Optimal design and defense of networks under link attacks*

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Abstract

Networks facilitate the exchange of goods and information and create benefits. We consider a network with n complementary nodes, i.e. nodes that need to be connected to generate a positive payoff. This network may face intelligent attacks on links. To study how the network should be designed and protected, we develop a strategic model inspired by Dziubiński and Goyal (2013) with two players: a Designer and an Adversary. First, the Designer forms costly protected and non-protected links. Then, the Adversary attacks at most k links given that attacks are costly and that protected links cannot be removed by her attacks. The Adversary aims at disconnecting the network shaped by the Designer. The Designer builds a protected network that minimizes her costs given that it has to resist the attacks of the Adversary. We establish that in equilibrium the Designer forms a minimal 1-link-connected network which contains only protected links, or a minimal $(k+1, n)$-link-connected network which contains only non-protected links, or a network which contains one protected link and $(n-1)(k+1)/2$ non-protected links. We also examine situations where the Designer can only create a limited number of protected links and situations where protected links are imperfect, that is, protected links can be removed by attacks with some probabilities. We show that if the available number of protected links is limited, then, in equilibrium, there exists a network which contains several protected and non-protected links. In the imperfect defense framework, we provide conditions under which the results of the benchmark model are preserved.

JEL Classification: D74, D85.
Key Words: Attacks on links, Network defense, Network design.

1 Introduction

Networks can be seen as communication structures. They are composed of nodes and links, where links represent the flow of information. Networks represent a crucial feature in our society, and are of particular interest in different fields such as military defense, telecommunication or computer networks. Some networks can be damaged by natural disasters or intelligent attacks. Attacks can affect nodes (agents, computers, telecommunication antennas, ...) or links (roads, communications flows, ...), and may disconnect a network.[1] In this paper, we examine situations where attacks target links. To illustrate this type of situations, consider a firm which has several production units (nodes of the network). Each production unit produces a part of the product and the pieces are assembled by a given production unit. The links of the network allow the parts of the product to be transferred among the units. If one unit is not connected to the rest of the units, its production cannot be transferred and the production

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[1] A network is connected if no set of nodes is isolated from the others.

has no value. Recall that during the Second World War, the production units for the weapons (nodes) were buried, so they were impossible to target, and attacks had to target the roads (links) in order to destroy the production process of the enemy. Therefore, the issue was to design a communication network between the production units that the enemy could not disconnect.

Our goal is to examine how to design and protect the network in an optimal way, such that the network remains connected after an intelligent link attack.[2] We say that a network is designed and protected in an optimal way if the costs associated with the design and the protection of the network are minimized.

We consider a two-stage game with two players: a Designer (D) and an Adversary (A).

- Stage 1. The Designer moves first and chooses both a set of protected, and a set of non-protected links. Protected links cannot be removed by the attacks of the Adversary.
- Stage 2. After observing the protected network (strategy) formed by the Designer, the Adversary attacks the network by allocating attacks to specific links. Since the attacks are costly, the Adversary has an incentive to attack at most k links.

Creating protected and non-protected links is costly for the Designer. The benefits obtained by the Designer at the end of the game depend on the connectivity of the residual network, that is the network obtained after the attack of the Adversary. If the residual network is connected, then the Designer wins the game: her benefits are equal to 1 and the benefits of the Adversary are 0. If the residual network is not connected, then the Adversary wins the game: her benefits are equal to 1 and the benefits of the Designer are 0. The payoff obtained by the players are equal to the difference between their benefits and the costs associated with their strategies.[3]

We are interested in the Sub-game Perfect Equilibrium (SPE) of the two-stage game. We assume that the cost of protected links and non-protected links are sufficiently low so that the Designer has some profitable strategies which allow the residual network to be connected. First, we provide for each number of protected links, the minimal number of non-protected links that the Designer has to form in order to avoid the Adversary to disconnect the network as well as a method to construct a solution network. Second, we establish that only three polar non-empty networks may arise in equilibrium in the benchmark model.

1. A minimal $(k+1,n)$-link-connected network which contains no protected links.[4]
2. A minimal $(1,n)$-link-connected network which contains $n-1$ protected links.
3. A network which contains one protected link and $\lceil (n-1)(k+1)/2 \rceil$ non-protected links.

The first family of networks constitutes the optimal strategy of the Designer when the cost of forming non-protected links is sufficiently low relative to that of forming protected links. The second one is the optimal strategy when the cost of forming non-protected links is sufficiently high relative to that of forming protected links. The third one is optimal for intermediate relative costs (cost of a protected link / cost of a non-protected link) when the number of nodes is odd and the number of attacks is even.

Additionally to the benchmark model described above, we study some variations of the game to develop a larger understanding of optimal design of protected networks. We take into account two types of limitations concerning protections. First, we consider that D cannot create as many protected links as in the benchmark model.[5] Then, we consider situations where each protected link has a probability π to be removed when it is attacked by A.[6]

In the framework where the number of protected links available for D is limited, we show that for intermediate relative costs (cost of a protected link / cost of a non-protected link), the optimal strategy

[2]Note that an intelligent attack can also be seen as the worst case scenario.

[3]If we take again our military example, and assume that for any i node $i-1$ is the supplier of node i, then the Designer has to maintain a path between each pair of nodes $i-1$ and i to obtain some end goods. In other words, the residual network has to be connected to allow some production.

[4]A network g, which contains n nodes, is a minimal $(k+1,n)$-link-connected network, if it is not possible to disconnect it by removing k links, and such that there is no network which cannot be disconnected by removing k links and contains a smaller number of links.

[5]If we take again our military example, the Designer may not have enough resources to protect the whole network.

[6]Despite the effort of the Designer (of the army) to protect the communication flow, the Adversary (the enemy) may still be able to succeed in destroying protected links with some probabilities.

of D consists in designing a network which contains both protected links and non-protected links. In the framework where protected links are removed by attacks with some probabilities, we provide conditions under which the results obtained in our benchmark model are preserved.

We now relate our paper to the existing literature on networks. This literature has become broader in the recent years (*Jackson* [18], *Goyal* [10] and *Vega-Redondo* [24]). The two seminal papers on the formation of social and economic networks are the paper of *Jackson and Wolinsky* [19] and the paper of *Bala and Goyal* [3]. *Bala and Goyal* [4] and *Haller and Sarangi* [14] introduce imperfectly reliable links in the *Bala and Goyal* [3] model. *Bala and Goyal* [4] show that for certain ranges of linking cost and probability of failure, the equilibrium network is at least $(2, n)$-link-connected, i.e. any two nodes are connected by at least two paths. *Haller and Sarangi* [14] extend the model of *Bala and Goyal* [4] by allowing heterogeneity in probabilities of link failure. These authors model random link failure but not an intelligent attack that seeks to interrupt the communication flow. In the present paper, we study the robustness of a network that must be designed and protected to resist an intelligent attack on links.

A growing literature on attacked networks studies situations where the Adversary attacks the nodes. *Dziubinski and Goyal* [9](DG) study the optimal design and defense of network under an intelligent attack. In their framework, there are two players: the Designer and the Adversary; the Designer can form links between n nodes, and/or protect these nodes to ensure their survival. The model we propose is close to the model of DG, with the following major differences:

- The Adversary attacks nodes in the DG's framework while she attacks links in our framework;
- In our framework, the Designer wins the game if every node is able to communicate with any other node in the residual network. In the DG's framework, the Designer wins the game if the residual network is connected regardless of the number of nodes removed by the Adversary. Thus, our setting is based on the complementarity of nodes while DG assume that nodes are substitutable.

DG show that in an SPE, the Designer protects 0 or 1 node. If the Designer protects 0 node, then she designs a minimal $(k+1, n)$-node-connected network.[7] We obtain the same type of networks when the Designer uses no protection. At first sight, this result seems intriguing since the Adversary attacks nodes in DG's paper and links in our paper. However, a minimal $(k+1, n)$-node-connected network defined as in DG is also a network that contains the minimal number of links and resists the Adversary who attacks links. In DG, if the Designer uses protections, she designs a star network[8] and protects 1 node, the central node. In our framework, when D uses protections, she designs either a network which contains 1 protected link and $\lceil (n-1)(k+1)/2 \rceil$ non-protected links, or a network which contains $n-1$ protected links. The results differ because in our framework every node needs to be connected with any other node in the residual network. Moreover, we establish that if we limit the number of available protections, then there exist situations where D designs networks which contain several protected and non-protected links. This result follows the fact that the number of non-protected links that each protected link allows the Designer to save is not constant.

DG examine imperfect defense through an example. They assume that the protections used by the Designer can fail when they are attacked by the Adversary. More precisely, an attack on an unprotected target always destroys the target, and an attack on a protected target destroys the target with a positive probability. A recent independent work of *Landwehr* [21] extends the analysis of imperfect defense. It shows that for a certain range of protection cost and link formation cost, strategies that use both protections and several links are equilibria.

Hoyer and De Jaegher [17] consider a framework where the Designer has to shape the network and form enough links in order to resist the attacks. In this framework, the Designer cannot protect specific parts of the network. The authors study the optimal way to design a network under link or node deletion with various cost levels. They show that if the costs of forming links are low, a regular network[9] with a sufficient number of links is the optimal network for the Designer. If costs are high and links are attacked, then a star network is optimal for the Designer. The difference with our paper (except for the fact that they do not use protected links) is that in our framework, nodes are complementary and the Designer

[7] A minimal $(k+1, n)$-node-connected network is a network, which contains n nodes, that cannot be disconnected by removing k nodes, and such that there is no network which cannot be disconnected by removing k nodes and contains a smaller number of links.

[8] A star network is a network where one node, the central one, is linked with all other nodes, and other nodes are only linked with the central node.

[9] A network where all nodes have the same number of links.

cannot sacrifice any node to minimize her costs. *Haller* [13] extends the model of *Hoyer and De Jaegher* by adding the possibility for two nodes to be connected by more than one link. In that case, it is harder for the Adversary to disconnect the network. Allowing multiple links between nodes can be seen as a different way to protect a connection between specific nodes than ours.

A part of the literature on attacked networks examines the role played by the contagion of attacks in networks. *Goyal and Vigier* [11] extend the work of DG by allowing the contagion of attacks (or threats). They find that the star network with a protected central node remains an equilibrium network. *Cabrales, Gottardi and Vega-Redondo* [6] and *Baccara and Bar-Isaac* [2] study the contagion of attacks in networks respectively in the field of financial firms where a financial risk can spread between connected firms and in the field of criminal networks where connectivity increases vulnerability because of external threats.[10] *Cerdeiro, Dziubinski and Goyal* [7] and *Acemoglu, Malekian and Ozdaglar* [1] identify nodes to players. Specifically, *Cerdeiro, Dziubinski and Goyal* [7] propose a three-stage game. First, the Designer chooses the network. Second, each player observes the network and chooses independently and simultaneously if she invests in protection or not. Third, the Adversary observes the protected network and chooses the players to infect. In *Acemoglu, Malekian and Ozdaglar* [1] nodes/players are connected in a random network. Players have to invest in protection to be immune. Their investment depends on their links and the probability of being infected in the random network. This model allows to examine for instance the impact of a contagious disease on the individual behavior. These papers are different from the present one for two reasons. First, we take into account situations where an attack on a link can remove only this specific link. Indeed, literature on contagious attacks reflects situations such as epidemics or virus spreading while our paper is focused on the study of specific link removal (for military strategies for instance). Second, in our model nodes cannot influence the architecture of the network by their decision.[11]

The rest of the paper is organized as follows. In section 2, we present the model setup. In section 3, we present our main results. In section 4, we extend our model by examining situations where the number of protected links available for the Designer is limited, and situations where protected links have some probabilities to be removed by an attack. In section 5, we conclude.

2 Model setup

To simplify the notations, we set $[\![a, b]\!] = \{i \in \mathbb{N}, a \leq i \leq b\}$. Moreover, $\lfloor x \rfloor$ and $\lceil x \rceil$ are respectively the largest integer smaller or equal to x and the smallest integer larger or equal to x, and $\text{abs}(x) = \max\{-x, x\}$. Further, for every set X, $\#X$ is its cardinality.

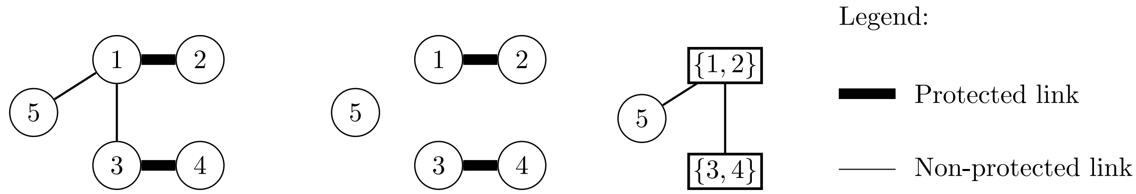
Network. For any integer $n \geq 4$, let $N = [\![1, n]\!]$ and $E(N)$ be the set of *unordered* pairs of N, i.e. $E(N) = \{(i, j) \in N \times N, i \neq j\}$. Throughout the paper, the elements of N are referred to as *nodes* while those of $E(N)$ are called *links*. An unordered pair $(i, j) \in E(N)$ is thus a link said to join the nodes i and j and the link is denoted by ij. We introduce the notion of *protected network* as a triplet $g = (N, E_P, E_{\text{NP}})$ with $E_P \subseteq E(N)$, $E_{\text{NP}} \subseteq E(N)$ and $E_P \cap E_{\text{NP}} = \emptyset$. We call *protected links* the elements of E_P and *non-protected links* the elements of E_{NP}. The significance of this refinement on the links will be made explicit in the two-player game formulation. To simplify the notations, we let $p = \#E_P$. In the rest of the paper, we will interchangeably use the term network or protected network.

[10] *McBride and Hewitt* [22] study the best way to dismantle a criminal network with imperfect information on its architecture. There also exists a literature which examines the particular cases of terrorist attacks, transportation network security, and homeland security (see *Brown, Carlyle, Salmeron and Wood* [5], *Tambe* [23], and *Hong* [16]).

[11] Additionally to economic, several fields investigate problems close to the one we deal with. In an early graph theoretic work, *Harary* [15] exhibits a family of (k, n)-node-connected networks with a total number of links that is minimal. This family of networks is crucial to establish our results. *Groetschel, Monma and Stoer* [12] study a situation where a firm has to prevent a communication network to be disconnected given that there exist possibilities of communication failure. As some connections may be interrupted, the firm has to design the least costly network that guarantees the best service for the consumers. Moreover, there also exists a literature on the design of survivable networks (see the survey of *Kerivin and Mahjoun* [20]) in Computer Science. *Cunningham* [8] studies network security and considers a model where the Designer allocates a different number of defense units to each link. A defended link has a level of resistance that depends on the number of defense units the Designer has allocated to it. The Adversary allocates attack units to remove a link. A link is removed if more attack units than defense units have been allocated to this link. The author proposes an algorithm which exhibits how some links have to be reinforced in order to protect the network.

