Incomplete Preferences and Learning over a Finite Choice Set*

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Abstract

A decision maker with incomplete preferences learns when they are able to order additional pairs of elements from a finite choice set. When learning satisfies a set of behavioural axioms, every new preference relation preserves the previous relation, but extends it (in the order-theoretic sense) until preferences settle on one - among finitely many - complete orderings in a finite number of learning periods. Thus, we associate with the resulting set of learning processes (the different ways in which learning occurs) the set of order-extensions of the preference relation of an indecisive decision maker. Several applications of the framework are then discussed, including the measurement of indecisiveness, how learning occurs from social interactions, the implications of learning for axioms of revealed preference, and for choice deferral.

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One does not argue about tastes for the same reason that one does not argue about the Rocky Mountains - both are there, and will be there next year too, and are the same to all men (Stigler and Becker, 1977).

Because endogenous preferences involve learning or genetic changes, behaviour in the same situation changes over time (Bowles, 2004).

1 Introduction

In the classical theory of choice, it is generally taken that the decision maker is rational if their preference relation over a choice set is transitive and complete. By the relation being *complete*, we mean that for any two alternatives x and y the decision-maker either prefers x to y, y to x or is indifferent between x and y. Though *completeness* is a generally accepted axiom, there is increasing evidence that decision makers are sometimes unable to order pairs of alternatives. A decision maker who is sometimes unable to order alternatives in the choice set is said to have *incomplete preferences*, or is said to be *indecisive*. Danan and Ziegelmeyer (2006) and Cettolin and Riedl (2015) provide evidence that preferences are incomplete in experiments involving choices under risk and uncertainty, respectively. Qiu and Ong (2017) design an experiment aimed at disentangling indifference from indecisiveness in choice and find strong evidence in favour of the latter. Costa-Gomes et al. (2016) find that forcing subjects to choose increases the extent to which their choice behavior is inconsistent. They report that a substantial fraction of subjects' decisions can be explained by preferences being incomplete.

Incompleteness of preferences raises many conceptual issues in relation to diverse questions such as the definition of transitivity (Mandler, 2005), utility representation (Richter, 1966; Peleg, 1970; Ok, 2002; Dubra, Maccheroni and Ok, 2004), the axioms of revealed preference (Arrow, 1959; Sen, 1971; Eliaz and Ok, 2006) or the measurement of decisiveness (Gorno, 2018), to mention a few such areas. Incompleteness of preferences also raises the question as to whether decision makers *can learn* to resolve their indecisiveness. Thus, the purpose of this paper is three-fold. The first purpose of the paper is to develop a theory of learning in the context of incomplete preferences. We naturally define *learning* as the process of being able to compare an increasing subset of all the pairs of alternatives in a choice set X. The central concern here is, starting from a situation where agents are completely indecisive, to formalize learning as the process of acquiring transitive and complete preferences.

The second purpose of the paper is to present the set of order extensions of an incomplete preference relation as a general framework for examining models of indecisiveness in economics. The set of order extensions of a finite ordered set was studied extensively by Brualdi, Jung and Trotter (1994) and Pouzet et al. (1995) in the mathematical sciences, but it would be fair to say that its importance in decision theory has not received the attention we feel it deserves. Specifically, we relate the type of learning studied in this paper to sequences of ordered sets in the set of order extensions of an incomplete preference relation. We also develop several applications that follow from the type of learning studied in this paper, that exploit the structure of the set of order extensions.

The third purpose is to provide a unified framework that reconciles the two radically different views of the decision-maker as rational versus boundedly rational, as emphasized by the two contrasted opening quotations of the paper. Specifically, we view learning as the connecting thread that transforms an indecisive decision-maker with time-varying preferences into the rational individual with complete and time-invariant preferences, where most textbook discussions of the theory of the rational decision-maker begin.

The context of bounded rationality considered in this paper is rooted in the decision maker's inability to order some pairs of alternatives. Throughout this paper the choice set (the grand set of alternatives) X is finite and there are three building blocks in the framework we develop: (i) the set of of reflexive and transitive preference relations on X is called *the set of tastes*, (ii) a sequence L of preference relations on the set of tastes is called a *learning process* and (iii) the t-th term of a learning process L is called the t-th *stage of learning* associated with L. The main result of the paper is to associate learning processes satisfying a set of behavioural axioms with sequences of preference relations, specifically maximal chains, in the set of order extensions of the preference relation of an indecisive decision maker. Particularly, when learning satisfies the behavioural axioms, every new preference relation is shown to preserve the previous preference relation, and extends it (in the order-theoretic sense). Ultimately, preferences are shown to settle in finite number of learning stages on one (among finitely many) transitive and complete preference relations.

Several additional results follow from the main result discussed above. Using a famous theorem in order theory (Dilworth, 1950), we exploit the property that the set of order extensions of the indecisive decision maker is *ranked* (Brualdi, Jung and Trotter, 1994; Pouzet et al., 1995) in order to partition this set into subsets corresponding to the various stages of learning. This decomposition of the set of order extensions is exploited to establish that the behavioural axioms of learning deliver a theory of *monotonic learning*. That is, along any learning processes described by the axioms, indecisiveness steadily decreases, until preferences settle in a finite number of learning periods on a complete preference relation. Thus, under the behavioural axioms, the possibility of learning eventually endows decision makers with rational (i.e. transitive and complete) preference orderings.

We provide several different applications of the framework that is developed. Firstly, following Ok (2002) and Gorno (2018), we revisit the problem of ordering decision makers by the extent of their indecisiveness. Here we provide two new orderings of indecisiveness, and we also discuss how these orderings relate to existing approaches in the literature. We then turn to the discussion on how preferences and norms are acquired (Carpenter and Nakamoto, 1989; Druckman, 2004; Bowles, 1998; Young, 2015), and we reflect on how the process of social interactions may shape preference formation in the context of learning as defined in this paper.

A further application of the paper consists in revisiting the axioms of revealed preference in the context of incomplete preferences (Eliaz and Ok, 2006), where, here, we allow for learning. Our discussion follows the framework of *extended choice problems* as developed by Bernheim and Rangel (2007) and Salant and Rubinstein (2008), allowing us to provide a possible formulation of the weak axiom of revealed non-inferiority (WARNI) in the context where decisions are subject to framing effects. It is clear that as a consequence of learning, the preference ordering of the decision maker changes, and as a result, we have to live with violations of the weak axiom of revealed preference (WARP) as well as WARNI. Though we can only state negative results in this context, we nonetheless provide some results about where in the various stages of learning the resulting weak axioms of revealed preference and non-inferiority may be violated. Our final application builds on the framework of extended choice correspondences, where we study the timing of choice deferral (Gerasimou, 2018) in the context of learning.

Clearly, learning takes many different meanings in the discipline of economics. Providing a detailed survey of this area is beyond the scope of this paper, and instead we mention a few diverse examples. Learning in growth theory enables producers to adopt more efficient technologies in the context of learning by doing (Arrow, 1962), learning by learning and other generalizations (Greenwald and Stiglitz, 2015). In the field of macroeconomics, learning pertains to the ability to form forecasts of fundamental economic aggregates such as prices, shadow prices, etc. (Evans and McGaw, 2015). In behavioural game theory for instance, learning enables players to implement Nash equilibrium by trial and error (Young, 2009). Additionally, the literature has investigated how, when, and to what the extent players following certain processes of learning, adaptation, and imitation will end up playing some alternative variants of equilibrium (Fudenberg and Levine, 2009). An important branch of the experimental economics literature on the other hand, has examined what type of learning rules subjects tend to adopt when they learn from experience, by identifying the robustness of behavioural regularities (Erev and Haruvy, 2016).

The paper is structured as follows: Section 2 contains definitions of key concepts pertaining to ordered sets. We also discuss the set of order extensions and illustrate choice in three popular models of bounded rationality in relation to the set of order extensions. Section 3 introduces the behavioural axioms of learning and presents the central result of this paper relating learning processes to sequences of preference relations in the set of order extensions. Section 4 develops the four different applications that follow from the main result of the paper. Section 5 concludes the paper. An appendix develops further examples and contains the proofs of the main results.

2 The Set of Order Extensions

2.1 Poset Concepts

Let Y be an l-element set. A binary relation on Y is reflexive if $x \preceq x$ for all $x \in Y$. The relation $\preceq$ is transitive if $x \preceq y$ and $y \preceq z$ jointly imply $x \preceq z$ for all $x, y, z \in Y$. A pre-order $(Y, \preceq)$ is a reflexive and transitive (not necessarily complete) relation $\preceq$ on a set Y. Two distinct elements x and y of Y are said to be *comparable* if either $x \preceq y$ or $y \preceq x$. The notation $x||y$ is used to denote that x and y are not comparable. The set of incomparable elements in Y associated with the binary relation $\preceq$ is defined as

$$\text{inc}(Y, \preceq) := \{(x, y) \in Y \times Y : x||y\}$$

and its complement $\text{comp}(Y, \preceq) := (Y \times Y) \backslash \text{inc}(Y, \preceq)$ is the set of comparable pairs. When every pair $(x, y) \in Y \times Y$ is $\preceq$-comparable we say that $\preceq$ is a complete relation.

