S_{j,t}$, or $D_{j,t}$ depending on the series that we use forecasting x_{t+h}. Let $\hat{y}_{t+h,t}$ be the forecast of y_t that is generated at time t for the time $t + h$ $(h \geq 1)$ by any forecasting model. In the RW model, $\hat{y}_{t+h,t}$ is equal to the value of y_t at time t.

The ARMA model is

$$y_t = \alpha + \sum_{i=1}^{p} \phi_{1,i} y_{t-i} + \sum_{i=1}^{q} \phi_{2,i} \varepsilon_{t-i} + \varepsilon_t, \tag{5}$$

where p and q are selected to minimize the Akaike information criterion (AIC) with a maximum lag of 24. After estimating the parameters of equation (5), one can easily produce h-step $(h \geq 1)$ forecasts through the following recursive equation:

$$\hat{y}_{t+h,t} = \alpha + \sum_{i=1}^{p} \hat{\phi}_{1,i} y_{t+h-i} + \sum_{i=1}^{q} \hat{\phi}_{2,i} \varepsilon_{t+h-i} \,. \tag{6}$$

When $h > 1$, we iterate a one-period forecasting model to obtain forecasts by feeding the previous period forecasts as regressors into the model. This finding means that when $h > p$ and $h > q$,, y_{t+h-i} is replaced by $\hat{y}_{t+h-i,t}$, ε_{t+h-i} is replaced by $\hat{\varepsilon}_{t+h-i,t} = 0$.

An obvious alternative to iterating forward on a single-period model would be to tailor the forecasting model directly to the forecast horizon; in other words, the following equation is estimated using the data to a maximum of t:

$$y_t = \alpha + \sum_{i=0}^{p} \phi_{1,i} y_{t-i-h} + \sum_{i=0}^{q} \phi_{2,i} \varepsilon_{t-i-h} + \varepsilon_t, \tag{7}$$

for $h \geq 1$. We use the fitted values of this regression to directly produce an h-step ahead forecast.[3]

Because it is a special case of ARMA, the estimation and forecasts of the AR model can be obtained by simply setting $q = 0$ in (5) and (7).

The AR-GARCH model is simply an AR model augmented by the following GARCH volatility component:

$$\sigma_t^2 = a_0 + \sum_{i=1}^{p} a_i \varepsilon_{t-i}^2 + \sum_{i=1}^{q} b_i \sigma_{t-i}^2 + \eta_t \tag{8}$$

[3]Deciding whether the direct or the iterated approach is better is an empirical matter because it involves a trade-off between the estimation efficiency and the robustness-to-model misspecification; see Elliott and Timmermann (2008). Marcellino et al. (2006) have addressed these points empirically using a dataset of 170 US monthly macroeconomic time series, and they found that the iterated approach generates the lowest RMSFE values, particularly if lengthy lags of the variables are included in the forecasting models and the forecast horizon is long.

where σ^2 represents the conditional variance of ε and η_t are i.d.d. errors. Once the AR-GARCH model is estimated, its forecasts can be obtained via the mean equation as in the AR case.

The LSTAR model is

$$y_t = \left(\alpha_1 + \sum_{i=1}^{p} \phi_{1,i}\, y_{t-i}\right) + d_t \left(\alpha_2 + \sum_{i=1}^{q} \phi_{2,i}\, y_{t-i}\right) + \varepsilon_t, \tag{9}$$

where $d_t = (1 + \exp\{-\gamma(y_{t-1} - c)\})^{-1}$. Whereas ε_t are regarded as normally distributed i.i.d. variables with zero mean, α_1, α_2, $\phi_{1,i}$, $\phi_{2,i}$, γ and c are simultaneously estimated using maximum likelihood methods.

In the LSTAR model, the direct forecast can be obtained in the same manner as with the ARMA model, which is the case for all subsequent nonlinear models.[4] However, any iterative scheme cannot be applied to obtain forecasts for multiple steps in advance as can be done for linear models. This impossibility follows from the general fact that the conditional expectation of a nonlinear function is not always equal to a function of that conditional expectation. In addition, one cannot iteratively derive the forecasts for the time steps $h > 1$ by plugging in the previous forecasts (e.g., Kock et al., 2011).[5] Therefore, we use the Monte Carlo integration scheme suggested by Lin and Granger (1994) to numerically calculate the conditional expectations, and we then produce the forecasts iteratively.

When $|\gamma| \to \infty$, the LSTAR model approaches the two-regime SETAR model, which is also included in our forecasting models. As with LSTAR and most nonlinear models, forecasting with SETAR does not permit the use of a simple iterative scheme to generate multiple-period forecasts. In this case, we employ a version of the Normal Forecasting Error (NFE) method suggested by Al-Qassam and Lane (1989) to generate multistep forecasts.[6] NFE is an explicit, form-recursive approximation for calculating higher-step forecasts under the normality assumption of error terms and has been shown by De Gooijer and De Bruin (1998)to perform with reasonable accuracy compared with the alternative methods numerical integration and Monte Carlo.

The two-regime MS-AR model that we consider here is as follows:

$$y_t = \alpha_s + \sum_{i=1}^{p} \phi_{s,i} y_{t-i} + \varepsilon_t, \tag{10}$$

where s_t is a two-state discrete Markov chain with $S = \{1, 2\}$ and $\varepsilon_t \sim$ i.i.d. $N(0, \sigma^2)$. We estimate MS-AR using the maximum likelihood expectation-maximization algorithm.

Although MS-AR models may encompass complex dynamics, point forecasting is less complicated than other nonlinear models. The h-step forecast from the MS-AR model is

$$\begin{aligned} \hat{y}_{t+h,t} &= P\left(s_{t+h} = 1 \mid y_t, ..., y_0\right) \left(\alpha_{s=1} + \sum_{i=1}^{p} \hat{\phi}_{s=1,i} y_{t+h-i}\right) \\ &\quad + P\left(s_{t+h} = 2 \mid y_t, ..., y_0\right) \left(\alpha_{s=2} + \sum_{i=1}^{p} \hat{\phi}_{s=2,i} y_{t+h-i}\right), \end{aligned} \tag{11}$$

where $P\left(s_{t+h} = i \mid y_t, ..., y_0\right)$ is the ith element of the column vector $\mathbf{P}^h \hat{\xi}_{t|t}$. In addition, $\hat{\xi}_{t|t}$ represents the filtered probability vector and $\mathbf{P}^h$ is the constant transition probability matrix (see, Hamilton (1994)). Hence, multistep forecasts can be obtained iteratively by plugging in $1, 2, 3, \ldots$-period forecasts, which are similar to the iterative forecasting method of the AR processes.

ARNN, which is the autoregressive single-hidden-layer feed-forward neural network model[7] suggested in Teräsvirta (2006), is defined as follows:

[4] This process involves replacing y_t with y_{t+h} on the left side of equation (4) and running the regression using data to a maximum time t to fitted values for corresponding forecasts.

[5] Indeed, d_t is convex in y_{t-1} whenever $y_{t-1} < c$, and $-d_t$ is convex whenever $y_{t-1} > c$. Therefore, by Jensen's inequality, naive estimations underestimate d_t if $y_{t-1} < c$ and overestimate d_t if $y_{t-1} > c$.

[6] A detailed exposition of approaches for forecasting from a SETAR model can be found in van Dijk et al. (2003)

[7] See Franses and Van Dijk (2000) for a review of feed-forward-type neural network models.

$$y_t = \alpha + \sum_{i=1}^{p} \phi_i y_{t-i} + \sum_{j=1}^{h} \lambda_j d \left(\sum_{i=1}^{p} \gamma_i y_{t-i} - c \right) + \varepsilon_t, \quad (12)$$

where d is the logistic function, which is defined above as $d = (1 + \exp\{-x\})^{-1}$. In general, the estimation of an ARNN model may be computationally challenging. Here, we act in accordance with the QuickNet method, which is a type of "relaxed greedy algorithm" that was originally suggested by White (2006). In contrast, the forecasting procedure for the ARNN model is identical to the procedure for the LSTAR model.

4 Data

We use Japanese/Euro Yen (JPY/EUR) tick by tick spot exchange rate data available from 00:00 February 1, 2011 to 23:45 29th April, 2011. The data are obtained from Thick Data, Inc .

During this period, we have a total of 258,275 data points at irregular intervals. We take 5-minute trade volume-weighted averages to perform our estimations; hence, we have 16,383 data points that can be used for estimation purposes. Figure 1 illustrates the exchange rate series. Figure 2 provides the average trading volumes used as weights in the calculation of the 5-minute averages. Figure 3 shows (log) return series, calculated based on 5-minute trade volume-weighted averages, which we use in the estimation of our forecasting models. We further calculate the volatility of the 5-minute return series in Figure 4, where the volatility is measured based on standard deviations calculated based on 90-point backwards rolling windows.

[Figure 1 - 4]

As shown in the figures, the exchange rate returns exhibit different volatility characteristics across the evaluation period. Certain jumps are also visible in periods where volatility appears to be higher.

5 Pseudo-out-of-sample forecasting and forecasting zones

The pseudo-out-of-sample forecasting exercise on which we base our forecast evaluations is conducted for different volatility regimes. We divide our data under 4 different regimes. Figure 5 illustrates these different volatility regimes on the time plot of the return series. Table 1 presents the average volatilities for each regime, which are calculated as the average of the rolling standard deviations, in addition to other details.