For any network g, let $E_P(g)$ (respectively $E_{\mathrm{NP}}(g)$) refer to the set of protected (respectively non-protected) links of g, and $E(g) = E_P(g) \cup E_{\mathrm{NP}}(g)$. If there exists a link between i and j in g (i.e. if $ij \in E(g)$), then i and j are said *adjacent*. For each node i, $d_i(g)$ is its *degree* in g, that is the number of links incident to i in g: $d_i(g) = \#\{ij \in E(g)\}$. A *path* between two nodes i_0 and i_L of a network g is a finite alternating sequence of nodes and *distinct* links: i_0, $i_0 i_1$, i_1, $i_1 i_2$, $i_2, \ldots, i_{L-1} i_L$, i_L where $i_\ell \in N$ for all $\ell \in [\![0, L]\!]$ and $i_\ell i_{\ell+1} \in E(g)$ for all $\ell \in [\![0, L-1]\!]$. A *cycle* is a path where $i_0 = i_L$. Finally, a network $g = (N, E_P, E_{\mathrm{NP}})$ is *connected* if there exists a path between any two nodes $i, j \in N$. We say that $g' = (N', E'_P, E'_{\mathrm{NP}})$ is a subnetwork of $g = (N, E_P, E_{\mathrm{NP}})$ if $N' \subseteq N$, $E'_P \subseteq E_P$ and $E'_{\mathrm{NP}} \subseteq E_{\mathrm{NP}}$. Subnetwork $g' = (N', E'_P, E'_{\mathrm{NP}})$ is a component of network g if g' is connected and if there is no connected subnetwork $g'' = (N'', E''_P, E''_{\mathrm{NP}})$ of g, such that $N' \subset N''$. By convention, a node $i \in N$ such that $d_i(g) = 0$ is a component.

Given a link $ij \in E_P(g)$, the network $g \oslash ij$ is obtained by contracting the link ij; that is, by merging the two nodes i and j into a single node $\{i, j\}$, and making any node a adjacent to the (new) node $\{i, j\}$ in $g \oslash ij$ if and only if a is adjacent either to i or j in the network g. In other words, all links, other than those incident to neither i nor j, are included in $E_P(g \oslash ij)$ (respectively $E_{\mathrm{NP}}(g \oslash ij)$) if and only if they are included in $E_P(g)$ (respectively $E_{\mathrm{NP}}(g)$). The E_P-contraction of network g, $\hat{g}^{E_P}$, is obtained from g by sequences of link contractions for all links in $E_P(g)$.[12] We illustrate the E_P-contraction of a network g in Figure 1.



(i) Network $g = (N, E_P, E_{\mathrm{NP}})$ (ii) Network $(N, E_P, \emptyset)$ (iii) Network $\hat{g}^{E_P}$

Figure 1: Illustration of the E_P-contraction: the links 12 and 34 are contracted.

Two-player game. The players are the Designer (D) and the Adversary (A). We consider a two-stage game where D plays first and A moves at the second stage. A strategy S^D for D is a mapping that assigns to each set of nodes N a protected network $(N, E_P, E_{\mathrm{NP}})$. In other words, D chooses to create some links from $E(N)$ and to protect some of them:

$$S^D : N \mapsto (N, E_P, E_{\mathrm{NP}}), \quad E_P \subseteq E(N), \quad E_{\mathrm{NP}} \subseteq E(N), \text{ and } E_P \cap E_{\mathrm{NP}} = \emptyset.$$

For ease of notations, in the following we write $E_P^D = E_P(S^D(N))$ and $E_{\mathrm{NP}}^D = E_{\mathrm{NP}}(S^D(N))$.

A strategy for the Adversary A is a mapping that assigns to each protected network g a subset of links $E^A(g)$ of $E(g)$. In other words, A chooses to attack some links of g:

$$S^A : g = (N, E_P, E_{\mathrm{NP}}) \mapsto E^A, \quad \text{with } E^A \subseteq E(g).$$

Residual network and benefits. The strategy $S^D(N)$ played at the first stage is a protected network. Then, the attack of A leads to a second protected network of the form $g^R = (N, E_P^D, E_{\mathrm{NP}}^D \backslash E^A)$,[13] which we call *residual network*. Note that, by construction, g^R is a subnetwork of $S^D(N)$. The benefits of D are given by

$$\phi(g^R) = \begin{cases} 1, & \text{if } g^R \text{ is connected,} \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

[12] In the E_P-contraction, two nodes may be joined by several links; in this case $\hat{g}^{E_P}$ is a *multigraph*. A formal definition of a multigraph is given in the appendix.

[13] We introduce the notation: $\forall A, B \subseteq E(N), A \backslash B = \{a \in A, a \notin B\}$. Note that the residual network does not depend on the attacks on protected links $E^A \cap E_P^D$.

Network and costs. One can consider that attacking a link has a unitary cost c_A. Therefore, the cost of a strategy $S^A(g) = E^A$ of the Adversary is

$$c^A(E^A) = c_A \# E^A,$$

where $c_A \in [1/(n-3), 1)$. Note that the cost of a strategy is less than 1 if and only if the Adversary attacks less than $k = \lfloor 1/c_A \rfloor$ links.

Similarly, both protected and non-protected links are costly to create: each protected link has a strictly positive cost $c_P > 0$ and each non-protected link has a strictly positive cost $c_L > 0$. We assume that $c_P > c_L$. The cost of a strategy $S^D(N)$ of the Designer is thus:

$$c^D(S^D(N)) = c_P \ \# E^D_P + c_L \ \# E^D_{\text{NP}}. \tag{2}$$

If the cost of creating protected or non-protected links is too large, then D cannot use a strategy where she forms protected or non-protected links. Therefore, to obtain non trivial results, we assume that the costs of creating protected and non-protected links are sufficiently low: $c_P < 1/(n-1)$ and $c_L < 1/(\lceil n(k+1)/2 \rceil)$.

Payoffs. The payoff of the Designer for choosing $S^D(N)$ when the Adversary responds with $S^A(S^D(N))$ is:

$$\Pi^D(S^D(N), S^A(S^D(N))) = \phi(g^R) - c^D(S^D(N)). \tag{3}$$

The payoff of the Adversary is

$$\Pi^A(S^D(N), S^A(S^D(N))) = 1 - \phi(g^R) - c^A(S^A(S^D(N))). \tag{4}$$

If there exist two strategies of A that lead to the same payoff, A chooses the one having the highest value of $\# E^A$.[14]

In a nutshell, in our framework the objective of the Designer is to obtain a connected residual network at a minimal cost. The objective of the Adversary is to obtain a residual network that is disconnected. Note that A does not attack strictly more than $n-3$ links since $k = \lfloor 1/c_A \rfloor \leq 1/c_A \leq n-3$.

We now provide some illustrations where Equation (1) captures the benefits of D. Suppose that D has n production units identified to nodes. Let y_i be the output of production unit i, and δ_i be such that $\delta_i = 1$ if there is a path between $i \in [\![2, n]\!]$ and production unit $i-1$, and $\delta_i = 0$ otherwise. Here, production unit $i-1$ can be interpreted as the *unique* supplier of production unit i. We assume that $y_1 = \gamma$, $\gamma > 0$, and $y_i = \delta_{i-1} y_{i-1}$ for $i \in [\![2, n]\!]$. If the total output obtained by D from the production units is $Y = y_n$, then the total output function is in line with the benefits function of D. The same conclusion occurs if we assume $Y = \min_{i \in N}\{y_i\}$ or $Y = \prod_{i \in N} (y_i)^{\rho_i}$ with $\rho_i > 0$.

Moreover, let nodes be identified to cities and links be identified to communication flows between cities. Public authorities may have an incentive to maintain communication between all the cities when some communication flows are broken because of a natural disaster or a strategic attack. Indeed, if some cities are isolated from the others, then it is difficult for the public authorities to rescue inhabitants of these cities.

Sub-game Perfect Equilibrium (SPE). An SPE is a pair $(S^D_\star(N), S^A_\star(S^D_\star(N)))$ that prescribes the following strategic choices. At Stage 2, A plays a best response $S^A_\star(S^D(N))$ to $S^D(N)$:

$$S^A_\star(S^D(N)) \in \underset{S^A \subseteq E^D_{NP}(N) \cup E^D_P(N)}{\text{argmax}} \{\Pi^A(S^D(N), S^A)\}.$$

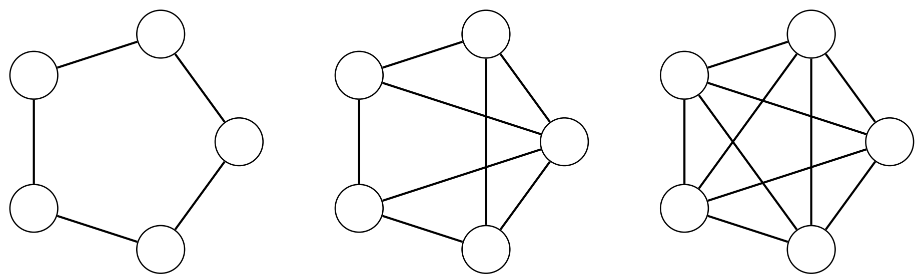
Note that $S^A_\star(S^D(N)) \subseteq E_{NP}$ since attacks cannot remove protected links. Let $g^R_\star(S^D(N))$ be the residual network obtained when D plays strategy $S^D(N)$ and A plays a best response to $S^D(N)$, that is $S^A_\star(S^D(N))$. Given the best response outcome $g^R_\star(S^D(N))$, D achieves payoff $\phi(g^R_\star(S^D(N))) -$

[14] In particular, note that the strategy $\emptyset$ for the Adversary leads to a payoff that equals zero. If $1/c_A$ is an integer, then there may exist a strategy such that $\# E^A = 1/c_A = k$ that disconnect the network. That strategy also has a payoff that equals zero and is chosen by the Adversary according to the tie-breaking rule.

$c^D(S^D(N))$ when choosing $S^D(N)$. By noting $\mathfrak{G}(N)$ the set of all protected networks with nodes in N, then, at stage 1, D plays $S^D_\star(N)$ such that

$$S^D_\star(N) \in \underset{S^D \in \mathfrak{G}(N)}{\text{argmax}} \{\Pi^D(S^D, S^A_\star(S^D))\}.$$

Specific architectures. The *empty network* is the network which contains no links. A *tree* is a connected and acyclic network. A network g which contains n nodes is a (κ, n)-link-connected network if any subnetwork g' obtained from g by removing $\kappa - 1$ non-protected links is connected, and there exists a subnetwork g' obtained from g by removing κ non-protected links that is not connected. Let $\mathcal{G}(\kappa, n)$ be the set of (κ, n)-link-connected networks with n nodes *and no protected links* which are minimal, i.e. if $g \in \mathcal{G}(\kappa, n)$, then there does not exist a (κ, n)-link-connected network, g', such that $\#E(g') < \#E(g)$. It is easy to see that every node i of a network $g \in \mathcal{G}(\kappa, n)$ satisfies $d_i(g) \geq \kappa$, as otherwise it could be separated by removing all links incident to i. Consequently, the number of links in a minimal (κ, n)-link-connected network is at least $\lceil n\kappa/2 \rceil$. As was shown by Harary [15], this condition is also sufficient. The proof of this result is constructive $-$ Harary describes how to obtain a family of solution graphs. The minimal (κ, n)-link-connected networks described by Harary are called (κ, n)*-Harary-networks*. To give the reader some idea of what (κ, n)-Harary-networks look like, we provide some examples in Figure 2 with 5 nodes. For full description of the construction the interested reader is referred to Harary [15].



(i) $(2, 5)$-Harary-network (ii) $(3, 5)$-Harary-network (iii) $(4, 5)$-Harary-network

Figure 2: Examples of (κ, n)-Harary-networks.

Specific strategies. We define specific strategies that play a crucial role in the rest of the paper. Strategy $(N, \emptyset, \emptyset)$ is the strategy where D forms the *empty* network. Let $S^D_{p,k}$ be the set of strategies where D designs a network that (i) cannot be disconnected by k attacks, (ii) contains p protected links, and (iii) contains a minimal number of non-protected links. Strategies which belong to the same set induce the same costs for D. The strategies associated with two specific values of p are:

- $S^D_{0,k}$ is the set of strategies where D designs a network which contains no protected link, and which is a minimal $(k+1, n)$-link-connected network. For instance, $(k+1, n)$-Harary-networks belong to $S^D_{0,k}$.
- $S^D_{n-1,k}$ is the set of strategies where D designs a network which only contains protected links, and such that $(N, E_P, \emptyset)$ is a tree.[15]

3 Model Analysis

Our first result provides, for any number of protected links, the minimal number of non-protected links that the Designer has to form in order to avoid the Adversary to disconnect the network.

To establish the first result, for any pair $(n, k) \in \mathbb{N} \times [\![1, n-3]\!]$, we set $p_1(k, n)$ and $p_2(k, n)$ as follows:

[15]Indeed, if $(N, E_P, \emptyset)$ is a tree and $E_{\text{NP}} = \emptyset$, then $S^D_{n-1,k}$ cannot be disconnected by k attacks. Otherwise $-$ that is if $(N, E_P, \emptyset)$ contains a cycle $-$ the network is not connected unless $\#E_{\text{NP}} > 0$. Therefore, that network does not satisfy condition (iii) and thus is not in $S^D_{n-1,k}$.

$$\begin{cases} \Delta & = (3k+5)^2 - 8n(k+1) \\ p_1(k,n) = & \left\lfloor \dfrac{4\,n - 3\,k - 5 - \sqrt{\Delta}}{8} \right\rfloor + 1 \quad \text{if } \Delta \geq 0, \qquad \text{and } -1 \text{ otherwise,} \\ p_2(k,n) = & \left\lceil \dfrac{4\,n - 3\,k - 5 + \sqrt{\Delta}}{8} \right\rceil - 1 \quad \text{if } \Delta \geq 0, \qquad \text{and } -1 \text{ otherwise.} \end{cases}$$

When no confusion is possible, we let $p_1 = p_1(k,n)$ and $p_2 = p_2(k,n)$.

Proposition 1 *Suppose that A attacks exactly k links in an optimal way. Let $n_1(p,k) = \left\lceil \frac{(n-p)(k+1)]}{2} \right\rceil$ and $n_2(p,k) = (n-2p)\left(k+1-\frac{n-2p-1}{2}\right)$. For $S^D \in S^D_{p,k}$,*

$$\#E_{NP}(S^D) = \begin{cases} n_1(p,k), & \textit{for } p \in [\![0, n-2]\!] \setminus [\![p_1(k,n), p_2(k,n)]\!], \\ n_2(p,k), & \textit{for } p \in [\![0, n-2]\!] \cap [\![p_1(k,n), p_2(k,n)]\!], \\ 0, & \textit{for } p = n-1. \end{cases} \tag{5}$$

So, if D forms p protected links and A attacks k links, then the optimal cost function associated with the pair (p,k) is

$$C^\star(p,k) = \#E_{NP}(S^D)\; c_L + pc_P, \;\; \textit{with } S^D \in S^D_{p,k}. \tag{6}$$

Proof The proof is given in Appendix. □

Let us provide the intuition of Proposition 1. If D forms $n-1$ protected links, then there exists a set of strategies for D that allows to resist k attacks without non-protected links, $S^D_{n-1,k}$. Otherwise, let D form $p \in [\![0, n-2]\!]$ protected links and build a protected network $S^D = (N, E_P, E_{\mathrm{NP}})$ in $S^D_{p,k}$.

