Stochastic petropolitics: The dynamics of institutions in resource-dependent economies

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Abstract

We provide an analysis of institutional dynamics under uncertainty by means of a stochastic differential game of lobbying with two main ingredients. The first one is uncertainty inherent in the institutional process itself. The second one has to do with the crucial role of resource windfalls in economic and political outcomes, shaping lobbying power and adding a second source of uncertainty. We show that the main consequence of the first source of uncertainty is the existence of multiple equilibria with very distinct features. First, we obtain symmetric equilibria that allow the economy to reach almost surely a stable pointwise institutional steady state, which is exactly at the center of the political spectrum. Second, there exist asymmetric equilibria that only show up under uncertainty and do no allow for stochastic convergence to a steady state, meaning that any political position may be reached asymptotically with nonzero probability. With resource revenue-dependent lobbying power, the economy converges to a conservative position in the absence of uncertainty. When accounting for the two sources of uncertainty, we obtain that revenue volatility tends to stabilize institutional dynamics compared to the deterministic counterpart, which weakens the case for Friedman's first law of petropolitics.

Keywords: Institutional dynamics, petropolitics, lobbying games, revenue-dependent lobbying power, stochastic dynamic games, stochastic stability.

JEL classification: D72, C73, Q32

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1 Introduction

The link between resource-dependence and quality of institutions is generally viewed through the prism of the resource curse problem: resource-dependent economies are then typically shown to misuse the revenues accruing from exporting the resources (see for example Gylfason, 2001). In particular, there is a lively debate on whether oil and natural resources have an impact on democratization: for example, Ross (2001) and Tsui (2011) argue that oil and natural resources tend to impede democracy. The same view is expressed quite provocatively by Friedman (2006): "Is it an accident that the Arab world's first and only real democracy (Bahrein) happens to not have a drop of oil?". Friedman ends up proposing what he calls *the first law of petropolitics*: "The price of oil and the pace of freedom always move in opposite directions in oil-rich petrolist states". While there is no compelling empirical evidence of such a law (see Alexeev and Conrad, 2009, and Haber and Menaldo, 2011), we are not aware of the existence of a theory exploring the impact of commodity price **volatility** on the internal functioning of the institutions in these countries, which is the main purpose of the present paper. We take petropolitics seriously and develop a stochastic framework reflecting political and oil price uncertainty in oil-rich countries to address the validity of Friedman-like laws in such a framework.

Our working example is Algeria. A striking feature of Algerian economic policy is that the legislations organizing the openness of the country to foreign goods, capital flows and multinationals have been closely driven by the price of the oil barrel, as detailed in Boucekkine and Bouklia (2011). For example, the Algerian economy has been fiercely closed in the 70s in the times of high barrel price levels, and turned to be significantly liberalized from the mid-80s after the 1986 oil counter-shock and a subsequent acute external debt crisis until 2008-2009 with the resurgence of strongly nationalist policy...coinciding with high price levels again for the oil barrel. The current oil counter-shock is now reversing the political line: the Algerian government has just announced that the 2016 budget will be much more FDI-friendly so as to limit the contractionary effects induced by the necessary adjustment of the balance of payments to the external shock.

As documented by Boucekkine and Bouklia, these sharp variations in economic policy relative to the scope of liberalization are the outcomes of a continuous struggle within the *nomenklatura* between the representatives of the nationalist (socialist) line and a minority reformist (liberal) wing which has emerged more clearly after the 1986 oil counter-shock. In periods of high oil prices, the nationalist wing is in better position to block the reforms (including political liberalization indeed) simply because the resulting massive inflows of capital (exports revenues) makes less urgent any further opening to foreign investment and the like.[1] A natural and broad research question is to inquire what could be the equilibrium outcome of such a struggle for a given (stochastic) law of motion for the commodity price (or the commodity revenues), and its long-term implications in terms of stochastic stability. Of course, we do not claim that the commodity price is the unique determinant of politico-economic equilibria in this type of countries. We also consider a second source of uncertainty, reflecting all the potential internal and/or external shocks affecting **directly** the political or constitutional state of the country. One crucial aspect, clearly motivated by the Algerian example, is that revenue volatility does not only add uncertainty, **it also affects the lobbying power of the players**. This ingredient is essentially consistent with the Friedman's law as high oil prices would give more power to the nomenklatura's wing blocking liberalization.

The struggle between two rival groups within the elite can be modelled with the so-called lobbying game, which is itself closely related to the rent-seeking literature (see Tullock, 1967, Kruger, 1974, Tulock again in 1980, or Becker, 1982). Dynamic deterministic versions of the game have been proposed by Leininger and Yang (1994) and Wirl (1994). We shall take the differential game avenue opened by Wirl (1994). A major departure from the original game-theoretic lobbying game developed by Tullock (1980) is that the players do not compete for a given prize but invest in rent-seeking to change the state of the institutional arrangements in their favor. Wirl computes the Markov per-

[1] In the case of Algeria, things could get even worse in periods of high oil prices, and some pro-liberalization legislations implemented in the past have been simply cancelled in the good times of the international oil markets (start and go). This is exactly what happened in 2009 when the Algerian government came to cancel the opening of domestic banks' capital decided in 2003, see Boucekkine and Bouklia (2011).

fect equilibrium and shows that the social costs of rent-seeking are rather low because the threat of retaliation refrains the (Markov-like) players from investing a lot in rent-seeking.

This framework is by construction best adapted to study institutional dynamics in the lobbying context outlined in our working example. To this end, we extend Wirl's game in two major aspects. We first introduce uncertainty in the dynamics of both resource revenue and the constitutional state itself.[2] We then explicitly model the impact of the stochastic resource revenue on the positions of the two players in the lobbying game: The revenue follows a given brownian motion, larger revenue making the anti-liberal player in better position to block the legislation in favor of economic openness or liberalization.

Our analysis extensively uses the concept of stochastic stability of equilibria (see Boucekkine et al., 2015), we proceed in two steps.

In a first step, we consider the particular case where players have the same constant bargaining power, that's resource prices may only intervene as noise to the political state of the economy (as all the other potential internal and external shocks). In this simplified framework, two main sets of results emerge. First, we show the occurrence of multiple Markov-perfect equilibria in sharp contrast to the deterministic counterpart studied by Wirl (1994), which only displays one (stable) equilibrium. The equilibria show very distinct features. On one hand, *symmetric* equilibria arise: these equilibria lead the economy to reach almost surely a stable pointwise institutional steady state in the long run which is exactly at the center of the political spectrum. Besides, *asymmetric* equilibria, which only show up under uncertainty, do emerge: these ones do not allow for stochastic convergence to a steady state, meaning that any position in the political spectrum may be reached asymptotically with nonzero probability. More importantly, the results obtained show that even in the case where the two players have equal (price-independent) bargaining

[2] An earlier attempt to build a stochastic version of Wirl (1994), to discuss the dynamics of liberalization in Arab countries, is due to Boucekkine et al. (2014). The authors consider a quite ad-hoc form of uncertainty in resource wealth. They indeed model it as a discrete random variable taking two values with given probabilities. Assuming that any realization of this variable affects the players' payoffs only, they solve the resulting piecewise deterministic differential game and examine the impact of this kind of uncertainty on the equilibrium.

power, the structure of Markov equilibria is particularly rich: not all equilibria are symmetric, and therefore the economy may not end up *centrist* despite players have identical power. That's to say uncertainty and risk aversion give rise to a new set of equilibria which are not generated by the simple retaliation-based mechanisms described in Wirl and in Leininger and Yang.

A second set of results has to do with the impact of uncertainty on lobbying effort, an issue which has been quite intensively addressed in the literature (see Treich, 2010, and Jullien et al., 1999, for example). This question is indeed essential in several important contexts. For example, Bramoullé and Treich (2009) consider the context of global commons problems like climate change, and study the effect of uncertainty on pollution emissions and welfare in a strategic context. They find that emissions are always lower under uncertainty than under certainty. To our knowledge, none of the papers, in this literature, have considered dynamic stochastic games, and the inherent stochastic stability issues, which are very important for long term issues such as climate change. In our stochastic lobbying game framework, we show that the variation of investment with the amount of uncertainty is far from simple, it can increase with uncertainty along certain Markov perfect equilibria and decrease along others.

In a second step, we account for the two sources of uncertainty together with resource revenue-dependent lobbying power consistently with our Algerian working example. To make our point neatly (that is, in an analytically tractable way), we select a special parameterizations of the augmented model. When shutting down the two sources of uncertainty, we show that the deterministic counterpart converges to a conservative position in the political spectrum whatever the nature of resources (renewable or not), the asymptotic political position being *infinitely* conservative in almost all the configurations considered.[3] With identical lobbying powers, one would get the Wirl's result: the economy converges to the center of the political spectrum. Here, by awarding the conservatives a better

[3]The political state or position is described by a real variable z in our model, with negative values for conservative positions and positive for liberal ones. In our setting, one can get z going to $-\infty$ at time passes, which corresponds to the infinitely conservative position mentioned just above.

bargaining position when the resource revenues go up consistently with the algerian case, we get conservative political states asymptotically. In such a case, Friedman's first law of petropolitics holds. However when the uncertainty sources are switched on, we show that revenue volatility tends to stabilize institutional dynamics compared to the deterministic counterpart, which weakens the case for Friedman's first law of petropolitics. Uncertainty and risk aversion lead the economy to a Markov perfect equilibrium which is less extreme than the one generated in the deterministic counterpart in the sense that liberal political positions are still possible asymptotically (with nonzero probability).