[Figure 5]

Table 1: Volatility regimes and periodization.

Period	Dates	Regime	Average Volatility	Data Points
1	2011/02/01 - 2011/03/12	Low	0.00035	6.709
2	2011/03/13 - 2011/03/21	High	0.00064	2.662
3	2011/03/22 - 2011/04/03	Low	0.00036	1.847

By using these volatility regimes, we further define 4 different estimation and forecasting zones for our pseudo-out-of-sample forecasting exercise. Table 2 depicts these estimation and forecasting zones. In Zone 1, the model forecasts and estimates are for the data points of the low volatility regime of Period 1 (Table 1). However, in Zone 2, although estimates are conducted for Period 1, which is the low volatility regime, forecasts are conducted for data points in Period 2, which is the high volatility zone. In Zone 3, both estimates and forecasts are conducted for the high volatility regime of Period 2. In Zone 4, however, although estimates are conducted for the high volatility regime, forecasts are solely confined to the low volatility regime of Period 3.

Table 2: Estimation and forecasting zones.

Zones	Estimation	Forecasting
1	Low (Period 1)	Low (Period 1)
2	Low (Period 1)	High (Period 2)
3	High (Period 2)	High (Period 2)
4	High (Period 2)	Low (Period 3)

In each of these zones, we perform pseudo-out-of-sample forecasts for the forecast horizons $h = 1$ (short) and $h = 5$ (long). When the estimation and forecasting zones both feature the same volatility nature, e.g., Zone 1 and 3, the models are estimated by running regressions with data no later than the date $t_i - \tau$, where τ is an integer that refers to the length of the training zone of the pseudo-out-of-sample forecast exercise, i.e., the number of pseudo-out-of-sample forecasts is repeated (see Figure 6).

[Figure 6]

In Zones 2 and 4, t_i refers to the final date for which the forecast evaluation is conducted. Alternatively, in Zone 2 and 4, the models are estimated using data no later than t_i, with $t_i + \tau$ referring to the final date used in the forecast evaluation. In this setting, t_{i-1} always refers to the date when the estimation is initialized (where $i = 1, 2, 3$), and $j = 1, 2, ..., \tau$ refers to the number of times for which the forecast error is calculated. For each of these 4 zones, when $j = 1$ and the first $h = 1$ horizon forecast is obtained using the coefficient estimates from the initial regressions, the regressions use data between t_{i-1} and $t_i - \tau$ (Zone 1, 3); t_i (Zone 2, 4). Next, after moving forward by one period, the procedure is repeated with $j = 2$, and the second $h = 1$ forecast is calculated from the coefficient estimates obtained from the re-estimation of the models after extending the data by one period. By repeating this procedure $j = \tau$ times, we calculate τ forecast errors for each of the models and wavelet components and the series used in our applications. For $h = 5$, the same procedure is repeated. Multistep forecasts are calculated in a recursive manner by feeding in the previous period forecast as the input.

It should be emphasized that the wavelet decomposition is applied to the available data before the estimation of each regression to prevent using the information in the forecast period. In the first round, we performed a wavelet decomposition for the data between t_{i-1}, $t_i - \tau$ (Zone 1, 3) and t_i (Zone 2, 4) and then moved forward by one-period and re-applied the wavelet decomposition.

6 Results

Tables 3 - 6 tabulate the RMSFE results for the pseudo-out-of-sample period of length $\tau = 100$ calculated at each volatility regime changing point for short forecast horizons ($h = 1$).

[Table 3 - 6]

Table 3 illustrates the RMSFE results for Zone 1. The second column shows the level, j, of the wavelet decomposition.[8] In the third column, the RMSFEs of the composite forecasts $\tilde{X}_j$ are presented. When $j = 0$, the data are used in the model estimations instead of wavelet components; hence, the forecasts become $\hat{x}$. The third column shows the RMSFE obtained from the smooth part of the data. The remaining columns display the RMSFE calculated from the details. For each row block, which represent each model, the smallest RMSFE is emboldened. For example, for the AR model, the smallest RMSFE belongs to S_8, whereas for ARMA, the smallest RMSFE is D_8. Furthermore, the smallest RMSFE for the entire table is indicated by a star. For example, in Table 3, the smallest RMSFE for Zone 1 where both estimations and forecasts are performed in the low volatility regime is indicated as the AR model for which forecasts are performed using the smooth part of the return series after the 8th level wavelet decomposition, S_8. Table 4 - 6 display the same results for the remaining estimation and forecast zones for $h = 1$. Table 7 - 10 illustrate the results for long forecast horizons when $h = 5$.

[8] $j = 0$ refers to $\hat{x}$, i.e. the forecast produced using unfiltered data

[Table 7 - 10]

The overall results are summarized in Table 11.

Table 11: Summary of Results

	$h = 1$		$h = 5$	
Zones	Model	Variable	Model	Variable
Zone 1 (Low - Low)	AR	S_8	ARNN	S_7
Zone 2 (Low - High)	ARMA	S_1	ARNN	S_7
Zone 3 (High - High)	LSTAR	D_3	LSTAR	D_2
Zone 4 (High - Low)	ARMA	S_1	LSTAR	$\tilde{X}_5$

Linear models have higher forecast abilities only for short forecasts and low volatility zones for both the estimation and the forecast window (Zone 1). In the other cases, when volatility is high either in the estimation or the forecast window, nonlinear models appear to be superior. When the forecast horizon is high, more sophisticated nonlinear models, such as the ARNN and LSTAR models, become more dominant over their simplistic counterparts, such as the ARMA model.

Wavelet decomposition becomes useful in all cases instead of the return series. The smooth part appears to be the most useful forecaster except when volatility is high in both the estimation and the forecast zones (Period 2), where the detail series becomes more useful for forecasting. This result indicates that when data exhibit nonstandard features, such as high volatility and jumps as in the Period 2 details series, they carry additional information on the noisy part of the data and become more useful for forecasting.

Usually either the details or the smooth part alone are sufficient to obtain suitable forecasting performance except for long horizon forecasting in Zone 4, where a composite series should be adopted for forecasting. Additionally, a higher level of decomposition proves to be more efficient when the forecasting horizon is long. The only exception for this letter result is the AR model in Zone 1, where the 8th level decomposition is performed to obtain the smooth part. This finding may indicate that for a simple linear model, such as AR, denoising should be performed several times to achieve a successful forecaster in a low volatility regime. Moreover, a higher level of decomposition is beneficial for longer forecast horizons.

7 Conclusions

We analyzed the effect of the wavelet decomposition, type models and volatility on forecasting performance using a high frequency exchange rate return series. The results indicate that because wavelets can partition data into homogeneous components, they have the ability to improve the forecast accuracy in high frequency settings. Conversely, the volatility characteristics of the data used for estimation and forecasting purposes become critical for the choice of the component used in time series models. Because smooth components of the series carry additional information on the general trends of the series, they usually become the most useful component for forecasting high frequency exchange rate returns. With a notable exception, when volatility is high, it is expected to remain high in the forecast horizon regardless of whether the forecasts are performed for short or long horizons, and the wavelet details component should be used instead of the smooth component. This finding indicates that when data are highly volatile and exhibit frequent jumps, wavelet details, which represent the noisier component of a series, carry more useful information for forecasting.

In terms of model choice, our results suggest that nonlinear models should be used unless volatility is low in both the estimation and forecast zone, and they should be used only if short-term forecasting is targeted. In these latter cases, a simple linear AR could be used on "cleaned" data after a higher level of decomposition.

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Tables and Figures

Table 3: RMSFEs, $h = 1$, Zone 1 (Low - Low) ($\times 10^{-7}$)