First, $(N, E_P, \emptyset)$ is acyclic. Indeed, if S^D contains a cycle, then there exists a protected link, say ij, that can be removed without altering the fact that S^D resists k attacks. Hence, it is be possible for D to remove the protected link ij and replace a non-protected link $i'j'$ by the protected link $i'j'$, and so reduce the number of non-protected links.

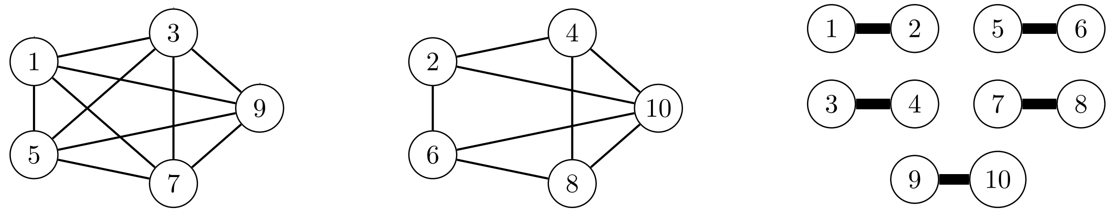
Second, consider the following sequence of networks: $g_0 = (N, \emptyset, \emptyset)$ and for any ℓ in $[\![1, p]\!]$, $g_\ell = (N, E_P(g_{\ell-1}) \cup ij, \emptyset)$ for some $ij \in E_P \backslash E_P(g_{\ell-1})$. Hence $g_p = S^D$. Since there is no cycle in $(N, E_P, \emptyset)$, then, for any ℓ, the extra link of $E_P(g_\ell)$ allows to merge two components of $E_P(g_{\ell-1})$. Since g_0 has n components, then by an immediate recurrence, $(N, E_P, \emptyset)$ has exactly $n-p$ components.

Third, observe that each component of $(N, E_P, \emptyset)$ has to be incident to at least $k+1$ non-protected links, otherwise A can disconnect the protected network S^D with k attacks. Since there are $n-p$ such components, $n_1(p,k)$ is the minimal number of non-protected links to form. There exist situations (that is some values of p and k) for which $n_1(p,k)$ non-protected links are sufficient to resist k attacks. We illustrate this type of situations in the following example.

Example 1 Suppose $N = [\![1, 10]\!]$, $k = 6$ and $p = 5$. We describe a strategy S^D where $\#E_{\mathrm{NP}}(S^D) = n_1(5,6)$. Consider the networks of Figure 3 and S^D the protected network such that g', g'', g''' are all subnetworks of S^D, with $E_P(S^D) = E_P(g''')$ and $E_{\mathrm{NP}}(S^D) = E_{\mathrm{NP}}(g') \cup E_{\mathrm{NP}}(g'')$. Note that subnetwork g' is a complete network and g'' is a $(3,5)$-Harary-network. Finally, we observe that each component in $(N, E_P, \emptyset)$ is incident to at least 7 non-protected links and there is no possibility for A to disconnect S^D with 6 attacks.

Recall that each component of $(N, E_P, \emptyset)$ has to be incident to at least $k+1$ non-protected links in order to resist k attacks. This fact implies that there exist some situations where $S^D_{p,k}$ contains $n_2(p,k)$ non-protected links. We illustrate these situations through the following example.

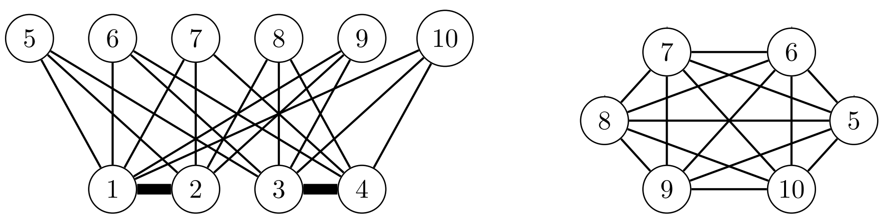
Example 2 Suppose $N = [\![1, 10]\!]$, $k = 7$, and $p = 2$. We assume that D forms a protected network S^D in $S^D_{p,k}$, with a protected link between nodes 1 and 2 and between nodes 3 and 4. Then S^D contains 8 components. We denote by $\mathcal{C}$ the component which contains nodes 1 and 2, and by $\mathcal{C}'$ the component which contains nodes 3 and 4. Each of these components has to be incident to 8 non-protected links. Networks g' and g'' given in Figure 4 are subnetworks of S^D, with $E_P(S^D) = E_P(g')$ and $E_{\mathrm{NP}}(S^D) =$



(i) Subnetwork g' (ii) Subnetwork g'' (iii) Subnetwork $g''' = (N, E_P, \emptyset)$

Figure 3: Subnetworks associated with Example 1: the solution graph is (N, E_P, E_{NP}) with $E_{\text{NP}} = E_{\text{NP}}(g') \cup E_{\text{NP}}(g'')$.

$E_{\text{NP}}(g') \cup E_{\text{NP}}(g'')$. Each node $a \in [\![5, 10]\!]$ can form at most 5 non-protected links with each other. Hence each of them has to form at least 3 non-protected links with nodes in $[\![1, 4]\!]$. Altogether, nodes 5, 6, 7, 8, 9 and 10 have to form a total of 18 non-protected links with nodes 1, 2, 3 and 4. As for components $\mathcal{C}$ and $\mathcal{C}'$, they should be incident to a total of 16 non-protected links. Hence, D cannot form a protected network with $n_1(2, 7)$ protected links that resists k attacks. More precisely, a network that cannot be disconnected by A with 7 attacks has to contain 33 non-protected links while $n_1(2, 7) = 32$.



(i) Subnetwork $g' = (N, E_P, E'_{\text{NP}})$ (ii) Subnetwork $g'' = (N, \emptyset, E''_{\text{NP}})$

Figure 4: Subnetworks associated with Example 2. The solution is (N, E_P, E_{NP}) with $E_{\text{NP}}(g) = E'_{\text{NP}} \cup E''_{\text{NP}}$.

In Example 2, we have assumed that D uses protected links to form two components of size two in $(N, E_P, \emptyset)$. Now, assume instead that $(N, E_P, \emptyset)$ has a unique component. For instance, D forms protected links between nodes 1 and 2 and between nodes 2 and 3. Then, each node $a \in [\![4, 10]\!]$ can form at most 6 non-protected links with other nodes in $[\![4, 10]\!]$. Consequently, since $k = 7$ each node $a \in [\![4, 10]\!]$ has to form at least 2 non-protected links with nodes in $[\![1, 3]\!]$. The component which contains nodes 1, 2 and 3 is incident to 14 non-protected links. It follows that D forms $21 + 14 = 35$ non-protected links instead of 33 links in Example 2. This example illustrates how D designs $(N, E_P, \emptyset)$ in order to minimize the costs of forming links.

We now generalize Example 2 to provide some intuition for $p_1(k, n)$ and $p_2(k, n)$. Consider a protected network in $S^D_{p,k}$ where each component of $(N, E_P, \emptyset)$ contains either one or two nodes. There are thus $n - 2p$ components of size 1 and p components of size 2. We observe that components which contain one node need to be incident to at least $k + 1$ non-protected links. Since the number of links between components which contain 1 node is at most $(n - 2p)(n - 2p - 1)/2$, then the total number of non-protected links between components which contain one node and those which contain two nodes is at least equal to $(n - 2p)((k + 1) - (n - 2p - 1)/2)$. Moreover, to minimize the number of links, the total number of non-protected links incident to components which contain two nodes should be equal to $(k + 1)p$. Equation $(n - 2p)((k + 1) - (n - 2p - 1)/2) = (k + 1)p$ is satisfied for x_1 and x_2. Since $p_1(k, n) = \lfloor x_1 \rfloor + 1$ and $p_2(k, n) = \lceil x_2 \rceil - 1$, the number of non-protected links required to resist k attacks is given by $n_2(p, k)$ when $p \in [\![p_1(k, n), p_2(k, n)]\!]$.

We now characterize the SPE according to the costs of forming protected and non-protected links.

Proposition 2 *Let the payoff functions be given by Equations* (3) *and* (4), *and let* $(S^D_\star(N), S^A_\star(S^D_\star(N)))$ *be an SPE.*[16]

1. *Suppose that* $n(k+1) = 0 \mod 2$.
   (a) *If* $\frac{c_P}{c_L} < \left(\frac{n}{n-1}\right)\left(\frac{k+1}{2}\right)$, *then* $S^D_\star(N) \in S^D_{n-1,k}$ *and* $S^A_\star(S^D_\star(N)) = \emptyset$.
   (b) *If* $\left(\frac{n}{n-1}\right)\left(\frac{k+1}{2}\right) < \frac{c_P}{c_L}$, *then,* $S^D_\star(N) \in S^D_{0,k}$ *and* $S^A_\star(S^D_\star(N)) = \emptyset$.
2. *Suppose that* $n(k+1) = 1 \mod 2$.
   (a) *If* $\frac{c_P}{c_L} < \left(\frac{n-1}{n-2}\right)\left(\frac{k+1}{2}\right)$, *then,* $S^D_\star(N) \in S^D_{n-1,k}$ *and* $S^A_\star(S^D_\star(N)) = \emptyset$.
   (b) *If* $\left(\frac{n-1}{n-2}\right)\left(\frac{k+1}{2}\right) < \frac{c_P}{c_L} < \frac{k+2}{2}$, *then* $S^D_\star(N) \in S^D_{1,k}$ *and* $S^A_\star(S^D_\star(N)) = \emptyset$.
   (c) *If* $\frac{k+2}{2} < \frac{c_P}{c_L}$, *then* $S^D_\star(N) \in S^D_{0,k}$ *and* $S^A_\star(S^D_\star(N)) = \emptyset$.

Proof Note that either $p_1(k,n) = p_2(k,n) = -1$, or $p_1(k,n) \geq \lfloor\frac{4n-3k-5}{8}\rfloor + 1 \geq \lfloor\frac{4n-3(n-3)-5}{8}\rfloor + 1 \geq \lfloor\frac{n-4}{8}\rfloor + 1 \geq 1$. Therefore, from Proposition 1, if D builds a protected network in $S^D_{0,k}$, then its cost equals $C^\star(0,k) = n_1(0,k)c_L$. Similarly, by Proposition 1, $C^\star(n-1,k) = (n-1)c_P$. Moreover, by Lemma 3, we know that $n_2(p,k) \geq n_1(p,k)$ when $p \in [\![p_1(k,n), p_2(k,n)]\!]$. Hence by Proposition 1, for all $p \in [\![1, n-2]\!]$, $C^\star(p,k) \geq n_1(p,k)c_L + pc_P$. We prove successively the two parts of the proposition.

1. Let $n(k+1) = 0 \mod 2$. For $p \in [\![1, n-2]\!]$, $C^\star(p,k) - C^\star(0,k) \geq p(c_P - c_L(k+1)/2) > 0$, if $c_P/c_L > (k+1)/2$. Moreover, $C^\star(n-1,k) - C^\star(0,k) > 0$ if $c_P/c_L > \left(\frac{n}{n-1}\right)\left(\frac{k+1}{2}\right)$. Assume that $c_P/c_L > \left(\frac{n}{n-1}\right)\left(\frac{k+1}{2}\right)$. Then, $c_P/c_L > (k+1)/2$ and $S^D_\star(N) \in S^D_{0,k}$.

   For $p \in [\![1, n-2]\!]$, $C^\star(p,k) - C^\star(n-1,k) > 0$ if $c_P/c_L < \left(\frac{n-p}{n-p-1}\right)\left(\frac{k+1}{2}\right)$. Assume that $c_P/c_L < \left(\frac{n}{n-1}\right)\left(\frac{k+1}{2}\right)$. Then, $c_P/c_L < \left(\frac{n-p}{n-p-1}\right)\left(\frac{k+1}{2}\right)$ for all $p \in [\![1, n-2]\!]$. Consequently, $S^D_\star(N) \in S^D_{n-1,k}$. By assumption, $C^\star(0,k) < 1$ and $C^\star(n-1,k) < 1$. It follows that D has an incentive to build a protected network in $S^D_{0,k}$ or $S^D_{n-1,k}$ since she obtains benefits equal to 1. Hence, A cannot disconnect g^R with k attacks. Consequently, $S^A_\star(S^D_\star(N)) = \emptyset$.

2. Let $n(k+1) = 1 \mod 2$. By Proposition 1, we know that:
   $C^\star(0,k) - C^\star(1,k) > 0$ if $c_P/c_L < (k+2)/2$,
   $C^\star(1,k) - C^\star(n-1,k) > 0$ if $c_P/c_L < \left(\frac{n-1}{n-2}\right)\left(\frac{k+1}{2}\right)$, and
   $C^\star(0,k) - C^\star(n-1,k) > 0$ if $c_P/c_L < \frac{1}{n-1}\left\lceil\frac{n(k+1)}{2}\right\rceil$.
   By using the same argument as in the previous point, we establish that for $p \in [\![2, n-2]\!]$, if $S^D(N) \in S^D_{p,k}$, then $S^D(N) \notin S^D_\star(N)$. Assume that $c_P/c_L < \left(\frac{n-1}{n-2}\right)\left(\frac{k+1}{2}\right)$. Then $c_P/c_L < \left(\frac{n}{n-1}\right)\left(\frac{k+1}{2}\right)$ and $S^D_\star(N) \in S^D_{n-1,k}$. Assume that $c_P/c_L > \left(\frac{n-1}{n-2}\right)\left(\frac{k+1}{2}\right)$. There are two possible cases. If $c_P/c_L < (k+2)/2$, then $C^\star(1,k) - C^\star(n-1,k) < 0$ and $C^\star(1,k) - C^\star(0,k) < 0$, and $S^D_\star(N) \in S^D_{1,k}$. If $c_P/c_L > (k+2)/2$, then $C^\star(0,k) - C^\star(n-1,k) < 0$ and $C^\star(0,k) - C^\star(1,k) < 0$, and $S^D_\star(N) \in S^D_{0,k}$. Since D has an incentive to build a network in $S^D_{0,k}$, $S^D_{1,k}$ or $S^D_{n-1,k}$, A cannot disconnect g^R with k attacks. Consequently, $S^A_\star(S^D_\star(N)) = \emptyset$.

□

Let us provide the intuition of Proposition 2. Note that if D builds a protected network that A cannot disconnect with k attacks (which is the maximal number of attacks that A has an incentive to choose),

[16]The case of equality follows the same pattern, that is:

1. If $n(k+1) = 0 \mod 2$ and $\frac{c_P}{c_L} = \left(\frac{n}{n-1}\right)\left(\frac{k+1}{2}\right)$, then $S^D_\star(N) \in S^D_{n-1,k} \cup S^D_\star(N)$ and $S^A_\star(S^D_\star(N)) = \emptyset$.
2. Suppose that $n(k+1) = 1 \mod 2$.
   (a) If $\frac{c_P}{c_L} = \left(\frac{n-1}{n-2}\right)\left(\frac{k+1}{2}\right)$, then $S^D_\star(N) \in S^D_{n-1,k} \cup S^D_{1,k}$ and $S^A_\star(S^D_\star(N)) = \emptyset$.
   (b) If $\frac{k+2}{2} = \frac{c_P}{c_L}$, then $S^D_\star(N) \in S^D_{0,k} \cup S^D_{1,k}$ and $S^A_\star(S^D_\star(N)) = \emptyset$.

then A attacks no link in an SPE since each attack is costly. Due to the costs of forming protected and non-protected links, in an SPE D always has an incentive to build a protected network that A cannot disconnect.

