The effect of externalities aggregation on network games outcomes

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April 2016

Abstract

We generalize results on the monotonicity of equilibria for network games with incomplete information. In those games players know the stochastic process of network formation and their own degree in the realized network, and decide an action whose payoff depends on the strategic interaction in the network between their own action and a statistic (as, for example, the mean, the maximum or the minimum) of neighbors' actions. We show that, even under degree independence, not only the distinction between *strategic complements* and *strategic substitutes* is important in determining the nature of Bayesian Nash equilibria, but also the nature itself of the statistic.

JEL classification: D85.
Keywords: Network Games, sample statistics, substitutes and complements.

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1 Introduction

Following the paper "Network Games" by Galeotti et al. (2010) (hencefort: NG), many recent models on games with local externalities assume that agents are nodes of a network environment, and that they have to take an action which has local externalities channeled trough the topology of the network. However, the agents have limited observability on the structure of the network and even on the identity of their peers. Essentially, nodes know only their own degree and have some information about the general network formation process that generated the whole social network. The realization could be such that the degree of neighbors is i.i.d., and independent also on the degree of the node itself. More generally, any stochastic process for the realization of the network could be formalized as in the recent paper by Acemoglu et al. (2013). The nodes, in these *network games* with limited observability, compute *expected statistics* over the sample of the actions of the neighbors that they will end up finding in the pool. From an applied point of view, such models are a good tool for analyzing many complex social phenomena, as peer effects, the spread of habits, marketing for goods with externalities, vaccination policies, and public good contributions, just to name a few.

However, it must be noted that up to now the theoretical predictions of these models are unclear when it comes to assign some correlation between the degree of nodes and their action: who will endogenously tend to vaccinate more during a flu pandemic, those with many or those with few links? when a bad habit as smoking spreads in a school and there are peer effects, who will be more likely to smoke? those with many or those with few links?[1] Since some of the theoretical prediction in those models differ from those of NG, it becomes important from a theoretical point of view to study what differs in the apparently similar assumptions.

In this note we point out that the distinctions that have been made up to now are not sufficient to understand the possible outcomes of those games. Apart from the degree correlation mentioned above, the literature has focused on the sign of the externality, distinguishing between *local public goods* and *local public bads*, and on the cross derivative between the externalities and own actions, distinguishing between games of *substitutes* and games of *complements*.[2] However, not much attention has been put on the nature of the *statistic* upon neighbors' actions that is affecting the payoff.

[1] On the vaccination example, see Goyal and Vigier (2014a,b), Galeotti and Rogers (2013) and again Acemoglu et al. (2013). On the smoking example, see Currarini et al. (2013).

[2] On this see also the discussion in Jackson and Zenou (2014) about *strategic complements* and *strategic substitutes*.

As an example, vaccination and acquisition of information are both activities with positive externalities and the substitute property (i.e. the more my neighbors contribute the less I will, in equilibrium). However, they differ in the statistic that affects payoffs. In the vaccination case I am influenced by the minimum contribution in my neighborhood (it is a *weakest–link* game), so that having more neighbors increases the probability of finding a non–vaccinated one, and agents with higher degree will be more likely to vaccinate. In the information acquisition case I am influenced by the neighbor who knows more (it is a *best–shot* game),[3] so that having more neighbors increases the probability of finding a well–informed one, and agents with higher degree will be more likely to free ride and not acquire information themselves

Best shot games fall in the cathegory of those statistics whose expectation is increasing with the degree of a node, which can be seen just as the size of sample from which the statistic is computed. For weakest link games this expectation is instead decreasing with the size of the sample.

Finally, we provide novel and non–trivial results also for those statistics that are expected to be constant in the size of the sample. This last category includes the mean and the mode, which are the statistics most used in the empirical literature on peer effects and reference groups (a good survey is given in Blume et al. 2010).

The paper is organized in the following way: in Section 2 we describe the model. Section 3 contains the main results. Section 4 provides the relation with the previous literature. Section 5 concludes the discussion and provides possible directions for further research.

2 The model

Let $\mathcal{N} = \{1, 2, ...n\}$ be a finite set of agents. Each agent $i \in \mathcal{N}$ obtains some partial information about the realization of a random network and then chooses an action $x_i \in \mathcal{X}$, where $\mathcal{X} \subseteq \mathbb{R}$ is a compact set. Payoffs are assigned in a way that depends on the realized network environment. The structure is the one of a Bayesian game, and we follow the notation of NG, integrating it with some of the formalization from Acemoglu et al. (2013).

Network: Our network environment is represented by a (possibly directed) *network* g, in which the set of nodes is the set of agents, and a link $ij \in g$ denotes that the action of

[3] Minimum and maximum contributions from neighbors relate respectively to *weakest–link* and *best–shot* games, as introduced by Hirshleifer (1983) in a non–network context. NG and Boncinelli and Pin (2012) discuss network best shot games, while classical *weakest–link* games are those related to contagion, as Galeotti and Rogers (2013).

agent j affects i's payoff. We denote by $N_i(g) = \{j \in \mathcal{N} : \ ij \in g\}$ the set of neighbors of i in g (excluding i) and by $k_{i,g} \in \mathcal{K}$ the number of such neighbors (i.e. i's out–degree in the network) where $\mathcal{K}$ is the set of natural numbers $\{0, 1, 2, \ldots, n-1\}$.. We call $V_{k,g}$ the set of nodes that have degree k in network g, and by $v_{k,g}$ the cardinality of this set. Network g is obtained from a probability distribution P over all the $2^{n(n-1)}$ possible networks. We call P the *network formation process*.

Before going on, let us consider as benchmark the *configuration model* proposed by Bender and Canfield (1978). It is a model P of random network configuration where a certain degree distribution is given, and as as the number n of nodes grows to infinity, knowing only own degree provides no additional information on the degree of neighbors, which can be supposed to be drawn uniformly and i.i.d. from that degree distribution. Following NG, we call *degree independence* this lack of correlation between own degree and the degree of neighbors. When instead, knowing own degree k changes the expectation on the degree of neighbors, we may have degree *assortativity* or *disassortativity* (on this, see also the discussion at the end of this section).

Payoffs: Payoffs are based on the realized network g. Player i's payoff function when she chooses x_i and her $k_{i,g}$ neighbors choose the action profile $\vec{x}_{i,g} = (x_1, ..., x_{k_i})$ is:

$$\Pi_{k_{i,g}}(x_i, \vec{x}_{i,g}) = f\left(x_i, s(\vec{x}_{i,g})\right) - c(x_i) \quad , \tag{1}$$

where s is an *aggregator* computed on the set of the neighbors' actions,[4] and $f(x, s)$ is continuously differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ with:

$$f_x > 0 \ , \ \text{and} \ f_{xx} \leq 0 \quad .$$

We say that when $f_s > 0$ we have *positive exterernalities*, when instead $f_s < 0$ we have *negative exterernalities*. Moreover, if $f_{xs} \leq 0$ or $f_{xs} \geq 0$, we say, respectively, that f has the *substitutes* or the *complements* property. Finally, $c(x)$ is a convex cost function such that $c_x > 0$ and $c_{xx} \geq 0$.