Model	level(j)	$\hat{C}_j$	$\hat{S}_j$	$\hat{D}_1$	$\hat{D}_2$	$\hat{D}_3$	$\hat{D}_4$	$\hat{D}_5$	$\hat{D}_6$	$\hat{D}_7$	$\hat{D}_8$
AR	0	0.7161									
AR	1	1.0393	0.8317	0.9178							
AR	2	1.0393	0.8299	0.9178	0.7337						
AR	3	1.0514	0.8295	0.9178	0.7337	0.7365					
AR	4	1.0444	0.7312	0.9178	0.7337	0.7365	0.807				
AR	5	1.0447	0.7256	0.9178	0.7337	0.7365	0.807	0.7263			
AR	6	1.0445	0.7184	0.9178	0.7337	0.7365	0.807	0.7263	0.7195		
AR	7	1.043	0.7104	0.9178	0.7337	0.7365	0.807	0.7263	0.7195	0.7194	
AR	8	1.0419	**0.7104***	0.9178	0.7337	0.7365	0.807	0.7263	0.7195	0.7194	0.7111
ARMA	0	0.7151									
ARMA	1	0.9174	0.9746	0.9879							
ARMA	2	0.9811	0.829	0.9879	0.8139						
ARMA	3	0.9613	0.8295	0.9879	0.8139	0.748					
ARMA	4	0.9557	0.7314	0.9879	0.8139	0.748	0.8064				
ARMA	5	0.955	0.7256	0.9879	0.8139	0.748	0.8064	0.7265			
ARMA	6	0.9551	0.7185	0.9879	0.8139	0.748	0.8064	0.7265	0.7193		
ARMA	7	0.9511	0.7148	0.9879	0.8139	0.748	0.8064	0.7265	0.7193	0.7194	
ARMA	8	0.9538	0.7161	0.9879	0.8139	0.748	0.8064	0.7265	0.7193	0.7194	**0.7113**
SETAR	0	0.7197									
SETAR	1	1.0195	0.8239	0.906							
SETAR	2	1.024	0.8282	0.906	0.7328						
SETAR	3	1.0368	0.8293	0.906	0.7328	0.7369					
SETAR	4	1.0264	0.7287	0.906	0.7328	0.7369	0.8057				
SETAR	5	1.0285	0.7252	0.906	0.7328	0.7369	0.8057	0.7269			
SETAR	6	1.0284	0.7183	0.906	0.7328	0.7369	0.8057	0.7269	0.7191		
SETAR	7	1.0282	0.7117	0.906	0.7328	0.7369	0.8057	0.7269	0.7191	0.7193	
SETAR	8	1.0283	0.7125	0.906	0.7328	0.7369	0.8057	0.7269	0.7191	0.7193	**0.7111**
LSTAR	0	0.7202									
LSTAR	1	0.868	0.855	0.9535							
LSTAR	2	0.9155	0.8285	0.9535	0.8058						
LSTAR	3	0.9332	0.828	0.9535	0.8058	0.7408					
LSTAR	4	0.9194	0.73	0.9535	0.8058	0.7408	0.8036				
LSTAR	5	0.9228	0.7249	0.9535	0.8058	0.7408	0.8036	0.7285			
LSTAR	6	0.9228	0.7184	0.9535	0.8058	0.7408	0.8036	0.7285	0.7188		
LSTAR	7	0.9228	0.712	0.9535	0.8058	0.7408	0.8036	0.7285	0.7188	0.7194	
LSTAR	8	0.9226	0.7125	0.9535	0.8058	0.7408	0.8036	0.7285	0.7188	0.7194	**0.7112**
ARNN	0	0.7284									
ARNN	1	1.0226	0.8518	0.8997							
ARNN	2	1.0133	0.8195	0.8997	0.7365						
ARNN	3	1.0359	0.8256	0.8997	0.7365	0.7363					
ARNN	4	1.0264	0.7291	0.8997	0.7365	0.7363	0.8055				
ARNN	5	1.0318	0.7256	0.8997	0.7365	0.7363	0.8055	0.727			
ARNN	6	1.0313	0.7188	0.8997	0.7365	0.7363	0.8055	0.727	0.7192		
ARNN	7	1.031	0.7116	0.8997	0.7365	0.7363	0.8055	0.727	0.7192	0.7194	
ARNN	8	1.0319	0.7125	0.8997	0.7365	0.7363	0.8055	0.727	0.7192	0.7194	**0.7115**
MSVAR	0	0.7184									
MSVAR	1	1.0387	0.8323	0.9177							
MSVAR	2	1.024	0.8246	0.9177	0.7326						
MSVAR	3	1.0428	0.8305	0.9177	0.7326	0.7364					
MSVAR	4	1.0356	0.7312	0.9177	0.7326	0.7364	0.8071				
MSVAR	5	1.0357	0.7256	0.9177	0.7326	0.7364	0.8071	0.7263			
MSVAR	6	1.0355	0.7184	0.9177	0.7326	0.7364	0.8071	0.7263	0.7195		
MSVAR	7	1.0355	0.7118	0.9177	0.7326	0.7364	0.8071	0.7263	0.7195	0.7194	
MSVAR	8	1.0355	0.7125	0.9177	0.7326	0.7364	0.8071	0.7263	0.7195	0.7194	**0.7111**
AR-GARCH	0	0.7159									
AR-GARCH	1	1.0228	0.8216	0.9113							
AR-GARCH	2	1.0271	0.8229	0.9113	0.7339						
AR-GARCH	3	1.0395	0.8251	0.9113	0.7339	0.7332					
AR-GARCH	4	1.0322	0.7303	0.9113	0.7339	0.7332	0.8029				
AR-GARCH	5	1.0327	0.7253	0.9113	0.7339	0.7332	0.8029	0.7255			
AR-GARCH	6	1.0324	0.7183	0.9113	0.7339	0.7332	0.8029	0.7255	0.7192		
AR-GARCH	7	1.0324	0.7118	0.9113	0.7339	0.7332	0.8029	0.7255	0.7192	0.7193	
AR-GARCH	8	1.0324	0.7125	0.9113	0.7339	0.7332	0.8029	0.7255	0.7192	0.7193	**0.7111**

Table 4: RMSFEs, $h = 1$, Zone 2 (Low - High) ($\times 10^{-7}$)

Model	level(j)	$\hat{C}_j$	$\hat{S}_j$	$\hat{D}_1$	$\hat{D}_2$	$\hat{D}_3$	$\hat{D}_4$	$\hat{D}_5$	$\hat{D}_6$	$\hat{D}_7$	$\hat{D}_8$
AR	0	1.0619									
AR	1	1.5778	1.015	1.557							
AR	2	1.5986	1.0515	1.557	0.9797						
AR	3	1.6083	1.0738	1.557	0.9797	**0.9751**					
AR	4	1.6044	1.0159	1.557	0.9797	**0.9751**	1.0317				
AR	5	1.6045	0.9954	1.557	0.9797	**0.9751**	1.0317	1.0055			
AR	6	1.6044	0.9942	1.557	0.9797	**0.9751**	1.0317	1.0055	0.9911		
AR	7	1.5962	0.9924	1.557	0.9797	**0.9751**	1.0317	1.0055	0.9911	0.9899	
AR	8	1.6024	0.9924	1.557	0.9797	**0.9751**	1.0317	1.0055	0.9911	0.9899	0.9943
ARMA	0	1.0584									
ARMA	1	1.2519	**0.9028***	1.8715							
ARMA	2	1.479	1.0408	1.8715	0.995						
ARMA	3	1.451	1.0738	1.8715	0.995	0.9618					
ARMA	4	1.4491	1.0178	1.8715	0.995	0.9618	1.0299				
ARMA	5	1.4488	0.9969	1.8715	0.995	0.9618	1.0299	1.006			
ARMA	6	1.4489	0.9951	1.8715	0.995	0.9618	1.0299	1.006	0.9918		
ARMA	7	1.4355	0.9884	1.8715	0.995	0.9618	1.0299	1.006	0.9918	0.9903	
ARMA	8	1.4435	0.9895	1.8715	0.995	0.9618	1.0299	1.006	0.9918	0.9903	0.9945
SETAR	0	1.0408									
SETAR	1	1.5794	1.0055	1.5593							
SETAR	2	1.6006	1.0496	1.5593	**0.974**						
SETAR	3	1.6075	1.0699	1.5593	**0.974**	0.9754					
SETAR	4	1.605	1.0162	1.5593	**0.974**	0.9754	1.0293				
SETAR	5	1.6046	0.9951	1.5593	**0.974**	0.9754	1.0293	1.0056			
SETAR	6	1.6046	0.9943	1.5593	**0.974**	0.9754	1.0293	1.0056	0.9909		
SETAR	7	1.6043	0.9961	1.5593	**0.974**	0.9754	1.0293	1.0056	0.9909	0.99	
SETAR	8	1.6045	0.9927	1.5593	**0.974**	0.9754	1.0293	1.0056	0.9909	0.99	0.9942
LSTAR	0	1.0803									
LSTAR	1	1.4022	1	1.7053							
LSTAR	2	1.4938	1.0378	1.7053	1.0924						
LSTAR	3	1.5164	1.0738	1.7053	1.0924	**0.9697**					
LSTAR	4	1.5099	1.018	1.7053	1.0924	**0.9697**	1.0277				
LSTAR	5	1.5115	0.9976	1.7053	1.0924	**0.9697**	1.0277	1.0073			
LSTAR	6	1.5122	0.9953	1.7053	1.0924	**0.9697**	1.0277	1.0073	1.0011		
LSTAR	7	1.5118	0.9967	1.7053	1.0924	**0.9697**	1.0277	1.0073	1.0011	0.9901	
LSTAR	8	1.5117	0.9929	1.7053	1.0924	**0.9697**	1.0277	1.0073	1.0011	0.9901	0.9944
ARNN	0	1.0355									
ARNN	1	1.5563	0.9962	1.5528							
ARNN	2	1.5741	1.0544	1.5528	0.977						
ARNN	3	1.6057	1.0771	1.5528	0.977	**0.9765**					
ARNN	4	1.5886	1.0132	1.5528	0.977	**0.9765**	1.0301				
ARNN	5	1.5939	0.9957	1.5528	0.977	**0.9765**	1.0301	1.0054			
ARNN	6	1.5928	0.9942	1.5528	0.977	**0.9765**	1.0301	1.0054	0.9898		
ARNN	7	1.5928	0.9963	1.5528	0.977	**0.9765**	1.0301	1.0054	0.9898	0.9898	
ARNN	8	1.5925	0.9928	1.5528	0.977	**0.9765**	1.0301	1.0054	0.9898	0.9898	0.9941
MSVAR	0	1.0649									
MSVAR	1	1.6033	1.028	1.5671							
MSVAR	2	1.6143	1.0542	1.5671	0.9825						
MSVAR	3	1.6161	1.0737	1.5671	0.9825	**0.9752**					
MSVAR	4	1.6101	1.0138	1.5671	0.9825	**0.9752**	1.0318				
MSVAR	5	1.6128	0.9954	1.5671	0.9825	**0.9752**	1.0318	1.0055			
MSVAR	6	1.6128	0.9942	1.5671	0.9825	**0.9752**	1.0318	1.0055	0.9911		
MSVAR	7	1.6129	0.9963	1.5671	0.9825	**0.9752**	1.0318	1.0055	0.9911	0.9899	
MSVAR	8	1.6129	0.9927	1.5671	0.9825	**0.9752**	1.0318	1.0055	0.9911	0.9899	0.9943
AR-GARCH	0	1.0787									
AR-GARCH	1	1.5576	1.0085	1.5428							
AR-GARCH	2	1.5782	1.0447	1.5428	0.9797						
AR-GARCH	3	1.5886	1.0698	1.5428	0.9797	**0.973**					
AR-GARCH	4	1.5847	1.0151	1.5428	0.9797	**0.973**	1.0293				
AR-GARCH	5	1.5849	0.9953	1.5428	0.9797	**0.973**	1.0293	1.005			
AR-GARCH	6	1.5849	0.9942	1.5428	0.9797	**0.973**	1.0293	1.005	0.991		
AR-GARCH	7	1.5849	0.9963	1.5428	0.9797	**0.973**	1.0293	1.005	0.991	0.9899	
AR-GARCH	8	1.5849	0.9927	1.5428	0.9797	**0.973**	1.0293	1.005	0.991	0.9899	0.9943