The focus in this paper is on indecisiveness with respect to the asymmetric part of a pre-order $\preceq$. That is, we are assuming that a decision maker recognizes all the elements that are equivalent to one particular element, but is unsure how to order a and b when a and b do not belong to the same indifference class. More formally: assume $(Y, \preceq)$ is a finite pre-ordered set and for $n \leq l$, let the collection of sets $\Phi = \{\Phi_1, ..., \Phi_n\}$ be a partition of Y into n indifference classes. A *System of Distinct Representatives* (SDR) of the set Y is a collection of elements $x_1, ..., x_n$ such that for each $i = 1, ..., n$ there is a unique $x_i \in \Phi_i$. Because $\Phi = \{\Phi_1, ..., \Phi_n\}$ is a partition of Y, we have that $x_i \nsim x_j$ for all $i, j \in \{1, ..., n\}$. Now gather the SDR into a set $X := \{x_1, ..., x_n\} \subseteq Y$. The restriction of the pre-order relation $\preceq$ to the subset X produces a ordered set with additional structure: It is clear that $\preceq$ is reflexive and transitive on the subset X. Now take a pair (x_i, x_j) in X^2 and assume that $x_i \preceq x_j$ and $x_j \preceq x_i$. Because X is constructed as an SDR, it follows that x_i and x_j can only be one unique element. We therefore conclude that $x_i \preceq x_j$ and $x_j \preceq x_i$ imply $x_i = x_j$. A pre-order with the additional property - known as antisymmetry - that for all (x_i, x_j) such that $x_i \preceq x_j$ and $x_j \preceq x_i$ there results $x_i = x_j$ is known as a *partially ordered set.*

Such ordered pairs $(X, \preceq)$ are often also called *ordered sets* or *posets*.

Our interest from here on then is in properties of an n-element partial order $(X, \preceq)$. An element x of $(X, \preceq)$ is a *minimal element* if there is no $y \in X$ such that $y \neq x$ and $y \preceq x$. If X has several minimal elements then these elements are pairwise incomparable. If $(X, \preceq)$ has a unique minimal element $\hat{x}$ then $\hat{x}$ is also called the *minimum* element. Dually we define maximal elements and maximum.

For any reflexive, antisymmetric and transitive relation $\preceq$ on X, we define a $\preceq$-chain to be a sequence of ordered elements $x_1 \preceq x_2 \preceq \cdots \preceq x_m$. We define a $\preceq$-antichain to be a sequence of elements that are pairwise incomparable by the $\preceq$ relation. We define y to be a cover of x, written $x \lhd y$, if $x \preceq y$ and additionally if there is a $z \in X$ such that $x \preceq z \preceq y$, then either $z = x$ or $z = y$. A maximal chain is a sequence of covers $x_1 \lhd x_2 \lhd \cdots \lhd x_m$ that is not a proper subset of any another chain of $(X, \preceq)$. A *complete* or *linear* order $(X, \preceq)$ in this paper refers to a partial order where all elements of X are comparable by the relation $\preceq$, so that $(X, \preceq)$ has the structure of a chain.

Two important summary statistics of finite posets are *height* and *width*. To define these concepts, we first observe that the set of chains of the poset $(X, \preceq)$ may be partially ordered by set inclusion. The maximal elements of this set are specifically maximal chains. Any chain C with a maximum number of elements is called a *maximum chain*. The height of the poset $(X, \preceq)$, denoted $\text{height}(X, \preceq)$ is equal to the number of elements $h := |C|$ of a maximum chain. Similarly, the set of antichains of the poset $(X, \preceq)$ may be ordered by set inclusion, and we may define maximal and maximum antichains. If S is a maximum size antichain of $(X, \preceq)$, the width of the poset is accordingly defined as the cardinality of S, and is denoted $\text{width}(X, \preceq)$. Height is a measure of the degree of completeness of the poset while the width of a poset is a measure of its incompleteness. We note for instance that for an n-element set X, when $(X, \preceq)$ has the structure of a chain, height $(X, \preceq) = n$ and $\text{width}(X, \preceq) = 1$. At the other end, when $(X, \preceq)$ has the structure of an antichain, $\text{height}(X, \preceq) = 1$ and $\text{width}(X, \preceq) = n$.

2.2 Choice and the Set of Order Extensions

An order-extension of a poset $(X, \preceq)$ is a relation $\preceq^e$ which preserves $\preceq$, but allows for more pairs of X to be compared, making $(X, \preceq^e)$ a more complete relation. Formally, the ordered set $(X, \preceq^e)$ is an *order-extension* of $(X, \preceq)$ whenever for all $x, y \in X$ such that $x \preceq y$ there holds $x \preceq^e y$. A *linear extension* of $(X, \preceq)$ is an order-extension of $(X, \preceq)$ that produces a complete ordering. Any such linear extension has $n^* := \begin{pmatrix} n \\ 2 \end{pmatrix}$ comparable pairs from the set $X \times X$, and Szpilrajn (1930)'s theorem ensures that every poset $(X, \preceq)$ has a linear extension.

In this paper, the set of order extensions of an n-element antichain $(X, \preceq_0)$ will play a prominent role. The set of order extensions of an n-element antichain $(X, \preceq_0)$ is a set of posets that is ordered by completeness (Brualdi, Jung and Trotter, 1994). Specifically, every order extension of the antichain $(X, \preceq_0)$ is an element of the set of order extensions, and $E_i := (X, \preceq_i)$ is ranked lower than $E_j := (X, \preceq_j)$ (written $E_i \lessdot E_j$) if $E_j := (X, \preceq_j)$ is an order extension of $E_i := (X, \preceq_i)$. Every order extension of $(X, \preceq_0)$ has $k \in \{0, ..., n^*\}$ comparable pairs from the set $X \times X$. The extension with $k = 0$ comparable pairs is the antichain $(X, \preceq_0)$, which defines the least element of the set of order extensions. Each maximal chain in the set of order extensions is a sequence of order covers, that has for minimum element $E_0 := (X, \preceq_0)$, and for maximum element a linear order $E_\lambda := (X, \preceq_\lambda)$. An element (i.e., a partial order) $E_q := (X, \preceq_q)$ is a cover of $E_i := (X, \preceq_i)$, written $E_i \lhd E_q$, if there is an order extension $E_j := (X, \preceq_j)$ whereby $E_i \lessdot E_j \lessdot E_q$, then either $E_i = E_j$ or $E_q = E_j$. Every maximal element of the set of order extensions of the n-element antichain $(X, \preceq_0)$ is a linear order, and there are $n!$ such maximal elements. Furthermore, given that the set of order extensions is the set of all extensions of the antichain $(X, \preceq_0)$, this set is exhaustive in the sense that it contains all posets that can be constructed from the choice set X.

We call $\text{ext}(X)$ the set of order extensions of the n-element antichain $(X, \preceq_0)$. Consider a maximal chain $M = E_0 \lhd E_1 \lhd \cdots \lhd E_{n^*}$ in the set of order extensions. For a specific order relation $E_i \in \text{ext}(X)$, the set of order extensions of E_i is a subset of $\text{ext}(X)$. The minimum element of this subset