We now compare the costs of protected networks that belong to different sets $S^D_{p,k}$, $p \in [\![0, n-1]\!]$, with respect to the values of c_L and c_P.

First, we consider point 1 of Proposition 2: $n(k+1) = 0 \mod 2$. In Figure 5, we draw lines $(d_{p,p'})$ whose slopes $s_{p,p'}$ can be interpreted as the *average number* of non-protected links that each protected link allows to save between a protected network in $S^D_{p,k}$, $p \in [\![0, n-1]\!]$, and a protected network that belongs to $S^D_{p',k}$, $p' \neq p$. We draw four such lines:

- Line $(d_{0,p'})$ whose slope $s_{0,p'}$ shows the average saving of non-protected link *per protected link* between networks in $S^D_{0,k}$ and those of $S^D_{p',k}$, for $p' \in [\![1, n-2]\!] \setminus [\![p_1, p_2]\!]$, and similarly:
- Line $(d_{0,\tilde{p}})$ of slope $s_{0,\tilde{p}}$ between networks in $S^D_{0,k}$ and in $S^D_{\tilde{p},k}$, with $\tilde{p} \in [\![p_1, p_2]\!]$
- Line $(d_{0,n-1})$ of slope $s_{0,n-1}$ between networks in $S^D_{0,k}$ and in $S^D_{n-1,k}$.
- Line $(d_{n-1,p'})$ of slope $s_{n-1,p'}$ between networks in $S^D_{n-1,k}$ and in $S^D_{p',k}$, with $p' \in [\![1, n-2]\!]$.

Observe that for any $p' \in [\![1, n-2]\!] \setminus [\![p_1, p_2]\!]$ and any $\tilde{p} \in [\![p_1, p_2]\!]$, $\text{abs}(s_{0,\tilde{p}}) < \text{abs}(s_{0,p'})$. Similarly, for any $p' \in [\![1, n-2]\!] \setminus [\![p_1, p_2]\!]$, $\text{abs}(s_{0,p'}) < \text{abs}(s_{0,n-1})$. Moreover, for any $p \in [\![1, n-2]\!]$, $\text{abs}(s_{0,n-1}) < \text{abs}(s_{n-1,p'})$.

Suppose $c_P/c_L > \text{abs}(s_{0,n-1})$. Then, costs of forming links with a strategy in $S^D_{0,k}$ are lower than the costs of forming links with a strategy in $S^D_{n-1,k}$. Moreover, $c_P/c_L > \text{abs}(s_{0,n-1}) > c_P/c_L > \text{abs}(s_{0,p'}) > \text{abs}(s_{0,\tilde{p}})$, with $p' \in [\![1, n-2]\!] \setminus [\![p_1, p_2]\!]$ and $\tilde{p} \in [\![p_1, p_2]\!]$. It follows that the costs of forming links are minimized for strategies in $S^D_{0,k}$. Conversely, suppose $c_P/c_L < \text{abs}(s_{0,n-1})$. Then $c_P/c_L < \text{abs}(s_{n-1,p'})$, for $p' \in [\![1, n-2]\!]$. It follows that the costs of forming links are minimized for strategies in $S^D_{n-1,k}$. Finally, note that $\text{abs}(s_{0,n-1}) = \left(\frac{n}{n-1}\right)\left(\frac{k+1}{2}\right)$.

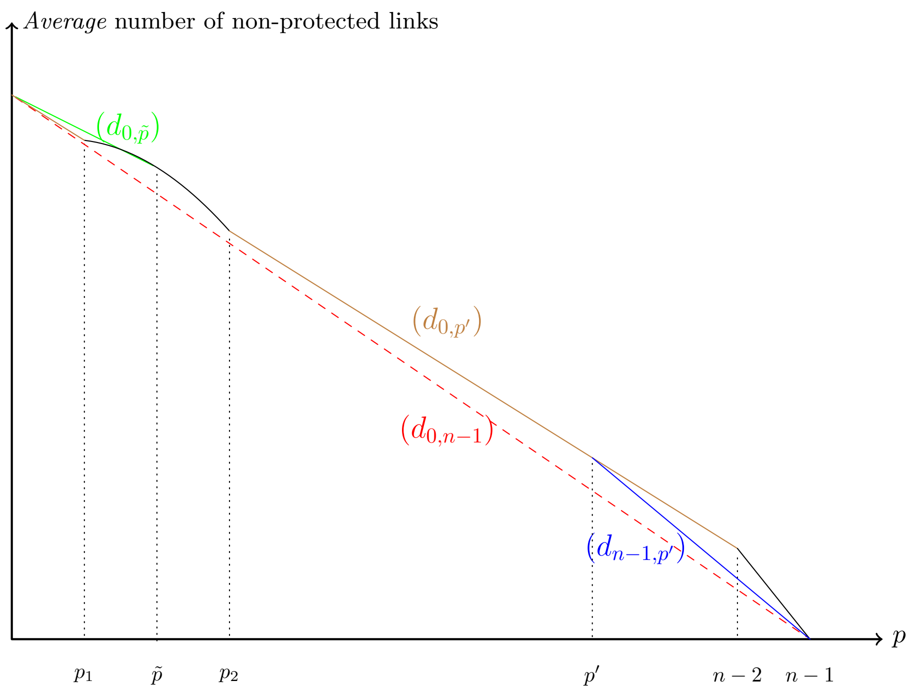


Figure 5: Intuition of Proposition 2 when $n(k+1) = 0 \mod 2$.

We now consider the second part of Proposition 2: $n(k+1) = 1 \mod 2$. The intuition is similar to the situation where $n(k+1) = 0 \mod 2$ except for protected networks which belong to $S^D_{0,k}$ and $S^D_{1,k}$.

Consequently, we focus only on three sets of protected networks: $S^D_{0,k}$, $S^D_{1,k}$ and $S^D_{n-1,k}$. In Figure 6, the slope $s_{0,1}$ of $(d_{0,1})$, can be interpreted as the number of non-protected links that the protected link allows to save between a protected network in $S^D_{0,k}$ and a protected network in $S^D_{1,k}$. The same interpretation is valid for the slope $s_{1,n-1}$ of $(d_{1,n-1})$, which relates a protected network in $S^D_{1,k}$ and a protected network in $S^D_{n-1,k}$, and for the slope $s_{0,n-1}$ of the line $(d_{0,n-1})$, which relates a protected network in $S^D_{0,k}$ and a protected network in $S^D_{n-1,k}$.

Suppose that $c_P/c_L < \text{abs}(s_{1,n-1})$. Then, $c_P/c_L < \text{abs}(s_{0,n-1})$, and networks in $S^D_{n-1,k}$ have a minimal link formation cost. Conversely, suppose that $c_P/c_L > \text{abs}(s_{1,n-1})$. If $k < n-3$,[17] then there are two possibilities. If $c_P/c_L > \text{abs}(s_{0,1})$, then protected networks in $S^D_{0,k}$ minimize the cost of forming links. If $c_P/c_L < \text{abs}(s_{0,1})$, then protected networks in $S^D_{1,k}$ minimize the cost of forming links. Finally, note that $\text{abs}(s_{0,1}) = (k+2)/2$, and $\text{abs}(s_{1,n-1}) = \left(\frac{n-1}{n-2}\right)\left(\frac{k+1}{2}\right)$.

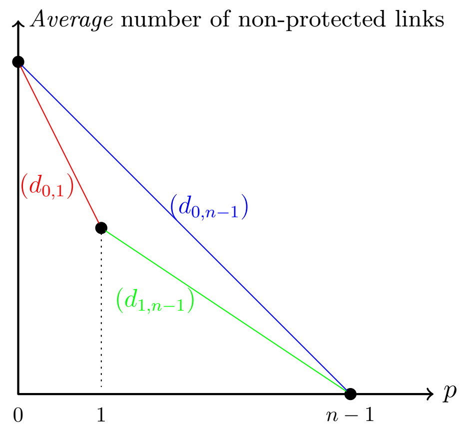


Figure 6: Intuition of Proposition 2 when $n(k+1) = 1 \mod 2$ and $k < n-3$.

We now compare the results obtained in our framework where A attacks links and the results obtained in DG's framework where A attacks nodes (Proposition 1 in [9]). Recall that in DG's paper, the non-empty networks formed by D at equilibrium are either a star network with a protected central node, or a minimal $(k+1,n)$-node-connected network without any protection. Hence, the strategy $(S^D_{0,k}, \emptyset)$ is an SPE when the cost of links (non-protected links in our case) is sufficiently low relatively to the cost of protection in both frameworks.

However, when A attacks nodes, D uses at most one protected node at equilibrium. The role played by the protections is different since in DG, if D builds a star network, one protection is sufficient to protect the network and resist any attack of A. In constrast, in our framework, D may use more than one protected link in an SPE: indeed, when the cost of protected links is sufficiently low relative to the cost of non-protected links, D designs a $(1,n)$-link-connected network which contains only protected links. Protecting a network under link-attack is thus more costly than protecting a network under node-attack. This is because our framework calls for the survival of every node, a requirement which does not hold in the DG's framework.

4 Limited number of protected links and imperfectly protected links

In this section, we consider two potential types of restrictions on the protection of the network for the Designer. First, we consider the case where D can only use a limited number of protected links and we focus on situations where this number is smaller than $p_2(k,n)$. Then, we consider situations where links

[17]For $k = n-3$, $\frac{k+2}{2} = \left(\frac{n-1}{n-2}\right)\left(\frac{k+1}{2}\right)$, and thus the case 2.(b) of Proposition 2 never occurs.

are imperfectly protected and can be removed by the Adversary with some a priori known probability $\pi \in (0, 1)$.

4.1 Limited number of protected links

We examine a situation where the maximal number of protected links, $\bar{p}$, that D can form is strictly smaller than $n-1$. More precisely, we are interested in the situation where $\bar{p} \in [\![p_1(k,n), p_2(k,n)]\!]$.[18]

Proposition 2 establishes that for $n(k+1) = 0 \mod 2$, there exists no SPE in which D uses both protected and non-protected links. In contrast, when $\bar{p} \in [\![p_1, p_2]\!]$, there exist situations where the SPE are of the form $(S^D, \emptyset)$ with $S^D \in S^D_{p_1-1,k}$, that is networks which contain both protected and non-protected links when $p_1 > 1$.[19] The following proposition gives a condition on the values of the parameters upon which such situations arise. To simplify the analysis, we restrict our attention to situations where $k+1$ is even.

Proposition 3 *Assume that* $n(k+1) = 0 \mod 2$. *Assume further that* $p_2 - p_1 \geq 2$ *and that the maximal number of protected links is* $\bar{p} \in [\![p_1, p_2]\!]$. *Let* $(S^D_\star(N), S^A_\star(S^D_\star(N)))$ *be an SPE. There exists* $\varepsilon > 0$ *such that if* $(k+1)/2 - \varepsilon < c_P/c_L < (k+1)/2$, *then* $S^D_\star(N) \in S^D_{p_1-1,k}$ *and* $S^A_\star(S^D_\star(N)) = \emptyset$.

Proof If $p_1 > 1$, for all $p \in [\![1, p_1 - 1]\!]$, $C^\star(0,k) - C^\star(p,k) = p(\frac{k+1}{2}c_L - c_P)$. If $c_P/c_L < (k+1)/2$, then $C^\star(0,k) - C^\star(p,k) \geq \left(\frac{k+1}{2}(p-p)\right)c_L = 0$. Therefore, $\underset{p \in [\![0,p_1-1]\!]}{\arg\min}\ \{C^\star(p,k)\} = p_1 - 1$.

Now, let $p \in [\![p_1, \bar{p}]\!]$. We have $C^\star(p,k) - C^\star(p_1-1,k) = (p - p_1 + 1)c_P + \Big(n_2(p,k) - n_1(p_1-1,k)\Big)c_L$. Consider $\varepsilon = \frac{k+1}{2} - \max_{p \in [\![p_1,\bar{p}]\!]}\{\frac{n_1(p_1-1,k)-n_2(p,k)}{p-p_1+1}\}$. If $(k+1)/2 - \varepsilon < c_P/c_L$, then $C^\star(p,k) - C^\star(p_1 - 1,k) \geq \Big((p - p_1 + 1)((k+1)/2 - \varepsilon) + n_2(p,k) - n_1(p_1-1,k)\Big)c_L \geq \Big((p-p_1+1)\frac{n_1(p_1-1,k)-n_2(p,k)}{p-p_1+1} + n_2(p,k) - n_1(p_1-1,k)\Big)c_L = 0$. Thus, $\underset{p \in [\![p_1-1,\bar{p}]\!]}{\arg\min}\ C^\star(p,k) = p_1 - 1$.

Since $[\![p_1 - 1, \bar{p}]\!]$ is a discrete non empty set, then ε is defined. It remains to show that $0 \leq \varepsilon \leq \frac{k+1}{2}$.

Note that by construction the number of non-protected links in any protected network of $S^D_{p,k}$ is non-increasing with p. Hence $n_1(p_1-1,k) \geq n_2(p,k)$. Thus, for all $p \in [\![p_1, \bar{p}]\!]$, $\frac{k+1}{2} - \varepsilon \geq \frac{n_1(p_1-1,k)-n_2(p,k)}{p-p_1+1} \geq 0$ which leads to $\varepsilon \leq \frac{k+1}{2}$.

Further, let $p \in [\![p_1, \bar{p}]\!]$. By Lemma 3, $n_1(p,k) \leq n_2(p,k)$. Thus

$$\frac{n_1(p_1-1,k) - n_2(p,k)}{p-p_1+1} \leq \frac{n_1(p_1-1,k) - n_1(p,k)}{p-p_1+1} \leq \frac{(k+1)(n-p_1+1-n+p)}{2(p-p_1+1)} = \frac{k+1}{2}.$$

Since this holds for any $p \in [\![p_1, \bar{p}]\!]$, then $\varepsilon \leq \frac{k+1}{2}$.

Finally, since (i) $S^D_\star(N) \in S^D_{p_1-1,k}$, (ii) any network in $S^D_{p_1-1,k}$ cannot be disconnected with k attacks and (iii) attacks are costly for A, then $S^A_\star(S^D_\star(N)) = \emptyset$. □

We illustrate this result with an example:

Example 3 Suppose that $n = 31$ and $k = 27$. Then, $p_1 = 3$ and $p_2 = 7$. Let $\bar{p} = 6$. Figure 7 shows the number of non-protected links in networks of $S^D_{p,k}$ as a function of the number of protected links (large dots in red).