Statistic: The effects of local interaction are aggregated by the function s. Formally s is a different k–dimensional function for every $k \in \mathcal{K}$. So, s is a family of n functions[5] and each of

[4]Formally, s is a function from $\mathbb{R}^{k_{i,g}}$ to $\mathbb{R}$ and, if $k_{i,g} = 0$, s is a constant.

[5]In principle we allow them to be even different functions for each k. We can write this dependence on k as $s : \bigcup_{k \in \mathbb{N}} \{k\} \times \mathcal{X}^k \to \mathbb{R}$. In this way, $s(\cdot, \cdot)$ is a function of two arguments: the degree k and a vector of dimension k. With this notation, the general expression for the payoff functions needs a small change and becomes: $\Pi_{k_{i,g}}(x_i, \vec{x}_{i,g}) = f\left(x_i, s(k_{i,g}, \vec{x}_{i,g})\right) - c(x_i)$.

them is anonymous on the arguments, which means that any permutation of the elements of $\vec{x}_{i,g}$ would give the same result. This very general specification includes measures of central tendency as well as measures of variability (or dispersion). Measures of central tendency include the mean, median and mode, while measures of variability include the standard deviation (or the variance), the minimum and maximum values of the variables, kurtosis and skewness. In the following we refer to any measure s as a *statistic*.

Strategy profiles: A strategy for player i is a mapping $\sigma_i : \mathcal{K} \to \Delta(\mathcal{X})$, where $\Delta(\mathcal{X})$ is the set of probability distributions on $\mathcal{X}$, i.e. $\sigma_i = [\sigma_{ik}]_{k \in \mathcal{K}}$ where σ_{ik} is the mixed strategy played by player i of degree k. Furthermore $\vec{\sigma}_{ig}$ is the strategy profile of i's neighbors in network g, $\vec{\sigma} = [\sigma_i]_{i \in \mathcal{N}}$ is the strategy profile of the game and $\vec{\sigma}_{-i} = [\sigma_j]_{j \in \mathcal{N}/i}$ is the set of strategy profiles of all players excluded i.

Information: The only piece of information that an agent i obtains before deciding her action, on top of the common prior P, is her own degree $k_{i,g}$ in the realized network g. Then players play a game of incomplete information described by the quadruple $(\mathcal{N}, \mathcal{X}, (\Pi_{k_{i,g}})_{k_{i,g} \in \mathcal{K}}, P)$.

We consider *symmetric Bayesian Nash equilibria* in which every agent with the same information and facing the same ex–ante conditions (i.e. each agent i with the same degree k) chooses the same strategy, i.e. $\sigma_{ik} = \sigma_{jk}$ for any $k \in \mathcal{K}$ and for any $i, j \in \mathcal{N}$.
We say that a strategy profile $\vec{\sigma}$ is *first order stochastic dominance (FOSD) increasing* if, for every $k \in \mathcal{K}\backslash\{n-1\}$ we have that σ_{k+1} FOSD σ_k (which is to say that the cumulative distribution of σ_{k+1} is always below the cumulative distribution of σ_k – in the context of pure strategies it means that $x_{k+1} \geq x_k$). Analogously, $\vec{\sigma}$ is *FOSD decreasing* if, for every $k \in \mathcal{K}\backslash\{n-1\}$ we have that σ_k FOSD σ_{k+1}.

Given a realized network g and a strategy profile $\vec{\sigma}$ the expected payoff of agent i of degree k, if she knew her own position and the positions of all other nodes, is given by:

$$\Pi^e_{i,g}(\sigma_i, \vec{\sigma}_{ig}) = \int_{\mathcal{X}^n} \Pi_{k_{i,g}}(x_i, \vec{x}_{i,g}) \; d\vec{\sigma} \quad . \tag{2}$$

On top of this, we have to include also the uncertainty about the realization of the network. Adding this, the expected payoff of agent i of degree k is

$$\Pi^e_k(\sigma_i, \vec{\sigma}_{-i}) = \frac{\sum_g P(g) \cdot \sum_{i \in V_{k,g}} \Pi^e_{i,g}(\sigma_i, \vec{\sigma}_{ig})}{\sum_g P(g) \cdot v_{k,g}} \quad . \tag{3}$$

In words, an agent evaluates all possible nodes i with degree k in any possible realized network g, updating priors with the information that her degree is actually k.[6] For each such node i's position and network g, and for each realization of $\vec{\sigma}$, there will be a vector $\vec{x}_{i,g}$ that lists the action of each neighbor, depending on their degree in network g.

The Bayesian Nash equilibria can be represented simply as a (mixed) strategy profile $\vec{\sigma}^*$, where every agent i, depending on her degree k_i, will choose an optimal strategy σ_k^*, that maximizes the individual expected payoff for agent i from (3).

Let $\phi_{ig}(s|\vec{\sigma}_{ig})$ be the probability density function of s exactly for node i in network g when the strategy profile of the i's neighbors is $\vec{\sigma}_{ig}$. For an agent observing only her own degree k the posterior distribution for the statistic s will be:

$$\phi_k(s|\vec{\sigma}, P) \equiv \frac{\sum_g P(g) \cdot \sum_{i \in V_{k,g}} \phi_{ig}(s|\vec{\sigma}_{ig})}{\sum_g P(g) \cdot v_{k,g}} \quad , \tag{4}$$

therefore, since the Bayesian updating based on the network structure is linear, the expected value of s for an agent of degree k is

$$E_k(s|\vec{\sigma}, P) = \frac{\sum_g P(g) \cdot \sum_{i \in V_{k,g}} E_{ig}\left(s|\vec{\sigma}_{ig}\right)}{\sum_g P(g) \cdot v_{k,g}} \quad , \tag{5}$$

where $E_{ig}\left(s|\vec{\sigma}_{ig}\right)$ is the expected value of s for node i in network g and when the strategy profile of the i's neighbors is $\vec{\sigma}_{ig}$.

We call $\Phi_k(s|\vec{\sigma}, P)$ the cumulative probability distribution on s from $\phi_k(s|\vec{\sigma}, P)$. Then $\Phi_k(s|\vec{\sigma}, P)$ summarizes all the information provided by P (the network formation process) and $\vec{\sigma}$ (the strategy profile), given k. Finally by $Var_k(s|\vec{\sigma}, P)$ we denote the variance of s for an agent of degree k when the strategy profile of the game is $\vec{\sigma}$ and the network formation process is P.

Definitions: Given that the type of statistic $s(\cdot, \cdot)$ affects the individual payoff and the optimal individual behavior, through the network formation process P and equilibrium strategy $\vec{\sigma}$, we highlight its relevant characteristics.