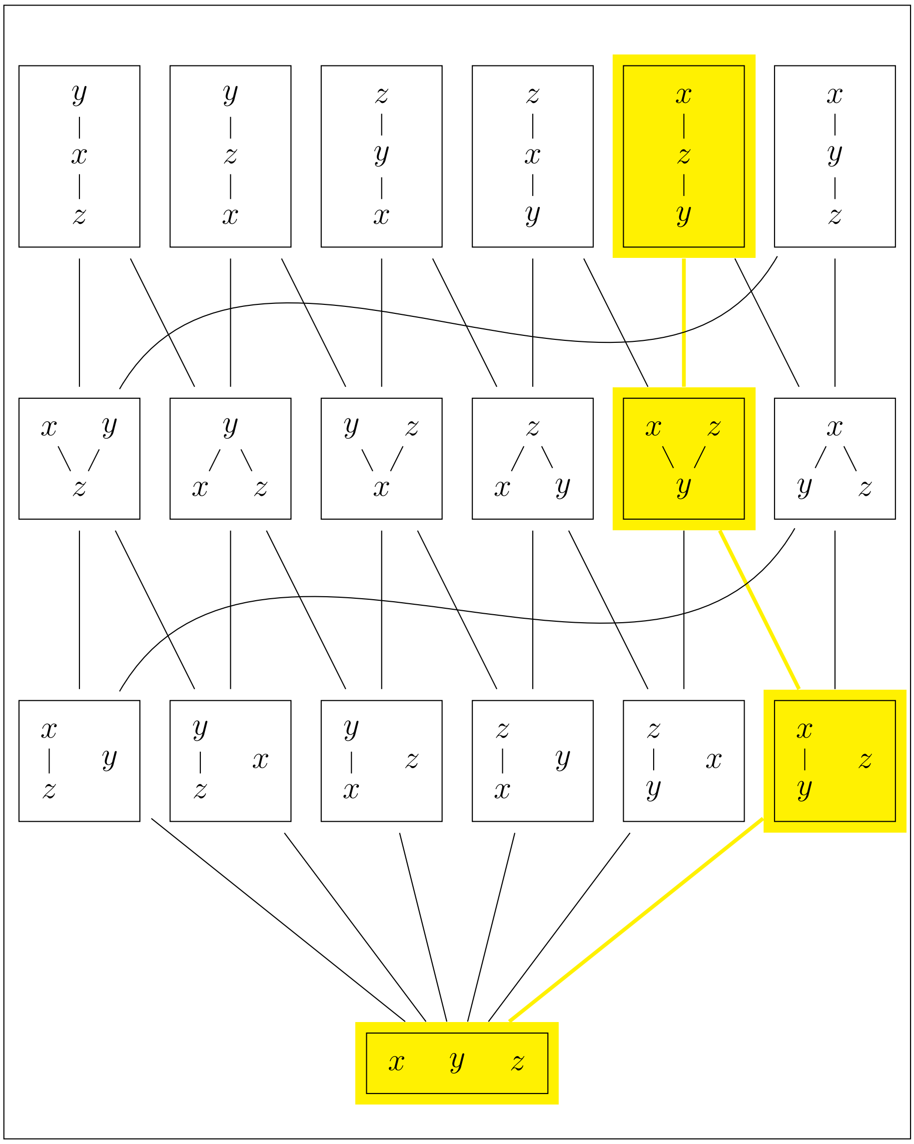

Figure 1: The Set of Order Extensions of Antichain $X = \{x, y, z\}$

is E_i, and the maximal elements associated with this subset are also linear orders. One purpose of this paper is to develop the relation between maximal chains in the set of order extensions and learning processes that stem from a behavioural theory of learning. In the simple case with $X = \{x, y, z\}$ we can construct the set of order extensions as shown in figure 1 of the paper. Each poset in figure 1 is represented by a Hasse diagram, where an edge connecting a lower element x_i to a higher element x_j indicates that x_j is a cover of x_i within the poset. Because the set of order extensions is a poset of posets, a single edge connecting a lower element E_i to a higher element E_j indicates that E_j is an order cover of E_i within the set of order extensions. Since the set of order extensions is exhaustive, each such poset that can be constructed form X is depicted in the Hasse diagram of figure 1. The sequence of four shaded posets in figure 1 constitutes a maximal chain in the set of order extensions of antichain $X = \{x, y, z\}$. In the case of a four-element poset $X = \{a, b, c, d\}$, we may similarly illustrate a maximal chain in the set of order extensions as in figure 2.

We see the set of order extensions serving a dual purpose in relation to the existing literature on bounded rationality. Firstly, by associating specific models of bounded rationality with particular elements of the set of order extensions, we may derive further results into how learning affects behaviour. The second purpose in relation to models of bounded rationality is to provide a natural framework for examining comparative statics of changes in decisiveness along maximal chains in the set of order extensions. We illustrates these two points in the context of several popular models of bounded rationality.

2.2.1 Satisficing

The literature has studied various versions of the satisficing heuristic (Simon, 1955). We here focus on a special case of Rubinstein and Salant (2006).[1]

For the sake of simplicity, let $X = \{x, y, z\}$ be a choice set. The decision-maker (DM) has in mind a partition of the grand set X into satisfactory and unsatisfactory alternatives. Formally, let S (resp., U) denote the set of

[1] See also Caplin and Dean (2011) and Papi (2012) as different approaches to satisficing.

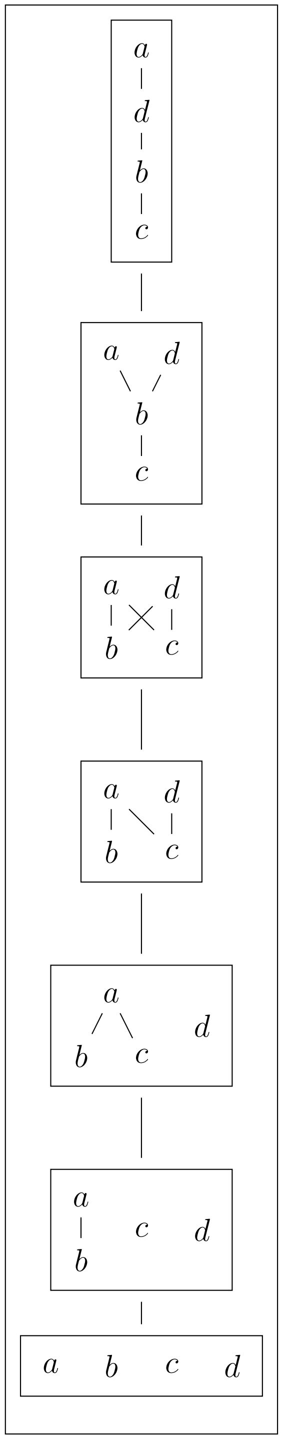

Figure 2: A Maximal Chain in the Set of Order Extensions of Antichain $X = \{a, b, c, d\}$

satisfactory (resp., unsatisfactory) alternatives in X. Assume, for simplicity, that $S \neq \emptyset$. The DM behaves as follows: the DM sequentially goes through the alternatives in X one by one (i.e, according to an exogenously given order/list $\langle x_{l(1)}, x_{l(2)}, x_{l(3)} \rangle$, where $x_{l(1)}$ is the first element examined by the DM), stops searching as soon as they find the first satisfactory alternative, and chooses the first satisfactory alternative.

Let $C = (\preceq_0^C, \preceq_1^C, \preceq_2^C, \preceq_3^C)$ denote a maximal chain in the set of order extensions of figure 1. Define $S_t^C := \max(X, \preceq_t^C)$ as the set of satisfactory alternatives at the preference relation $\preceq_t^C$ and $U_t^C := X \setminus S_t^C$ as the set of unsatisfactory alternatives. That is, we call satisfactory the set of $\preceq_t^C$-maximal alternatives in X and unsatisfactory any remaining alternatives. Note that S_t^C is an antichain. Note also that $\langle S_0^C, S_1^C, S_2^C, S_3^C \rangle$ is a sequence of sets with the properties that (i) $S_0^C = X$, (ii) $S_k^C \subseteq S_{k-1}^C$ for any $k = 1, \ldots, 3$ and (ii) $|S_3^C| = 1$. Hence, initially the DM regards all alternatives to be satisfactory. Then, their set of satisfactory alternatives progressively shrinks until it becomes a singleton at period 3. When the set of satisfactory alternatives is a singleton, the DM is a maximiser, in that they are happy only when they identify the unique best alternative in the choice set.

Throughout we omit to specify the reflexive parts of the posets. To illustrate, let C be the maximal chain in the set of order extensions shaded in figure 1 and characterised by the following sequence of relations.

$$\preceq_0^C = \{\}, \ \preceq_1^C = \{(y,x)\}, \ \preceq_2^C = \{(y,x),(y,z)\}, \ \preceq_3^C = \{(y,x),(y,z),(z,x)\} \tag{1}$$

Hence, $S_0^C = \{x, y, z\}$, $S_1^C = S_2^L = \{x, z\}$, and $S_3^C = \{x\}$.

Note that when the DM is at period 0, the DM chooses a from any list $\langle a, x_{l_i}, x_{l_j} \rangle$, because all alternatives are satisfactory. In contrast, when the DM is in periods 1 and 2, there exists no list from which the DM chooses y, but there exist lists from which the DM chooses x or z (e.g. the DM chooses x from $\langle y, x, z \rangle$). Finally, when the DM is in period 3, the DM chooses x from any list, so the order according to which alternatives are presented to the DM is irrelevant.

In summary, the set of order extensions provides a description of the process through which a satisficing DM learns to become maximiser by progressively turning satisfactory alternatives into unsatisfactory ones.

2.2.2 Limited Attention

We next consider the evolution of limited attention (Masatlioglu, Nakajima and Ozbay, 2012) along a maximal chain in the set of order extensions.[2] The purpose is to exploit the structure of the set of order extensions in order to provide a description of a process through which a limited-attention DM progressively expands their attention to consider more and more alternatives, until at the end of the process they consider every alternative in every menu.