The paper is organized as follows. Section 2 briefly presents the general stochastic differential game. Section 3 solves the special case where the outcomes of the stochastic lobbying game are independent of the resource revenues, this special case can be considered as the natural stochastic extension of Wirl's deterministic game, uncertainty being intrinsic to the lobbying process. Section 4 studies the Algerian case where the lobbying game also depends on the stochastic process driving resource revenues. In both Sections 3 and 4, we study the stochastic stability of the resulting Markov Perfect Equilibria (MPE). Section 5 concludes.

2 Model

We consider a differential game opposing two rival groups, $i = 1, 2$, who engage in lobbying efforts, $x_i \geq 0$, to push the legislation, $z \in (-\infty, \infty)$, in their preferred direction. The variable z can alternatively be interpreted as the state of (economic and/or political) liberalization. In both cases, z is an indicator of the quality of institutions, and by convention, the larger z, the better the institutions. Players have opposite views on how the legislation should evolve: Player 1 consists of the reformist group, i.e., wants z to be as high as possible, whereas player 2 exerts efforts to lower z. As in Wirl (1994), $z = 0$ is the neutral level of legislation, or liberalization. We extend his framework in two essential ways.

First, we take into account the uncertainty surrounding the evolution of z. The legislative process is uncertain in the (obvious) sense that the legislation z does not only depend on the investments made by the lobbyists: it also depends on other political, economic, and social circumstances that we account for by making stochastic the law of motion of state z. In addition, interpreting z as the level of liberalization, it is fair to say that there are many factors – internal or external shocks – that also affect the evolution of z. It is enough to mention the consequences of the Arab Spring events in countries such as Algeria and Morocco where the uprising of the citizens didn't lead to the overthrow of the ruling elite but changed the political system (legislation and policies) quite substantially.

Second, we incorporate the "Algerian story", which basically means that the economy relies on windfall revenues from natural resources, R. In the resource-dependent economy, these revenues play a crucial role since they determine the positions of the players in the lobbying game. To fit with the Algerian case, we assume that the larger R, the more efficient is the investment of player 2 in moving the legislation z. Note that accounting for the impact of resources windfall on the relative lobbying power is very much in line with the resource curse hypothesis according to which natural resources wealth tends to make political institutions less democratic, or worse (see Ross, 2001, and Tsui, 2010). Of course, considering the impact of resources revenues also rises the question of their evolution in time and requires the volatility of these rents be taken into account (just think about the volatility in the price of oil). This adds a second source of uncertainty to our problem.

In our setting, the two types of uncertainties and the link between z and R are incorporated by means of two stochastic state equations:

$$dz = [x_1 - g_z(R)x_2]dt + \sigma_z z \, dW, \tag{1}$$

$$dR = g_R(R)dt + \sigma_R R \, dW, \tag{2}$$

where $W = (W_t)_{t\geq 0}$ is a standard Wiener process, [4] and σ_i, $i = z, R$, measure volatilities

[4]The same Wiener process is used in the two equations. This is not essential in this study. In addition, considering two different Wiener processes with given correlation would complicate tremendously the algebra (in Section 4) without adding too much economic insight.

of z and R, respectively. Function $g_z(R)$ is increasing in R to reflect the fact that player 2 is more efficient in times of high windfalls. In general, function $g_R(R)$ may take any form, depending on whether the resources are renewable or not. The important point is that what matters to lobbyists is the resource revenues (they barely control extraction directly anyway), which typically have a deterministic time trend (which can be positive or negative) but are essentially stochastic because of the volatility of international (energy) prices and unpredictable technological innovations or resource discoveries.

Note, however, that to have a chance to solve the differential game analytically, we have no other option but to resort to specific functional forms for $g_z(.)$ and $g_R(.)$. In order to keep the well-known and very common linear-quadratic structure of the game, we will work with the following functions:

$$\begin{aligned} g_z(R) &= 1 + \varepsilon R, \\ g_R(R) &= \eta + \xi R, \end{aligned} \tag{3}$$

with $\varepsilon \geq 0$, and $\eta, \xi \in \mathbb{R}$. Despite their apparent simplicity, these forms are quite meaningful. In particular, it is quite easy to retrieve the expression of $g_R(R)$ in (3) and the dynamics of R given by (2) from two separate state equations in the resource stock, and the resource price. Indeed, define $R = pE$ as the resource rent, with p the price (in the absence of market power), and E the extraction (or harvesting) rate. For simplicity let us assume that the extraction rate takes the following form: $E = eS$ with S the stock of resource and e a constant effort representing the share of the stock extracted at each date. This is enough to capture the decreasing time path of extraction over time. Then, define the dynamics of both variables as follows:

$$\begin{aligned} dp &= \alpha p dt + \sigma_p p dW, \\ dS &= (a - eS)dt. \end{aligned}$$

This boils down to considering uncertainty in the evolution of the price only, which furthermore follows a constant deterministic and positive trend α (this is the simplest version of the Hotelling rule). Combining these two differential equations, we obtain the one characterizing the evolution of R:

$$dR = [(\alpha - e)R + eap]dt + \sigma_p R dW.$$

Now making a change of variable with $\eta = eap$, $\xi = \alpha - e$, and $\sigma_R = \sigma_p$ is sufficient to obtain equation (2), given the specification in (3). Taking $a = 0$, which implies $\eta = 0$, brings us to the analysis of the case of a non-renewable resource like oil, which is the relevant one for describing the Algerian economy. In this case, the sign of ξ basically depends on the relative size of α and e. Considering $\alpha > e$ means that price increases exceed the decreasing trend of extraction rates and result in ever growing resources rents, whereas when $\alpha < e$ resources revenues are driven down since the fall in extraction dominates the upward tendency of the price. The case with $a > 0$ (and furthermore η constant) is a very simple representation of the evolution of rents from a renewable resource. However, we keep considering this case to be as general as possible.

Let us now turn to the definition of players' payoffs. Players maximize the present value of benefit from their efforts of liberalization minus the associated cost:

$$\max_{x_i} \int_0^\infty e^{-rt} \left[\omega_i(z) - \beta(x_i)\right] dt, \tag{4}$$

with $r > 0$ the (same) rate of time preference, subject to state constraints (1) and (2), with $z(0) = z_0$ and $R(0) = R_0$ given. Still motivated by our will to keep things as simple as possible, players' instantaneous benefit, $\omega_i(z)$, from the level of legislation or liberalization, takes a quadratic form: $\omega_i(z) = a_0 \pm a_1 z + \frac{a_2}{2} z^2$, with $a_0, a_1 > 0$, and $a_2 \leq 0$. The opposite sign of the term in z reflects players' opposite interests with respect to the legislation. By convention, player 1 payoff is increasing in z, i.e, we put a + in front of a_1. Moreover, exerting lobbying is a costly activity and we shall use a quadratic lobbying cost: $\beta(x_i) = \frac{b}{2} x_i^2$. Last but not least, notice that R does not affect the payoff functions directly. Corruption motives or office rents, which would imply that part of the revenues is captured by player 2 shows in her payoff, are left aside. This allows to focus on a game where the players are entirely devoted to push the legislation in the direction they wish, which is the essence of lobbying.

It is worth closing this section with a summary of the differences between our framework and the ones considered in the related literature. Our model is similar to Wirl (1994) except that he works with $\varepsilon = \sigma_z = 0$ in (3), i.e., no uncertainty, identical lobbying power, and no attention paid to the role of resources revenues. Boucekkine et al. (2014)

do account for stochastic resources revenues but choose a very different and somehow elementary approach. Actually, they extend Wirl's lobbying game by assuming that a_1 is a discrete random variable that can take two values, with given probabilities. This is their unique source of uncertainty in the model, and the lobbying power are kept identical as in Wirl.

As explained above, the model entails two types of uncertainties: one affecting the legislation state, z, say *legislative uncertainty*, and the other resulting from *resource revenues volatility*. Of course, uncertainty on resource revenues may itself affect the political and legislative processes. An interesting special case is when resource revenues do not affect the lobbying power of the players: In such a case, the lobbyists only consider legislative uncertainty (that's resource revenues volatility has not impact on their decisions) in determining their lobbying efforts. The analysis of such a case, that is conducted in Section 3, brings out some important and striking results on the impact of uncertainty on both the properties (existence, stability and uniqueness) of the equilibrium and the shape of lobbying efforts and legislation levels. We shall consider both uncertainties in Section 4 and deal with the issue of stochastic stability and its implications on the long run behavior of the economy. In these two sections, our analysis will mainly be based on the comparison between the deterministic benchmark and its stochastic counterpart. All the proofs are relegated in the Appendix.

3 Dynamics of lobbying under legislative uncertainty

Let us start with the case where resources revenues R do not play any direct role in the institutional dynamics, that is: $g_z(R) = 1$. The stochastic game, characterized by the objective (4) and the constraint (1), only has a linear quadratic structure, which allows for the use of Markov perfect Nash equilibrium (MPE) as the solution concept. Solving for the MPE, we show that there exist multiple equilibria with very distinct features. Among the important features that differentiate the equilibria, two are of particular interest.