Let us consider the following function:

$$\widetilde{E_{\text{NP}}} : x \mapsto \begin{cases} \widetilde{n_1}(x) = \dfrac{(n-x)(k+1)}{2} & \text{if } x \in [0, p_1) \cup (p_2, n-2], \\ \widetilde{n_2}(x) = (n-2x)\left(k+1-\dfrac{n-2x-1}{2}\right) & \text{otherwise.} \end{cases}$$

Note that since $k+1 = 0 \mod 2$, then for any $p \in [\![1, n-2]\!]$ we have $\widetilde{n_1}(p) = n_1(p,k)$, $\widetilde{n_2}(p) = n_2(p,k)$ and $\widetilde{E_{\text{NP}}}(p) = \#E_{\text{NP}}(S)$ for any $S \in S^D_{p,k}$. Therefore the functions $\widetilde{n_1}$, $\widetilde{n_2}$ and $\widetilde{E_{\text{NP}}}$ can be interpreted as

[18] This interval is non empty if $p_2 > p_1$ which implies that $\Delta > 0$, that is $n < \frac{(3k+5)^2}{8(k+1)}$.

[19] Indeed, note that $p_1 \leq \frac{4n}{8} + 1 \leq n - 1$ since $n \geq 4$ and therefore $S^D_{p_1-1,k}$ always contains non-protected links.

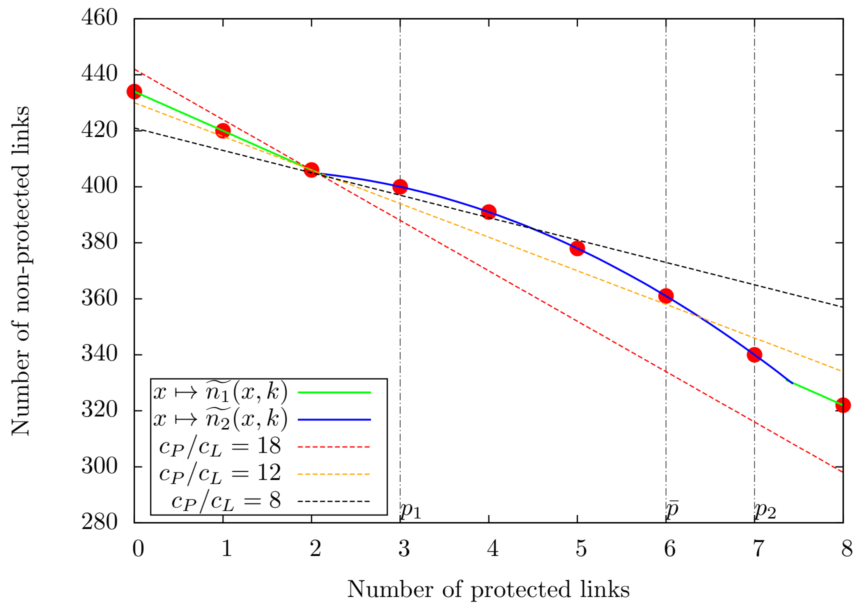


Figure 7: Example with $n = 31$, $k = 27$, $p_1 = 3$, $p_2 = 7$. The optimal strategy for D depends on the value of c_P/c_L.

the natural continuous extensions of $p \mapsto n_1(p,k)$, $p \mapsto n_2(p,k)$ and $p \mapsto \#E_{\text{NP}}(S \in S^D_{p,k})$ respectively. The functions $\widetilde{n_1}$ and $\widetilde{n_2}$ are ploted in green and blue plain lines respectively in Figure 7.

Let us consider a value of c_P and c_L. The lines of equal costs (isocost lines) have slope c_P/c_L, thus have equations of the type

$$y = c\, c_L - p\, c_P, \tag{Iso}$$

with $c\, c_L$ being the y-coordinate of the y-intercept. The value of c corresponds to the associated cost for the Designer normalized by c_L.

The optimal cost for D corresponds to the smallest value c such that the associated element of (Iso) intersects with the plot of $\widetilde{E_{\text{NP}}}$ for some integer value p. The optimal strategies of D are then the networks of $S^D_{p,k}$.

Figure 7 shows situations for three different values of c_P/c_L, namely 18, 12 and 8 (in dashed lines):

For large values of c_P/c_L (value 18 in Figure 7) the slope of the line of (Iso) is larger (in absolute value) than that of $\widetilde{n_1}$ and thus the optimal strategy for D is obtained with $p = 0$ protected links. In other words, if $c_P/c_L \geq \frac{k+1}{2}$ then the optimal strategies for D are the networks of $S^D_{0,k}$.

For small values of c_P/c_L (value 8 in Figure 7) the slope of the line (Iso) is low, hence favoring strategies with maximal values of p. In Figure 7, one can see that for $c_P/c_L = 8$, the optimal strategy for the Designer is obtained for $p = \bar{p}$ protected links.

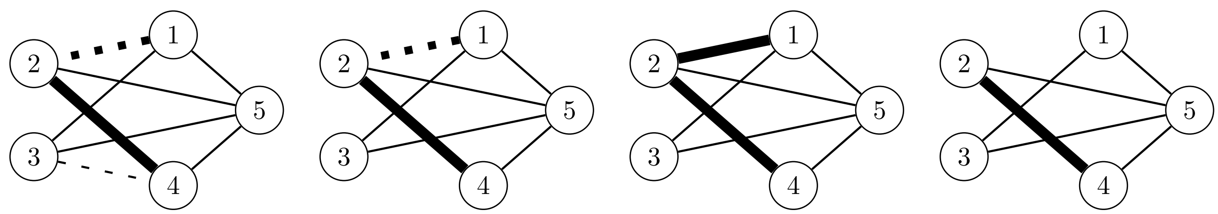
For intermediate values of c_P/c_L (value 12 in Figure 7) the optimal strategy for the Designer is obtained when using $p_1 - 1$ protected links, which is the inflection point of $\widetilde{E_{\text{NP}}}$.

DG [9] show that when A attacks nodes, there exist situations where the optimal strategy of D is a star network with a protected central node. In this case, D uses both node protections and link creations to protect her network. In our framework, D may use both protected and non-protected links to protect her network if the number of protected links available to D belongs to $[\![p_1, p_2]\!]$. This result is a consequence of the discontinuity in the number of non-protected links that each protected link allows the Designer to save.

4.2 Imperfectly protected links

We now assume that each protected link has a probability $\pi \in (0,1)$ to be removed when it is attacked by A. Let $g = (N, E_P, E_{\rm NP})$ be an (imperfectly) protected network, and E^A an attack over the links of g. In the benchmark model, g^R is obtained by removing the links of $E_{\rm NP}$ that are targeted by A, i.e. $g^R = (N, E_P, E_{\rm NP} \backslash E^A)$. Now, a realization of the attack is a subnetwork of g^R of the form $g^\varrho = (N, E_\varrho, E_{\rm NP}\backslash E^A)$ with $E_\varrho \subseteq E_P \backslash E^A$. We illustrate these networks in the following example.

Example 4 Suppose $N = [\![1,5]\!]$, $E_{\rm NP}(g) = \{13, 15, 25, 34, 35, 45\}$, $E_P(g) = \{12, 24\}$ and $E^A = \{12, 34\}$. The subnetwork g^R obtained when removing the non-protected links that are attacked (i.e. $E^A \cap E_{\rm NP}$), is drawn in Figure 8(ii). The two possible realizations are drawn in Figures 8(iii) and 8(iv). Note that g_1^ϱ occurs with probability $1-\pi$, and g_2^ϱ occurs with probability π.



(i) g and attacked links (E^A, in dashed lines) (ii) Subnetwork g^R (iii) A possible realization g_1^ϱ (iv) The other realization g_2^ϱ

Figure 8: Networks of Example 4. Thick lines represent protected links and dashed lines represent links that are targetted by A.

Let g^ϱ be a realization and $\lambda(g^\varrho | g^R, E^A)$ be the probability that g^ϱ is realized given g^R and E^A. We have

$$\lambda(g^\varrho | g^R, E^A) = \prod_{\substack{ij \in E_P(g^\varrho), \\ ij \in E_P^D \cap E^A}} (1-\pi) \prod_{\substack{ij \notin E_P(g^\varrho), \\ ij \in E_P^D \cap E^A}} \pi.$$

The expected benefits obtained by D, $\mathbb{E}\phi(g^R, E^A)$, when she chooses strategy g and A chooses E^A is

$$\mathbb{E}\phi(g^R, E^A) = \sum_{g^\varrho = (N, E_\varrho, E_{\rm NP}^D \backslash E^A)} \lambda(g^\varrho | g^R, E^A)\phi(g^\varrho).$$

We assume that the costs incurred by D when she chooses a strategy are given by Equation (2). The expected payoffs obtained by D, $\mathbb{E}\Pi^D$, is the difference between the expected benefits and the costs of forming protected and non-protected links:

$$\mathbb{E}\Pi^D = \mathbb{E}\phi(g^R, E^A) - c^D(S^D(N)). \tag{7}$$

Finally, the expected payoffs obtained by A are

$$\mathbb{E}\Pi^A = 1 - \mathbb{E}\phi(g^R, E^A) - c^A \# E^A. \tag{8}$$

Recall that $c^A \geq 1/(n-3)$.

Proposition 4 *Let the payoff functions be given by Equations* (7) *and* (8). *Suppose that* $\pi < c_A$. *Then, results provided in Proposition 2 are preserved.*

Proof Let $S_\star^D(N) = (N, E_P, E_{\rm NP})$ and let $S_\star^A(S_\star^D(N)) = E^A$ with $E^A = (E_P^A, E_{\rm NP}^A)$, where $E_P^A \subseteq E_P$ and $E_{NP}^A \subseteq E_{\rm NP}$.

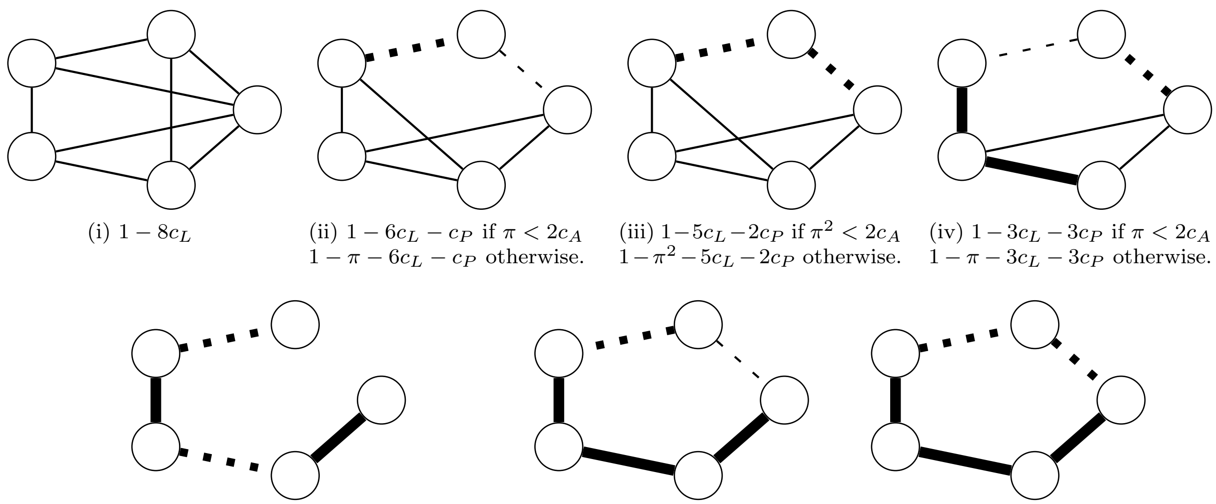
If A can disconnect the protected network $S_\star^D(N) = (N, E_P, E_{\rm NP})$ with strategy $E^A = (\emptyset, E_{\rm NP}^A)$, then since attacks are costly, her best response is to not attack any protected links, i.e. $E_P^A = \emptyset$.

If A cannot disconnect the protected network $S_\star^D(N) = (N, E_P, E_{\rm NP})$ with strategy $(\emptyset, E_{\rm NP}^A)$, then the strategy $(E_P^A, E_{\rm NP}^A)$ should disconnect the network with a strictly positive probability (otherwise

A would not be playing a best response). The highest probability to disconnect network (N, E_P, E_{NP}) occurs when the deletion of any protected link implies that (N, E_P, E_{NP}) is disconnected. This probability is $1-(1-\pi)^{\#E_P^A}$. Since $1-\pi \in (0,1)$, by Taylor's expansion $1-(1-\pi)^{\#E_P^A} = \sum_{\ell=0}^{\infty}\binom{\#E_P^A}{\ell}(-1)^{\ell+1}\pi^{\ell}$. Then by Leibniz's rule on alternating series $1-(1-\pi)^{\#E_P^A} \leq \pi\#E_P^A$. Hence, strategy E^A disconnects the network with a probability lower or equal to $\pi\#E_P^A$. So, the expected benefits associated with E^A are at most $\pi\#E_P^A$ with a cost of at least $c_A\#E_P^A$. Hence, if $E_P^A \neq \emptyset$, then the payoff associated to strategy E^A is (strictly) negative and thus is not a best response. Therefore $E_P^A = \emptyset$.

Since no optimal strategy of A targets any protected link, the situation is equivalent to the one examined in Proposition 2. □

Proposition 4 examines situations where probability π is low relative to c_A, the cost of attacking links. We now examine other situations through an example.



(i) $1-8c_L$

(ii) $1-6c_L-c_P$ if $\pi<2c_A$; $1-\pi-6c_L-c_P$ otherwise.

(iii) $1-5c_L-2c_P$ if $\pi^2<2c_A$; $1-\pi^2-5c_L-2c_P$ otherwise.

(iv) $1-3c_L-3c_P$ if $\pi<2c_A$; $1-\pi-3c_L-3c_P$ otherwise.

(v) $1-4c_P$ if $\pi(1-\pi/2)<c_A$; $1-\pi-4c_P$ if $\pi(1-\pi)<c_A\leq\pi(1-\pi/2)$; $(1-\pi)^2-4c_P$ if $c_A\leq\pi(1-\pi)$

(vi) $1-c_L-4c_P$ if $\pi<2c_A$; $1-\pi-c_L-4c_P$ otherwise.

(vii) $1-5c_P$ if $\pi^2<2c_A$; $1-\pi^2-5c_P$ otherwise.

Figure 9: Networks of Example 5: $n=5$, $k=2$. Optimal strategies for D for different numbers of protected links and when A attacks (up to) 2 links. The captions of the figures give the Designer's expected benefit.

Example 5 Suppose that $N = [\![1,5]\!]$ and $k=2$, thus the maximum number of attacks that A has an incentive to do is 2. Figure 9 shows the networks that maximize the expected payoff of D for different numbers of protected links $p=1,\ldots,5$ and all of A potential best response attacks. As in the previous figures, the protected links are represented by thick lines and dashed lines identify the links potentially attacked by A. The captions represent the Designer's expected benefit. Depending on the value of c_A, the Adversary may attack 0, 1 or 2 links leading to different values of D's expected benefit.