The main idea behind this class of models is that the DM does not pay attention to all alternatives in a menu A, but only a subset of it, called *attention filter*. The DM then chooses by selecting the best alternative in the attention filter according to their rational preferences. For the sake of simplicity, assume that $\succ$ is a linear order on X representing the DM's rational preferences over alternatives. Define $\Gamma : 2^X \setminus \emptyset \to 2^X \setminus \emptyset$ as an attention filter with the two key properties that, for any menu $A \subseteq X$, (i) $\Gamma(A) \subseteq A$ is the set of alternatives the DM pays attention to at menu A and (ii) $\Gamma(A) = \Gamma(A \setminus \{x\})$ whenever $x \notin \Gamma(A)$. That is, the removal of an alternative that is not paid attention does not alter the attention filter. Then, the DM's choice behaviour is a *choice with limited attention* whenever there exists a linear order $\succ$ on X and an attention filter $\Gamma(\cdot)$ such that, for any menu $A \subseteq X$, the DM chooses the $\succ$-best alternative from $\Gamma(A)$.

By a famous result in order theory (Dilworth, 1950), a poset can be partitioned into a minimum number h of chains and this number equals the height of the poset. Our suggestion here is to construct attention filters by intersecting menus with chains of the chain partition. Formally, let $\mathcal{P}_t^C$ denote a partition into h chains of the tth element of the sequence of preference relations C in the set of order extensions given by equation 1 and illustrated in figure 1. Then, for any menu $A \subseteq X$, define $\Gamma_t^C(A) := A \cap P$ for some $P \in \mathcal{P}_t^C$ such that (i) $A \cap P \neq \emptyset$ and (ii) $\Gamma(\cdot)$ satisfies the attention filter property.

$\Gamma_t^C(A)$ is the set of alternatives the DM pays attention to at menu A when

[2] See also Manzini and Mariotti (2014) for a stochastic generalisation of this model.

their attention filter is constructed in relation to $E_t^C = (X, \preceq_t)$. Since along a maximal chain C of the set of order extensions, the partition $\mathcal{P}_t^C$ becomes coarser and coarser, the DM's pays attention to more and more alternatives. To illustrate, assume that $X = \{x, y, z\}$. Return to the maximal chain C of the above section, where $\preceq_0^C = \{\}$, $\preceq_1^C = \{(y, x)\}$, $\preceq_2^C = \{(y, x), (y, z)\}$, and $\preceq_3^C = \{(y, x), (y, z), (z, x)\}$. Focus on $\preceq_1^C$. Define the DM's rational preferences $\succ$ to be a linear extension of $\preceq_1^C$ (for example, $\preceq_3^C$). A chain partition of $\preceq_1^C$ is $\mathcal{P}_1^C = \{\{x, y\}, \{z\}\}$. Since $\Gamma(\{z\}) = \{z\}$ for any $z \in X$, it remains to define an attention filter for the non-singleton menus. Let $\Gamma(\{x, y\}) = \{x, y\} \cap \{x, y\} = \{x, y\}$, $\Gamma(\{x, z\}) = \{x, z\} \cap \{z\} = \{z\}$, $\Gamma(\{y, z\}) = \{y, z\} \cap \{z\} = \{z\}$, and $\Gamma(X) = X \cap \{x, y\} = \{x, y\}$. Hence, given the constructed attention filter $\Gamma(\cdot)$ and $\succ$, if the DM chooses $\max(\Gamma(A), \succ)$ for any menu $A \subseteq X$, then their choice behaviour is a choice with limited attention.

In summary, the set of order extensions provides a process of evolution of limited attention, in which the DM progressively expands their attention, until they consider every alternative in every menu in a way that their choice becomes rational.

2.2.3 Shortlisting

In the shortlisting literature (Manzini and Mariotti, 2007; Apesteguia and Ballester, 2013), the DM sequentially applies a finite list of rationales (criteria) according to a fixed ordering to determine their final choice. Let $P_0, \ldots, P_K$ be a sequence of binary relations. Given a menu $A \subseteq X$, let $M_i^j(A)$ with $i \leq j$ denote the set $M_i^j(A) = \max(\max(\ldots \max(\max(A, P_i), P_{i+1}), \ldots, P_{j-1}), P_j)$. The DM uses a rational shortlist method whenever there exists a sequence of asymmetric binary relations $P_0, \ldots, P_K$ such that the DM chooses a unique alternative contained in $M_0^K(A)$ from any menu A.

Here we explore a rational shortlist method that uses the sequence of rationales along a maximal in the set of order extensions. Returning to the choice set $X = \{x, y, z\}$ and assume that $K = 3$. Let C be given by the maximal chain in the set of order extensions highlighted in figure 1 and defined in equation 1. That is, each P_i coincides with a particular element of the maximal chain, i.e., $C_i = P_i$ for $i = 0, \ldots, 3$. Note that $M_0^0(X) = \{x, y, z\}$,

$M_1^1(X) = M_2^2(X) = \{x, z\}$, and $M_3^3(X) = \{x\}$. Moreover, $M_0^3(X) = M_t^3(X)$ for any $t = 0, 1, 2$. Similarly, $M_0^3(A) = M_t^3(A)$ for any $t = 0, 1, 2$ and any $A \subseteq X$. Hence, when the sequence of rationales is given by a maximal chain in the set of order extensions, a behavioural shortlisting DM could disregard the first terms of the sequence of rationales and directly apply the maximal element of the chain ($\preceq_3^C$).

We also discuss in sections 4.4 and 4.5 choice along maximal chains of the set of order extensions in the context of two contrasting models of incomplete preferences, namely Eliaz and Ok (2006) and Gerasimou (2018).

3 The Behavioural Learning Set

3.1 The Framework

Let $X = \{x_1, \ldots, x_n\}$ be a finite n-element set of alternatives available to the DM. We call X the choice set. Let Σ_X denote the set of all reflexive and transitive binary relations on X. We call Σ_X the *set of tastes* and interpret it as the set of preferences that the DM may be endowed with at each learning stage. We will be precise later about what we mean by learning stage. We do not impose completeness on the binary relations $\preceq \in \Sigma_X$ and interpret it as the DM being indecisive over particular pairs of alternatives in the choice set.

Below we illustrate an example of an element of Σ_X.

Example 3.1 (Paretian Consumer)**.** *Let $X = \{0, 1\}^3$, where each $x = (x^1, x^2, x^3) \in X$ is a binary three-attribute product. Given $x, y \in X$, let $\preceq^P$ denote the **Paretian relation** over X, i.e., $x \preceq^P y$ if and only if $x_i \leq y_i$ for all i. Note that $\preceq^P$ is transitive and anti-symmetric, but not complete. E.g. the consumer is not able to rank* 100 *and* 011.[3]

Definition 3.1 (Learning Set, Learning Process, Learning Stage)**.** *Let T be a finite natural number, where $T \geq \frac{n(n-1)}{2}$.*

[3] Behavioural transitivity requires that if $x \succ y$ and $y \succ z$, then either the consumer is indecisive over the pair (x, z) or $x \succ z$ (Mandler, 2005). Behavioural transitivity is equivalent to transitivity if and only if preferences are complete. Therefore, in the context of incomplete preferences behaviourally transitive preference relations do not belong to Σ_X.

i We call ***set of learning processes*** *the set* $\Sigma_X \times \cdots \times \Sigma_X = \Sigma_X^{T+1}$.

ii A sequence $L := (\preceq_0, \cdots, \preceq_T) \in \Sigma_X^{T+1}$ *is called a* ***learning process****.*

iii The t*-th term of the sequence* L *is called the* t*-th* ***learning stage*** *associated with the learning process* L*.*

The interpretation is that the DM goes through $T + 1$ learning opportunities in their life. A learning opportunity could be interpreted as attending a lecture, being subject to persuasive marketing, being subject to framing, becoming aware of a social norm, etc. A learning process L describes how the DM's preferences evolve while being subject to $T + 1$ learning opportunities. As the example below illustrates, we consider both the polar cases in which, on the one extreme, the DM's preferences change at every learning opportunity and, on the other extreme, their preferences remain constant throughout, and all cases in between. The set of learning processes Σ_X^{T+1} is the set of all learning processes, and the t-th stage of learning process L refers to the DM's preferences - often denoted by $\preceq_t^L$ - at the t-th learning opportunity in learning process L.

Example 3.2 (Learning Process)**.** *Let* $X = \{x, y, z\}$ *and* $T = 4$*. For the purpose of this example, in defining the posets we will change notation by specifying the direction of preference. Define the following posets:* $Q := \{\}$*,* $Q' := \{(y \prec x)\}$*,* $P' := \{(x \prec y)\}$*,* $Q'' := \{(y \prec x), (y \prec z)\}$*,* $Q''' := \{(y \prec x), (y \prec z), (x \prec z)\}$*, and* $P''' := \{(y \prec x), (y \prec z), (x \sim z)\}$*.*[4] *Consider the following learning processes.*

1. $\bar{L} = (Q, Q', Q'', P''', P''')$*. The DM's starts by being indecisive and progressively learns how to compare an extra pair of alternatives at a time, until at learning stage 3 their preference become complete. Note that at learning stages 3 and 4 their preferences do not satisfy anti-symmetry.*

2. $\hat{L} = (Q', Q'', Q''', Q''', Q''')$*. The DM does not start by being completely indecisive, as at learning stage 0 they are able to compare a pair of alternatives.*

[4]Recall that for brevity throughout the paper we omit to specify the reflexive parts of the preference relations.