The first property refers to the symmetric vs asymmetric nature of the equilibrium.

We define symmetric MPE as follows:

Definition 1 *An MPE is said symmetric if the corresponding state z converges almost surely to zero. Otherwise, the MPE is said asymmetric.*

In his deterministic game, Wirl says that a MPE is symmetric if the equilibrium lobbying efforts lead to the state variable to the neutral (or central) level $z = 0$ along the symmetric MPE. In other words, $z = 0$ is asymptotically stable along the MPE, which incidentally can only hold if the lobbying efforts are equal asymptotically (by the deterministic counterpart of equation (1)). Definition 1 is a direct extension of Wirl's approach to a stochastic environment. As explained in the introduction, symmetric MPEs refer to situations where the strategies played lead to the center of the political spectrum asymptotically. Such equilibria are particularly natural when lobbying powers are equal as in Wirl's set-up. We show, among others, in this paper that equality of lobbying powers is not sufficient to have symmetric MPEs in a stochastic frame.

The second important ingredient is the stochastic stability of the equilibrium. Here, we follow Merton (1975, Page 378) on the stability of stochastic dynamic processes and inquire whether *there is a unique distribution which is time and initial condition independent and toward which the stochastic process tends.*

Definition 2 *A stochastic process $X(t)$ is [called] stable if there is stationary time invariant distribution of $X(t)$ for $t \to \infty$.*[5]

Hereafter two striking differences with the deterministic case are put forward. We start with a comparison between stochastic MPE and their deterministic counterpart(s). This requires to focus on symmetric solutions since Wirl (1994) has proved the existence of a unique stable symmetric MPE. In contrast, we show that there generically exist two stable symmetric MPE and discuss the implications of multiplicity on the dynamic and

[5]If this density distribution degenerates into a Dirac function, then the stochastic process converges to a unique point.

long term behavior of the economy. Then, we leave the symmetric world and push the analysis of the impact of uncertainty further by showing the existence of a new type of equilibrium: The asymmetric MPE. Finally, the economic implications of this new and important result are disclosed.

3.1 First impact of uncertainty: The multiplicity of stable MPE

At the least for the sake of (direct) comparison, we first concentrate the analysis on the symmetric case. Our findings are summarized in the following proposition.

Proposition 1 *Under legislative uncertainty (only), there exist two symmetric MPE, indexed by $j = 1, 2$.*

(i) Players' lobbying strategies are given by the following linear feedback rules:

$$x_1^{(j)} = \frac{1}{b}\left(\frac{a_1}{br - C^{(j)}} + C^{(j)}z\right), \quad x_2^{(j)} = \frac{1}{b}\left(\frac{a_1}{br - C^{(j)}} - C^{(j)}z\right), \quad j = 1, 2 \tag{5}$$

where,

$$C^{(j)} = \frac{-b(\sigma_z^2 - r) \pm \sqrt{b^2(\sigma_z^2 - r)^2 - 12ba_2}}{6}, \text{ with } C^{(1)} < 0, \text{ and } C^{(2)} > 0. \tag{6}$$

(ii) The stochastic process $z(t)$, whose dynamic behavior is given by

$$dz = \frac{2C^{(j)}}{b} z\, dt + \sigma_z z\, dW,$$

almost surely converges to the steady state $z_\infty = 0$ if and only if

$$\frac{2C^{(j)}}{b} - \frac{\sigma_z^2}{2} < 0. \tag{7}$$

Analogy with Wirl (1994) comes immediately when making σ_z going to zero: One recovers exactly the solutions to the deterministic counterpart of the problem. With $\sigma_z = 0$, the steady state $\dot{z} = 0$ exists if and only if $x_1 = x_2$ at equilibrium. But the MPE

$j = 2$ is obviously unstable because $C^{(2)} > 0$. Henceforth, the MPE $j = 1$ is the unique symmetric MPE in his study. Therefore, the first striking impact of uncertainty shows itself in the fact that uniqueness of the stable MPE is not necessarily the rule because of the stabilizing effects of the noise term. In the stochastic environment, the necessary and sufficient condition for stability (7) is always satisfied by the first MPE featuring $C^{(1)} < 0$, provided $\sigma_z \geq 0$ (see Boucekkine et al. 2015). But one can observe that even though $C^{(2)} > 0$, condition (7) can also hold for the second MPE if σ_z is not too small.[6] So, in some sense, the stability condition is weaker under uncertainty, which gives rise to the second symmetric MPE.

To understand why and how the stabilization by noise mechanism works, it is useful to examine players' reactions to a change in z as well as the differences in terms of these reactions between our two solutions. For $j = 1$, and since $C^{(1)}$ is always negative (for any value of $\sigma_z \geq 0$), player 1's feedback rule is decreasing in z whereas player's 2 feedback is increasing in the state. The reason why player 1 behaves this way while she is interested in large values of z is the fear that player 2 would exert an opposite lobbying effort in retaliation. So one gets the retaliation motive invoked by Wirl to argue that the social cost of lobbying is likely to be low.[7] When $j = 2$, $C^{(2)}$ turns positive, which means that the retaliation argument put forward by Wirl is no longer valid in the deterministic world: The player interested in large z increases her effort with z while the player vowing to push z down decreases her effort, which is incompatible with a stable legislative state in the long run.

In our stochastic environment, this kind of behavior may nevertheless lead the economy to a stable solution. The intuition behind the reversal in the reactions to changes in z can be found in Boucekkine et al. (2014). In their piecewise deterministic differential game with discrete uncertainty and two states of the world only, they also obtain that player 1's effort may be increasing in z in the anticipation of future tough times. Extending their argument to our setting with continuous uncertainty (affecting z, not R), we claim that player 1 may want to push further for the increase in the level of legislation even if

[6] Straightforward computations yield the lower bound on σ_z such that (7) is met.

[7] The same logic is at work for player 2.

its current level is good. This cautious behavior provides the equilibrium strategy when uncertainty, and the probability of a negative realization of the random part of z, are high. The symmetric reasoning applies to player 2.

The conditions under which uncertainty promotes stability are the following. When legislative uncertainty is too low, risk aversion is not enough to offset the destabilizing effect resulting from the absence of the retaliation motive. However, this effect gets dominated by the new effect channelling through uncertainty when σ_z is large enough. This result is quite consistent with the economic literature pointing at the negative impact of uncertainty and risk aversion on effort intensity (see Treich, 2010). It's possible to assess more precisely the impact of uncertainty on the lobbying efforts at the MPE. For simplicity, we restrict attention to the steady state equilibria, which allows us to establish that:

Corollary 1 *At the MPE j=1, the larger the uncertainty, the lower the effort exerted by lobbyists. In addition, stochastic efforts are lower than their deterministic counterparts. The opposite property holds at the MPE j=2.*

Quite interestingly, we observe that (more) uncertainty is not always associated with lower lobbying effort. This again depends on the type of MPE considered. At the first MPE steady state (and, by continuity, in its neighborhood), uncertainty supplements the retaliation effect and tends to lower the social cost of lobbying. By contrast, at the second MPE, uncertainty stimulates lobbying. This result somehow differs from the one obtained by the literature that studies the impact of uncertainty on static common resources problems (Bramoullé and Treich, 2009). Indeed, the general message conveyed by this literature is that uncertainty alleviates the tragedy of the commons. This illustrates another interesting result brought by our analysis.

In the next section, we will show that uncertainty is not always so sharply stabilizing. The discussion will be based on the analysis of the features of a new class of MPE, which emerges under uncertainty. This class of equilibria has the very characteristic of being asymmetric, in the sense of Definition 1. In particular, this will imply that none of them lead the system to the neutral level $z = 0$ almost surely.

3.2 Second impact of uncertainty: The asymmetry of MPE

Here we need to decouple the issue of existence (and uniqueness) from the one of stability, which deserves much more attention than in the previous symmetric situation. As for existence, Proposition 2 states that:

Proposition 2 *Under legislative uncertainty, the game of lobbying also exhibits two asymmetric MPE, indexed by $j = 3, 4$, that are characterized by the following lobbying efforts:*

$$x_1^{(j)} = \frac{B_1^{(j)} + C_1^{(j)} z}{b}, \quad x_2^{(j)} = -\frac{B_2^{(j)} + C_2^{(j)} z}{b}, \tag{8}$$

where

$$\begin{aligned} C_1^{(3)} &= \tfrac{-b(\sigma_z^2 - r) - \sqrt{b^2(\sigma_z^2 - r)^2 + 4a_2 b}}{2} = C_2^{(4)} (< 0), \\ C_2^{(3)} &= \tfrac{-b(\sigma_z^2 - r) + \sqrt{b^2(\sigma_z^2 - r)^2 + 4a_2 b}}{2} = C_1^{(4)} (> 0), \end{aligned} \tag{9}$$

and,

$$B_1^{(j)} = \frac{a_1 b \left(b\sigma_z^2 - C_1^{(j)}\right)}{b^2 \sigma_z^4 - C_1^{(j)} C_2^{(j)}}, \quad B_2^{(j)} = \frac{a_1 b \left(C_2^{(j)} - b\sigma_z^2\right)}{b^2 \sigma_z^4 - C_1^{(j)} C_2^{(j)}} \tag{10}$$

provided the square roots above are real.