DEFINITION **1.** *Given a network formation process P, a statistic s is* stable *with respect to P if for every $\vec{\sigma}$ and $k \in \mathcal{K} \backslash \{n-1\}$ we have that $E_{k+1}(s|\vec{\sigma}, P) = E_k(s|\vec{\sigma}, P)$.*

DEFINITION **2.** *Given a network formation process P, a statistic s is* increasing *(or* decreasing*) with respect to P if for every $\vec{\sigma}$ and $k \in \mathcal{K} \backslash \{n-1\}$ we have that $E_{k+1}(s|\vec{\sigma}, P) \geq$*

[6] Note that $\Pi_k^e(\sigma_i, \vec{\sigma}_{-i}) = \sum_g P(g|k) \cdot \frac{\sum_{i \in V_{k,g}} \Pi_{ig}^e(\sigma_i, \vec{\sigma}_{ig})}{v_{k,g}}$ where $P(g|k) = \frac{v_{k,g} \cdot P(g)}{\sum_g v_{k,g} \cdot P(g)}$ is the updated probability of network g.

$E_k(s|\vec{\sigma}, P)$ *(or respectively, if for every* $\vec{\sigma}$ *and* $k \in \mathcal{K}\backslash\{n-1\}$ *we have that* $E_{k+1}(s|\vec{\sigma}, P) \leq E_k(s|\vec{\sigma}, P)$*). We say that statistics s is strictly increasing (or decreasing) when all inequalities are strictly satisfied.*

Definition **3.** *Given a network formation process P, a statistic s is* FOSD increasing *(or* FOSD decreasing*) with respect to P if for every* $\vec{\sigma}$, $k \in \mathcal{K}\backslash\{n-1\}$, *and* $x \in \mathbb{R}$, *we have that* $\Phi_{k+1}(x|\vec{\sigma}, P) \leq \Phi_k(x|\vec{\sigma}, P)$ *(or respectively, if for every* $\vec{\sigma}$, $k \in \mathcal{K}\backslash\{n-1\}$, *and* $x \in \mathbb{R}$, *we have that* $\Phi_{k+1}(x|\vec{\sigma}, P) \geq \Phi_k(x|\vec{\sigma}, P)$*). We say that statistics s is strictly FOSD increasing (or decreasing) when all inequalities are strictly satisfied.*

Definition **4.** *Given a network formation process P, a statistic s satisfies second order stochastic dominance (SOSD) with respect to P if for every* $\vec{\sigma}$ *and* $y \in \mathbb{R}$ *we have the following inequality:*

$$\int_{-\infty}^{y} \Phi_{k+1}(x|\vec{\sigma}, P)\ dx \leq \int_{-\infty}^{y} \Phi_k(x|\vec{\sigma}, P)\ dx \quad . \tag{6}$$

The following two results are useful to understand the meaning of the assumptions made in our propositions.

1. It is directly verifiable that a strictly FOSD increasing statistic (Definition 3) implies an increasing one (Definition 2).

2. If s is stable and satisfies SOSD (Definitions 1 and 4), then s is *converging*, in the sense that for every σ and $k \in \mathcal{K}\backslash\{n-1\}$, we have that $Var_{k+1}(s|\vec{\sigma}, P) < Var_k(s|\vec{\sigma}, P)$ (we prove this in Appendix A as a corollary to Lemma A).

When P has degree independence, for any strategy profile $\vec{\sigma}$, many standard statistics as the mean, the median, or the sample variance, are both stable and converging. Still under degree independence, examples of increasing and decreasing statistics are instead, respectively, the maximum and the minimum (whenever the strategy profile $\vec{\sigma}$ is not constant for each $k \in \mathcal{K}$).

How can we classify a network formation process without degree independence? In the literature on complex networks, stemming from Newman (2002), a network exhibits degree *assortativity* or *disassortativity*, depending on the sign of the Pearson correlation coefficient of the degree between pairs of linked nodes, computed over all links. When the sign is positive (negative) the network is characterized by assortativity (disassortativity).[7] In general, when

[7]However, in NG this notion is related to the function that rules the network externalities of the game (i.e. f), and they talk about *positive* or *negative neighbour affiliation.*

P does not show degree independence, the value of the statistics s for an agent of given degree k will depend both on P and on the strategy profile $\vec{\sigma}$. Consider a stable statistic under degree independence, for example the mean. Suppose that players play a symmetric strategy profile $\vec{\sigma}$ increasing with the degree and that P is characterized by degree assortativity. It is direct to check that in this case the expected value of the mean over the neighbours actions is increasing with the degree.

3 Results

In this game the existence of a symmetric Bayesian Nash equilibrium follows directly from Kakutani fixed point theorem, as mixed equilibria on a compact set $\mathcal{X}$ form themselves a convex compact set.

Our first proposition provides a general result, in which we aggregate in a single expression the function f that determines payoffs together with all the information that we have about the network structure and the statistics from $\phi_k(y|\vec{\sigma}, P)$. This result provides a general check to see if there is monotonicity in the equilibrium of the game.

Proposition **1.** *Consider the expected marginal revenues given by quantity*

$$E_k(f_x|x, \vec{\sigma}, P) \equiv \int_{-\infty}^{\infty} f_x(x, y) \cdot \phi_k(y|\vec{\sigma}, P)\ dy \tag{7}$$

where f_x is the derivative of f with respect to x. Then:

1. *if (7) is strictly decreasing in k for any $\vec{\sigma}$, then in every symmetric Bayesian Nash equilibrium of the network game the equilibrium strategy σ_k^* is FOSD decreasing in k;*

2. *if (7) is strictly increasing in k for any $\vec{\sigma}$, then in every symmetric Bayesian Nash equilibrium of the network game the equilibrium strategy σ_k^* is FOSD increasing in k.*

The formal proof is in Appendix B, as all the proofs of the following results. Figure 1 provides a visual interpretation of the argument in the proof, which is classical and is based simply on best responses: from the payoff function (1) the optimal response for a player is when marginal costs (increasing by assumptions) intersect expected marginal revenues (decreasing by assumption) – so, if an increase/decrease in k has a monotonic effect on those expected marginal revenues, also the intersection point will move monotonically.[8]

[8] In Appendix C we consider the case, which is not excluded by Proposition 1, in which all agents, independently on their degree, adopt the same action in equilibrium.

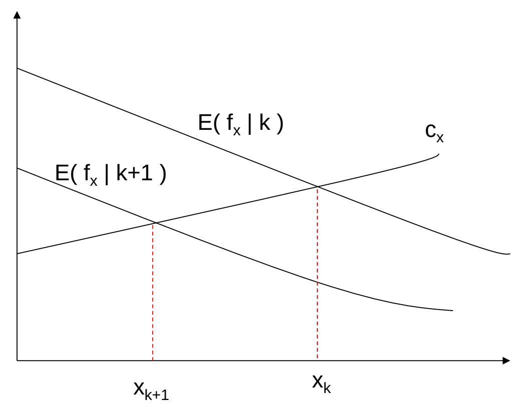

Figure 1: Intuition for Proposition 1 when $x^*(k) \geq x^*(k+1)$.