Table 5: RMSFEs, $h = 1$, Zone 3 (High - High) ($\times 10^{-7}$)

Model	level(j)	$\hat{C}_j$	$\hat{S}_j$	$\hat{D}_1$	$\hat{D}_2$	$\hat{D}_3$	$\hat{D}_4$	$\hat{D}_5$	$\hat{D}_6$	$\hat{D}_7$	$\hat{D}_8$
AR	0	6.78									
AR	1	8.8608	6.6177	9.0213							
AR	2	9.1512	7.0948	9.0213	6.5941						
AR	3	9.2756	7.522	9.0213	6.5941	**6.4571**					
AR	4	9.2465	6.9162	9.0213	6.5941	**6.4571**	7.357				
AR	5	9.2463	6.7721	9.0213	6.5941	**6.4571**	7.357	6.9086			
AR	6	9.2469	6.811	9.0213	6.5941	**6.4571**	7.357	6.9086	6.7458		
AR	7	9.2469	6.7547	9.0213	6.5941	**6.4571**	7.357	6.9086	6.7458	6.8267	
AR	8	9.2611	6.8375	9.0213	6.5941	**6.4571**	7.357	6.9086	6.7458	6.8267	6.7249
ARMA	0	6.78									
ARMA	1	7.1284	6.58	10.8538							
ARMA	2	7.8421	7.0373	10.8538	6.7025						
ARMA	3	8.0661	7.4943	10.8538	6.7025	**6.5268**					
ARMA	4	8.0573	6.9162	10.8538	6.7025	**6.5268**	7.3348				
ARMA	5	8.0534	6.767	10.8538	6.7025	**6.5268**	7.3348	6.9086			
ARMA	6	8.0559	6.811	10.8538	6.7025	**6.5268**	7.3348	6.9086	6.7414		
ARMA	7	8.0561	6.7547	10.8538	6.7025	**6.5268**	7.3348	6.9086	6.7414	6.8267	
ARMA	8	8.0487	6.8281	10.8538	6.7025	**6.5268**	7.3348	6.9086	6.7414	6.8267	6.7249
SETAR	0	7.0549									
SETAR	1	9.1994	6.6877	9.2566							
SETAR	2	9.3641	7.0251	9.2566	6.5874						
SETAR	3	9.5113	7.4621	9.2566	6.5874	**6.4343**					
SETAR	4	9.5326	6.9139	9.2566	6.5874	**6.4343**	7.3493				
SETAR	5	9.5501	6.771	9.2566	6.5874	**6.4343**	7.3493	6.9168			
SETAR	6	9.5488	6.8103	9.2566	6.5874	**6.4343**	7.3493	6.9168	6.7454		
SETAR	7	9.5512	6.7539	9.2566	6.5874	**6.4343**	7.3493	6.9168	6.7454	6.8266	
SETAR	8	9.5502	6.8395	9.2566	6.5874	**6.4343**	7.3493	6.9168	6.7454	6.8266	6.7244
LSTAR	0	7.1823									
LSTAR	1	8.183	6.6433	10.0767							
LSTAR	2	8.7241	6.993	10.0767	7.1614						
LSTAR	3	8.7851	7.5043	10.0767	7.1614	**6.3997***					
LSTAR	4	8.6838	6.9387	10.0767	7.1614	**6.3997***	7.2786				
LSTAR	5	8.4784	6.7713	10.0767	7.1614	**6.3997***	7.2786	6.785			
LSTAR	6	8.4741	6.8116	10.0767	7.1614	**6.3997***	7.2786	6.785	6.7356		
LSTAR	7	8.4735	6.7517	10.0767	7.1614	**6.3997***	7.2786	6.785	6.7356	6.8268	
LSTAR	8	8.4767	6.8412	10.0767	7.1614	**6.3997***	7.2786	6.785	6.7356	6.8268	6.7232
ARNN	0	7.0725									
ARNN	1	9.0784	6.6549	9.3289							
ARNN	2	9.1314	6.968	9.3289	6.5638						
ARNN	3	9.5214	7.4333	9.3289	6.5638	**6.4829**					
ARNN	4	9.5133	6.9437	9.3289	6.5638	**6.4829**	7.2593				
ARNN	5	9.5105	6.7862	9.3289	6.5638	**6.4829**	7.2593	6.9038			
ARNN	6	9.5131	6.8162	9.3289	6.5638	**6.4829**	7.2593	6.9038	6.768		
ARNN	7	9.4809	6.7524	9.3289	6.5638	**6.4829**	7.2593	6.9038	6.768	6.8095	
ARNN	8	9.4855	6.8393	9.3289	6.5638	**6.4829**	7.2593	6.9038	6.768	6.8095	6.7261
MSVAR	0	7.1801									
MSVAR	1	8.7498	6.5729	9.0174							
MSVAR	2	9.0453	7.0831	9.0174	6.5784						
MSVAR	3	9.1706	7.5168	9.0174	6.5784	**6.4629**					
MSVAR	4	9.162	6.9161	9.0174	6.5784	**6.4629**	7.3536				
MSVAR	5	9.1576	6.7672	9.0174	6.5784	**6.4629**	7.3536	6.9088			
MSVAR	6	9.1761	6.8184	9.0174	6.5784	**6.4629**	7.3536	6.9088	6.746		
MSVAR	7	9.1626	6.7549	9.0174	6.5784	**6.4629**	7.3536	6.9088	6.746	6.8266	
MSVAR	8	9.1557	6.832	9.0174	6.5784	**6.4629**	7.3536	6.9088	6.746	6.8266	6.7245
AR-GARCH	0	7.097									
AR-GARCH	1	8.9104	6.5806	9.0983							
AR-GARCH	2	9.1043	6.9711	9.0983	6.5874						
AR-GARCH	3	9.2664	7.4461	9.0983	6.5874	**6.449**					
AR-GARCH	4	9.2349	6.9069	9.0983	6.5874	**6.449**	7.2862				
AR-GARCH	5	9.2362	6.7722	9.0983	6.5874	**6.449**	7.2862	6.8985			
AR-GARCH	6	9.237	6.8112	9.0983	6.5874	**6.449**	7.2862	6.8985	6.7458		
AR-GARCH	7	9.2373	6.7553	9.0983	6.5874	**6.449**	7.2862	6.8985	6.7458	6.8265	
AR-GARCH	8	9.2371	6.842	9.0983	6.5874	**6.449**	7.2862	6.8985	6.7458	6.8265	6.7249

Table 6: RMSFEs, $h = 1$, Zone 4 (High - Low) ($\times 10^{-7}$)