It is fair to say that expressions above are quite ugly. But a simple examination of how players' adapt to a change in z (this is given by the sign of the $C_i^{(j)}$ coefficients in (9)) is enough to emphasize the first distinction between asymmetric MPE and symmetric (deterministic and stochastic) ones.

Corollary 2 *At the asymmetric MPE, lobbyists' efforts to change the legislation move along the same direction.*

In sharp contrast to the symmetric equilibria studied above, players' equilibrium efforts display the same response to a change in z. Quite naturally, asymmetric equilibria cannot arise in a deterministic world because players, that have both opposing interests and perfect foresight, cannot in principle respond in the same way to changes in z. In our setting, the stochastic evolution of z introduces a noise that may result in different players adopting the same kind of strategies. Let's consider, for example, the MPE $j = 3$ where both players' efforts decrease in z. Recall that Wirl type of MPE is such that player 1's effort decreases with z: The larger z, the lower the effort to maintain the legislation to an acceptable (high) level. On the contrary, player 2's effort is increasing in z. Since high values of z are damaging to player 2, high enough investment levels are required to limit the extent of the damage. These reactions show the existence of a *strategic effect*. Now, under uncertainty, another effect comes into play. Indeed, future benefits from the effort to push the legislation in one's favorite direction are far from granted because of the stochastic dynamics of z whereas the (current) costs associated with this activity are felt for sure. Under risk aversion, this implies that both players tend to devote less resources to change the legislation at any level of z. This *uncertainty effect* clearly adds to the strategic effect for player 1. The larger z the lower player 1's incentives to invest in lobbying because the resulting marginal gains are both quite low (strategic effect) and uncertain (uncertainty effect). Things are different for player 2, i.e., these two effects work in opposite directions. Then it turns out that when uncertainty is large enough (this is required for having a non-negative discriminant in (9)), the latter effect dominates the former which implies that the logic prevailing in the deterministic benchmark no longer holds. In this situation, player 2's effort is decreasing in z, as for player 1, despite their competing objectives.

The second related important implication of Corollary 2 is the existence of a self-enforcing mechanism, that may play in both directions, that in essence constitutes a destabilizing force. This leads to the second (strong) difference with symmetric equilibria: By definition, none of these two new equilibria bring the legislative state almost surely to $z = 0$. To get this point, a few computations are needed. Substituting the equilibrium efforts $x_1^{(j)}$ and $x_2^{(j)}$ into the state equation and simplifying yield

$$dz^{(j)} = [x_1^{(j)} - x_2^{(j)}]dt + \sigma_z z^{(j)}\, dW = \left[\frac{a_1(C_1^{(j)} - C_2^{(j)})}{b(b\sigma_z^4 + a_2)} + (r - \sigma_z^2) z^{(j)} \right] dt + \sigma_z z^{(j)}\, dW, \quad (11)$$

with $C_1^{(3)} - C_2^{(3)} = -\sqrt{b^2(\sigma_z^2 - r)^2 + 4a_2 b} = -(C_1^{(4)} - C_2^{(4)})(< 0)$.

Define $\Gamma^{(j)}$ as: $\Gamma^{(j)} = \frac{a_1(C_1^{(j)} - C_2^{(j)})}{b(b\sigma_z^4 + a_2)}, j = 3, 4$. Since $\Gamma^{(j)} \neq 0$, it is clear that $z = 0$ is not a solution to (11), hence $z = 0$ cannot be a steady state for the deterministic part of the equation above. Instead, it can be shown, following the same strategy as Merton (1975), that while z cannot converge almost surely to 0, it admits a stationary invariant distribution which density is given in the following proposition.

Proposition 3 *At the asymmetric MPE, the density function of stochastic process $z(t)$ almost surely converges to its long-run steady state density function $q(z)$, which is given by*

$$q(z^{(j)}) = \frac{M}{\sigma_z^2\, (z^{(j)})^2} \exp\left\{ 2 \int \frac{\Gamma^{(j)} + (r - \sigma_z^2) z^{(j)}}{\sigma_z^2\, (z^{(j)})^2} dz \right\}, \quad j = 3, 4, \quad (12)$$

where positive parameter M is chosen such that $\int_{-\infty}^{+\infty} q(z^{(j)}) dz = 1$.

Propositions 1, second item, and 3 provide with highly interesting and sharply contrasted results as to the asymptotic implications of symmetric vs asymmetric MPE. The main reason is that, in the symmetric case, the two players' efforts move along opposite directions, see (8), that's one's efforts increase with z and the rival player's efforts decrease with z. The two efforts end up balancing each other and the legislative state converges almost surely to $z_\infty = 0$. However, in the asymmetric case, the two players efforts move along the same direction, see the Corollary 2, and such a race can hardly result in any type of compensation mechanism pushing the economy towards $z_\infty = 0$. Instead, the z-process has an invariant distribution on the whole real line. As a consequence, uncertainty and risk aversion are much less stabilizing in a sense than at symmetric equilibria, and the cost of lobbying can be much higher than in the latter class of equilibria.

A more politically-oriented interpretation of the result is that uncertainty in the dynamics of legislation, or institutions, itself may induce opposing groups to adopt strategies that are not generally compatible with the convergence to the status quo $z_\infty = 0$, that arises at the symmetric MPE. Indeed, following the lobbying strategies of the asymmetric MPE, anything can happen asymptotically, that is, any level of legislation can be achieved with a positive probability, and the economy can end up with very bad or very good political and economic institutions.

The next section deals with the extended version of our political game where legislative uncertainty is combined with the uncertainty surrounding the evolution of natural resources rents.

4 Revenue-dependent lobbying power and institutional dynamics

We now introduce the second ingredient discussed in Section 2. To fit with the Algerian case, we consider that *(i)* relative lobbying powers are determined by resource revenues, and *(ii)* the dynamics of resource wealth are stochastic (because of price volatility). This boils down to working with the dynamical system (1)-(2), with the specification of $g_R(.)$ and $g_z(.)$ given in (3). Moreover, due to the (more) complex structure of the differential game, that encompasses now two state variables and nonlinear state functions, we have to resort to a further simplification of the model. We set coefficient a_2 to zero in the payoff functions, thus removing the quadratic term in z from these functions. Though the resulting MPE are particular, they ultimately say a lot on how resource revenue volatility matters in the lobbying game.[8] It's enough to make our point: while the first law of petropolitics advocated by Friedman may seem obvious in a deterministic framework, it may not by robust to uncertainty and the induced risk aversion.

[8] In fact, as it will be apparent soon, taking $a_2 = 0$ is a mean to neutralize the effect of z – and thus to isolate the specific effect of R – on the equilibrium.

4.1 Deterministic vs stochastic MPE

Using the same methodology as in Section 3, we start by solving for the MPE (see the details in Appendix A.2). Then, we turn to the most interesting and difficult part of the problem where stability properties of both the deterministic and the stochastic MPE are established and compared. Finally, we discuss the implications of the results in terms of the dynamics of the legislation and of the institutions.

As far as existence is concerned, we show that:

Proposition 4 *Under legislative and resource revenue uncertainties, there exists a unique MPE. Players' lobbying efforts are defined by the following feedback rules:*

$$x_1(t) = \frac{a_1}{br}, \;\; x_2(t) = \frac{a_1(1+\varepsilon R(t))}{br}. \tag{13}$$

The corresponding canonical dynamic system is given by

$$\begin{cases} dz = -\dfrac{a_1\varepsilon}{br}(2+\varepsilon R)R dt + \sigma_z z \, dW, \\ dR = (\eta + \xi R)dt + \sigma_R R \, dW. \end{cases} \tag{14}$$

Thus player 1's equilibrium effort is constant while player 2's effort linearly depends on the resource rent R. More precisely, one can see that the equilibrium strategies are independent of the legislative state, z. Clearly, these linear feedbacks are obtained thanks to zeroing a_2 in the payoff functions. Still, this case is most useful as it allows us to emphasize the pure impact of resource revenues on the lobbying game: Player 1, which is by assumption not directly affected by these windfalls, has a constant feedback, whereas player 2 does care about resource revenue because they increase her lobbying power. Then it appears that player 2's lobbying effort goes up as the economy gets more revenue from natural resources. One may expect that player 2 rests on (high) resource revenue to change the legislation toward her preferred direction, without investing much in lobbying. This is not the case here. In fact, rising resource revenues is a positive signal for the second lobbyist, who knows that in this situation her effort is more efficient. So the larger R, the larger the lobbying effort of this player. In a deterministic world, one would therefore conclude that the cost of lobbying is linearly increasing in the resource revenues, abstracting away from corruption and other distortions.

Still it remains to study the dynamic and asymptotic implications of the optimal feedback rules identified just above. At first glance, at least the parameters determining the stochastic resource revenue law of motion should matter.