Our second result is about a characterization of Bayesian Nash equilibria based on the definitions provided in previous section. The following proposition describes the equilibrium strategies and their relation with the characteristics of the statistics s, both for strategic complements and strategic substitutes.

Proposition **2.** *Consider some network formation process P, then:*

1. *if s is stable and satisfies SOSD with respect to P, then for any symmetric Bayesian Nash equilibrium σ^*, (i) if $f_{xss} > 0$ then σ^* is FOSD decreasing, if (ii) $f_{xss} < 0$ then σ^* is FOSD increasing, if finally (iii) $f_{xss} = 0$ then $\sigma^*_k = \sigma^*_{k'}$ for each $k, k' \in \mathcal{K}$;*

2. *if s is FOSD increasing with respect to P, then for any symmetric Bayesian Nash equilibrium σ^*, (i) if $f_{xs} > 0$ then σ^* is FOSD increasing, if instead (ii) $f_{xs} < 0$ then σ^* is FOSD decreasing;*

3. *if s is FOSD decreasing with respect to P, then for any symmetric Bayesian Nash equilibrium σ^*, (i) if $f_{xs} > 0$ then σ^* is FOSD decreasing, if instead (ii) $f_{xs} < 0$ then σ^* is FOSD increasing.*

Point (1) follows directly from a result that we prove in Lemma A in Appendix A. The proofs of Points (2) and (3) are analogous to the proof of Proposition 2 in NG. [9]

Point (1) of the proposition shows that when the statistics s is stable and satisfies $SOSD$, equilibrium strategies do not depends on the characteristics of complementarity or substitutability of the utility function but on the sign of the third partial derivative f_{xss}. The equilibrium strategies are increasing (decreasing) with respect to the degree when $f_{xss} < 0$ ($f_{xss} > 0$) irrespective of the properties of complementarity or substitutability of the utility function. This result arises from the fact the the assumptions of stability and $SOSD$ of the statistics s together with $f_{xss} < 0$ ($f_{xss} > 0$) imply that the marginal utility of action x_i is increasing (decreasing) in k. Furthermore, we want to stress that this is not only a theoretical and abstract case but an effective one. Indeed, under degree independence, the mean represents a case of statistic satisfying the assumptions in point 1 and, as described in Blume et al. (2010), it is largely used in the empirical literature on peer effects and reference groups.

Point (2) (and (3), respectively) shows that when the statistic s is increasing (decreasing) with respect to the degree, then the equilibrium strategies are increasing (decreasing) with respect to the degree in the case of strategic complements and decreasing (increasing) with respect to the degree in the case of strategic substitute. Examples of statistics that satisfy the properties in point (2) and (3) under degree independence are, respectively, the maximum (as in *one shot games*, see for example in NG and Boncinelli and Pin 2012) and the minimum (as in *weakest links* or minimum effort games).

In order to give an intuition of the forces at work, consider individuals arranged on a random network characterized by degree independence. Suppose their utility function is affected only by the higher action across their neighbors (in this case s is $FOSD$ increasing). Then for individuals with high degree it is more profitable to play higher (lower) actions when the utility function is characterized by strategic complement (substitutes).

The typical statistics used in many empirical applications is the mean. Point (1) of Proposition 2 tells us that if, under degree independence, we have $f_{xss} = 0$, then we will have only the *all equal* equilibrium, but it does not tell us when this *all equal* equilibrium can be excluded in the other cases. Because of this we derive some results in Appendix C for which we provide an example here below.

[9] Note that the assumptions on the properties of statistic s are not so restrictive. Indeed many of the statistics used in the literature of strategic interaction satisfy these assumptions. Ideally, we would like to take the less restrictive assumption of SOSD in points (2) and (3) but it requires to have more assumptions on the shape of the function f.

Example 1. *Suppose that P has degree independence and that s is the mean. Assume that $\mathcal{X} = \{0,1\}$, that*

$$f(x,s) = x\left(-\alpha s^2 - \beta s + 1\right) \quad ,$$

and that $c(x) = c \cdot x$, with $\alpha + \beta > 1 - c > 0$. These conditions tell us that f has the substitute property, because $f_{xs} = -2\alpha s - \beta$ will always be negative – but this is not what determines the characterization of the equilibrium. We also don't impose conditions on the sign of α. First of all, under these conditions we cannot have an all equal equilibrium where all play pure strategy 0 or pure strategy 1.
Suppose now that there is an equilibrium in which all players mix between 0 and 1, which means that all players are indifferent between the two actions and, for any k with positive support from P, we have

$$\begin{aligned} E_k(\alpha s^2 + \beta s) &= \\ \alpha Var_k(s) + \alpha\left(E_k(s)\right)^2 + \beta E_k(s) &= 1 - c \quad . \end{aligned}$$

However, as k increases, $Var_k(s)$ decreases, and then we cannot have all players indifferent, independently on their degree k. So, we exclude the all equal equilibrium and point (1) of Proposition 2 tell us that we have a monotonic equilibrium that will be increasing or decreasing depending on the sign of $f_{xss} = 2\alpha$. ▮

Even if Propostion 1 seems less intuitive to apply than Proposition 2, it turns out to be more general. From the next example it is clear that the standard distinction between complements and substitute is not enough to determine the characterization of equilibria.

Example 2. *Consider P such that networks are undirected, nodes can have only degree 1 or 2, and they face ex–ante symmetric probability $0 < p < 1$ to find all neighbors of the same degree, and $1 - p$ of finding all neighbors of the other degree (so, when $p \to 1$ we have a network made almost only by circles and disconnected couples, as p decreases we have more and more triplets of nodes in a line). Suppose that nodes play symmetric pure strategies $y_1, y_2 \in \mathcal{X} \subseteq \mathbb{R}_+$, and that the statistic s is the sum of neighbors' actions.*
Proposition 1 tells us that we need to consider the relation between

$$E_1(f_x|x,\vec{\sigma},P) = pf_x(x,y_1) + (1-p)f_x(x,y_2)$$

and

$$E_2(f_x|x,\vec{\sigma},P) = pf_x(x,2y_2) + (1-p)f_x(x,2y_1) \quad .$$

Assume that $1/3 < p < 2/3$*, so that* $2p > (1-p)$ *and* $2(1-p) > p$*. Assume also that* f_x *is increasing (*$f_{xs} > 0$*, complementarity between* x *and* s*) and convex (*$f_{xss} > 0$*) in* s*, so that* $f_x(x, 2s) - f_x(x, 0) > 2\Big(f_x(x, s) - f_x(x, 0)\Big)$*. Then, for any* $y_1, y_2 \geq 0$*, the following holds*

$$\begin{aligned} E_2(f_x|x, \vec{\sigma}, P) &= p\Big(f_x(x, 2y_2) - f_x(x, 0)\Big) + (1-p)\Big(f_x(x, 2y_1) - f_x(x, 0)\Big) + f_x(x, 0) \\ &\geq 2p\Big(f_x(x, y_2) - f_x(x, 0)\Big) + 2(1-p)\Big(f_x(x, y_1) - f_x(x, 0)\Big) + f_x(x, 0) \\ &> (1-p)\Big(f_x(x, y_2) - f_x(x, 0)\Big) + p\Big(f_x(x, y_1) - f_x(x, 0)\Big) + f_x(x, 0) \\ &= E_1(f_x|x, \vec{\sigma}, P) \end{aligned}$$