Model	level(j)	$\hat{C}_j$	$\hat{S}_j$	$\hat{D}_1$	$\hat{D}_2$	$\hat{D}_3$	$\hat{D}_4$	$\hat{D}_5$	$\hat{D}_6$	$\hat{D}_7$	$\hat{D}_8$
AR	0	18.7402									
AR	1	23.193	**16.0463**	25.8188							
AR	2	23.2864	16.35	25.8188	18.3991						
AR	3	23.7108	17.7087	25.8188	18.3991	17.7575					
AR	4	23.6286	17.9878	25.8188	18.3991	17.7575	18.4715				
AR	5	23.6177	17.6803	25.8188	18.3991	17.7575	18.4715	19.0519			
AR	6	23.6099	18.4955	25.8188	18.3991	17.7575	18.4715	19.0519	17.9279		
AR	7	23.6974	18.3667	25.8188	18.3991	17.7575	18.4715	19.0519	17.9279	18.4719	
AR	8	24.0098	18.7929	25.8188	18.3991	17.7575	18.4715	19.0519	17.9279	18.4719	18.1445
ARMA	0	18.7402									
ARMA	1	17.4302	**13.4608***	30.2768							
ARMA	2	17.3499	16.084	30.2768	17.3293						
ARMA	3	17.1755	17.5215	30.2768	17.3293	17.5415					
ARMA	4	17.2225	17.9686	30.2768	17.3293	17.5415	18.3377				
ARMA	5	17.1792	17.6364	30.2768	17.3293	17.5415	18.3377	19.0497			
ARMA	6	17.1921	18.4955	30.2768	17.3293	17.5415	18.3377	19.0497	17.9045		
ARMA	7	17.0388	18.0914	30.2768	17.3293	17.5415	18.3377	19.0497	17.9045	18.4719	
ARMA	8	17.0983	18.5276	30.2768	17.3293	17.5415	18.3377	19.0497	17.9045	18.4719	18.1062
SETAR	0	18.4336									
SETAR	1	24.7703	17.052	25.2398							
SETAR	2	23.6124	**16.3062**	25.2398	18.1873						
SETAR	3	24.1996	17.8274	25.2398	18.1873	17.7463					
SETAR	4	24.0889	17.9979	25.2398	18.1873	17.7463	18.5609				
SETAR	5	24.1084	17.6854	25.2398	18.1873	17.7463	18.5609	19.0479			
SETAR	6	24.0927	18.5045	25.2398	18.1873	17.7463	18.5609	19.0479	17.9216		
SETAR	7	24.1002	18.2989	25.2398	18.1873	17.7463	18.5609	19.0479	17.9216	18.4715	
SETAR	8	24.0943	18.4232	25.2398	18.1873	17.7463	18.5609	19.0479	17.9216	18.4715	18.2004
LSTAR	0	18.2958									
LSTAR	1	21.0197	**15.881**	27.1077							
LSTAR	2	20.2002	16.2892	27.1077	18.1192						
LSTAR	3	20.7667	17.8055	27.1077	18.1192	18.2545					
LSTAR	4	20.6636	17.9948	27.1077	18.1192	18.2545	18.4917				
LSTAR	5	20.5653	17.5868	27.1077	18.1192	18.2545	18.4917	19.0665			
LSTAR	6	20.5796	18.4896	27.1077	18.1192	18.2545	18.4917	19.0665	17.8842		
LSTAR	7	20.5909	18.3001	27.1077	18.1192	18.2545	18.4917	19.0665	17.8842	18.4699	
LSTAR	8	20.5884	18.4354	27.1077	18.1192	18.2545	18.4917	19.0665	17.8842	18.4699	18.1983
ARNN	0	18.9037									
ARNN	1	26.7655	18.5306	26.3635							
ARNN	2	23.9453	**16.2643**	26.3635	18.419						
ARNN	3	24.1399	17.448	26.3635	18.419	17.5056					
ARNN	4	23.9414	17.7909	26.3635	18.419	17.5056	18.7082				
ARNN	5	23.9762	17.5258	26.3635	18.419	17.5056	18.7082	18.9889			
ARNN	6	23.9467	18.4484	26.3635	18.419	17.5056	18.7082	18.9889	17.8173		
ARNN	7	23.9085	18.2117	26.3635	18.419	17.5056	18.7082	18.9889	17.8173	18.4724	
ARNN	8	24.0475	18.4242	26.3635	18.419	17.5056	18.7082	18.9889	17.8173	18.4724	18.1846
MSVAR	0	18.5284									
MSVAR	1	21.9535	**15.2797**	25.8418							
MSVAR	2	22.9883	16.2563	25.8418	18.3594						
MSVAR	3	23.7168	17.8656	25.8418	18.3594	17.6872					
MSVAR	4	23.3813	17.9329	25.8418	18.3594	17.6872	18.4605				
MSVAR	5	23.3803	17.6551	25.8418	18.3594	17.6872	18.4605	19.0436			
MSVAR	6	23.3855	18.4909	25.8418	18.3594	17.6872	18.4605	19.0436	17.9263		
MSVAR	7	23.3921	18.2922	25.8418	18.3594	17.6872	18.4605	19.0436	17.9263	18.4717	
MSVAR	8	23.4139	18.4505	25.8418	18.3594	17.6872	18.4605	19.0436	17.9263	18.4717	18.2002
AR-GARCH	0	20.0689									
AR-GARCH	1	23.4084	15.9874	26.0277							
AR-GARCH	2	23.0791	**15.9358**	26.0277	18.4351						
AR-GARCH	3	23.4481	17.3047	26.0277	18.4351	17.7561					
AR-GARCH	4	23.4405	17.7993	26.0277	18.4351	17.7561	18.3403				
AR-GARCH	5	23.5067	17.6509	26.0277	18.4351	17.7561	18.3403	18.9903			
AR-GARCH	6	23.744	18.6823	26.0277	18.4351	17.7561	18.3403	18.9903	17.9234		
AR-GARCH	7	25.7613	19.9958	26.0277	18.4351	17.7561	18.3403	18.9903	17.9234	18.4697	
AR-GARCH	8	25.8251	19.7831	26.0277	18.4351	17.7561	18.3403	18.9903	17.9234	18.4697	19.0582

Table 7: RMSFEs, $h = 5$, Zone 1 (Low - Low) ($\times 10^{-7}$)

Model	level(j)	$\hat{C}_j$	$\hat{S}_j$	$\hat{D}_1$	$\hat{D}_2$	$\hat{D}_3$	$\hat{D}_4$	$\hat{D}_5$	$\hat{D}_6$	$\hat{D}_7$	$\hat{D}_8$
AR	0	0.7114									
AR	1	0.7190	0.7180	0.7124							
AR	2	0.7563	0.7552	0.7124	0.7114						
AR	3	0.7590	0.7445	0.7124	0.7114	0.7245					
AR	4	0.7469	0.7115	0.7124	0.7114	0.7245	0.7355				
AR	5	0.7488	0.7268	0.7124	0.7114	0.7245	0.7355	**0.705**			
AR	6	0.7482	0.7173	0.7124	0.7114	0.7245	0.7355	**0.705**	0.7206		
AR	7	0.7476	0.7104	0.7124	0.7114	0.7245	0.7355	**0.705**	0.7206	0.7157	
AR	8	0.7476	0.7104	0.7124	0.7114	0.7245	0.7355	**0.705**	0.7206	0.7157	0.7114
ARMA	0	0.7114									
ARMA	1	0.7190	0.7184	0.7119							
ARMA	2	0.7317	0.7310	0.7119	0.7114						
ARMA	3	0.7414	0.7348	0.7119	0.7114	0.7162					
ARMA	4	0.7287	0.7079	0.7119	0.7114	0.7162	0.7274				
ARMA	5	0.7313	0.7257	0.7119	0.7114	0.7162	0.7274	**0.7024**			
ARMA	6	0.7306	0.7167	0.7119	0.7114	0.7162	0.7274	**0.7024**	0.7200		
ARMA	7	0.7297	0.7104	0.7119	0.7114	0.7162	0.7274	**0.7024**	0.7200	0.7150	
ARMA	8	0.7304	0.7104	0.7119	0.7114	0.7162	0.7274	**0.7024**	0.7200	0.7150	0.7147
SETAR	0	**0.7022**									
SETAR	1	0.7105	0.7093	0.7122							
SETAR	2	0.7500	0.7486	0.7122	0.7109						
SETAR	3	0.7524	0.7424	0.7122	0.7109	0.7219					
SETAR	4	0.7414	0.7117	0.7122	0.7109	0.7219	0.7332				
SETAR	5	0.7382	0.7104	0.7122	0.7109	0.7219	0.7332	0.7161			
SETAR	6	0.7387	0.7147	0.7122	0.7109	0.7219	0.7332	0.7161	0.7086		
SETAR	7	0.7392	0.7136	0.7122	0.7109	0.7219	0.7332	0.7161	0.7086	0.7143	
SETAR	8	0.7391	0.7099	0.7122	0.7109	0.7219	0.7332	0.7161	0.7086	0.7143	0.7154
LSTAR	0	0.7116									
LSTAR	1	0.7203	0.7215	0.7108							
LSTAR	2	0.7127	0.7207	0.7108	**0.6975**						
LSTAR	3	0.7139	0.7317	0.7108	**0.6975**	0.7071					
LSTAR	4	0.7054	0.7120	0.7108	**0.6975**	0.7071	0.7218				
LSTAR	5	0.7042	0.7099	0.7108	**0.6975**	0.7071	0.7218	0.7176			
LSTAR	6	0.7041	0.7145	0.7108	**0.6975**	0.7071	0.7218	0.7176	0.7074		
LSTAR	7	0.7041	0.7135	0.7108	**0.6975**	0.7071	0.7218	0.7176	0.7074	0.7136	
LSTAR	8	0.7041	0.7098	0.7108	**0.6975**	0.7071	0.7218	0.7176	0.7074	0.7136	0.7156
ARNN	0	0.7108									
ARNN	1	0.7166	0.7161	0.7119							
ARNN	2	0.7218	0.7208	0.7119	0.7119						
ARNN	3	0.7111	0.7159	0.7119	0.7119	0.7053					
ARNN	4	0.7265	0.7180	0.7119	0.7119	0.7053	0.7268				
ARNN	5	0.7451	0.7184	0.7119	0.7119	0.7053	0.7268	0.7262			
ARNN	6	0.7805	0.7407	0.7119	0.7119	0.7053	0.7268	0.7262	0.7397		
ARNN	7	0.7509	**0.6944***	0.7119	0.7119	0.7053	0.7268	0.7262	0.7397	0.7184	
ARNN	8	0.8137	0.7259	0.7119	0.7119	0.7053	0.7268	0.7262	0.7397	0.7184	0.7556
MSVAR	0	0.7105									
MSVAR	1	0.7170	0.7159	0.7125							
MSVAR	2	0.7539	0.7526	0.7125	0.7115						
MSVAR	3	0.7586	0.7440	0.7125	0.7115	0.7245					
MSVAR	4	0.7474	0.7119	0.7125	0.7115	0.7245	0.7354				
MSVAR	5	0.7483	0.7263	0.7125	0.7115	0.7245	0.7354	**0.7047**			
MSVAR	6	0.7478	0.7172	0.7125	0.7115	0.7245	0.7354	**0.7047**	0.7204		
MSVAR	7	0.7477	0.7141	0.7125	0.7115	0.7245	0.7354	**0.7047**	0.7204	0.7156	
MSVAR	8	0.7477	0.7114	0.7125	0.7115	0.7245	0.7354	**0.7047**	0.7204	0.7156	0.7145
AR-GARCH	0	**0.7099**									
AR-GARCH	1	0.7142	0.7120	0.7135							
AR-GARCH	2	0.7348	0.7331	0.7135	0.7114						
AR-GARCH	3	0.7376	0.7298	0.7135	0.7114	0.7168					
AR-GARCH	4	0.7330	0.7178	0.7135	0.7114	0.7168	0.7201				
AR-GARCH	5	0.7350	0.7252	0.7135	0.7114	0.7168	0.7201	0.7113			
AR-GARCH	6	0.7337	0.7145	0.7135	0.7114	0.7168	0.7201	0.7113	0.7214		
AR-GARCH	7	0.7339	0.7150	0.7135	0.7114	0.7168	0.7201	0.7113	0.7214	0.7122	
AR-GARCH	8	0.7339	0.7119	0.7135	0.7114	0.7168	0.7201	0.7113	0.7214	0.7122	0.7148