Before moving to this analysis, it proves useful to solve the canonical system (14) to obtain the expressions of z and R at the MPE:

$$\begin{aligned} R(t) &= e^{\left(\xi-\frac{\sigma_R^2}{2}\right)t+\sigma_R W_t}\left[R_0+\eta\int_0^t e^{-\left(\xi-\frac{\sigma_R^2}{2}\right)s-\sigma_R W_s}ds\right], \\ z(t) &= e^{-\frac{\sigma_z^2}{2}t+\sigma_z W_t}\left[z_0+\int_0^t A(s)\, e^{\frac{\sigma_z^2}{2}s-\sigma_z W_s}ds\right] \end{aligned} \tag{15}$$

with $A(t)=-\frac{a_1\varepsilon}{br}(2+\varepsilon R(t))R(t)$. In addition, from the dynamic system (14), it is clear that the stability analysis can be performed sequentially, studying the stability of $R(t)$ first, then moving to the one of $z(t)$.

4.2 Long term implications of the resource revenue volatility

Let us start with a quick investigation of what is going on in the deterministic benchmark. Then we examine the stochastic stability properties of the MPE and bring out some conclusions on how the volatility of resource revenue affects the outcomes of the lobbying game asymptotically.

First, consider the deterministic case, that is, $\sigma_R=\sigma_z=0$. It is easy to show that the solutions in (15) reduce to

$$\begin{aligned} R(t) &= \left(R_0+\frac{\eta}{\xi}\right)e^{\xi t}-\frac{\eta}{\xi}, \\ z(t) &= -\frac{a_1\varepsilon}{br}\left[\left(R_0+\frac{\eta}{\xi}\right)\frac{e^{\xi t}-1}{\xi}-\frac{\eta}{\xi}t\right]-\frac{a_1\varepsilon^2}{br}\left[\left(R_0+\frac{\eta}{\xi}\right)^2\frac{e^{2\xi t}-1}{2\xi}+\frac{\eta^2}{\xi^2}\,t-\left(R_0+\frac{\eta}{\xi}\right)\frac{2\eta(e^{\xi t}-1)}{\xi^2}\right]. \end{aligned} \tag{16}$$

Then we have,

Proposition 5 *Suppose that $\eta \geq 0$. Then the unique solution of the deterministic dynamic system, which is given by (15) with $\sigma_R = \sigma_z = 0$, exhibits the following properties:*

(1) Along the MPE, z and R follow the trajectories reported in (16).

(2) In the long-run, there are three possible situations, depending on the sign of the coefficients of the function $g_R(.)$:

(2.a) If $\xi > 0$ and $\eta \geq 0$, we have $\lim_{t\to\infty} R(t) = +\infty$ and $\lim_{t\to\infty} z(t) = -\infty$.

(2.b) If $\xi < 0$ and $\eta > 0$, we have $\lim_{t\to\infty} R(t) = -\frac{\eta}{\xi}$ and $\lim_{t\to\infty} z(t) = -\infty$.

(2.c) If $\xi < 0$ and $\eta = 0$, we have $\lim_{t\to\infty} R(t) = 0$ while

$$\lim_{t\to\infty} z(t) = \frac{a_1 \varepsilon R_0}{br\xi} + \frac{a_1 \varepsilon^2 R_0^2}{2br\xi} < 0. \tag{17}$$

Not surprisingly, given the shape of the optimal feedback rules given in Proposition 4, the deterministic differential game yields an explosive steady state as the legislative state z goes to ∞, with the exception of the parametric case $\eta = 0$ and $\xi < 0$. Even in the case where resource revenues are finite asymptotically ($\eta > 0$ and $\xi < 0$), the system leads to an economy which is infinitely conservative. Only in the case where the resource rents vanish in the long-run, that is the case $\eta = 0$ and $\xi < 0$, the economy can converge to a steady state, thereby implying that the corresponding costs of lobbying get limited asymptotically.

As mentioned in Section 2, the cases with $\eta = 0$ and $\xi \lesseqgtr 0$ describe the dynamic behavior of an economy that relies on non-renewable resources rents. For this kind of economy, we observe that both situations are possible. In particular, the economy will reach in the long run a steady state with finite R and z if and only if the effect of decreasing extraction rates exceeds the positive trend of the resource price (case *2.c*).[9] In the opposite situation however, resource wealth goes to $+\infty$, which in turn brings the level of legislation to $-\infty$ (case *2.a* with $\eta = 0$). Finally, the cases with $\eta > 0$

[9] Actually, in this case, the resource will be exhausted asymptotically.

and $\xi \lesseqgtr 0$ better represent economies that own and sell renewable resources. In these cases, regardless of the level to which resource rents converge, the economy ends up in an infinitely conservative system, with $z = -\infty$, which implies ever increasing costs of lobbying. It's worth pointing out that if players' lobbying powers were equal, one would get the Wirl's result: the economy converges to the center of the political spectrum. If instead we assume as in this section that the proponents of the conservative line get their bargaining position improved when the resource revenues go up as in Algeria, we get conservative political states asymptotically, which is fully consistent with Friedman's first law of petropolitics.

Next, the question is: Could uncertainty and its inherent stabilization by noise mechanism via risk aversion reduce these costs? To answer this question, note that the expression of R given in (15) can be rewritten as

$$R(t) = R_0 e^{\left(\xi - \frac{\sigma_R^2}{2}\right)t} + \eta \int_0^t e^{\left(\xi - \frac{\sigma_R^2}{2}\right)(t-s) + \sigma_R (W_t - W_s)} ds. \tag{18}$$

From Boucekkine et al. (2015), we know that the first term is (stochastically) stable if and only if

$$\xi - \frac{\sigma_R^2}{2} < 0. \tag{19}$$

Thus, a straightforward result is that if $\eta = 0$, the resource revenue process, as given by (18), is stochastically stable if and only if (19) holds. Here it is worth comparing this result with the ones obtained in the deterministic counterpart of our problem, as stated in Proposition 5. If $\eta = 0$ (non-renewable resource case), then either the resource revenue process goes to zero when $\xi < 0$ or it goes to ∞ when $\xi > 0$. Under uncertainty, we show that noise is fully stabilizing in the latter case provided the revenue volatility, or uncertainty, is large enough. But this is far from a definitive result since the endogenous variable is the legislative state, z. The following proposition proves that (19) is enough to ensure the boundedness of the z-process whatever $\eta \geq 0$.

Proposition 6 *Suppose that condition (19) and $\sigma_z > 0$ hold, then there exist $M_R = M_R(\xi, \sigma_R) > 0$ and $M_z = M_z(\xi, \sigma_R, \sigma_z) > 0$, such that, both stochastic processes R and z*

are almost surely bounded in the sense of absolute values:

$$0 \leq R(t) \leq M_R, \ \mid z(t) \mid \leq M_z, \ \forall t \geq 0.$$

Thus in contrast to the deterministic case, one can assure the almost sure boundedness of the legislative state even in the case where $\xi > 0$, that is even when the deterministic part of resource revenue dynamics leads to explosive rents. In our working example, the MPE feedback rules, $x_i(t)$, $i = 1, 2$, only depend on resource revenues (because $a_2 = 0$), and therefore the deterministic part of the dynamics of the legislative state only depends on the latter variable. Proposition 6 establishes that when the process of resource revenue is stochastically stable, the stochastic legislative state is almost surely bounded for any level of political uncertainty σ_z.

With this in mind, we can take a step further, and proceed to the computation of asymptotic invariant distribution. The results are summarized in the next proposition (again we use the mathematical apparatus developed in the Appendix B of Merton (1975), see the Appendix A.4).

Proposition 7 *Consider the stochastic differential game above.*

(7.1) Under condition (19) and $\sigma_z > 0$, both R and z will converge to their long-run stationary distributions which are time invariant and independent of initial states.

(7.2) Let $\eta > 0$. The long run density function of the legislative state z is given by

$$\pi_z(z) = \frac{m}{\sigma_z^2 z^2} \exp\left\{ -\frac{2a_1\varepsilon R(2+\varepsilon R)}{br\sigma_z^2} \frac{1}{z} \right\}, \tag{20}$$

where $m > 0$ is chosen such that $\int_{-\infty}^{+\infty} \pi_z(z)dz = 1$, and the long run density function of revenue R is characterized by:

$$\pi_R(R;\eta) = \frac{n\, R^{\frac{2\xi}{\sigma_R^2}}}{\sigma_R^2 R^2} \exp\left\{ -\frac{2\eta}{\sigma_R^2\, R} \right\} \tag{21}$$

with $n > 0$ chosen such that $\int_0^{+\infty} \pi_R(R;\eta)dR = 1$.