3. $\tilde{L} = (Q, P', Q'', Q''', Q''')$. *At learning stage 1 the DM prefers y to x and then at learning stage 2 they change their mind by preferring x to y.*

4. $\dot{L} = (Q, Q, Q, Q, Q)$. *The DM's preferences remain constant throughout and equal Q.*

5. $L^* = (Q, Q', Q'', Q''', Q''')$. *$L^*$ is similar to $\bar{L}$, apart from the fact that the DM's preferences are anti-symmetric throughout.*

We next define the concept of settling preference, i.e. a situation whereby the DM's preferences do not change any more, starting from a certain learning stage onwards within a learning process. The approach we follow in this paper is to think of a learning process as a finite sequence of preference relations. Let $\mathbb{N}_n = \{1, \ldots, n\}$ the set of the first n natural numbers. In this respect, suppose that Z is an n-element set (so that Z is non-empty and finite), the set $\{z_i : i \in \mathbb{N}_n,\, i \geq k\}$ is called the k-th tail of the sequence, and is denoted $\text{tail}_k(z)$. We also define *the set of m first terms* of the sequence as the set, $\{z_i : i \in \mathbb{N}_n,\, i \leq m\}$.

Definition 3.2 (Settling Preferences)**.** *A learning process $L \in \Sigma_X^{T+1}$ is said to* ***settle*** *at the tth learning stage whenever t is the smallest index such that $tail_t(L)$ is a constant sequence. The learning process L is said to settle on $\preceq^*$ whenever $tail_t(L) = (\preceq^*, \ldots, \preceq^*)$.*[5] [6]

Note that, when the choice set is finite, every learning process settles at some learning stage. The interesting questions are to be able to predict (i) at which learning stage a learning process settles and (ii) what are the features of the preferences on which a learning process settles.

As discussed above, the paper studies and axiomatises learning on a set $(X, \preceq)$, where we abstract from the important question of the distinction between indecisiveness and indifference that have been a subject of many papers, such as Eliaz and Ok (2006), Qiu and Ong (2017), and Gerasimou (2018). For this purpose, the following remark is useful.

[5] Given the finiteness imposed on our framework, the notion of convergence is not a suitable one, because - by definition - the limit of a learning process $L = (\preceq_0, \ldots, \preceq_T) \in \Sigma_X^{T+1}$ is equal to the last term $\preceq_T^L$ of learning process L.

[6] Note that in example 3.2 $\bar{L}$ settles on P''' at learning stage 3, $\hat{L}$ settles on Q''' at learning stage 2, $\dot{L}$ settles on Q at learning stage 0, etc.

Remark 3.1. *Let $(Y, \preceq)$ be a finite preordered set. Then,*

i There exists a system of distinct representatives $X = \{x_1, \dots, x_n\} \subseteq Y$, where each x_i is a unique element of an indifference class Φ_i and $\Phi_1, \dots, \Phi_n$ is a partition of Y.

ii The restriction of $\preceq$ to X is an anti-symmetric preorder, that is, a partially ordered set $(X, \preceq)$.

In other terms, when the DM compares two distinct elements x_i and x_j from the SDR three possibilities can occur, namely, either x_i is strictly preferred to x_j, x_j is strictly preferred to x_i or x_i and x_j are incomparable. The construction of the partially ordered set $(X, \preceq)$ as an SDR is a convenient way of separating issues of indifference versus indecisiveness in the context of preorder Y.

We can think of Y being an orchestra comprising three groups: cellists, violinists, and flutists. The maestro is indifferent between any pair of musicians that play the same instrument. In other words, groups of cellists, violinists, and flutists induce a partition of the orchestra Y and any SDR X of Y precisely consists of one cellist, one violinist, and one flutist. The learning processes examined in this paper assume that at period 0 the maestro is indecisive on how to order any pair of musicians who perform on different instruments and in each learning episode acquires information on how to order a new pair of elements from the SDR.

3.2 The Behavioural Axioms of Learning

We now present a list of axioms that impose reasonable restrictions on how the DM can learn by characterising a subset of the set of learning processes Σ_X^{T+1}.

Specifically, let $L = (\preceq_0^L, \dots, \preceq_T^L)$ be a $(T+1)$-sequence of preference relations on a finite set X. We consider the following behavioural axioms of learning.

- $[A1]$ $(X, \preceq_t^L)$ is a partially ordered set, for any $t = 0, \dots, T$.
- $[A2]$ At learning stage 0, $|\text{comp}(X, \preceq_0^L)| = 0$.

- $[A3]$ For any $x, y \in X$, if $x \prec^L_{s-1} y$, then $x \prec^L_s y$ for all $s = 1, \ldots, T$.
- $[A4]$ If $|\text{inc}(X, \preceq^L_{t-1})| = k > 0$, then $|\text{inc}(X, \preceq^L_t)| = k - 1$ for all $t = 1, \ldots, T$.

From here on, we define the subset $\mathcal{L}(X) := \{L \in \Sigma^{T+1}_X :$ axioms $[A1] - [A4]$ hold$\}$ as the **behavioural learning set**.

Axiom $[A1]$ defines the context of bounded rationality in that the preference relation is assumed to be anti-symmetric and transitive, but not necessarily complete. Anti-symmetry of $(X, \preceq^L_t)$ arises here because each element of $X = \{x_1, ..., x_n\}$ is a unique representative of one of the n indifference classes of a superset Y of X. Equivalently, $x_1, ..., x_n$ is an SDR of a larger choice set Y.

Axiom $[A2]$ is a normalization axiom: we begin our story at stage 0, where the DM is completely indecisive. As an example, imagine an 18-year old that for the first time is entitled to vote for the general elections. Presumably, such individual has not been interested in politics up to that point in time and just begins to form an opinion about the candidates.

Axiom $[A3]$ is a dynamic consistency axiom that ensures that the evolution of learning is not erratic. So if the DM prefers y to x at some learning stage, they continue to do so at the next learning stage. This axiom captures durable changes of preferences arising from learning that Bowles (2004: Ch. 3) refers to as the endogeneity of preferences.

Axiom $[A4]$ captures the incentives an indecisive decision maker has for learning. So if the DM is unable to compare $k > 0$ pairs of alternatives at some learning stage, axiom [A4] ensures that at the next learning stage they are indecisive over $k - 1$ alternatives[7].

We note that the above axioms are logically independent as demonstrated in example A1 of the appendix of paper. Anticipating on the results below, we also mention several properties that the axioms imply. Firstly, together the four axioms allow us, starting from the completely indecisive preference

[7]For instance, in the consumer search literature purchasers keep searching if the expected benefit of exploring an additional firm outweighs the search cost (Stahl, 1989). Similarly, axiom 4 may be thought of as a reduced form, for a decision-maker who engages at every stage of learning in a cost-benefit analysis, where the benefit of learning to compare an additional pair of alternatives outweighs the cost.

ordering on X, to construct all the different partial orders on the choice set X. Thus the learning we obtain from the four axioms is in this sense exhaustive (Theorem 3.1 below). Secondly, the learning that the four axioms produce is *monotonic* in a sense that we shall define in Proposition 3.2 below. That is, the DM learns in a way as to reduce their indecisiveness steadily in every period, until their preferences become complete, and from there on time invariant.

Theorem 3.1. *Let $M = (E_0, E_1, \ldots, E_{n^*})$ denote a sequence of preference relations in the set of order extensions of the antichain $E_0 = (X, \preceq_0)$, where $n^* \leq T$. Then, M is a maximal chain of the set of order extensions if and only if the sequence $(E_0, E_1, \ldots, E_{n^*})$ constitutes the first $n^* + 1$ terms of a learning process L in the behavioural learning set $\mathcal{L}(X)$.*

Theorem 3.1 suggests that if a DM satisfies the four axioms of learning, a given learning process L may be described by a sequence of ordered sets and vice versa any maximal chain in the set $\text{ext}(X)$ of order extensions provides the first n^* terms of a sequence defining a learning process in $\mathcal{L}(X)$. The various terms of the sequence belong to a poset of posets - that is the set of order extensions - where the sequence: begins with the antichain E_0 (as minimal element), constructs every subsequent learning stage t as an order extension of the ordered set of stage $t-1$; until they have exhausted all n^* incomparabilities and work with a complete preference relation for $T - n^*$ residual periods. That is, theorem 3.1 delivers a story whereby in stages 0 to n^* the DM behaves like the learners in the opening quotation of the paper to Bowles (2004) and from then onwards their preferences are not subject to change *for the same reason that one does not argue about the Rocky Mountains* (Stigler and Becker, 1977). We believe that the axiomatization of this paper is useful in that starting from extreme indecisiveness, the axioms provide a general process for arriving at a complete, time-invariant, and transitive preference relation, where most textbook discussions of the theory of the rational DM begin.