Table 8: RMSFEs, $h = 5$, Zone 2 (Low - High) ($\times 10^{-7}$)

Model	level(j)	$\hat{C}_j$	$\hat{S}_j$	$\hat{D}_1$	$\hat{D}_2$	$\hat{D}_3$	$\hat{D}_4$	$\hat{D}_5$	$\hat{D}_6$	$\hat{D}_7$	$\hat{D}_8$
AR	0	0.9912									
AR	1	1.0020	1.0016	0.9916							
AR	2	1.0572	1.0565	0.9916	0.9913						
AR	3	1.0513	1.0303	0.9916	0.9913	1.0115					
AR	4	1.0571	1.0226	0.9916	0.9913	1.0115	0.9949				
AR	5	1.0593	1.0159	0.9916	0.9913	1.0115	0.9949	0.9975			
AR	6	1.0587	0.9977	0.9916	0.9913	1.0115	0.9949	0.9975	1.0081		
AR	7	1.0513	0.9925	0.9916	0.9913	1.0115	0.9949	0.9975	1.0081	0.9912	
AR	8	1.0513	0.9925	0.9916	0.9913	1.0115	0.9949	0.9975	1.0081	0.9912	**0.9912**
ARMA	0	0.9912									
ARMA	1	1.0040	1.0029	0.9921							
ARMA	2	1.0256	1.0244	0.9921	0.9913						
ARMA	3	1.0319	1.0213	0.9921	0.9913	1.0005					
ARMA	4	1.0400	1.0220	0.9921	0.9913	1.0005	0.9919				
ARMA	5	1.0426	1.0155	0.9921	0.9913	1.0005	0.9919	0.9978			
ARMA	6	1.0418	0.9978	0.9921	0.9913	1.0005	0.9919	0.9978	1.0075		
ARMA	7	1.0346	0.9925	0.9921	0.9913	1.0005	0.9919	0.9978	1.0075	**0.9911**	
ARMA	8	1.0412	0.9925	0.9921	0.9913	1.0005	0.9919	0.9978	1.0075	**0.9911**	0.9973
SETAR	0	1.0065									
SETAR	1	1.0017	1.0002	0.9925							
SETAR	2	1.0527	1.0506	0.9925	**0.9914**						
SETAR	3	1.0696	1.0581	0.9925	**0.9914**	1.0026					
SETAR	4	1.0647	1.0232	0.9925	**0.9914**	1.0026	1.0126				
SETAR	5	1.0645	1.0148	0.9925	**0.9914**	1.0026	1.0126	0.9957			
SETAR	6	1.0664	0.9997	0.9925	**0.9914**	1.0026	1.0126	0.9957	1.0077		
SETAR	7	1.0668	0.9986	0.9925	**0.9914**	1.0026	1.0126	0.9957	1.0077	0.9932	
SETAR	8	1.0667	0.9935	0.9925	**0.9914**	1.0026	1.0126	0.9957	1.0077	0.9932	0.9958
LSTAR	0	0.9912									
LSTAR	1	0.9952	0.9954	0.9917							
LSTAR	2	1.0348	1.0090	0.9917	1.0090						
LSTAR	3	1.0577	1.0396	0.9917	1.0090	**0.985**					
LSTAR	4	1.0539	1.0192	0.9917	1.0090	**0.985**	1.0057				
LSTAR	5	1.0577	1.0175	0.9917	1.0090	**0.985**	1.0057	0.9936			
LSTAR	6	1.0561	1.0001	0.9917	1.0090	**0.985**	1.0057	0.9936	1.0064		
LSTAR	7	1.0571	0.9987	0.9917	1.0090	**0.985**	1.0057	0.9936	1.0064	0.9942	
LSTAR	8	1.0572	0.9937	0.9917	1.0090	**0.985**	1.0057	0.9936	1.0064	0.9942	0.9958
ARNN	0	0.9924									
ARNN	1	1.0021	1.0032	0.9901							
ARNN	2	1.0081	1.0106	0.9901	0.9898						
ARNN	3	1.0090	1.0088	0.9901	0.9898	0.9935					
ARNN	4	1.0106	1.0202	0.9901	0.9898	0.9935	0.9831				
ARNN	5	1.0880	1.0123	0.9901	0.9898	0.9935	0.9831	1.0681			
ARNN	6	1.1399	1.0490	0.9901	0.9898	0.9935	0.9831	1.0681	1.0140		
ARNN	7	1.1645	**0.9779***	0.9901	0.9898	0.9935	0.9831	1.0681	1.0140	1.0668	
ARNN	8	1.2730	1.0065	0.9901	0.9898	0.9935	0.9831	1.0681	1.0140	1.0668	1.0422
MSVAR	0	0.9925									
MSVAR	1	1.0037	1.0033	0.9916							
MSVAR	2	1.0582	1.0573	0.9916	0.9914						
MSVAR	3	1.0559	1.0345	0.9916	0.9914	1.0115					
MSVAR	4	1.0566	1.0222	0.9916	0.9914	1.0115	0.9950				
MSVAR	5	1.0582	1.0155	0.9916	0.9914	1.0115	0.9950	0.9970			
MSVAR	6	1.0583	0.9978	0.9916	0.9914	1.0115	0.9950	0.9970	1.0081		
MSVAR	7	1.0584	0.9985	0.9916	0.9914	1.0115	0.9950	0.9970	1.0081	**0.9912**	
MSVAR	8	1.0582	0.9920	0.9916	0.9914	1.0115	0.9950	0.9970	1.0081	**0.9912**	0.9970
AR-GARCH	0	0.9935									
AR-GARCH	1	0.9994	0.9967	0.9938							
AR-GARCH	2	1.0275	1.0247	0.9938	0.9912						
AR-GARCH	3	1.0242	1.0138	0.9938	0.9912	0.9991					
AR-GARCH	4	1.0336	1.0189	0.9938	0.9912	0.9991	**0.9902**				
AR-GARCH	5	1.0378	1.0164	0.9938	0.9912	0.9991	**0.9902**	0.9956			
AR-GARCH	6	1.0370	0.9993	0.9938	0.9912	0.9991	**0.9902**	0.9956	1.0066		
AR-GARCH	7	1.0373	0.9988	0.9938	0.9912	0.9991	**0.9902**	0.9956	1.0066	0.9926	
AR-GARCH	8	1.0372	0.9923	0.9938	0.9912	0.9991	**0.9902**	0.9956	1.0066	0.9926	0.9972

Table 9: RMSFEs, $h = 1$, Zone 3 (High - High) ($\times 10^{-7}$)