Note that in Proposition 7, we voluntarily pay attention to the case $\eta > 0$ only. The reason for this is quite simple. If $\eta = 0$, the density function (21) simplifies to

$$\pi_R(R; \eta = 0) = \frac{n}{\sigma_R^2} R^{\frac{2}{\sigma_R^2}(\xi - \sigma_R^2)},$$

but then it is impossible that

$$\int_0^\infty \pi_R(R; \eta = 0) dR = 1.$$

In other word, when $\eta = 0$, the limit of the density function (21) can not serve as a density function. Nonetheless in this case, it is easy to show that[10]

Proposition 8 *Suppose that* $\eta = 0$ *and* $\xi - \frac{\sigma_R^2}{2} < 0$*. Then both stochastic processes* R *and* z *converge to their long run steady state* (R_∞, z_∞) *with*

$$R_\infty = z_\infty = 0. \tag{22}$$

Finally, we can put all of these elements together to emphasize the role of uncertainty in our differential game of lobbying. Legislative uncertainty and the resource revenue volatility mostly play through the long run properties of the MPE. More precisely, the comparison between Proposition 5 on the one hand, and Propositions 6-8 on the other, clearly highlights the stabilization power of uncertainty. First, when the deterministic MPE is stable (case *2.c*), we obtain that the stochastic MPE is necessarily stable as well. More importantly, when the conditions are such that the deterministic economy follows

[10]With $\eta = 0$, the process R follows a linear homogenous stochastic differential equation:

$$dR(t) = \xi R(t)dt + \sigma_R R dW_t.$$

Boucekkine et al. (2015) show that the R process almost surely converges to its steady state $R_\infty = 0$, provided $\xi - \frac{\sigma_R^2}{2} < 0$. This means that, at the limit, the z process in turn follows

$$dz = \mu_z z dt + \sigma_z z dW,$$

with $\mu_z = 0$, and hence $\mu_z - \frac{\sigma_z^2}{2} = -\frac{\sigma_z^2}{2} < 0$ is always true. Thus, by Boucekkine et al. (2015) again, we have that process z converges to its long run steady state $z_\infty = 0$.

an explosive path (at least in terms of z, cases *2.a* & *2.b*), a sufficient level of uncertainty in the R-process ensures that the stochastic system will reach a stationary state in the long run, that is characterized by the density functions in Proposition 7. Here again, risk aversion plays a stabilization role.

As far as the policy implications of our results are concerned, let us focus first on the implications of uncertainties on the steady state level of legislation or institutions (case *2.c*). Here we observe that beside stabilization, uncertainties endow the economy with better institutions in the long run. Therefore, our results tend to weaken the case for the resource curse hypothesis (Ross, 2001) and for the first law of geopolitics advocated by Frieman. Resource wealth might lead to inefficient institutions (this is the sense of our results in the deterministic case). But, what ultimately matters is not the level of this wealth but its volatility. Once we account for such volatility, we obtain that resource wealth may indeed improve the institutions (at least when $z_0 < 0$). At least, it is apparent from (17) and (22) that it drives the system to a higher level of z, i.e. to a higher quality of institutions. The same kind of conclusion can be drawn from the analysis of the more general cases *2.a* & *2.b*, except that we have now to discuss about long run densities rather than steady states. Indeed, in these situations, considering the impact of resource rents introduces a bias in the interaction between lobbyists toward the conservatives and necessarily leads, in the long run, to a terribly bad outcome in terms of the legislation, or the institutions, in the deterministic world. This is no longer the case with uncertainties. Again, anything can happen because of the convergence to time invariant and independent of initial states distributions for both z and R. But this in particular implies that the economy might achieve a long run equilibrium characterized by a finite and even positive state of legislation (with positive probability).

5 Conclusion

This paper proposes a dynamic analysis of the economic and political liberalization process in resource-dependent countries, motivated by the empirical debate on the role of oil abundance in the (non)-emergence of democracy and associated writings about the so-called petropolitics (Friedman, 2006). For that purpose, Wirl (1994)'s differential game of lobbying is extended in two major directions. The basic structure of Wirl (1994) is retained: We model the interaction between two groups with opposing interests with regard to the state of the legislation. When it comes to the analysis of the dynamics of institutions in Arab countries, such as Algeria, two more ingredients, absent from Wirl's analysis, seem to be particularly important. The first ingredient is the uncertainty in the process of liberalization itself. The second one has to do with the crucial role of resource windfalls in economic and political outcomes, which also rises the question of the role of a second source of uncertainty playing through the dynamics of resource rents. Our study precisely aims at investigating the impact of these two uncertainties on the players' strategies, and more generally, on the properties of the equilibrium.

In the first place, we focus on the uncertainty surrounding the liberalization process only and show that this element alone sheds new light on the results. The main consequence of uncertainty is the existence of multiple stable equilibria that have very distinct features. The analysis of symmetric equilibria allows us to identify a new mechanism – directly linked to the size of uncertainty and players' risk aversion – that helps to explain why the economy can reach a stable state in the long run even in the absence of the retaliation motive put forward by Wirl in his deterministic framework. But a second type of asymmetric equilibria may also arise where players offer the same response to a change in the level of liberalization despite their opposing interests. In this situation, the compensation mechanisms no longer come into operation, which tends to destabilize the whole system.

In the second place, it is assumed that the player who controls resource revenue benefits from this because it increases her relative lobbying power. Moreover, we put the two sources of uncertainty together in the same picture (the second coming from price volatility). Here, the true impact of uncertainty shows itself in the stability property of the equilibrium. Quite interestingly we obtain under a fairly general condition that uncertainties not only tend to stabilize the behavior of the economy (at least) in the long run, but also promote the convergence toward a state of liberalization that is better than its deterministic counterpart. Taking liberalization as a good indicator of the quality of economic and political institutions, this result helps to explain the mixed support for the oil impedes democracy hypothesis: If resource wealth may tend to worsen institutions, what ultimately matters to understand the impact of resource rents on resource-dependent countries is less their level then their volatility.

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A Appendix

A.1 Proof of Propositions 1, 2, and 3

The proof of Propositions 1, 2, and 3 can be done altogether, since they all come from the solution of the same stochastic dynamic game. Thus, in the following, we first present the general calculation and then show one by one of the proofs of the propositions.

To make it clear, we restate the the dynamic game as following: objective of player 1 is

$$\max_{x_1} \int_0^{+\infty} F_1(x_1, z)e^{-rt}dt = \int_0^{+\infty} e^{-rt} \left[a_0 + a_1 z + \frac{a_2}{2}z^2 - \frac{b}{2}x_1^2\right] dt$$

and the Objective of player 2 is

$$\max_{x_2} \int_0^{+\infty} F_2(x_2, z)e^{-rt}dt = \int_0^{+\infty} e^{-rt} \left[a_0 - a_1 z + \frac{a_2}{2}z^2 - \frac{b}{2}x_1^2\right] dt,$$

with constants a_0, a_1, b positive and a_2 non-positive. The common state constraint is

$$dz = (x_1 - x_2)dt + \sigma_z z dW.$$

It is easy to see this is a standard linear-quadratic stochastic differential game. To obtain the stationary MPE, we define the value function of player i as

$$V_i(z) = A_i + B_i z + \frac{C_i}{2}z^2, \quad i = 1, 2,$$

with A_i, B_i, C_i undetermined coefficients. Thus these value functions must check the following Hamilton-Jacob-Bellman equations:

$$rV_i(z) = \max_{x_i} \left[F_i(x_i, z) + \frac{dV_i}{dz}(x_1 - x_2) + \frac{\sigma_z^2 z^2}{2}\frac{d^2 V_i}{dz^2}\right], \quad i = 1, 2. \tag{23}$$

The standard first order (necessary and sufficient) conditions on the right hand sides of (23) yield the optimal choice of player 1 and 2:

$$x_1^* = \frac{1}{b}\frac{dV_1}{dz} = \frac{B_1 + C_1 z}{b} \text{ and } x_2^* = -\frac{1}{b}\frac{dV_2}{dz} = -\frac{B_2 + C_2 z}{b}.$$

Substituting these optimal choices into the right hand sides of equation (23) and comparing the coefficients of term z on both left and right hand sides of (23), we obtain the following equation system for parameters:

$$\begin{cases} rA_1 = a_0 + \frac{B_1^2}{2b} + \frac{B_1 B_2}{b}, \\ rB_1 = a_1 + \frac{B_1 C_1}{b} + \frac{B_1 C_2 + B_2 C_1}{b}, \\ rC_1 = a_2 + \frac{C_1^2}{b} + \frac{2C_1 C_2}{b} + \sigma_z^2 C_1 \end{cases} \tag{24}$$

and

$$\begin{cases} rA_2 = a_0 + \frac{B_2^2}{2b} + \frac{B_1 B_2}{b}, \\ rB_2 = -a_1 + \frac{B_2 C_2}{b} + \frac{B_1 C_2 + B_2 C_1}{b}, \\ rC_2 = a_2 + \frac{C_2^2}{b} + \frac{2C_1 C_2}{b} + \sigma_z^2 C_2. \end{cases} \tag{25}$$

Combining the last equation of (24) and (25) together and rearranging terms, it yields

$$\left(r - \sigma_z^2\right) \ (C_1 - C_2) = \frac{(C_1 - C_2)(C_1 + C_2)}{b}.$$

Thus, two groups of solutions are possible:

$$C_1 = C_2$$

and

$$C_1 \neq C_2, \ \text{ then } \ C_2 = b(r - \sigma_z^2) - C_1.$$

A.1.1 Proof of Proposition 1

Substituting $C_1 = C_2 = C$ into the last equation of (24) (or (25)), it yields that

$$C_1 = C_2 = \frac{-b(\sigma_z^2 - r) \pm \sqrt{b^2(\sigma_z^2 - r)^2 - 12ba_2}}{6},$$

which is always real, given $a_2 < 0$. For shortening the notation, we denote the above $C_1 = C_2$ as $C^{(j)}$, $j = 1, 2$, with $C^{(1)}$ taking positive in front of the square root term, while $C^{(2)}$ taking the negative one.