It is clearly the case that by relaxing any combination of the axioms, one can obtain generalizations of behavior. We discuss below some implications of changing each separate axiom in turn when we consider the simple case where the number of learning episodes, T, coincides with n^*.

We first discuss axiom [$A1$]. If we consider a broader role for learning whereby DMs are allowed to rank pairs of alternatives in the choice set as belonging to the same indifference class, we will need to impose further restrictions on the type of preferences we are working with. For instance, we may want to restrict learning to preferences that are regular in the sense of Eliaz and Ok (2006) so as to have a criterion that enables us to distinguish learning that produces a strict order of pairs of alternatives versus learning about indifference.

Consider next relaxing [$A2$] to *at learning stage* 0*,* $|comp(\preceq_0^L)| = k \in \{0, ..., n^*\}$. The resulting behavioural learning set would be equal to a subset of $\mathrm{ext}(X)$.[8] When $k = 0$, we would be back with Theorem 3.1. In this sense axiom [$A2$] adds generality by modelling the initial stages of learning.

Axiom [$A3$] captures the role of learning as a means shaping endogenous preference formation in the perspective of Bowles (2004). Consider then the following variation on axiom [$A3$]: *for any* $x, y \in X$*, if* $x \prec_{s-1}^L y$*, then* $(x, y) \in$ *comp*$(X, \prec_s^L)$ *for all* $s = 1, \ldots, T$. The resulting behavioural learning set would have for generic element a learning process L that occasionally jumps from one chain of the set of order extensions to other ones. As such the learning process L would be obtained by gluing together (i.e., constructing unions of) subsets of maximal chains of the set of order extensions.

Consider relaxing Axiom [$A4$] to *if* $|inc(\preceq_{t-1}^L)| = k > 0$ *then* $0 \leq |inc(\preceq_t^L)| \leq k$ *for all* $t = 1, \ldots, T$. As such, the learning process L is obtained by having a combination of episodes with no learning, or some learning, or learning involving more than one additional comparable pair (consider the case where $|\mathrm{inc}(\preceq_t^L)| = k - 2$). The resulting behavioural learning set would have for generic element a learning process L that is a chain of the set of order extensions (but not necessarily a maximal chain). As such, at time T, the DM may then have an incomplete, or possibly complete, preference relation.[9]

There are two further results that we discuss in relation to theorem 3.1

[8]Specifically, the behavioural learning set would be equal to the *up-set* of $\preceq_0^L$ in the set of order extensions.

[9]This variant of axiom [$A4$] would be relevant to the literature on cognitive load and decision-making in the context of bounded rationality (Allred, Duffy and Smith, 2016).

above. We begin with constructing partitions of the set of order extensions.

3.3 Partitions of the Set of Order Extensions

The set of order extensions of an antichain exhibits in fact a lot of structure that may be related to the way DMs learn.

Let $(Z, \preceq)$ be an ordered set and $\theta : Z \to \mathbb{N}$ be a function such that, if y is a cover of x, then $\theta(y) = \theta(x) + 1$ for all $x, y \in Z$. The function θ is a rank function. If such a function θ exists, then $(Z, \preceq)$ is said to be a *ranked ordered set*.

Let $\rho : \text{ext}(X) \to \{0, ..., n^*\}$ be a function counting the number of elements that are comparable in the extension $E_t := (X, \preceq_t)$ of $E_0 = (X, \preceq_0)$. That is, for any $E_t \in \text{ext}(X)$, we define $\rho(E_t) := |\text{comp}(E_t)|$.

From Brualdi, Jung and Trotter (1994) and Pouzet et al. (1995), the following result follows.

Remark 3.2. *The ordered set ext(X) is ranked by the function $\rho(\cdot)$.*

We define learning in this paper as a process where we begin as totally indecisive and gradually increase our capacity to compare alternatives until our preferences are complete. It is useful to be able to construct the set of order extensions $\text{ext}(X)$ in a way that relates to our understanding of learning. The following result follows straightforwardly from Dilworth (1950)[10], Brualdi, Jung and Trotter (1994), and Theorem 3.1 of this paper.

Proposition 3.1. *In the specific case of Theorem 3.1 where $T = n(n-1)/2$, such that the behavioural learning set is equal to the set of maximal chains of ext(X), the stages of learning associated with the learning processes of $\mathcal{L}(X)$ provide a partition of the set of order extensions ext(X) into $n(n-1)/2+1$ antichains $S_0, ..., S_{n(n-1)/2}$, where*

$$S_i := \{E \in ext(X) : |comp(E)| = i\}$$

for $i = 0, 1, ..., n(n-1)/2$.

[10] For the sake of precision, the version of Dilworth's theorem involving the partition of a poset into antichains is attributed to Leon Mirsky.

The significance of this proposition is three-fold. Firstly, it provides a constructive approach to deriving the set of order extensions: we may begin by constructing all covers of the antichain. This provides the DMs at stage 1 of learning. Next, all successive covers of order relations with one comparability provide DMs at stage 2 of learning etc. This construction provides a partition of DMs on the basis of the stages of learning. In turn, this construction will be used to suggest different orderings of decisiveness and to clarify their relation with existing orderings (see proposition 4.1).

Secondly, the resulting decomposition points to an explanation for the existence of preference heterogeneity of DMs. Specifically, given that the learning stages provide a partition, we can think of heterogeneity in a population of DMs arising due to heterogeneity within a given stage of learning and between different stages of learning. A given rank set exhausts all the possibilities of constructing a poset with a fixed number of comparable pairs and the union of the rank sets accounts for the variation in the sizes of the comparability sets.

Thirdly, note that in general antichain partitions of ordered sets would not be as easily obtained without the rank property of remark 3.2. That is, the rank property guarantees that all maximal chains of the set of order extensions have the same length and, therefore, that the stage of learning the DM is at can be determined independently of the learning process the DM is choosing to follow.

3.4 Settling Preferences

Because the set of order extensions $\text{ext}(X)$ is ranked, all maximal chains of this set (a poset of posets) have equal length. It is therefore possible to predict that under the axioms of learning, all learning processes *settle* (recall definition 3.2 in section 3.1) on a complete ordering in exactly $n^* + 1 = n(n-1)/2 + 1$ learning periods.

Under these behavioural axioms, learning is equivalently characterized by two monotonic sequences, and by preferences evolving from incompleteness, to settling on a complete preference ordering.

Proposition 3.2. *Let $\mathcal{L}(X)$ denote the behavioural learning set. If $L :=$*

$(\preceq_0, \cdots, \preceq_T)$ *is a learning process in* $\mathcal{L}(X)$*,* $h_t := height(X, \preceq_t)$*,* $w_t := width(X, \preceq_t)$*, and* $n^* := n(n-1)/2 \leq T$*, the following three conditions are equivalent:*

i *For all* $t = 1, ..., n^*$ *the preference relation* $\preceq_t$ *is a cover of* $\preceq_{t-1}$ *in the set of order extensions of the antichain* $(X, \preceq_0)$ *and the* n^**-th tail of the sequence of preference relations* $\{\preceq_t\}$ *associated with the learning process* L *is a constant sequence with generic element the complete preference relation* $\preceq = \preceq_{n^*}$*.*

ii *The sequence of ordered set heights* $\{h_t\}$ *associated with the learning process* L *is an increasing bounded sequence, where the* n^**-th tail of the sequence* $\{h_t\}$ *is a constant sequence with generic element* $h = n$*.*

iii *The sequence of ordered set widths* $\{w_t\}$ *associated with the learning process* L *is a decreasing bounded sequence, where the* n^**-th tail of the sequence* $\{w_t\}$ *is a constant sequence with generic element* $w = 1$*.*

This proposition helps to clarify how the axioms of the behavioural learning set jointly deliver a theory of monotonic learning. The proposition states that under the axioms of learning (i) the DM preserves prior knowledge (in terms of how pairs of alternatives in the choice set are ordered) and acquires the knowledge to order an additional pair of alternatives in each new stage of learning until their preferences settle on a complete relation in exactly $n^* + 1$ learning episodes. Under point (ii), the above result is equivalently stated by characterising the sequence of ordered set heights corresponding to the $n^* + 1$ stages of the given learning process. Such a sequence of ordered set heights is increasing and settles on height n precisely in $n^* + 1$ periods (when the preferences of the DM become complete). Likewise, under point (iii), the monotonicity of learning property may be stated in terms of width of the sequence of posets defining a learning process. Such a sequence is decreasing and settles on unit width in precisely $n^* + 1$ periods.