So, according to Proposition 1, in every symmetric Bayesian Nash equilibrium of the network game, optimal best responses are increasing, such that $x_1^* < x_2^*$*.* ▮

Last example combines results from points (1) and (2) of Proposition 2 assuming a not too strong degree correlation,. It is possible to show that when P has not degree independence the characteristics of complementarity and substitutability still play a role in shaping the equilibrium strategies even if the statistics s is stable and converging. In general, Proposition 1 uses simple best response arguments to establish a necessary and sufficient condition under which all equilibria are monotone either increasing or decreasing. When this condition fails, e.g. because positive or negative degree correlation is too strong, a monotone equilibrium may still always exist, but both increasing and decreasing equilibria may coexist. On this see the following example and the discussion that we provide in the conclusion.

Example **3.** *Consider the network in Figure 2. Even if it is a unique network, a node knowing only own degree may not know in which of the disconnected components she will be placed. A node with degree 1 attributes probability* 0.1 *on the event that her only neighbor also has degree 1; a node with degree 2 knows for sure that both her neighbors will have the same degree, and the probability that they also have degree 2 is* 0.25*. So, this network shows degree disassortativity.*

The action space of the nodes is $\mathcal{X} = \{0, 1\}$*, the statistic* s *is just the sum of neighbors actions (a case of increasing statistic) and the payoff is*

$$\Pi_{k_{i,g}}(x_i, s) = \frac{x_i}{1+s} - c \cdot x_i \quad ,$$

with $c = 0.6$*. This is a game of substitutes with negative externalities.*

This network game has the two following equilibria: a decreasing one in which degree–1 nodes play 1 and degree–2 nodes play 0, but also an increasing one in which degree–1 nodes *play 0 and degree–2 nodes play 1.* ∎

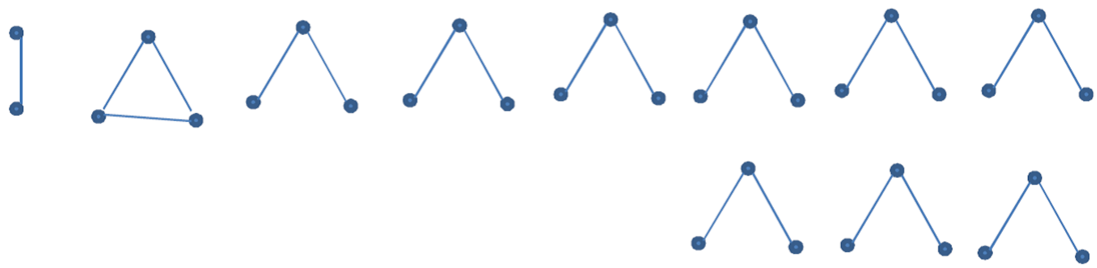

Figure 2: The network considered in Example 3.

4 Relation with previous literature

The results provided in Propositions 1 and 2 include all the cases that we are aware of in the literature on Bayesian network games. In particular, they generalize those of NG in several ways. First, they depend on whatever *statistics* that enters in the strategic interaction, and on its relation with the game structure. Then, the classical distinction between substitutability and complementarity holds only when the statistic is naturally monotonic, otherwise we need to check for third cross-derivatives or for monotonicity of expected marginal profits.

So, what are the analogies with NG? Most of the results in that paper (from their Proposition 2 on) are based on what they call Property A: the value of a statistic s computed on a vector does not change when the vector size is increased by adding a null element. The formal definition is easy if we remember that s is actually a class of function from $\mathbb{R}^k$ to $\mathbb{R}$, for any $k \in \mathbb{N}$, and that they are all anonymous on the arguments.

Definition **5.** *A statistic s satisfies property A if for every $\vec{x} \in \mathbb{R}^k$ and $\vec{x}_{+0} = (0, \vec{x})' \in \mathbb{R}^{k+1}$, we have that $s(k, \vec{x}) = s(k+1, \vec{x}_{+0})$.*

The easiest example of statistic that does not satisfy this property is the average. Also, Property A alone is not enough to guarantee a monotonic equilibrium in the context of our generalized model. The following example shows that if s is not monotonic with respect to its arguments, we may not have monotonic equilibria. Then we show that, assuming monotonicity of s, property A leads to the case of FOSD increasing statistics.

Example **4.** *Consider the case in which $\mathcal{X} = \{0, 1\}$, with the statistic s defined on every vector of at least two elements, as the difference between its two greatest elements. This* *statistic clearly satisfies Property A from NG, but none of the Definitions from 1 to 3. Since* $\mathcal{X} = \{0, 1\}$, *we have that* s *is* 1 *if and only if there is one and only one element* 1 *in the vector, otherwise it is* 0. *Consider the case of degree independence, so that the matching process is i.i.d.. So, if a fraction* p *of the nodes plays* 1, *then the probability that* s *is* 1 *is*

$$p_k = k \cdot p(1-p)^{k-1} \quad ,$$

which can be non–monotonic in k. *Imagine that the degree distribution is such that a fraction* .15 *of the nodes have degree* 2, *a fraction* .7 *have degree* 3, *and the remaining fraction* .15 *of nodes have degree* 4. *Payoff is* $\Pi_{k_{i,g}}(x_i, s) = \sqrt{x+s} - c \cdot x_i$ *(a case of substitutes) where* $c = .75$.

In this case there is an equilibrium in which nodes with degree 2 *and* 4 *contribute with* 1, *while nodes with degree* 3 *contribute by* 0. *With this strategy profile* $p = .3$, $p_2 = .42$, $p_3 = .441$ *and* $p_4 = .4116$. *The expected net value of contributing is given by*

$$\Delta_k = p_k \left(\sqrt{2} - 1\right) + (1 - p_k) \quad ,$$

and this is above .75 *for* $k \in \{2, 4\}$, *but not for* $k = 3$, *proving that this strategy profile is an equilibrium.* ∎

In the last example the statistic s produces a different ordering of vectors of the neighbours' actions with respect to the model in NG, where the definitions of *complements* and *substitutes* are implicitely based on the natural partial ordering between vectors.[10] Therefore, our definitions of strategic substitutes and strategic complements do not coincide with those of NG when the statistic does not respect the natural partial ordering of vectors. However, even if in the NG's payoff function there is not an explicit statistic s but only a vector of the neighbors' actions, it is possible to check that the main results in NG are a specific case of our framework.