Note that the (3,5)-Harary-network (in $S_{0,k}^D$ and represented in Figure 9(i)) contains 0 protected link 8 non-protected links and is fully protected against any attack of 2 links. Therefore, all other strategies that can be optimal for D contain at most 7 links, which in turn implies that there exists at least one node of degree lower or equal to 2. Further, the expected benefit associated to any network for the Designer is always bounded by the probability of its weakest node to be disconnected. Hence, the expected benefits of any protected network not in $S_{0,k}^D$ are lower or equal to $1-\pi^2$. Figure 9(vii) shows a network with such expected benefits and 5 protected links. Since $c_P > c_L$, D has no incentive to form networks which contains 6 protected links or more. Hence, Figure 9 contains all potential optimal strategies of D under imperfectly protected links.

Let us now focus on the optimal strategies of A. Networks 9(ii)-9(iv) and 9(vi)-9(vii) can only be disconnected if at least two links fail. Therefore the Adversary has no incentive to attack only 1 link in these networks. Further, note that $3c_L+3c_P > 6c_L+c_P$, and thus the network 9(iv) is never an optimal strategy for D. Further, suppose that for some value of π, the strategy of D depicted in Figure 9(iii) is optimal for D, then it has a greater expected payoff than that of Figure 9(vii) and thus $5c_L < 3c_p$. But then its expected payoff is no more than $1 - 5c_L - 10c_L/3 < 1 - 8c_L$ and thus is strictly dominated by strategy of Figure 9(i) which is a contradiction. Thus the network of Figure 9(iii) is never an optimal strategy. Similarly, if strategy of D depicted in Figure 9(ii) is optimal for some π, then for that π it strictly dominates network of Figure 9(vi) and thus $5c_L < 3c_P$. But then it is dominated by the strategy of Figure 9(i). Thus the network of Figure 9(ii) is never an optimal strategy.

In the case of perfectly protected links, from Proposition 2, since $n(k+1) = 1 \mod 2$, three potential SPE could occur, resulting in the networks of Figure 9(i), 9(ii) and 9(v). However, as in this case $k = n - 3$, then protected networks in $S^D_{1,k}$ are never optimal when $\pi = 0$ (as explained in Footnote 17). For small values of π, the networks of Figure 9(i) and 9(v) can occur, for instance for $\pi = c_A = 0.1$ and $c_P = 0.2$ and $c_L = 0.075$ then the networks of $S^D_{0,k}$ are optimal, while for $c_P = 0.02$ and $c_L = 0.1$ the networks of $S^D_{n-1,k}$ are optimal. Note that in that cases, A does not attack any link in SPEs.

Suppose that $c_P/c_L > (k + 2)/2$, then for perfectly protected links (i.e. π=0), the network of Figure 9(i) is an equilibrium strategy for D and an equilibrium strategy for A is to attack no link. Observe now that the expected payoff of D associated with this network is not modified when π changes, while her expected payoff associated with all other networks drawn in Figure 5 decreases with π and all converge to negative values. Consequently, given c_P and c_L there exists a probability $\bar{\pi}$ such that for $\pi > \bar{\pi}$ networks in $S^D_{0,k}$ are optimal strategies for D.

Moreover, for $\pi = 0.2$, $c_P = 0.02$, and $c_L = 0.1$ and $c_A = 0.1$, the network of Figure 9(vii) is an optimal strategy for D given that A chooses an optimal attack. Note that D never builds this protected network in a situation where links are perfectly protected.[20] Further, for $c_P = 0.1$, $c_L = 0.075$, $\pi = 0.3$ and $c_A = 0.2$, the network of Figure 9(vi) is an optimal strategy for D given that A plays an optimal strategy.

Example 5 establishes three main insights. First, $S^D_{0,k}$ are the unique optimal strategies in situation where π is sufficiently high. Second, there exist situations where the Designer's best strategy is to build a network where each node is incident to m protected links, with $m = k$. Note that since $c_P > c_L$, D has no incentive to form a protected network where each node is incident to $k + 1$ protected links. Third, there exist situations where nodes which belong to the same component in $(N, E_P, \emptyset)$ are linked with a non-protected link (see Figure 9(vi)) in an optimal strategy for D.

DG [9] examine the impact of imperfect defense in a framework where D protects nodes instead of links. They use an example and provide two insights. First, there exist parameters such that the SPEs obtained in the perfect defense model remain equilibria in the imperfect defense model, namely the empty network, the center protected star, and the minimal $(k + 1, n)$-node-connected networks. Second, they establish that richer strategies than those played by D in the perfect defense model may appear in equilibrium. In particular, there exist situations where an optimal strategy for D is to protect several nodes and create a network which generalizes the center protected star network, or to design a $(2, n)$-node-connected network and to protect all the nodes.

It is worth noting that imperfect defense has the same type of impact in the framework of DG and in our framework. First, if the probability of success of attacks π is sufficiently high and the cost of forming non-protected links is sufficiently low, then strategies in $S^D_{0,k}$ are the unique optimal strategies for D. Second, the set of strategies candidate to be an equilibrium is larger in the imperfect defense framework than in the perfect defense framework. In particular, for sufficiently high π, D has an incentive to use more protections than in a situation where $\pi = 0$: there exist situations where D protects all the nodes in DG's framework, and there exist situations where D designs a network where each node is incident to k protected links in our framework. Third, in both frameworks it is difficult to obtain general results when imperfect defense is introduced. However, Landwehr [21] provides equilibrium strategies for D when the number of attacks is very small. In particular, he establishes that if $k = 2$, then there exist 6 types of strategies that D may play in equilibrium according to the value of π, c_P, and c_L.

[20] Note that in Figure 9(vii), each node is incident to 2 protected links and $k = 2$. Interestingly, DG [9] and Landwehr [21] establish that in models with imperfect defense, there exist situations where D designs a $(2, n)$-Harary-networks in equilibrium.

5 Conclusion

In this paper, we have studied the optimal way to design and protect a network under strategic link attacks. In our benchmark model, the number of protected links available for the Designer is not bounded, and protected links cannot be removed by the Adversary. Our main findings in this model are the following. There exists three types of networks that are equilibrium strategies according to the value of the parameters of the model (which are the number of nodes and the costs of link creation and attack). First, if the relative cost of protections (i.e. the ratio of the cost of a protected link over the cost of a non-protected link) is low comparatively to the number of attacks, then D forms a $(1, n)$-link-connected network which contains only protected links. Second, if the relative cost of protection is high in regards to to the number of attacks, then the Designer forms a minimal $(k+1, n)$-connected network which contains only non-protected links. Third, for intermediate relative costs of protection, there exist situations where the Designer forms a network which contains one protected link and $(n-1)(k+1)/2$ non-protected links. To sum up, in this paper we provide the minimal costs that D incurs to protect her network against an intelligent attack (i.e. the worst attack).

We also examine situations where the number of protected links available for the Designer is limited. In that case, we establish that for intermediate relative costs, the Designer forms a network which contains several protected and non-protected links. Finally, we discuss the case of imperfectly protected links. We cannot provide a full characterization of the SPEs in the imperfect defense model, but we give conditions under which results obtained in the framework with perfect defense are preserved. Moreover, we establish through an example that the set of equilibria is larger in the framework with imperfect defense links than in the framework with perfect defense.

In this paper, we have assumed that the Designer incurs the same costs if she forms protected links that are adjacent and if the that are not adjacent. It would be interesting to examine a situation where it is more costly for the Designer to form protected links that are not adjacent. As we explained after Example 2, if D protects adjacent links, it can lead to situations where D has to create more non-protected links than in the optimal strategies described in Proposition 1. It can be interesting to study situations where protecting non adjacent links increases the costs incurred by the Designer.

Adding constraints on the location of protected links can be applied in different contexts. Indeed, it is more costly for a company to protect some cables (by reinforcing them or replacing them with new equipment) in different locations. Indeed, the company has to send several teams of workers to protect cables which are far from each other instead of one team when they are close to each other.

Appendix: Proof of Proposition 1

Lemma 1 establishes that if D forms p protected links in $S^D \in S^D_{p,k}$, then S^D contains $n-p$ components.

Lemma 1 *Let $S^D = (N, E_P, E_{NP})$, $S^D \in S^D_{p,k}$. The subnetwork $(N, E_P, \emptyset)$ of S^D is acyclic.*

Proof If $p = n-1$, the result holds from Footnote 15. Otherwise, to introduce a contradiction, suppose that the subnetwork $(N, E_P, \emptyset)$ of $S^D_{p,k}$ contains a cycle. Then, there exists a link $ij \in E_P$ such that $(N, E_P \setminus \{ij\}, E_{\text{NP}})$ cannot be disconnected by an optimal attack of A. Moreover, since $p \in [\![1, n-2]\!]$ and g^R has to be connected, $E_{\text{NP}} \neq \emptyset$. Let $i'j' \in E_{\text{NP}}$. Network $(N, E_P \setminus \{ij\} \cup \{i'j'\}, E_{\text{NP}} \setminus \{i'j'\})$ contains p protected links and $\#E_{\text{NP}}(S^D \in S^D_{p,k}) - 1$ non-protected links, a contradiction. □

The proof of Proposition 1 is organized in the following way. First, given a number of nodes and the number of components, we provide an alternative optimization problem whose optimum corresponds to the optimal Designer's strategy and we provide the minimal number of non-protected links in a protected network $S^D \in S^D_{p,k}$ (Appendix A). We then provide the solutions of the optimization problem, both in terms of value (i.e. the minimum number of non-protected links) as well as a constructive method for the Designer to obtain an optimal set of non-protected links according to the number of protected links formed by D and the number of attacks (Appendices B and C).

A An equivalent optimization formulation

The equivalent problem formulation relies on the concept of multigraph which we now develop. A *multigraph* is a graph where multiple links and loops are allowed. Formally, an (undirected) multigraph $\hat{g}$ is an ordered triple $(\hat{N}, \hat{E}, \hat{\psi})$ consisting of a non-empty set of nodes, $\hat{N}$, a set of links, $\hat{E}$, disjoint of $\hat{N}$, and an incidence function $\hat{\psi} : \hat{E} \to \hat{N}^2$ that associates to each link an *unordered* pair of nodes of $\hat{g}$. If e is a link and i and j are nodes such that $\hat{\psi}(e) = (i, j)$, then e is said to join i and j.[21]

We define the adjacency matrix $\mathcal{M}(\hat{g})$ of a multigraph $\hat{g} = (\hat{N}, \hat{E}, \hat{\psi})$ as: $\forall (a, b) \in \hat{E}^2,\ \mathcal{M}_{a,b}(\hat{g}) = \#\{e \in \hat{E} : \hat{\psi}(e) = (a, b)\}$, i.e. the number of links between nodes a and b in $\hat{g}$. Note that the adjacency matrix of an undirected multigraph is symmetric.

Let $g = (N, E_P, E_{\text{NP}})$ be a protected network and $\Gamma_1(g), \ldots, \Gamma_\ell(g), \ldots, \Gamma_\nu(g)$ be the components of the subnetwork $(N, E_P, \emptyset)$ with $\gamma_\ell(g)$ the number of nodes of the component $\Gamma_\ell(g)$. By construction, the E_P-contraction of g is $\hat{g}^{E_P} = (\llbracket 1, \nu \rrbracket, \hat{E}_g, \hat{\psi}_g)$ with $\hat{E}_g = \{e_{ij} : ij \in E_{NP}(g)\}$ and

$$\forall (a, b) \in \llbracket 1, \nu \rrbracket^2,\ \forall i \in \Gamma_a(g),\ \forall j \in \Gamma_b(g), \quad \hat{\psi}_g(e_{ij}) = (a, b).$$

Note that a protected network g induces one and only one E_P-contraction $\hat{g}^{E_P}$ (up to ordering). However the converse is not true: a multigraph can be the E_P-contraction of two (or more) distinct protected networks. However, these graphs have the same number of non-protected links (which is given by $\sum_{a,b \in \llbracket 1,\nu \rrbracket^2} \mathcal{M}_{a,b}(\hat{g}^{E_P})$), and the same minimum number of protected links (which is equal[22] to $n - \nu$). Therefore, all protected networks resulting in a given E_P-contraction have the same minimal cost.

Then, an optimal strategy for the Designer is the choice of vector $(\Gamma_\ell(g))_{1 \le \ell \le \nu}$ and matrix $\mathcal{M}(g)$, under some constraints, which we develop below:

Lemma 2 *For a given number of components ν, an optimal strategy for D is a solution of the following optimization problem:*

$$\min_{\substack{\gamma \in \mathbb{N}^\nu, \\ \mathcal{M} \in \mathbb{N}^\nu \times \mathbb{N}^\nu}} \frac{1}{2} \sum_{i \in \llbracket 1,\nu \rrbracket} \sum_{j \in \llbracket 1,\nu \rrbracket} \mathcal{M}_{i,j} \quad s.t. \begin{cases} \forall (i,j) \in \llbracket 1, \nu \rrbracket^2,\ \mathcal{M}_{i,j} = \mathcal{M}_{j,i}, & \text{(CS-1)} \\ \forall (i,j) \in \llbracket 1, \nu \rrbracket^2,\ \mathcal{M}_{i,j} \le \gamma_i \gamma_j, & \text{(CS-2)} \\ \forall i \in \llbracket 1, \nu \rrbracket, \quad \mathcal{M}_{i,i} \le \gamma_i(\gamma_i - 1)/2 - (\gamma_i - 1), & \text{(CS-3)} \\ \forall I \subseteq \llbracket 1, \nu \rrbracket, \quad \sum_{i \in I} \sum_{j \in \llbracket 1,\nu \rrbracket \setminus I} \mathcal{M}_{i,j} \ge k + 1, & \text{(CS-4)} \\ \quad n = \sum_{i \in \llbracket 1,\nu \rrbracket} \gamma_i. & \text{(CS-5)} \end{cases}$$

Proof Consider any matrix $\mathcal{M} = (\mathcal{M}_{i,j})_{i \in \llbracket 1,\nu \rrbracket,\ j \in \llbracket 1,\nu \rrbracket}$. Build $\hat{N} = \llbracket 1, \nu \rrbracket$, $\hat{E} = \cup_{(i,j) \in \hat{N} \times \hat{N}} \hat{E}_{ij}$, with $\hat{E}_{ij} = \{e^1_{ij}, \ldots, e^{\mathcal{M}_{i,j}}_{ij}\}$, and $\hat{\psi}(e) = (i, j)$ if and only if $e \in \hat{E}_{ij}$. The triplet $(\hat{N}, \hat{E}, \hat{\psi})$ is an (undirected) multigraph if and only the links are undirected (constraint (CS-1)).

In turns, this multigraph is the E_P-contraction of a protected network $g = (N, E_P, E_{\text{NP}})$, such that $(N, E_P, \emptyset)$ has ν components of size $\gamma_1, \gamma_2, \ldots, \gamma_\nu$, if and only if all nodes of g belong to exactly one component (constraint (CS-5)), if each node in one component is connected to any other node in a different component by at most one link (constraint (CS-2)), and if the network does not contain any loop. The latter requires that each node in a component Γ_i is linked with at most $\gamma_i - 1$ nodes which belong to Γ_i. From Lemma 1, each component of size γ_i contains exactly $\gamma_i - 1$ protected links (since it is connected and does not contain any loops). This is reflected in constraint (CS-3).