Furthermore, substituting $C_1 = C_2$ into the second equations of (24) and (25), it yields

$$B_1 + B_2 = 0, \quad \text{or} \quad B_1 = -B_2.$$

Thus, by the second equation of (24) again, we have

$$B_1 = \frac{a_1}{br - C} = -B_2.$$

Substituting B_i, C_i $(i = 1, 2)$ into the optimal choice, we obtain the results of Proposition 1.

Substituting now the above two equilibrium strategies into the stochastic state equation, we have

$$dz = [x_1 - x_2]dt + \sigma_z z\, dW = \frac{2C^{(j)}}{b} z\, dt + \sigma_z z\, dW,$$

which is a linear homogenous stochastic differential equation with $z = 0$ as one long-run solution. From the $AK-$type model of Boucekkine et al. (2015), it is easy to check that $z = 0$ is almost surely stochastically stable if and only if

$$\frac{2C^j}{b} - \sigma_z^2 < 0.$$

A.1.2 Proof of Proposition 2

Substituting $C_2 = b(r - \sigma_z^2) - C_1$ into the last equation of (24) and rearranging terms, it follows:

$$C_1^{(j)} = \frac{-b(\sigma_z^2 - r) \pm \sqrt{b^2(\sigma_z^2 - r)^2 + rba_2}}{2}, \quad j = 3, 4$$

with $C_1^{(3)}$ taking positive sign of the square root term and $C_1^{(4}$ taking negative one. Thus,

$$C_2^{(j)} = b(r - \sigma_z^2) - C_1^{(j)} = \frac{-b(\sigma_z^2 - r) \mp \sqrt{b^2(\sigma_z^2 - r)^2 + rba_2}}{2}, \quad j = 3, 4.$$

Remark. To guarantee that the square root term is real, some conditions on the parameters are needed, however, it is not essential for the current study, for example we can assume the absolute value of a_2 is not too large.

Combining the above explicit $C_i^{(j)}$ into the second equations of (24) and (25)), we obtain the $B_i^{(j)}$, $i = 1, 2$ and $j = 3, 4$, as presented in the Proposition 2.

A.1.3 Proof of Proposition 3

Similar as the proof of Proposition 1, substituting the above optimal efforts of both players into the stochastic differential equation, and simplifying terms, we have

$$dz^{(j)} = [x_1^{(j)} - x_2^{(j)}]dt + \sigma_z z^{(j)}\, dW = \left[\frac{a_1(C_1^{(j)} - C_2^{(j)})}{b(b\sigma_z^4 + a_2)} + (r - \sigma_z^2)z^{(j)}\right] dt + \sigma_z z^{(j)}\, dW.$$

Following Merton (1975, Page 390) that "*a steady state distribution will always exist in the sense that*" $z(t)$ "*will either (1) be absorbed at one of the natural boundaries (i.e. a degenerate distribution with a dirac function for a density)" or (2) it will have a finite density function of the interval*" $(-\infty, +\infty)$ " *or (3) it will have a discrete probability mix of (1) and (2)*".

The same arguments as Merton (1975) by applying the results of Cox and Miller (1968, Page 223-225), that both natural boundaries $\pm\infty$ are inaccessible, provided some simple parameter conditions are imposed on a_1, a_2, b, r and σ_z.

The rest of the proof can be done straightforward following the same arguments as Merton (1975, Page 389-390) by applying the Kolmogorov-Fokker-Planck "forward" equation.

Given $z(t)$ is a diffusion process, its transition density function will satisfy the Kolmogorov-Foller-Planck "forward" equation. Let $Q(z, t; z(0))$ be the conditional probability density for process $z(t)$ at time t, given initial condition $z(0)$. Then the corresponding Kolmogorov-Foller-Planck "forward" equation would be

$$\frac{\partial Q(z,t)}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial z^2}\left(\sigma_z^2 z^2 Q(z,t)\right) - \frac{\partial}{\partial z}\left[\left(\frac{a_1(C_1 - C_2)}{b(b\sigma_z^4 + a_2)} + (r - \sigma_z^2)z\right) Q(z,t)\right].$$

Suppose $z(t)$ has a steady state distribution, which is independent of initial condition $z(0)$. then

$$\lim_{t\to+\infty} Q(z,t) = q(z), \quad \lim_{t\to+\infty} \frac{\partial Q(z,t)}{\partial t} = 0,$$

and $q(z)$ satisfies

$$\frac{1}{2}\frac{d^2}{dz^2}(\sigma_z^2 z^2 Q(z,t)) - \frac{d}{dz}\left[\left(\frac{a_1(C_1 - C_2)}{b(b\sigma_z^4 + a_2)} + (r - \sigma_z^2)z\right) Q(z,t)\right] = 0.$$

The last equation can be easily solved by the standard variation of coefficient method in ordinary differential equation. Combining with the above arguments that boundaries $\pm\infty$ are inaccessible, then following Merton (1975, Page 390), the steady state density function must be given as in the Proposition 3.

That completes the proofs of the three Propositions displayed in Section 3.

A.2 Proof of Proposition 4

Player 1 considers

$$\max_{x_1} \int_0^\infty e^{-rt} F_1(x_1, z) dt = \int_0^\infty e^{-rt} \left[a_0 + a_1 z - \frac{b}{2} x_1^2 \right] dt$$

and Player 2 considers

$$\max_{x_2} \int_0^\infty e^{-rt} F_2(x_2, z) dt = \int_0^\infty e^{-rt} \left[a_0 - a_1 z - \frac{b}{2} x_2^2 \right] dt.$$

Both players are subject to the following general constraints:

$$dz(t) = [x_1(t) - g_z(R) x_2(t)] dt + \sigma_z z dW_t = [x_1(t) - (1 + \varepsilon R(t)) x_2(t)] dt + \sigma_z z dW_t$$

and

$$dR(t) = g_R(R) dt + \sigma_R R dW_t = (\eta + \xi R) dt + \sigma_R R dW_t,$$

where process $(W_t)_{t \geq 0}$ read standard Brownian motion.

Denote the stationary value function of player i as $V_i(z, R)$, then the Hamilton-Jacob-Bellman equation of player i $(i = 1, 2)$ is

$$rV_i(z, R) = \max_{x_i} \left[F_i(x_i, z) + \frac{\partial V_i}{\partial z} \cdot (x_1 - (1 + \varepsilon R) x_2) + \frac{\partial V_i}{\partial R} (\eta + \xi R) + \frac{1}{2} \sigma D^2 V_i \sigma' \right],$$

with $\sigma = (\sigma_z z \;\; \sigma_R R)$, its transpose $\sigma^T = \begin{pmatrix} \sigma_z z \\ \sigma_R R \end{pmatrix}$ and $D^2 V_i$ the second order derivative.

Consider linear-quadratic value function

$$V_i(z, R) = A_i + B_i z + \frac{C_i}{2} z^2 + D_i R + \frac{E_i}{2} R^2 + H_i z R,$$

then we have

$$\frac{\partial V_i(z,R)}{\partial z} = B_i + C_i z + H_i R,$$
$$\frac{\partial V_i(z,R)}{\partial R} = D_i + E_i R + H_i z,$$

and

$$D^2 V_i = \begin{pmatrix} \frac{\partial^2 V_i}{\partial z^2} & \frac{\partial^2 V_i}{\partial z \partial R} \\ \frac{\partial^2 V_i}{\partial z \partial R} & \frac{\partial^2 V_i}{\partial R^2} \end{pmatrix} = \begin{pmatrix} C_i & H_i \\ H_i & E_i \end{pmatrix}.$$

Thus

$$\Sigma_i = \frac{1}{2}\sigma\ D^2 V_i\ \sigma^T = \frac{1}{2}\left[C_i(\sigma_z z)^2 + 2H_i \sigma_z z \sigma_R R + E_i(\sigma_R R)^2\right].$$

Substituting the above functions into the HJB equation, it follows, for $i = 1, 2$

$$\begin{aligned} rV_i(z,R) \ &= \max_{x_i} [F_i(x_i, z) + (B_i + C_i z + H_i R) \cdot (x_1 - (1+\varepsilon R)x_2) \\ &+ (D_i + E_i + H_i z)(\eta + \xi R) + \Sigma_i]. \end{aligned}$$

The standard first order conditions yields the following optimal efforts:

$$x_1(t) = \frac{1}{b}\ \frac{\partial V_1}{\partial z} = \frac{(B_1 + C_1 z + H_1 R)}{b}$$

and

$$x_2(t) = -\frac{1+\varepsilon R}{b}\ \frac{\partial V_2}{\partial z} = \frac{-(1+\varepsilon R)(B_2 + C_2 z + H_2 R)}{b}.$$