The next proposition provides a link between our formulation and the results from NG.

PROPOSITION **3.** *Suppose that* $\mathcal{X} \subseteq \mathbb{R}_+$, *and that* $0 \in \mathcal{X}$. *If* s *is a monotonically increasing function and satisfies Property A, then* s *is FOSD increasing with respect to a network formation process* P *with degree independence.*

Therefore, when s is monotonically increasing, satisfies property A from NG, and the

[10] As an example, in the case of degree equal three the ordering of the possible vectors of neighbours' actions (from the smaller to the larger) under statistics s is: $(0,0,0) \sim (0,1,1) \sim (1,1,1) \prec (0,0,1)$. Using the criterion NG the ordering would be: $(0,0,0) \prec (0,0,1) \prec (0,1,1) \prec (1,1,1)$.

network formation process P is characterized by degree independence, then the characteristics of Bayesian Nash equilibria are described in point 2 of our Proposition 2. We note that choosing a statistic s that is monotonically increasing in all its arguments, i.e. respecting the natural partial ordering,[11] the definitions of strategic substitutes/complements coincide between our framework and NG. So it is straightforward that all the results from NG are specific cases of our model.

5 Conclusion

In many applications, externalities, peer effects, learning and/or strategic interactions between individuals, can all be easily modeled as network games between agents of a social network. The neighbors of a node are in one to one correspondence with the peers of the individual, and the actions of those neighbors enter in the payoff function of the individual. The way in which the neighbors' actions affect the individual payoff can be, for example, the average, as in most of peer effects framings or non–Bayesian learning models, the maximum, as in local public goods games, or the minimum, as in vaccination games against the risk of pandemic contagion. The existing literature points out the influence that the nature of this payoff function, and in particular whether there is a complementarity or a substitute effect between own action and the statistic on the actions of neighbors, has on the correlation between degree in the network and action taken in equilibrium. However, in this note we have shown that in all the above cases it is important to know also the nature of the statistics.

The table in Figure 3 summarizes the classification that we apply and how our results integrate with those in the literature. Our Proposition 1 and 2 provide a characterization of the domain in which monotonic equilibria are the only possible ones, also when there is some positive or negative degree correlation, as in Example 2. Example 3 shows instead that under too extreme correlation (a negative one in the example) both increasing and decreasing equilibria can coexist.[12] Finally, with Proposition 3 we have included all the results from NG into the domain of increasing statistics.

Additional comments are the following. First of all, a statistic may not be stable or monotone, as in our Example 4. In such cases equilibria that are non–monotonic in the degree may exist. Finally, the case of stable statistic is the most intuitive and used in the empirical literature on peer effects and reference groups. We have shown that in this case it

[11] This relates naturally to standard utility theory and to the assumption of non–satiation.

[12] It should be noted that Example 3 falls exactly in the assumptions of Proposition 4 in NG, which is about existence of monotonic equilibria (decreasing in the case of the example), but does not claim that all equilibria are of that form.

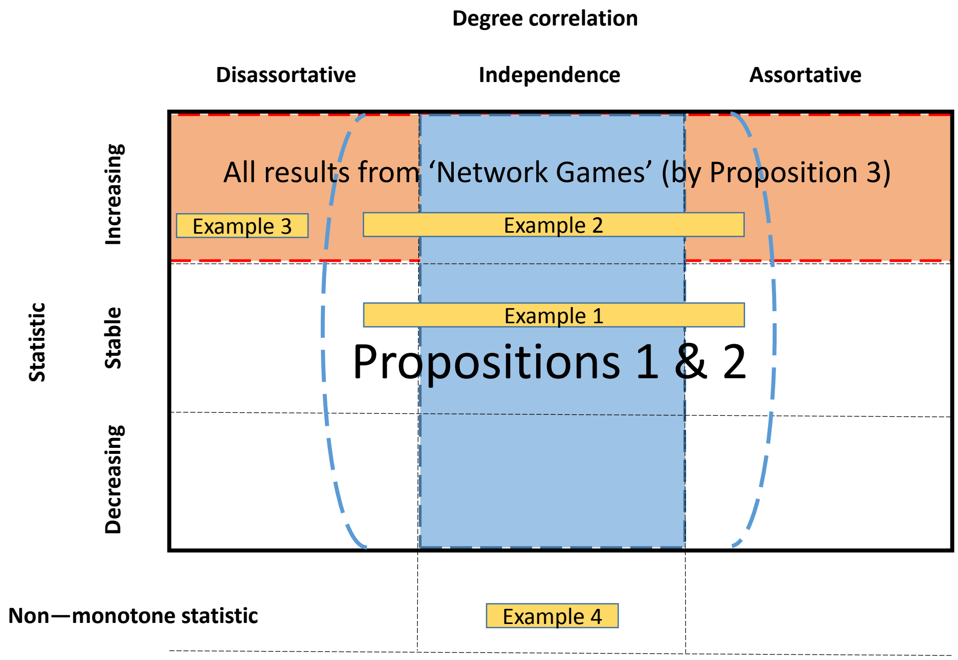

Figure 3: Classification of the statistic (rows) adds a dimension on the analysis of network games on top of degree correlation (columns), and the one on complements/substitutes games (not shown here).

becomes important to look also at third order derivatives of the payoff function. However, we leave to future research the identification of necessary conditions that are more easily identifiable on real data, and with a clearer economic interpretation.

We believe that our results will turn out to be useful for both theorists studying specific models, and for applied researchers studying the interactions of economic agents.

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Appendix A Some Lemmas

We extend results from utility theory (see e.g. section 4.2 in the notes from Levin 2006) to our context with the following corollary.

We define by $\phi_k(x)$ a bounded probability distribution function on $\mathbb{R}$ that depends on $k \in \mathcal{K}$, and we call $\Phi_k(x)$ its cumulative distribution. Following Definition 4, we say that

$\phi_k(x)$ satisfies second order stochastic dominance (SOSD) if for every $y \in \mathbb{R}$ we have:

$$\int_{-\infty}^{y} \Phi_{k+1}(x)\ dx \leq \int_{-\infty}^{y} \Phi_k(x)\ dx \quad . \tag{8}$$

Lemma **A.** *If statistic s is stable and satisfies SOSD, and $u(\cdot)$ is a positive valued concave function, then*

$$\int u(s) \cdot \phi_{k+1}(s)\ ds \geq \int u(s) \cdot \phi_k(s)\ ds \quad . \tag{9}$$

If instead $u(\cdot)$ is a positive valued convex function, then

$$\int u(s) \cdot \phi_{k+1}(s)\ ds \leq \int u(s) \cdot \phi_k(s)\ ds \quad . \tag{10}$$