Model	level(j)	$\hat{C}_j$	$\hat{S}_j$	$\hat{D}_1$	$\hat{D}_2$	$\hat{D}_3$	$\hat{D}_4$	$\hat{D}_5$	$\hat{D}_6$	$\hat{D}_7$	$\hat{D}_8$
AR	0	6.7800									
AR	1	6.8598	6.8489	6.7910							
AR	2	7.4285	7.4166	6.7910	6.7796						
AR	3	7.4000	7.2059	6.7910	6.7796	6.9643					
AR	4	7.4293	7.1100	6.7910	6.7796	6.9643	6.9038				
AR	5	7.4178	6.8524	6.7910	6.7796	6.9643	6.9038	7.0213			
AR	6	7.4195	6.8124	6.7910	6.7796	6.9643	6.9038	7.0213	6.8294		
AR	7	7.4158	6.7279	6.7910	6.7796	6.9643	6.9038	7.0213	6.8294	6.8506	
AR	8	7.4487	6.8400	6.7910	6.7796	6.9643	6.9038	7.0213	6.8294	6.8506	**6.7157**
ARMA	0	6.7800									
ARMA	1	6.8463	6.8370	6.7884							
ARMA	2	7.1982	7.1879	6.7884	6.7797						
ARMA	3	7.1697	7.1258	6.7884	6.7797	6.8142					
ARMA	4	7.2144	7.0978	6.7884	6.7797	6.8142	6.8519				
ARMA	5	7.2007	6.8502	6.7884	6.7797	6.8142	6.8519	7.0085			
ARMA	6	7.2023	6.8133	6.7884	6.7797	6.8142	6.8519	7.0085	6.8257		
ARMA	7	7.1983	6.7282	6.7884	6.7797	6.8142	6.8519	7.0085	6.8257	6.8506	
ARMA	8	7.2316	6.8400	6.7884	6.7797	6.8142	6.8519	7.0085	6.8257	6.8506	**6.7158**
SETAR	0	6.8084									
SETAR	1	6.8714	6.8089	6.8412							
SETAR	2	7.2499	7.1376	6.8412	6.8367						
SETAR	3	7.4918	7.2485	6.8412	6.8367	6.9173					
SETAR	4	7.5482	7.1190	6.8412	6.8367	6.9173	6.9829				
SETAR	5	7.5633	6.8516	6.8412	6.8367	6.9173	6.9829	7.0645			
SETAR	6	7.5661	6.8142	6.8412	6.8367	6.9173	6.9829	7.0645	6.8367		
SETAR	7	7.5749	6.7478	6.8412	6.8367	6.9173	6.9829	7.0645	6.8367	6.8401	
SETAR	8	7.5706	6.8293	6.8412	6.8367	6.9173	6.9829	7.0645	6.8367	6.8401	**6.7217**
LSTAR	0	6.6989									
LSTAR	1	6.7781	6.7955	6.7713							
LSTAR	2	6.8970	6.9734	6.7713	**6.667***						
LSTAR	3	7.0391	7.1636	6.7713	**6.667***	6.7485					
LSTAR	4	7.1151	7.1288	6.7713	**6.667***	6.7485	6.9010				
LSTAR	5	7.1057	6.8599	6.7713	**6.667***	6.7485	6.9010	7.0238			
LSTAR	6	7.1164	6.8120	6.7713	**6.667***	6.7485	6.9010	7.0238	6.8526		
LSTAR	7	7.1189	6.7430	6.7713	**6.667***	6.7485	6.9010	7.0238	6.8526	6.8423	
LSTAR	8	7.1215	6.8302	6.7713	**6.667***	6.7485	6.9010	7.0238	6.8526	6.8423	6.7187
ARNN	0	6.8411									
ARNN	1	6.9027	6.9042	6.7777							
ARNN	2	7.0406	7.0394	6.7777	6.7822						
ARNN	3	7.0491	6.9812	6.7777	6.7822	6.8466					
ARNN	4	6.8356	6.8090	6.7777	6.7822	6.8466	**6.7416**				
ARNN	5	7.1940	7.0155	6.7777	6.7822	6.8466	**6.7416**	6.9251			
ARNN	6	7.5352	7.0357	6.7777	6.7822	6.8466	**6.7416**	6.9251	7.0622		
ARNN	7	7.6649	7.0472	6.7777	6.7822	6.8466	**6.7416**	6.9251	7.0622	6.8883	
ARNN	8	7.5211	6.7473	6.7777	6.7822	6.8466	**6.7416**	6.9251	7.0622	6.8883	6.9504
MSVAR	0	6.8345									
MSVAR	1	6.9032	6.8923	6.7908							
MSVAR	2	7.4121	7.4023	6.7908	6.7772						
MSVAR	3	7.3554	7.1597	6.7908	6.7772	6.9678					
MSVAR	4	7.4377	7.1036	6.7908	6.7772	6.9678	6.9129				
MSVAR	5	7.3998	6.8343	6.7908	6.7772	6.9678	6.9129	7.0155			
MSVAR	6	7.4173	6.7990	6.7908	6.7772	6.9678	6.9129	7.0155	6.8275		
MSVAR	7	7.4217	6.7311	6.7908	6.7772	6.9678	6.9129	7.0155	6.8275	6.8494	
MSVAR	8	NaN	NaN	6.7908	6.7772	6.9678	6.9129	7.0155	6.8275	6.8494	**6.7171**
AR-GARCH	0	6.7842									
AR-GARCH	1	6.8432	6.8311	6.7920							
AR-GARCH	2	7.0556	7.0438	6.7920	6.7796						
AR-GARCH	3	7.0567	6.9740	6.7920	6.7796	6.8509					
AR-GARCH	4	7.1134	6.9782	6.7920	6.7796	6.8509	6.8301				
AR-GARCH	5	7.1018	6.8302	6.7920	6.7796	6.8509	6.8301	6.9091			
AR-GARCH	6	7.1038	6.7940	6.7920	6.7796	6.8509	6.8301	6.9091	6.8250		
AR-GARCH	7	7.1007	6.7336	6.7920	6.7796	6.8509	6.8301	6.9091	6.8250	6.8270	
AR-GARCH	8	7.0999	6.8274	6.7920	6.7796	6.8509	6.8301	6.9091	6.8250	6.8270	**6.7168**

Table 10: RMSFEs, $h = 5$, Zone 4 (High - Low) ($\times 10^{-7}$)

Model	level(j)	$\hat{C}_j$	$\hat{S}_j$	$\hat{D}_1$	$\hat{D}_2$	$\hat{D}_3$	$\hat{D}_4$	$\hat{D}_5$	$\hat{D}_6$	$\hat{D}_7$	$\hat{D}_8$
AR	0	18.7399									
AR	1	18.4039	**18.3667**	18.7770							
AR	2	18.7929	18.7540	18.7770	18.7340						
AR	3	20.2455	20.3106	18.7770	18.7340	18.6402					
AR	4	19.9313	19.0721	18.7770	18.7340	18.6402	19.7179				
AR	5	19.9808	18.8768	18.7770	18.7340	18.6402	19.7179	18.9587			
AR	6	19.9073	18.8113	18.7770	18.7340	18.6402	19.7179	18.9587	18.6358		
AR	7	19.5996	18.4775	18.7770	18.7340	18.6402	19.7179	18.9587	18.6358	18.6366	
AR	8	19.9656	18.8157	18.7770	18.7340	18.6402	19.7179	18.9587	18.6358	18.6366	18.5520
ARMA	0	18.7402									
ARMA	1	18.6846	18.6929	18.7291							
ARMA	2	18.4083	18.4071	18.7291	18.7399						
ARMA	3	19.6866	19.6512	18.7291	18.7399	18.7699					
ARMA	4	19.4714	18.8836	18.7291	18.7399	18.7699	19.3104				
ARMA	5	19.5497	18.8031	18.7291	18.7399	18.7699	19.3104	18.8994			
ARMA	6	19.5223	18.9315	18.7291	18.7399	18.7699	19.3104	18.8994	18.6101		
ARMA	7	19.1033	18.3819	18.7291	18.7399	18.7699	19.3104	18.8994	18.6101	18.6233	
ARMA	8	19.5502	18.8141	18.7291	18.7399	18.7699	19.3104	18.8994	18.6101	18.6233	**18.3592**
SETAR	0	18.7797									
SETAR	1	18.2845	18.1571	18.8753							
SETAR	2	18.3244	**18.1509**	18.8753	18.8609						
SETAR	3	19.0819	19.3224	18.8753	18.8609	18.4707					
SETAR	4	18.7819	18.8441	18.8753	18.8609	18.4707	19.0507				
SETAR	5	18.8330	18.5915	18.8753	18.8609	18.4707	19.0507	19.0765			
SETAR	6	18.7856	18.8835	18.8753	18.8609	18.4707	19.0507	19.0765	18.4306		
SETAR	7	18.7852	18.5688	18.8753	18.8609	18.4707	19.0507	19.0765	18.4306	18.5686	
SETAR	8	18.7521	18.5158	18.8753	18.8609	18.4707	19.0507	19.0765	18.4306	18.5686	18.3780
LSTAR	0	18.6067									
LSTAR	1	18.7681	18.9413	18.5855							
LSTAR	2	18.2587	19.3853	18.5855	18.1508						
LSTAR	3	18.1362	18.6668	18.5855	18.1508	18.8355					
LSTAR	4	18.0695	18.6826	18.5855	18.1508	18.8355	18.5158				
LSTAR	5	**18.0567***	18.3566	18.5855	18.1508	18.8355	18.5158	18.9186			
LSTAR	6	18.2951	18.8552	18.5855	18.1508	18.8355	18.5158	18.9186	18.2808		
LSTAR	7	18.3071	18.5689	18.5855	18.1508	18.8355	18.5158	18.9186	18.2808	18.5685	
LSTAR	8	18.2913	18.5110	18.5855	18.1508	18.8355	18.5158	18.9186	18.2808	18.5685	18.3649
ARNN	0	18.8211									
ARNN	1	18.6770	18.6731	18.7447							
ARNN	2	**18.4453**	18.4509	18.7447	18.7318						
ARNN	3	19.3287	19.3095	18.7447	18.7318	18.7765					
ARNN	4	18.9894	18.8688	18.7447	18.7318	18.7765	18.8361				
ARNN	5	18.8922	18.7918	18.7447	18.7318	18.7765	18.8361	18.7230			
ARNN	6	18.8975	18.6979	18.7447	18.7318	18.7765	18.8361	18.7230	18.8611		
ARNN	7	19.0808	18.7214	18.7447	18.7318	18.7765	18.8361	18.7230	18.8611	18.9055	
ARNN	8	19.1473	18.8501	18.7447	18.7318	18.7765	18.8361	18.7230	18.8611	18.9055	18.6383
MSVAR	0	18.7626									
MSVAR	1	18.5294	18.4896	18.7803							
MSVAR	2	18.5517	18.5150	18.7803	18.7259						
MSVAR	3	19.5052	19.5713	18.7803	18.7259	18.6391					
MSVAR	4	19.6142	18.8182	18.7803	18.7259	18.6391	19.6217				
MSVAR	5	19.6695	18.6792	18.7803	18.7259	18.6391	19.6217	18.9358			
MSVAR	6	19.8039	18.9469	18.7803	18.7259	18.6391	19.6217	18.9358	18.6228		
MSVAR	7	19.7841	18.5855	18.7803	18.7259	18.6391	19.6217	18.9358	18.6228	18.6303	
MSVAR	8	NaN	NaN	18.7803	18.7259	18.6391	19.6217	18.9358	18.6228	18.6303	**18.3989**
AR-GARCH	0	18.7524									
AR-GARCH	1	18.5115	18.4747	18.7763							
AR-GARCH	2	18.3367	18.3024	18.7763	18.7365						
AR-GARCH	3	18.5139	18.5283	18.7763	18.7365	18.7022					
AR-GARCH	4	18.4719	**18.2842**	18.7763	18.7365	18.7022	18.9696				
AR-GARCH	5	18.8401	18.6538	18.7763	18.7365	18.7022	18.9696	18.7660			
AR-GARCH	6	19.1491	19.1560	18.7763	18.7365	18.7022	18.9696	18.7660	18.5603		
AR-GARCH	7	20.2778	19.9659	18.7763	18.7365	18.7022	18.9696	18.7660	18.5603	18.6148	
AR-GARCH	8	21.2187	20.0161	18.7763	18.7365	18.7022	18.9696	18.7660	18.5603	18.6148	19.2816