[21] By definition a simple graph does not contain a loop, that is a link joining a node to itself; neither does it contain multiple links, that is, several links joining the same two nodes. Therefore, it is a multigraph for which $\hat{\psi}$ is injective and for which there is no $e \in E$ such that $\hat{\psi}(e) = (i, i)$ with $i \in \hat{N}$.

[22] Since $\hat{g}^{E_P}$ results of the contraction of p links, then $\nu \ge \nu - p$. The equality is attained when the subnetwork $(N, E_P, \emptyset)$ contains no cycle.

The goal of the Designer is to minimize her number of non-protected links, which are given by $\frac{1}{2}\sum_{i\in[\![1,\nu]\!]}\sum_{j\in[\![1,\nu]\!]}\mathcal{M}_{i,j}$.

Finally, no component of $(N, E_P, \emptyset)$ should be vunerable to an attack of A, that is, every component of $(N, E_P, \emptyset)$ should be incident to at least $k+1$ non-protected links in g: this means that $\forall i \in [\![1,\nu]\!]$, $\sum_{j\in[\![1,\nu]\!]\setminus\{i\}} \mathcal{M}_{i,j} \geq k+1$. This should also hold for any group of components, as reflected by constraint (CS-4). □

This formulation allows us to directly derive a bound on the number of (non-protected) links that are necessary in the construction. We will show in the subsequent paragraphs that this bound can be reached under some assumptions on the values of n, ν and k.

Lemma 3 *The number of non-protected links induced by any strategy is at least* $N_1 = \left\lceil \frac{\nu(k+1)}{2} \right\rceil$.

Proof This is a direct consequence of constraint (CS-4). Indeed, for any i, Eq. (CS-4) implies that $\sum_{j\in[\![1,\nu]\!],j\neq i} \mathcal{M}_{i,j} \geq k+1$. Therefore, $\sum_{i\in[\![1,\nu]\!]}\sum_{j\in[\![1,\nu]\!]}\mathcal{M}_{i,j} \geq \sum_i (k+1) = \nu(k+1)$. □

In the rest of the proof, we provide, for each value of components ν, the optimal value of the optimization problem (9) as well as the corresponding optimal vector γ and matrix $\mathcal{M}$. The constructions will rely heavily on the following definitions:

$$\alpha_1 = \left\lfloor \frac{n}{\nu} \right\rfloor, \qquad \alpha_2 = n \mod \nu, \qquad \beta_1 = \left\lfloor \frac{k+1}{\nu-1} \right\rfloor, \qquad \text{and } \beta_2 = k+1 \mod \nu-1. \tag{10}$$

Also, one specific vector of component sizes, $\overline{\gamma}$, will come in handy in the proofs. Consider

$$\overline{\gamma}_i = \begin{cases} \alpha_1 & \text{if } i \leq \nu - \alpha_2 \text{ and} \\ \alpha_1 + 1 & \text{otherwise.} \end{cases} \tag{11}$$

Then $\sum_{i\in[\![1,\nu]\!]}\overline{\gamma}_i = \alpha_1(\nu-\alpha_2) + (\alpha_1+1)\alpha_2 = \alpha_1\nu + \alpha_2 = n$, which satisfies constraint (CS-5).

We distinguish two cases, depending on the values of α_1 and β_1, that would lead to different construction and optimal numbers of non-protected links.

B Solution of the case where $\alpha_1^2 \geq \beta_1 + 1$.

In this case, an optimal matrix $\mathcal{M}$ that reaches the bound given by Lemma 3 can be built. It relies on the matrices of (a,b)-Harary network $\mathcal{H}(a,b)$ (for any $a < b$).

To construct such matrices, consider the squared matrices $\mathcal{D}(b)$ of size b which are defined by $\mathcal{D}(b)_{i,j} = 1$ if $j - i = 1$ and 0 otherwise. That is,

$$\mathcal{D}(b) = \begin{pmatrix} 0 & 1 & & & & \\ & 0 & 1 & \mathbf{0} & & \\ & & \ddots & \ddots & & \\ & \mathbf{0} & & & 0 & 1 \\ 1 & & & & & 0 \end{pmatrix}.$$

Note that the powers of $\mathcal{D}(\nu)$ have a specific structure: they satisfy $\mathcal{D}(\nu)^{(z)}_{i,j} = 1$ if $j - i = z \mod b$ and 0 otherwise. Let us also define for any $z \in \mathbb{N}$, the matrix $\mathcal{F}(z,\nu) = \mathcal{D}(\nu)^{(z)} + (\mathcal{D}(\nu)^{(z)})^\top$ (with $.^\top$ the transpose operator).

Define also by $\mathcal{E}(i,j,b)$ the squared matrix of size b whose elements are all zero except for the one at line i and column j and its symmetric element (at line j and column i) whose value is 1. Then, a matrix of a (a,b)-Harary network is given by:

$$\mathcal{H}(a,b) = \begin{cases} \mathcal{F}(1,\nu) + \mathcal{F}(2,\nu) + \cdots + \mathcal{F}(a/2,\nu) & \text{if } a = 0 \mod 2, \\ \mathcal{F}(1,\nu) + \mathcal{F}(2,\nu) + \cdots + \mathcal{F}(\left\lfloor \frac{a}{2} \right\rfloor,\nu) + \mathcal{F}(\frac{b}{2},\nu) & \text{if } a = 1 \mod 2 \text{ and } b = 0 \mod 2, \\ \mathcal{F}(1,\nu) + \mathcal{F}(2,\nu) + \cdots + \mathcal{F}(\left\lfloor \frac{a}{2} \right\rfloor,\nu) + \mathcal{F}(\frac{b+1}{2},\nu) & \text{if } \beta_2 = 1 \mod 2 \text{ and } b = 1 \mod 2. \\ \qquad + \mathcal{E}(1, \frac{b-1}{2}, b) & \end{cases}$$

Note that $\mathcal{H}(b-1,b) = \begin{pmatrix} 0 & & & & \\ & 0 & & 1 & \\ & & \ddots & & \\ & 1 & & 0 & \\ & & & & 0 \end{pmatrix}$. In the following, for ease of notations, we note $|.|$ the sum of elements of a matrix or a vector, that is, for any matrix $\mathcal{A} \in [\![1,a]\!] \times [\![1,b]\!]$, $|\mathcal{A}| = \sum_{i \in [\![1,a]\!], j \in [\![1,b]\!]} \mathcal{A}$. Note that, for any a and b, we have

$$|\mathcal{H}(a,b)| = \begin{cases} ab & \text{if } ab = 0 \mod 2, \\ ab+1 & \text{if } ab = 1 \mod 2. \end{cases}$$

Lemma 4 *Suppose that $\alpha_1^2 \geq \beta_1 + 1$. Then, any optimal strategy has exactly N_1 non-protected links.*

Proof Consider vector $\overline{\gamma}$ as defined in Eq. (11), so (CS-5) is satisfied, and the matrix $\mathcal{M}$ which is the sum of the incidence matrix of β_1 (ν-1,ν)-Harary networks and one (β_2,ν)-Harary network. More precisely, let $\mathcal{M}$ be given by:

$$\mathcal{M} = \beta_1 \mathcal{H}(\nu-1,\nu) + \mathcal{H}(\beta_2,\nu).$$

Let us first check that $\mathcal{M}$ is an admissible matrix (i.e. that it satisfies the constraints (CS-1)-(CS-4)):

CS-1 is satisfied by construction.

CS-2 Further, $\forall i,j, \mathcal{M}_{i,j} \leq \beta_1 + 1 \leq \alpha_1^2 \leq \overline{\gamma_i}\ \overline{\gamma_j}$, which complies with constraint (CS-2).

CS-3 By construction Harary networks do not contain loops. Hence $\forall i, \mathcal{M}_{i,i} = 0$ which satisfies constraint (CS-3).

CS-4 Finally, let $I \subseteq [\![1,\nu]\!]$:

- If I is the singleton $\{i\}$, by definition of a Harary network, $\sum_{j \in [\![1,\nu]\!] \setminus \{i\}} \mathcal{M}_{i,j} = \beta_1(\nu-1) + \beta_2 = k+1$.
- Similarly, if $\#I = \nu - 1$, then $\exists j \in [\![1,\nu]\!],\ I = [\![1,\nu]\!] \setminus \{j\}$. Then $\sum_{i \in I} \sum_{j \in [\![1,\nu]\!] \setminus I} \mathcal{M}_{i,j} = \sum_{i \in [\![1,\nu]\!],\, i \neq j} \mathcal{M}_{i,j} = k+1$.
- Finally, suppose that $2 \leq \#I \leq \nu-2$. Since, $\forall i,j,\ i \neq j,\ \mathcal{M}_{i,j} \geq \beta_1$, then $\sum_{i \in I} \sum_{j \in [\![1,\nu]\!] \setminus I} \mathcal{M}_{i,j} \geq \#I(\nu - \#I)(\min_{i \in I, j \in [\![1,\nu]\!] \setminus I} \mathcal{M}_{i,j}) \geq 2(\nu-2)\beta_1 \geq k+1$.

Hence constraint (CS-4) is satisfied.

Let us now compute the number of links induced by this strategy. We have:

$$|\mathcal{M}| = \begin{cases} \beta_1\nu(\nu-1) + \beta_2\nu & \text{if } \beta_2\nu = 0 \mod 2 \\ \beta_1\nu(\nu-1) + \beta_2\nu + 1 & \text{if } \beta_2\nu = 1 \mod 2 \end{cases} = \begin{cases} \nu(k+1) & \text{if } \beta_2\nu = 0 \mod 2 \\ \nu(k+1) + 1 & \text{if } \beta_2\nu = 1 \mod 2. \end{cases}$$

But then, $k+1 \mod 2 = (\beta_1(\nu-1) \mod 2) + (\beta_2 \mod 2) \mod 2$. Hence $\beta_2\nu = 1 \mod 2 \Rightarrow k+1 = 1 \mod 2 \Rightarrow \nu(k+1) = 1 \mod 2 \Rightarrow \left\lceil \frac{\nu(k+1)+1}{2} \right\rceil = \left\lceil \frac{\nu(k+1)}{2} \right\rceil$. Finally, in all cases, the number of links is N_1, which is optimal from Lemma 3. □

Therefore, in this case, the bound on the number of links is attained. An important special case is when $2\nu \leq n$, that is when the number of protections is large:

Lemma 5 *If $2\nu \le n$, then $\alpha_1^2 \ge \beta_1 + 1$.*
Thus, if $2\nu \le n$, then any optimal strategy has N_1 non-protected links.

Proof Indeed,
$$\begin{aligned} 2\nu \le n \quad & \Rightarrow \nu(2\nu - n) \le 2(2\nu - n) \Rightarrow 2\nu^2 + 2n \le n\nu + 4\nu \\ & \Rightarrow \nu n - 2\nu \le 2n\nu - 2n - 2\nu^2 + 2\nu \Rightarrow \nu(n-2) \le 2(n-\nu)(\nu - 1) \\ & \Rightarrow \frac{n-2}{\nu-1} \le \frac{2(n-\nu)}{\nu}. \end{aligned}$$
Hence $\dfrac{k+1}{\nu-1} \le \dfrac{n-2}{\nu-1} \le 2\dfrac{n}{\nu} - 2$. Thus, if $2\nu \le n$, then $\left\lfloor \frac{n}{\nu} \right\rfloor \ge 2$ and finally $\left\lfloor \frac{n}{\nu} \right\rfloor^2 \ge 2\left\lfloor \frac{n}{\nu} \right\rfloor \ge \left\lfloor \frac{k+1}{\nu-1} \right\rfloor + 1$. □

C Solution of the case where $\alpha_1^2 < \beta_1 + 1$.

Under the condition that $\alpha_1^2 < \beta_1 + 1$, the construction based on Harary networks is no longer valid (it violates constraint (CS-2)). In this case, we therefore propose a variant of the construction in which the large values $\mathcal{M}_{i,j}$ are shifted towards the nodes of larger values of γ_i.

The proofs rely on the properties of a type of matrices which we denote as $\mathcal{Z}(a, b, c) \in [\![1, b]\!] \times [\![1, c]\!]$, with $a \le c$, that are such that
$$\forall i \in [\![1, b]\!], \sum_{j \in [\![1,c]\!]} \mathcal{Z}(a, b, c)_{i,j} = a$$
$$\forall d \in [\![1, b]\!],\ \forall x, y \in [\![1, c]\!], \left| \sum_{i \in [\![1,d]\!]} \mathcal{Z}(a, b, c)_{i,x} - \sum_{i \in [\![1,d]\!]} \mathcal{Z}(a, b, c)_{i,y} \right| \le 1.$$

These matrices differ from Harary's since they are not symmetric. They can be constructed in different ways. A possible construction is to pick, for each line i, exactly a indices among those who have the lower partial sum $\sum_{i' \in [\![1,i-1]\!]} \mathcal{Z}(a, b, c)_{i',\cdot}$. On these indices, the elements are equal to 1 while the others are 0. Finally, we let
$$\bar{\mathcal{Z}} = \mathcal{Z}((k+1) - (\nu - 1), a, \nu - a).$$

Lemma 6 *If $\alpha_1^2 < \beta_1 + 1$ and $(2\nu - n)(2\nu - n - 1) < (3\nu - 2n)(k+1)$ then, the optimal strategy requires exactly $(2\nu - n)\left((k+1) - \frac{2\nu - n - 1}{2}\right)$ non-protected links.*

Proof We start the proof by showing that this is a necessary condition. Let γ be any (non-decreasing) vector satisfying constraint (CS-5). By Lemma 5, we have $2\nu > n$. Then $\gamma_1 = 1$, otherwise we would have $\sum_{i \in [\![1,\nu]\!]} \gamma_i \ge \sum_{i \in [\![1,\nu]\!]} 2 \ge 2\nu > n = \sum_{i \in [\![1,\nu]\!]} \gamma_i$ which is impossible. Let a be such that for all $i \le a, \gamma_i = 1$. Note that $n = \sum_{i \in [\![1,\nu]\!]} \gamma_i = a + \sum_{i \in [\![a+1,\nu]\!]} \gamma_i \ge a + 2(\nu - a)$. Hence $a \ge 2\nu - n$. Any solution matrix $\mathcal{M}$ can be written on the form:

$$\mathcal{M} = \left(\begin{array}{c|c} \overbrace{\mathcal{A}}^{a} & \overbrace{\mathcal{B}}^{\nu - a} \\ \hline \mathcal{B}^\top & \mathcal{C} \end{array} \right) \begin{array}{l} \}\, a \\ \}\, \nu - a \end{array} . \tag{12}$$

Then, constraint (CS-4) implies that $|\mathcal{A}| + |\mathcal{B}| \ge a(k+1)$ and $|\mathcal{B}| + |\mathcal{C}| \ge (\nu - a)(k+1)$. But constraint (CS-2) imposes that $|\mathcal{A}| \le a(a-1)$. Therefore $|\mathcal{B}| \ge a\left((k+1) - (a-1)\right)$ which leads to the bound $|\mathcal{M}| \ge |\mathcal{A}| + 2|\mathcal{B}| \ge a\left(2(k+1) - (a-1)\right) \ge (2\nu - n)(2(k+1) - (2\nu - n - 1))$.