Proof: Let us start by assuming that $\phi_k(x)$ is stable and satisfies SOSD, and that u is positive valued and concave, i.e. that $u > 0$ and $u_{xx} \leq 0$. Let us call $I(x) \equiv \int_{-\infty}^{x} \Phi_k(y)\ dy - \int_{-\infty}^{x} \Phi_{k+1}(y)\ dy$, which is non–negative by inequality (8). Also, integrating by parts

$$\int_{-\infty}^{x} \Phi_k(y)\ dy = [y \cdot \Phi_k(y)]_{-\infty}^{x} - \int_{-\infty}^{x} y\ d\Phi_k(y)$$

Replacing into the expression for $I(x)$ and taking its limit to ∞, the stability of $\phi_k(x)$ implies that

$$\lim_{x\to\infty} I(x) = \int_{-\infty}^{\infty} y\ d\Phi_{k+1}(y)) - \int_{-\infty}^{\infty} y\ d\Phi_k(y) = 0 \quad . \tag{11}$$

Since $I(x)$ is non–negative, also

$$-\int_{-\infty}^{\infty} u_{xx}(x) I(x) dx \geq 0 \tag{12}$$

Integrating by parts, the expression (12) is equivalent to

$$[-u_x(x) \cdot I(x)]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} u_x(x)\Big(\Phi_k(x) - \Phi_{k+1}(x)\Big)dx \geq 0 \tag{13}$$

By (11) the first term is equal to 0. Then again integrating by parts we get

$$\Big[u(x)\Big(\Phi_k(x) - \Phi_{k+1}(x)\Big)\Big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u(x)\Big(\phi_k(x) - \phi_{k+1}(x)\Big)dx \geq 0 \tag{14}$$

It is directly verifiable that the first term is equal to 0. Therefore inequality (12) can be rewritten as:

$$-\int_{-\infty}^{\infty} u(x)\Big(\phi_k(x) - \phi_{k+1}(x)\Big)dx \geq 0 \quad , \tag{15}$$

so $\int u(x)\Big(\phi_k(x) - \phi_{k+1}(x)\Big)dy$ is non–positive, which proves the statement. With the same reasoning, the case in which u is positive valued and convex, i.e. that $u > 0$ and $u_{xx} \geq 0$, leads to the reverse inequality. ▮

Corollary **A.** *If statistic s is stable and satisfies SOSD, then* $Var_{k+1}(s|\sigma, P) < Var_k(s|\sigma, P)$.

Proof: We have that

$$Var_k(s|\sigma, P) = \int s^2 \cdot \phi_k(s)\ ds - (E_k(s))^2 \quad .$$

So, when s is stable $(E(s))^2$ remains constant, and since s^2 is convex we get the result from previous Lemma A. ▮

Lemma **B.** *If the statistic s is FOSD increasing, and* $u(\cdot)$ *is a positive valued non–decreasing (non–increasing) function, then*

$$\int u(s) \cdot \phi_{k+1}(s)\ ds \geq \Big(\leq \text{, respectively}\Big) \int u(s) \cdot \phi_k(s))\ ds \quad . \tag{16}$$

If the sample statistic s is FOSD decreasing, and $u(\cdot)$ *is a positive non–decreasing (non–increasing) function, then*

$$\int u(s) \cdot \phi_{k+1}(s)\ ds \leq \Big(\geq \text{, respectively}\Big) \int u(s) \cdot \phi_k(s))\ ds \quad . \tag{17}$$

Proof: from the proof of Lemma A we have that:

$$\int_{-\infty}^{\infty} u_x(x)\Big(\Phi_k(x) - \Phi_{k+1}(x)\Big)dy = -\int_{-\infty}^{\infty} u(x)\Big(\phi_k(x) - \phi_{k+1}(x)\Big)dx$$

So, a sufficient condition to determine the sign of

$$\int_{-\infty}^{\infty} u(x)\Big(\phi_{k+1}(x) - \phi_k(x)\Big)dx$$

is the sign of the integral in the left–hand side.

When statistc s is FOSD increasing and $u(\cdot)$ is non-decreasing (non-increasing) we have that $\Big(\Phi_k(x) - \Phi_{k+1}(x)\Big) \geq 0$ and $u_x(x) \geq 0$ ($u_x(x) \leq 0$) for every x so the integral on the left hand side is non-negative (non-positive). The second part of the lemma is proved in a similar way and it is omitted ∎

Appendix B Proof of the Propositions

We first prove the technical result of Proposition 1, and then we use it as a lemma to prove Proposition 2. Finally, we prove Proposition 3.

Proof of Proposition 1 (page 8): Suppose the quantity in (7) is strictly decreasing in k. In order to compute x_k^*, the first order conditions are:

$$E\left[\frac{\partial}{\partial x} f\left(x^*, s(\vec{x}_{i,g})\right)\right] = \frac{\partial}{\partial x} c(x^*) \quad ,$$

or equivalently

$$\int_{-\infty}^{\infty} f_x\left(x^*, y\right) \cdot \phi_k(y|\vec{\sigma}, P))\ dy = c_x(x^*) \quad .$$

Since f_x and c_x are both strictly positive, and they are both strictly monotone with different sign, there is a unique $x_k^* \in \mathbb{R}$ that satisfies the equality. If this $x_k^* \in \mathcal{X}$, then $\sigma_k^* = x_k^*$ is a pure strategy. However, this x_k^* could not be an element of $\mathcal{X}$. In this last case the optimal σ_k^* should play one of the two (possibly both), left–most x_k^{*-} or rightmost x_k^{*+}, elements of $\mathcal{X}$ closest to x_k^* in $\mathbb{R}$. If x_k^{*-} and x_k^{*+} give different expected payoffs, then σ_k^* would be a pure strategy playing the best one of the two. Only in the case in which x_k^{*-} and x_k^{*+} give the same payoff, then any randomization σ_k^* between those two points would be an optimal best response.

By assumption, for an agent with degree $k+1$, we have that

$$\int_{-\infty}^{\infty} f_x\left(x^*, y\right) \cdot \phi_{k+1}(y|\vec{\sigma}, P))\ dy < c_x(x^*) \quad . \tag{18}$$

Left–hand part of (18) is decreasing in x, right–hand part is non–decreasing, and then to balance it back we need $x_{k+1}^* < x_k^*$. Equality of best response strategies may hold only when $x_k^* \notin \mathcal{X}$ but not when σ_k^* is a randomization between two points, because if the expected payoff of x_k^{*-} and x_k^{*+} is the same for an agent with degree k, then x_k^{*+} will provide a lower payoff than x_k^{*-} for an agent with degree $k+1$.

This proves that if the quantity in (7) is strictly decreasing in k, for every $x \in \mathcal{X}$, then in every symmetric Bayesian Nash equilibrium of the network game the optimal action σ_k^* is

FOSD decreasing in k.