In addition, the proposition specifies the way in which learning changes the DMs' preferences over time. This result is further exploited in the application section. Firstly, when we examine decisiveness orderings, one implication of monotonic learning is that a necessary condition for DM i to be more decisive than DM j is that i is at higher learning stage than j. This condition applies to the decisiveness ordering proposed by Gorno (2018) as well as the two additional orderings that we discuss in the applications. Secondly, this result will prove useful when we explore the implications of learning for violations of alternative versions of the weak axiom of revealed preference as discussed in Eliaz and Ok (2006). Likewise, the result is used to shed light on how learning impacts on choice deferral in the context of Gerasimou (2018).

4 Applications

In this section of the paper we consider four applications of the above framework. These include (i) the measurement of (in)decisiveness, (ii) social interactions as a source of learning, (iii) the implications of learning for the axioms of revealed preference, and (iv) learning and the timing of choice deferrals. In all four applications the set of order extensions provides a unifying role in characterising the effects of learning on preferences and choices.

4.1 Measurement of Indecisiveness

It has been emphasized (Danan and Ziegelmeyer, 2006; Cettolin and Riedl, 2015; Qiu and Ong, 2017) that once that the completeness axiom is relaxed, it is of utmost importance to investigate the extent to which DMs are decisive. Most experimental work has investigated indecisiveness over lottery choices. Indecisiveness has broadly been measured by the number of times a subject defers their decisions (Danan and Ziegelmeyer, 2006) and by the number of times they have made their choices by use of a random device (Cettolin and Riedl, 2015; Qiu and Ong, 2017).

From an order-theoretic point of view, Ok (2002) discusses the use of height and width of the preference relation $\preceq_t^L$ as potentially useful measures of decisiveness. Specifically, under the behavioural learning axioms, proposition 3.2 informs us that learning is monotonic in that along a learning process the sequence of poset heights is weakly increasing and likewise the sequence of poset widths is weakly decreasing. Furthermore, Gorno (2018) introduces a very natural decisiveness ordering that follows from the structure of incomplete preferences. Namely, Gorno (2018) suggests that E_i is less decisive than E_j if E_j is an order extension of E_i. In this context, theorem 3.1 indicates that along a learning process satisfying the behavioural axioms of learning each preference relation E_t is an order extension of E_s and, more particularly, an order cover of E_{t-1}. In equivalent terms a chain in the set order extensions defines a sequence of increasingly decisive preference relations in the perspective of Gorno (2018)'s ordering. We note that this approach delivers what we may call an *order-theoretic* perspective, in that individual i is more decisive than individual j if i's preference ordering is an order extension of j's. We choose to relax Gorno (2018)'s order-theoretic criterion of indecisiveness by exploring a *graph-theoretic* perspective. The meaning we attach to E_i being less decisive than E_j in the graph-theoretic perspective is that if E_i can compare x and y then E_j also knows how to compare x and y. We furthermore complement the partial orders based on the order-theoretic and graph-theoretic perspectives by introducing a complete order , i.e., a measure of decisiveness:

Definition 4.1 (Three Indecisiveness Orderings)**.** *Assume that* $E_i = (X_, \preceq_i)$ *and* $E_j := (X_, \preceq_j)$ *are some posets in the set of order extensions of the antichain* $(X, \preceq_0)$*.*

- $E_i \preceq_{d^{ext}} E_j$ *if* E_j *is an order extension of* E_i *(Gorno, 2018).*
- $E_i \preceq_{d^{comp}} E_j$ *if* $comp\,(E_i) \subseteq comp\,(E_j)$*.*
- $E_i \preceq_{d^{rank}} E_j$ *if* $\rho\,(E_j) \geq \rho\,(E_i)$*.*

In the graph-theoretic perspective, E_i is less decisive than E_j if the comparability set of E_i is a subset of the comparability set of E_j. On the other hand, $E_i \preceq_{d^{rank}} E_j$ if the comparability set of E_i is of a smaller magnitude than that of E_j. The next proposition explores the logical relation between the above three definitions of decisiveness.

Proposition 4.1 (Relations between the Three Indecisiveness Orderings)**.** *In the set of order extensions of the n-element antichain* $E_0 = (X, \preceq_0)$*, the following implications hold:*

$$E_i \preceq_{d^{ext}} E_j \Longrightarrow E_i \preceq_{d^{comp}} E_j \Longrightarrow E_i \preceq_{d^{rank}} E_j$$

Therefore, in the design of experiments aimed at exploring decisiveness, the three orderings discussed above would allow the investigator to detect finer differences in decisiveness. By this we mean that experimental research could also elicit decisiveness orders, such as $\preceq_{d^{ext}}$ and $\preceq_{d^{comp}}$, alongside the existing measures discussed earlier. Furthermore, looking at several decisiveness relations that are logically related (such as the trilogy of orders of the above proposition) could provide additional testable hypotheses about decision making.

The distinction between the order-theoretic and graph-theoretic perspectives on decisiveness is also put to good use in the next subsection, where we explore the possibility of learning through social interactions.

4.2 Learning from Social Interactions

The traditional view of the DM as someone with exogenous and stable preferences has been challenged in the context of the literatures on endogenous preferences (Bowles, 1998; 2004), norm dynamics (Young, 2015), and experiments on risk taking (Lejarraga and Müller-Trede, 2016). For instance, the opening quotation in the paper to Bowles (2004) identifies learning as mechanism through which preferences are shaped and become endogenous. In the literature on norm dynamics where individuals interact with their geographic neighbours, preferences tend to be homogeneous within well defined neighbourhoods clusters and social networks (Young, 2015). In the context of experiments on risk taking, Lejarraga and Müller-Trede (2016) find that subjects with differential knowledge benefit from social interactions in that they learn from one another to make more informed choices in a decision-theoretic context.

The graph-theoretic and order-theoretic perspectives can provide some additional insights into how social interactions shape the learning discussed in this paper over pairs of alternatives in the choice set.

Example 4.1 (Behavioural Learning over a Coffee Discussion)**.** *Individual i at some stage of learning s has a poset $E_s^i = (X, \preceq_s^i)$ and individual j with a poset $E_t^j := \left(X, \preceq_t^j\right)$ get together for coffee, where E_s^i and E_t^j are some posets in the set of order extensions of the antichain $(X, \preceq_0)$, and $x_1, x_2, y_1, y_2 \in X$.*

Assume $(x_1, x_2) \in comp\left(E_s^i\right) \backslash comp\left(E_t^j\right)$ *while* $(y_1, y_2) \in comp\left(E_t^j\right) \backslash comp\left(E_s^i\right)$.

Assuming the coffee discussion has brought learning to both individuals, i *is now at stage* $s+1$ *of learning and* j *has moved up to stage* $t+1$ *of learning.*

Since both individuals satisfy the axioms of learning, and assuming that before the coffee i *preferred* x_1 *to* x_2 *while* j *preferred* y_2 *to* y_1*, we may assume that after the coffee discussion,* i *has learned to compare* y_1 *and* y_2 *and likewise* j *has learned to compare* x_1 *and* x_2.

We highlight two possible outcomes from the social interaction: (i) if we follow the literature on evolutionary norm dynamics, i *and* j *want to reinforce their conformity, and this mechanism acts to suppress preference heterogeneity. As such,* i *will copy* j *in a way that* $y_1 \preceq_{s+1}^i y_2$ *in* E_{s+1}^i*. Likewise* j *will order* x_1 *and* x_2 *by copying* i *so that* $x_2 \preceq_{t+1}^j x_1$ *in* E_{t+1}^j*; (ii) we allow* i *and* j *to disagree on their acquired preferences between the new alternatives. That is, we may still assume* $comp(E_{s+1}^i) = comp(E_s^i) \cup \{(y_1, y_2)\}$ *and* $comp(E_{t+1}^j) = comp(E_t^j) \cup \{(x_1, x_2)\}$ *without specifying how the two agents will order the new pair of alternatives in their respective comparability sets.*

The first perspective is order-theoretic in that i replicates the ordering of y_1 and y_2 from j's initial preference relation E_t^j and similarly for j. Likewise, the second perspective is graph-theoretic in the sense that the coffee discussion allows new connections between pairs of alternatives to be made, without making assumptions about how the two individuals will order the new pairs of their respective comparability sets.

The coffee discussion also shapes preference formation and your immediate neighbours shape your preferences as emphasized by Young (2015). We conclude however by noting that the graph-theoretic perspective does not suppress preference heterogeneity to the same extent as the order-theoretic perspective.

4.3 Learning and the Axioms of Revealed Preference

Suppose that - as external analysts - we are not able to observe the learning stage at which the DM's choice has been made. The purpose of this section is to explore some implications of learning for the axioms of revealed preference.

In the classical theory of the consumer, a choice problem involves a subset

$A \subseteq X$ and there is a choice correspondence $c(\cdot)$ that assigns to the choice problem a set of maximal elements $c(A)$.