Consider now the following construction: let $\overline{\gamma}$ be as in Eq. (11), $a = 2\nu - n$ and

$$\left\{ \begin{array}{l} \mathcal{A} = \mathcal{H}(a-1, a), \\ \\ \mathcal{B} = \underbrace{\left(\ 1 \ \right)}_{\nu - a} \Big\}\, a \quad + \bar{\mathcal{Z}}, \end{array} \right. \tag{13}$$

and $\mathcal{C}$ the zero matrix: $\forall i, j,\ \mathcal{C}_{i,j} = 0$.

$\overline{\mathcal{Z}}$ is well defined as $(k+1)-(\nu-1) \geq 0$ (since $\beta_1 > 0$ from $\alpha_1^2 < \beta_1+1$) and $(k+1)-(\nu-1) \leq \nu - a$ (since $k+1 \leq n-1$).

It is important to note that the asumption of the lemma imposes that $3\nu - 2n > 0$. Hence, in this construction, $a > \nu - a$. Let us show that this construction satisfy the problem constraints altogether with the desired value of $\mathcal{M}$.

CS-1 is verified since by construction, $\mathcal{A}$ and $\mathcal{C}$ are symmetric.

CS-2 Further, $\forall i, j \in [\![1, a]\!],\ \mathcal{A}_{i,j} \leq 1 = \gamma_i\gamma_j,\ \forall i, j \in [\![1, a]\!],\ \mathcal{B}_{i,j} \leq 2$ and $\forall i, j,\ \mathcal{C}_{i,j} = 0 \leq 4$ and thus constraint (CS-2) is satisfied.

CS-3 is verified by construction since $\forall i, \mathcal{A}_{i,i} = \mathcal{C}_{i,i} = 0$.

CS-4 Let $I \subseteq [\![1, \nu]\!]$. Let I_1 and I_2 be such that $I = I_1 \cup I_2$ with $I_1 \subseteq [\![1, a]\!]$ and $I_2 \subseteq [\![a+1, \nu]\!]$.

- If I is a singleton $\{i\} \in [\![1, a]\!]$, then $\sum_{j \neq i} \mathcal{M}_{i,j} = (a-1) + (\nu - a) + ((k+1) - (\nu - 1)) = k+1$.
- If I is a singleton $\{i\} \in [\![a+1, \nu]\!]$, then $\sum_{j \neq i} \mathcal{M}_{i,j} \geq a + \left\lfloor \frac{a((k+1)-(\nu-1))}{\nu - a} \right\rfloor = (2\nu - n) + \left\lfloor \frac{(2\nu-n)((k+1)-(\nu-1))}{n-\nu} \right\rfloor = \left\lfloor (2\nu - n)\frac{(k+1)-(2\nu-1-n)}{n-\nu} \right\rfloor \geq \left\lfloor \frac{(2\nu-n)(k+1)-(k+1)(3\nu-2n)}{n-\nu} \right\rfloor$ by the lemma's assumption. Hence $\sum_{j \neq i} \mathcal{M}_{i,j} \geq \left\lfloor \frac{(-\nu+n)(k+1)}{n-\nu} \right\rfloor = k+1$.
- Otherwise, suppose first that $\nu - a = 0$. Then $n = \nu$ which violates Lemma 5. Suppose now that $\nu - a = 1$. Then $n = \nu + 1$. Since we have that $n < 2\nu$ then $n = 1$ and $\nu = 0$ which is impossible. Therefore $\nu - a \geq 2$. Then, $a > \nu - a \geq 2$, i.e. $a > 3$, $\nu - a \geq 2$. Hence $\nu \geq 5$ and thus $k \geq 3$. Then, note that for all $i, j \in [\![1, \nu]\!]$, we have $\mathcal{M}_{i,j} \leq 2$. Hence $\sum_{i \in I} \sum_{j \in [\![1,\nu]\!] \setminus I} \mathcal{M}_{i,j} \geq \#I((k+1) - (\#I - 1)\max_{i,j} \mathcal{M}_{i,j}) \geq 2(k+1-2) = (k+1)+(k-3) \geq k+1$.

Hence constraint (CS-4) is satisfied.

Finally, $|\mathcal{M}| = |\mathcal{A}| + 2|\mathcal{B}| + |\mathcal{C}| = (a-1)a + 2a((\nu - a) + (k+1) - (\nu - 1)) = (a-1)a + 2a(1 - a + (k+1)) = a(1 - a + 2(k+1))$ which leads to the result by substituting $a = 2\nu - n$. □

Lemma 7 *If $\alpha_1^2 < \beta_1 + 1$ and $(2n - 3\nu)(k+1) + (2\nu - n)(2\nu - n - 1) \geq 0$, then optimal strategies have N_1 non-protected links.*

Proof Note that from Lemma 5, we have $\alpha_1 = 1$. Define further $\overline{\gamma}$ as in Eq. (11). We construct a solution matrix $\mathcal{M}_{(i,j)\in[\![1,\nu]\!]^2}$ of the shape of Eq. (12) with $a = 2\nu - n$, and $\mathcal{A}$ and $\mathcal{B}$ defined as in Eq. (13). We now build matrix $\mathcal{C}$. We need the following additional definitions.

Let

$$\begin{aligned} f &= (k+1) - \left\lceil \frac{\lfloor \overline{\mathcal{Z}} \rfloor}{n-\nu} \right\rceil - (2\nu - n), \\ f_1 &= \lfloor f/(n - \nu - 1) \rfloor, \qquad f_2 = f \mod (n - \nu - 1), \\ K &= \left\{ j,\ \textstyle\sum_{i \in [\![1,a]\!]} \mathcal{B}_{i,j} = \left\lfloor \frac{\lfloor \overline{\mathcal{Z}} \rfloor}{n-\nu} \right\rfloor + (2\nu - n) \right\}. \end{aligned}$$

K can be interpreted as the set of lines of matrix $\mathcal{C}$ for which the sum of elements should be equal to $f+1$ so that each singleton $i \in [\![a+1, \nu]\!]$ has a degree of $k+1$. The other lines of the matrix should have a sum of f.

$\mathcal{C}$ is constructed as the sum of 2 matrices. The first one $\mathcal{C}_1$ is such that each line sums to f while in the second one, $\mathcal{C}_2$, each line of the set K sums to 1 while each line out of this set sums to 0:

$$\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2.$$

$\mathcal{C}_1$ can be constructed in an analogous way as $\mathcal{M}$ in Lemma 4, that is:

$$\mathcal{C}_1 = f_1 \mathcal{H}(n - \nu - 1, n - \nu) + \mathcal{H}(f_2, n - \nu).$$

Let σ be an ordering of the elements of K, that is $K = \{\sigma_1, \sigma_2, ..., \sigma_{\#K}\}$, with $\sigma_1 < \sigma_2 < \cdots < \sigma_{\#K}$. Recall that $\mathcal{E}(i, j, b)$ is the squared matrix of size b whose elements are all zero except for the one at line i and column j and its symmetric element (at line j and column i) whose value is 1. We define:

$$\mathcal{C}_2 = \sum_{i \in [\![1, \lfloor \#K/2 \rfloor]\!]} \mathcal{E}(\sigma_{2\times i}, \sigma_{2\times i+1}, n - \nu) + \begin{cases} 0 & \text{if } \#K = 0 \mod 2, \\ \mathcal{E}(\sigma_{\#K}, \sigma_1, n - \nu) & \text{otherwise.} \end{cases}$$

Note that $\mathcal{C}_2$ is symmetric as a sum of symmetric matrices. Further, since $\sigma_1 < \sigma_2 < \cdots < \sigma_{\#K}$, then each row as at most one element whose value is 1 while the other elements are all equal to 0. Each row has such non zero element if and only if it is in the set K. Therefore all lines in the set K have elements summing to 1 while lines out of the set K have elements summing to 0.

Let us now show that this construction satisfies the system's constraints:

CS-1 is satisfied since by construction, $\mathcal{A}$ and $\mathcal{C}$ are both symmetric.

CS-2 From the choice of $\overline{\gamma}$, the constraint (CS-2) translates into: $\forall i \in [\![1, 2\nu - n]\!], j \in [\![2\nu - n + 1, \nu]\!]$, $\gamma_{i,j} \leq 2$ and $\forall i, j \in [\![2\nu - n+1, \nu]\!]$, $\gamma_{i,j} \leq 4$. The construction of $\mathcal{M}$ induces that $\forall i, j$ $\mathcal{A}_{i,j} \leq 1$, $\mathcal{B}_{i,j} \leq 2$ and $\mathcal{C} \leq f_1 + 1 + 1$. Then,

$$\begin{aligned} f_1 &\leq \frac{(k+1) - \frac{\lfloor \overline{\mathcal{Z}} \rfloor}{n-\nu} - (2\nu - n)}{n - \nu - 1} \\ &\leq \frac{(k+1)(n-\nu) - (2\nu - n)((k+1) - (\nu - 1)) - (2\nu - n)(n - \nu)}{(n - \nu - 1)(n - \nu)} \\ &= \frac{(k+1)(2n - 3\nu) + (2\nu - n)(2\nu - n - 1)}{(n - \nu - 1)(n - \nu)} \\ &\leq \frac{(n-2)(2n - 3\nu) + (2\nu - n)(2\nu - n - 1)}{(n - \nu - 1)(n - \nu)} = 4 - \frac{n}{n - \nu} < 3. \end{aligned}$$

Since f_1 is an integer, then $f_1 \leq 2$ and thus $\mathcal{C}_{i,j} \leq 4$, hence, satisfying constraint (CS-2).

CS-3 $\mathcal{A}$ and $\mathcal{C}_1$ have a zero diagonal as they are sum of matrices with zero diagonals. Since $\sigma_1 < \sigma_2 < \cdots < \sigma_{\#K}$ then all $\mathcal{E}$ matrices involved in the construction of $\mathcal{C}_2$ have zero diagonals. Therefore $\forall i, \mathcal{M}_{i,i} = 0$ and constraint (CS-3) is satisfied.

CS-4 Finally, let $I \subseteq [\![1, \nu]\!]$.

- If I is a singleton, then by construction $\sum_{j \neq i} \mathcal{M}_{i,j} = k + 1$.
- Otherwise, note that $\forall i, j, \mathcal{M}_{i,j} \leq \lceil (k+1)/(n - \nu) \rceil$. Then, $\sum_{i \in I} \sum_{j \notin I} \mathcal{M}_{i,j} \geq \#I(k + 1 - (\#I - 1)(\max_{i,j} \mathcal{M}_{i,j})) \geq 2((k+1) - (\max_{i,j} \mathcal{M}_{i,j})) \geq k + 1$.

Hence, constraint (CS-4) is satisfied.

We now compute the number of induced links by $\mathcal{M}$:

- By construction $|\mathcal{A}| + |\mathcal{B}| = (2\nu - n)(k + 1)$.
- Let $y = \begin{cases} 0 & \text{if } f.(n-\nu) = 0 \mod 2, \\ 1 & \text{otherwise.} \end{cases}$
- Then $|\mathcal{C}| = (n - \nu)(f_1(n - \nu - 1) + f_2) = (n - \nu)f = (n - \nu)((k + 1) - \left\lceil \frac{|\overline{\mathcal{Z}}|}{n-\nu} \right\rceil - (2\nu - n)) = (n - \nu)((k + 1) - \left\lceil \frac{(2\nu-n)((k+1)-(\nu-1))}{n-\nu} \right\rceil - (2\nu - n)) = (n - \nu)((k + 1) - (2\nu - n)) - (2\nu - n)((k + 1) - (\nu - 1)) + y = (2n - 3\nu)(k + 1) - (2\nu - n)(n - 2\nu + 1) + y$.
- Thus, $|\mathcal{B}| + |\mathcal{C}| = (2\nu - n)(n - 2\nu + (k + 1) + 1) + (2n - 3\nu)(k + 1) - (2\nu - n)(n - 2\nu + 1) + y = (n - \nu)(k + 1) + y$.

Thus, we obtain that $|\mathcal{M}| = |\mathcal{A}| + 2|\mathcal{B}| + |\mathcal{C}| = (2\nu - n)(k + 1) + (n - \nu)(k + 1) + y = \nu(k + 1) + y$ which concludes the proof. □

We now conclude the proof of Proposition 1. First, let us observe the conditions of Lemma 6. These are:

$$\left(\left\lfloor \frac{n}{\nu} \right\rfloor\right)^2 < \left\lfloor \frac{k+1}{\nu - 1} \right\rfloor + 1 \tag{14a}$$

$$(2\nu - n)(2\nu - n - 1) < (3\nu - 2n)(k + 1) \tag{14b}$$

From Lemma 5, Eq. (14a) requires that $n < 2\nu$, which leads to $\left\lfloor \frac{n}{\nu} \right\rfloor = 1$ (since $n \geq \nu$). This, in turns leads to $\frac{k+1}{\nu-1} \geq 1$, that is $\nu - 1 \leq k + 1$.

Let us now look at the solutions of Eq. (14b). For any n and ν, we have $(2\nu - n)(2\nu - n - 1) \geq 0$. Therefore $3\nu - 2n > 0$, which implies that $n < 2\nu$.

Suppose that $\nu-1>k+1$. Then:

$(2\nu-n)(2\nu-n-1)<(3\nu-2n)(k+1) \Rightarrow (2\nu-n)(2\nu-n-1)<(3\nu-2n)(\nu-1) \Rightarrow (2\nu-n)^2-(2\nu-n)<3\nu^2-2n\nu-3\nu+2n \Rightarrow 4\nu^2+n^2-4\nu n+n-2\nu<3\nu^2-2n\nu-3\nu+2n \Rightarrow \nu^2+n^2-2\nu n+\nu<n \Rightarrow (n-\nu)^2<n-\nu \Rightarrow n-\nu=0$, which is impossible. Therefore, the constraint (14a) is induced by constraint (14b). Second, we have $\sum_{\ell=1}^{\nu}\gamma_\ell=n$ and by Lemma 1, $\sum_{\ell=1}^{\nu}(\gamma_\ell-1)=p$ since $(N,E_P,\emptyset)$ is acyclic. It follows that $\nu=n-p$. Since $\nu=n-p$, $\left\lceil\frac{\nu(k+1)}{2}\right\rceil=\left\lceil\frac{(n-p)(k+1)}{2}\right\rceil$, so $N_1=n_1(p,k)$. Similarly, $(2\nu-n)\left((k+1)-\frac{2\nu-n-1}{2}\right)$ is equal to $(n-2p)\left(k+1-\frac{n-2p-1}{2}\right)$, and so $N_2=n_2(p,k)$. Finally, note that Equation $(n-2p)((k+1)-(n-2p-1)/2)=(k+1)p$ is a second order equation in p. Let x_1 and x_2 be the two real roots of p when they exist (which occurs when $\Delta\geq 0$, i.e. when $n\leq\frac{(3k+5)^2}{8(k+1)}$). Then, $p_1(k,n)=\lfloor x_1\rfloor+1$ and $p_2(k,n)=\lceil x_2\rceil-1$.

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