Figure 1: JPY/EUR price series (5 minutes volume weighted averaged).

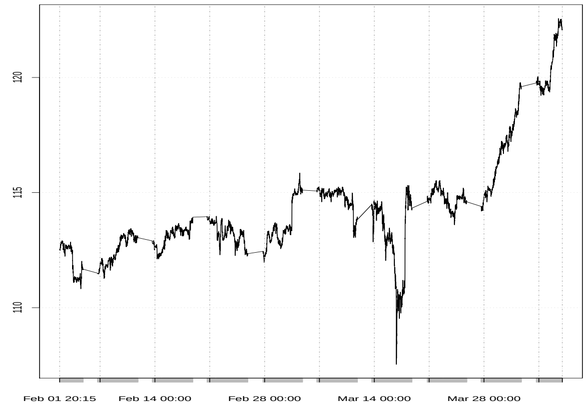

Figure 2: Average trading volume by hours of the day.

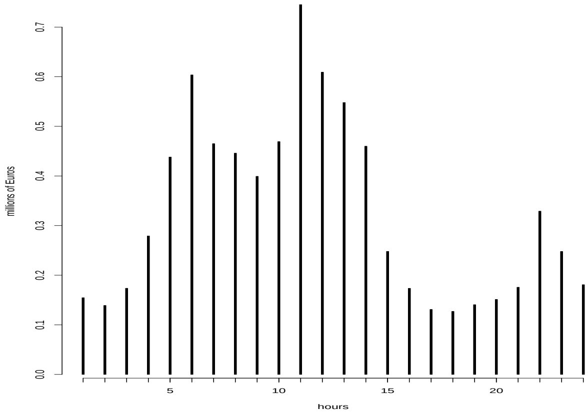

Figure 3: Returns of JPY/EUR

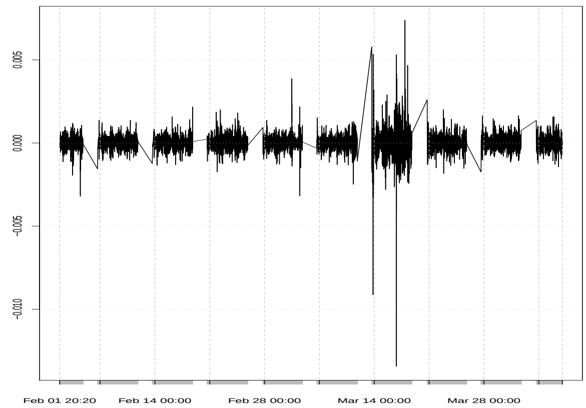

Figure 4: Volatility of JPY/EUR Returns

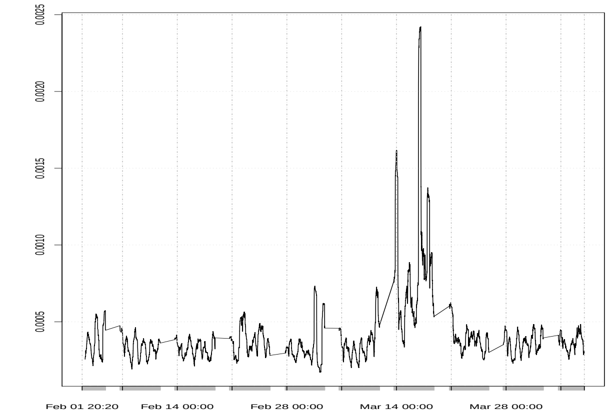

Figure 5: Volatility of returns.

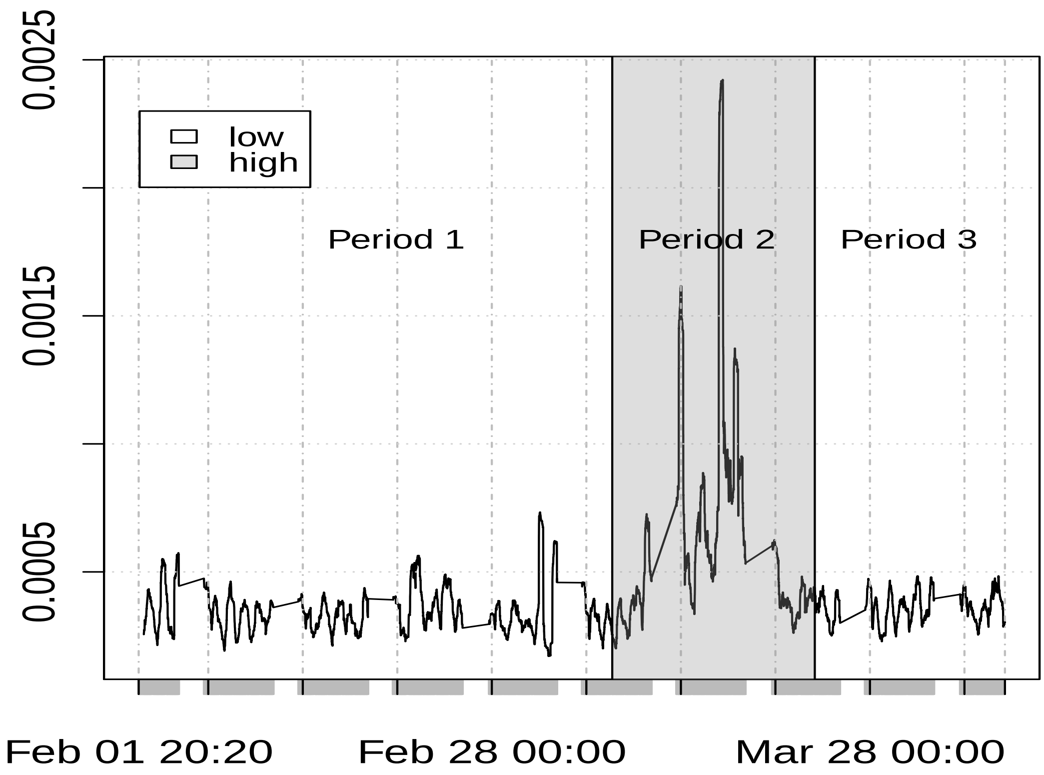

Figure 6: Estimation and forecasting periodization.