Substituting x_1 and x_2 into the right hand side of HJB equations for both players and comparing the coefficients of the left and right hand sides, we have

$$\begin{cases} rA_1 = a_0 + \dfrac{B_1^2}{2b} + \dfrac{B_1 B_2}{b} + \eta D_1, \\ rB_1 = a_1, \\ C_1 = 0, \\ rD_1 = \dfrac{2\varepsilon B_1 B_2}{b} + \eta E_1 + \xi D_1, \\ rE_1 = \dfrac{\varepsilon^2 B_1 B_2}{b} + \left(\xi + \dfrac{\sigma_R^2}{2}\right) E_1, \\ H_1 = 0, \end{cases}$$

and

$$\begin{cases} rA_2 = a_0 + \dfrac{B_2^2}{2b} + \dfrac{B_1 B_2}{b} + \eta D_2, \\ rB_2 = -a_1, \\ C_2 = 0, \\ rD_2 = \dfrac{\varepsilon B_2^2}{b} + \eta E_2 + \xi D_2, \\ rE_2 = \dfrac{\varepsilon^2 B_2^2}{b} + (\sigma_R^2 + 2\xi) E_2, \\ H_2 = 0. \end{cases}$$

Solving these systems yields

$$\begin{cases} C_1 = 0, \quad H_1 = 0, \quad B_1 = \dfrac{a_1}{r}, \quad E_1 = -\dfrac{2\varepsilon^2 a_1^2}{br^2(r - 2\xi - \sigma_R^2)}, \\ D_1 = \dfrac{2\varepsilon a_1^2}{br^3(r-\xi)} + \dfrac{\eta\, E_1}{r-\xi}, \\ A_1 = \dfrac{a_0}{r} - \dfrac{a_1^2}{2br^3} + \dfrac{\eta D_1}{r}, \end{cases}$$

and

$$\begin{cases} C_2 = 0, \quad H_2 = 0, \quad B_2 = -\dfrac{a_1}{r}, \quad E_2 = \dfrac{\varepsilon^2 a_1^2}{br^2(r - 2\xi - \sigma_R^2)}, \\ D_2 = \dfrac{\varepsilon a_1^2}{br^2(r-\xi)} + \dfrac{\eta\, E_2}{r-\xi}, \\ A_2 = \dfrac{a_0}{r} - \dfrac{a_1^2}{br^3} + \dfrac{\eta D_2}{r}. \end{cases}$$

The current value functions are

$$V_1(z, R) = A_1 + B_1 z + D_1 R + \frac{E_1}{2} R^2, \;\; V_2(z, R) = A_2 + B_2 z + D_2 R + \frac{E_2}{2} R^2.$$

The optimal choice of player 1 is thus

$$x_1(t) = \frac{1}{b}\frac{\partial V_1}{\partial z} = \frac{B_1}{b} = \frac{a_1}{br}$$

and the optimal choice of player 2 is

$$x_2^*(t) = -\frac{1 + \varepsilon R(t)}{b}\frac{\partial V_2}{\partial z} = \frac{a_1(1+\varepsilon R)}{br}.$$

Substituting the optimal choices into the two state equations, we obtain the dynamic system (14). That finishes the proof.

A.3 Proof of Proposition 6

In the following we keep condition (19) and hence, we only need to study the second term of (18). Let $X(t)$ be the solution of the following homogenous stochastic equation

$$\begin{cases} dX(t) = \xi X(t)dt + \sigma_R X(t)dW_t, \forall t \geq s, \\ X(s) = 1. \end{cases}$$

By Ito's Lemma, the solution satisfies

$$\ln(X(t;s)) = \left(\xi - \frac{\sigma_R^2}{2}\right)(t-s) + \sigma_R(W_t - W_s),$$

and

$$\lim_{t\to\infty} \frac{\mathbf{E}\ln(X(t;s))}{t} = \xi - \frac{\sigma_R^2}{2} < 0$$

under condition (19). Therefore, for any $\epsilon \in \left(0, \frac{\sigma_R^2}{2} - \xi\right)$, there exits $\delta = \delta(\epsilon)$, such that

$$\mid X(t;s) \mid \leq \delta e^{\left(\xi - \frac{\sigma_R^2}{2} + \epsilon\right)(t-s)}, \quad \forall t \geq s.$$

Hence, we have

$$\int_0^t X(t;s)ds \leq \delta \int_0^t e^{\left(\xi - \frac{\sigma_R^2}{2} + \epsilon\right)(t-s)} ds = \frac{\delta}{\xi - \frac{\sigma_R^2}{2} + \epsilon}\left[1 - e^{\left(\xi - \frac{\sigma_R^2}{2} + \epsilon\right)t}\right].$$

Furthermore, taking limits on both sides, we have

$$\lim_{t\to\infty} \eta \mathbf{E} \int_0^t X(t;s)ds \leq \frac{\eta\delta}{\xi - \frac{\sigma_R^2}{2} + \epsilon}.$$

Take $\epsilon = \frac{1}{2}\left(\xi - \frac{\sigma_R^2}{2}\right)$, then $\delta = \delta(\xi, \sigma_R)$ and

$$\lim_{t\to\infty} \eta \int_0^t X(t;s)ds \leq \eta H(\xi, \sigma_R) < +\infty,$$

where $H(\xi, \sigma_R)$ is a constant which depends on ξ and σ_R only. In other world, the second part of (18) is bounded under condition (19).

Combining the first and second parts together, it yields that condition (19) guarantees that function $R(t)$ is finite for any $t \geq 0$.

Substituting the above bounded results of $R(t)$ into the expression of $z(t)$ given in 15, and applying the same analysis, we could conclude that $z(t)$ is also bounded given $-\sigma_z < 0$. That finishes the proof.

A.4 Proof of Proposition 7

The proof of existence of steady state density distribution of stochastic process $R(t)$ is exactly the same as Merton (1975).

Given $R(t)$ is a diffusion process, its transition density function will satisfy the Kolmogorov-Foller-Planck "forward" equation. Let $P(R,t)$ as the conditional probability density for process $R(t)$ at time t, given initial condition $R(0) = R_0$. Then the corresponding Kolmogorov-Foller-Planck "forward" equation would be

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial R}[(\eta + \xi R)P(R,t)] + \frac{\partial^2}{\partial R^2}\left(\frac{\sigma_R^2 R^2}{2} P(R,t)\right).$$

The above equation can be rewritten as

$$\frac{\partial P}{\partial t} = (\sigma_R^2 - \xi)P(R,t) + (4\sigma_R R - \xi R - \eta)\frac{\partial P}{\partial R} + \frac{\sigma_R^2 R^2}{2}\frac{\partial^2}{\partial R^2}P(R,t). \tag{26}$$

Suppose that R has a steady state distribution, independent of R_0, then

$$\lim_{t\to\infty} P(R,t) = \pi_R(R)$$

and

$$\lim_{t\to\infty} \frac{\partial P}{\partial t} = 0.$$

Thus, the stationary density function $\pi(R)$ is the solution of the following second order differential equation:

$$0 = \frac{d}{dR}\left[-(\eta + \xi_R R)\pi(R)] + \frac{d}{dR}\left(\frac{\sigma_R^2 R^2}{2}\pi_R(R)\right)\right]. \tag{27}$$

The rest will follow the same arguments as Appendix B of Merton (1975), except the inaccessible of one natural boundary $R = 0$, where we recall the stochastic differential equation

$$dR(t) = (\eta + \xi R)dt + \sigma_R R dB_t, \tag{28}$$

with $R \in [0, M_R]$. To finish this part of proof, we follow the method of Merton (1975, Page 390-391) that we "compare the stochastic process generated by" (28) "with another process which is known to have inaccessible boundaries and then to show that the probability that" R "reaches its boundary" $R = 0$ "is no larger than the probability that the comparison process reaches its" boundary.

Define a new process $X(t) = \ln(R)$. By Ito's Lemma, it follows from condition (19) that

$$dX = \left(\frac{\eta}{R} + \xi - \frac{\sigma_R^2}{2}\right)dt + \sigma_R dB_t, \tag{29}$$

with $\xi - \frac{\sigma_R}{2} < 0$.

Noticing that if $M_R > 1$ and $R \in [1, M_R]$, by continuity, it is impossible that $R(t) \to 0$ as $t \to \infty$. Thus, we only need to consider the case $R \in [0, \min\{1, M_R\}]$.

Take $\delta = \frac{1}{2}\left(\frac{\sigma_R^2}{2} - \xi\right) > 0$, then, provided $\eta > 0$

$$\frac{\eta}{R} + \xi - \sigma_R^2 \geq \delta > 0$$

if and only if,

$$R \leq \frac{2\eta}{3\left(\frac{\sigma_R^2}{2} - \xi\right)} = \underline{R}.$$

Consider a Wiener process $W(t)$ with drift δ and variance σ_R^2 defined on the interval $[-\infty, \underline{R}]$ where $\underline{R}$ is a reflecting barrier. I.e.,

$$dW(t) = \delta dt - \sigma_R dB_t$$

for $W \in [-\infty, \underline{R}]$. Merton (1975, Page 391) and Cox and Miller (1968, page 223-225) have shown that such a process with $\delta > 0$ has a non-degenerate steady state. Thus, $-\infty$ is an inaccessible boundary for $W-$process. Therefore, $-\infty$ is also an inaccessible boundary for $X-$process. Given $X(t) = \ln(R)$, thus, 0 is an inaccessible boundary for $R-$process, provided $\eta > 0$.

For the process z, the same arguments apply as well. That completes the proof.