The reverse inequality can be proved analogously. ∎

Proof of Proposition 2 (page 9): Point 1. Let s be stable and converging. The derivative with respect to x_i of i's expected payoff $\frac{\partial}{\partial x_i}\Pi^e_k(x_i, \vec{\sigma}_{-i}) = \int_{-\infty}^{\infty} f_x(x,s) \cdot \phi_k(s|\vec{\sigma}, P)\ ds$. If $f_{xss} > 0$ by Lemma A we have that the derivative is decreasing in k. Then, by Proposition 1, it directly follows that σ^* is FOSD decreasing. If $f_{xss} < 0$ by Lemma A we have that the derivative is increasing in k. Then, by Proposition 1 directly follows that σ^* is FOSD increasing.

Point 2. Let s be FOSD increasing . The derivative with respect to x_i of i's expected payoff $\frac{\partial}{\partial x_i}\Pi^e_k(x_i, \vec{\sigma}_{-i}) = \int_{-\infty}^{\infty} f_x(x,s) \cdot \phi_k(s|\vec{\sigma}, P)\ ds$. If $f_{xs} \geq 0$ by Lemma B we have that the derivative is increasing in k. Then, by Proposition 1 directly follows that σ^* is FOSD increasing. If $f_{xs} \leq 0$ by Lemma B we have that the derivative is decreasing in k. Then, by Proposition 1 directly follows that σ^* is FOSD decreasing.

Point 3. Let s be FOSD decreasing . The derivative with respect to x_i of i's expected payoff $\frac{\partial}{\partial x_i}\Pi^e_k(x_i, \vec{\sigma}_{-i}) = \int_{-\infty}^{\infty} f_x(x,s) \cdot \phi_k(s|\vec{\sigma}, P)\ ds$. If $f_{xs} \geq 0$ by Lemma B we have that the derivative is decreasing in k. Then, by Proposition 1 directly follows that σ^* is FOSD decreasing. If $f_{xs} \leq 0$ by Lemma B we have that the derivative is increasing in k. Then, by Proposition 1 directly follows that σ^* is FOSD increasing. ∎

Finally, we prove Proposition 3, that relates our result with those in NG.

Proof of Proposition 3 (page 14): For every y in $\mathbb{R}$ we have that

$$\Phi[y|s,k] = Prob\left[\vec{x} \in \mathcal{X}^k : s(\vec{x}) \leq y\right] \quad ,$$

and

$$\Phi[y|s,k+1] = Prob\left[\vec{x} \in \mathcal{X}^{k+1} : s(\vec{x}) \leq y\right] \quad ,$$

Consider the operator $\sigma_0 : \mathcal{X}^{k+1} \to \mathcal{X}^{k+1}$ that takes a random element of $\vec{x}$ (with uniform probabilities) and puts it to 0. Then, also $E\left[s \circ \sigma_0(\cdot)\right]$ is a statistic (as it is anonymous), and by monotonicity of s it is always the case that $s(\vec{x}) \geq E\left[s \circ \sigma_0(\vec{x})\right]$. Note also that it is probabilistically the same to extract with uniform probabilities k elements, or to extract $k+1$ elements, and then remove randomly one of them. So, we have that for every y

$$\Phi[y|s,k+1] \leq \Phi[y|E\left[s \circ \sigma_0(\cdot)\right], k+1] = \Phi[y|s,k] \quad ,$$

which proves the statement. ∎

Appendix C Non *all equal* strategies

One concern arising from Proposition 1 is that it allows for non–monotone equilibrium strategies, so that in principle an equilibrium could have all the players playing the same strategy, independently on their degree. We address this issue here, starting with some definitions.

Definition **A.** *A symmetric strategy $\vec{\sigma}$ is* all equal *if $\sigma_k = \sigma_h$ for any two different $k, h \in \mathcal{K}$.*

Next definition is a general case of Definition 2.

Definition **B.** *Given a network formation process P, a statistic s is* weakly increasing *(or* weakly decreasing*) with respect to P if for every non* all equal *$\vec{\sigma}$ and $k \in \mathcal{K}\backslash\{n-1\}$ we have that $E_{k+1}(s|\vec{\sigma}, P) > E_k(s|\vec{\sigma}, P)$ (or respectively, if for every non* all equal *$\vec{\sigma}$ and $k \in \mathcal{K}\backslash\{n-1\}$ we have that $E_{k+1}(s|\vec{\sigma}, P) < E_k(s|\vec{\sigma}, P)$).*

Proposition **A.** *If s is is* weakly increasing *or* weakly decreasing *with respect to P, then there cannot exist and all equal equilibrium in mixed strategies.*

Proof: In equilibrium nodes face a realization of the mixed strategy $\vec{\sigma}$, and this realization is not all equal in most of the cases. So, the expected s from a mixed strategy $\vec{\sigma}$ varies if s is weakly decrasing or decreasing. ▮

Next result builds on proposition 1, showing when the conditions for having all equal equilibria are non generic.

Proposition **B.** *Consider some P. Suppose that $\mathcal{X} = \{x_1, x_2, \ldots x_z\}$ has cardinalty z. Write the cost function as $c(x) + \gamma x$, and call*

$$\bar{\kappa} \equiv f_x\left(x_1, s(x_1 \cdot \vec{1})\right) - c'(x_1) \quad ,$$

and

$$\underline{\kappa} \equiv f_x\left(x_z, s(x_z \cdot \vec{1})\right) - c'(x_z) \quad ,$$

and suppose that $\underline{\kappa} < \bar{\kappa}$ Suppose finally that s is is weakly increasing *or* weakly decreasing *with respect to P. Then:*

1. *if (7) is strictly decreasing in k for any $\vec{\sigma}$, then in every symmetric Bayesian Nash equilibrium of the network game the equilibrium strategy σ_k^* is FOSD decreasing in k, and there are at most $z-2$ values for $\gamma \in (\underline{\kappa}, \bar{\kappa})$ such that there exist an all equal equilibrium.*

2. *if (7) is strictly increasing in* k *for any* $\vec{\sigma}$*, then in every symmetric Bayesian Nash equilibrium of the network game the equilibrium strategy* σ_k^* *is FOSD increasing in* k*, and there are at most* $z-2$ *values for* $\gamma \in (\underline{\kappa}, \bar{\kappa})$ *such that there exist an all equal equilibrium.*

Proof: For $\gamma \in (\underline{\kappa}, \bar{\kappa})$, we cannot have an all equal equilibrium where all play x_1 or all play x_z.
We can have an all equal equilibrium where all play some x_i, with $i \in \{2, \dots z-1\}$, if $\gamma = f_x\left(x_i, s(x_i \cdot \vec{1})\right) - c'(x_i)$, and there are at most $z-2$ values for $\gamma \in (\underline{\kappa}, \bar{\kappa})$ such that this can happen.
Otherwise, Proposition 1 tells us that every symmetric Bayesian Nash equilibrium of the network game the equilibrium strategy σ_k^* is FOSD decreasing (or increasing) in k, while Proposition A tells us that we cannot have an all equal equilibrium. ∎