To capture the effect of learning it is useful to generalize the classical choice problem to the context of an ordered pair (A, f), where $A \subseteq X$ is the standard choice problem, and f is a frame belonging to a set of frames (Bernheim and Rangel, 2007; Salant and Rubinstein, 2008). (A, f) now becomes an *extended choice problem*: the frame influences choice as a result of psychological but also procedural factors.[11]

Let $\mathcal{T}$ denote the set of integers $\{0, \ldots, T\}$. The frame we have in mind here will take a particular form. Let $\mathcal{F} := \left\{f : 2^X \backslash \{\emptyset\} \longrightarrow \mathcal{T}\right\}$ denote the set of functions that assign to each choice problem A a unique learning stage $t := f(A)$. The choice correspondence $c(\cdot)$ now assigns to the extended choice problem a maximal set $c(A, f(A)) := \max\left(A, \preceq^L_{f(A)}\right)$. What form does the weak axiom of revealed preference take here in the presence of indecisiveness and learning? We attempt to complete the discussion in Eliaz and Ok (2006) as follows:

WARP-EXT For any extended choice problem $(A, f) \in 2^X \backslash \{\emptyset\} \times \mathcal{F}$ and $y \in A$, if there exists an $x \in c\,(A, f(A))$ such that $y \in c\,(B, f(B))$ for some other extended choice problem (B, f) with $x \in B$, then $y \in c\,(A, f(A))$.

Note that if there is no learning, f is a constant function and the above definition specializes to WARP in Eliaz and Ok (2006). We propose to generalise the Weak Axiom of Revealed Non-Inferiority (WARNI) as follows (Eliaz and Ok, 2006).

WARNI-EXT For any extended choice problem $(A, f) \in 2^X \backslash \{\emptyset\} \times \mathcal{F}$ and $y \in A$, if for every $x \in c\,(A, f(A))$ there exists some other extended choice problem (B, f) with $y \in c\,(B, f(B))$ and $x \in B$, then $y \in c\,(A, f(A))$.

Note also here that if there is no learning, f is a constant function and the above definition specializes to WARNI in Eliaz and Ok (2006).

[11] Examples of (A, f) models involve choice with status-quo bias (Masatlioglu and Ok, 2005), choice from lists (Rubinstein and Salant, 2006), market competition with frames (Piccione and Spiegler, 2012), etc.

We know that learning changes the preference ordering. On the other hand, learning is very structured in this paper, and changes in the preferences of the DM along a learning process are monotonic: every new preference relation preserves the previous ones, but extends it (in the order-theoretic sense) until preferences settle on a complete ordering. Because preferences are not stable, we have to live with violations of WARP and WARNI when there is learning.

Though we can only state negative results, the following proposition informs about where in the various stages of learning (recall Dilworth (1950)'s decomposition theorem) WARP-EXT and WARNI-EXT may be violated.

Proposition 4.2 (Proposition 4.3)**.** *Consider an extended choice problem* $(A, f) \in 2^X \backslash \{\emptyset\} \times \mathcal{F}$ *for a DM with preferences evolving along a learning process* $L \in \mathcal{L}(X)$*,*

i $c(A, f(A))$ *satisfies WARP-EXT for any function* $f \in \mathcal{F}$ *and any* $A \in 2^X \setminus \{\emptyset\}$ *such that* $f(A) \geq n^*$*.*

ii $c(A, f(A))$ *satisfies WARNI-EXT for any function* $f \in \mathcal{F}$ *and any* $A \in 2^X \setminus \{\emptyset\}$ *such that (a)* $f(A) \geq n^*$*, or (b)* $f(A) = \bar{t}$ *for some* $\bar{t} \in \mathcal{T}$*.*

iii The following two statements are equivalent:

- *a There are distinct menus* $A, B \in 2^X \backslash \{\emptyset\}$ *and* $x, y \in A \cap B$ *and a function* $f \in \mathcal{F}$ *with* $f(A) \neq f(B)$ *and* $f(A) < n^*$*.*
- *b There is a learning process* $L \in \mathcal{L}(X)$ *and a function* $f \in \mathcal{F}$ *such that* $c(\cdot, f(.))$ *violates WARNI-EXT.*

Because at n^* the DM's preferences are complete, then it follow from Arrow (1959) that for any $f(A) \geq n^*$ the extended choice criterion satisfies WARP-EXT. Likewise, for a fixed learning stage $\bar{t} \in \mathcal{T}$, it follows from Eliaz and Ok (2006) that the extended choice correspondence satisfies WARNI-EXT. Finally, as long as we observe choices at two different learning stages, it will be possible to observe violations of WARNI-EXT. Example A2 in the appendix illustrates these points.

4.4 Learning and Choice Deferral

We owe to the experimental literature on choice behaviour the insight that DMs manifest their indecisiveness by selecting several maximal elements, but also by deferring choices (Danan and Ziegelmeyer, 2006; Costa-Gomes et al., 2016). In this respect it is convenient to follow the *maximal dominant choice* approach introduced by Gerasimou (2018), whereby the DM chooses the best alternative from any menu $A \subseteq X$, if there exists one, and chooses to defer, otherwise.[12] Formally, let $Best(A, \preceq_t^L) := \{x \in A : y \prec_t^L x \text{ for all } y \in A \setminus \{x\}\}$ denote the singleton set of the best element in menu A according to $\preceq_t^L$ for some learning process L in the behavioural learning set. Accordingly, for some learning process $L \in \mathcal{L}(X)$, the choice correspondence takes the form

$$c(A,t) \quad := \quad \begin{cases} Best(A, \preceq_t^L) & \text{if } Best(A, \preceq_t^L) \neq \emptyset \\ \emptyset & \text{otherwise.} \end{cases} \tag{2}$$

for any $t \in \mathcal{T}$ and any $A \subseteq X$. Given that indecisiveness steadily decreases in the behavioural learning set, it is interesting to explore how choice deferral evolves with learning.

Proposition 4.3. *Let $c(A,t)$ denote the choice correspondence (2) of the maximal dominant choice procedure. Then, for any nonempty menu $A \subseteq X$ and some learning process $L \in \mathcal{L}(X)$,*

i $c(A,0) \neq \emptyset$ *if and only if* $|A| = 1$.

ii If $c(A,t) \neq \emptyset$, *then* $c(A,s) \neq \emptyset$ *for any* $s > t$.

iii If $c(A,t) = \emptyset$, *then there exists* $s > t$ *such that* $c(A,s) \neq \emptyset$.

As a consequence of the structure of monotonic learning, proposition 4.3 informs us that (i) at the initial stages of learning the DM defers at all menus, but the singleton menus; (ii) as soon as a DM stops deferring they

[12] Recall that the choice correspondence in Eliaz and Ok (2006) selects the set of maximal elements. When the set of maximal elements has a unique element, both approaches coincide. However, when there are multiple maximal elements, the Gerasimou (2018)'s choice rule is to defer the decision to the future.

will no longer choose to defer thereafter; (iii) for any menu, there is a learning stage at which the DM switches from choice deferral to best choice; this is a consequence of the fact that they learn to resolve one extra incomparability at a time.

Table 2 illustrates the results of proposition 4.3 in an example that assumes the DM to follow the learning process of figure 2. Choice deferral is highlighted in red and $Best(A, \preceq_t^L)$ is highlighted in green.

5 Conclusions

Using a set of behavioural axioms, this paper has studied how a decision maker with incomplete preferences over a finite choice set acquires complete and transitive preferences via the process of learning. We have provided conditions under which each given learning process contains a specific maximal chain of the set of order-extensions of the preference relation of an indecisive decision maker (see theorem 3.1). Several applications of the framework were discussed, including the measurement of indecisiveness, how learning occurs from social interactions, the implications of learning for axioms of revealed preference, and for choice deferral.

We next discuss some future research directions. The behavioural axioms of this paper were chosen to produce a theory of monotonic learning. But experimental tests may well document more erratic forms of behaviour. Decision makers may well want to revise the way they order two elements in the choice set. This form of preference reversal would of course entail that maximal chains of the set of order extensions would not provide the required description of learning processes in the behavioural learning set. In a greater degree of generality, it may then be needed to work with a *graph* of preference relations, where for instance we may allow for cycles of preference reversals. Another possibility is to consider in the behavioural axioms weaker forms of transitivity, such as those introduced by Mandler (2005).

As a further extension of the paper, it would be possible to adopt a probabilistic perspective on learning in the context of the indecisive decision maker of this paper. We can for instance follow Brightwell and Trotter (2002) in treating the set of order extensions as a probability space, where the decision maker has equal probability of adopting each complete preference ordering at stage 0 of learning. When the decision maker learns that (say) they prefer x over y, they accordingly update their probability distribution over the remaining linear extensions of their current incomplete preference ordering, and they continue to do so, until their preferences settle on a complete ordering.

Another extension would consist of associating a cost to learning to compare some specific pairs of alternatives. This would enable researchers to derive more predictions as to which learning processes individuals may choose to follow. This extension may open the door for applications of the framework in diverse areas, such as political science (Druckman, 2004) and industrial organization (Spiegler, 2011).

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A Examples and Tables

Example A.1 (Independence of the Axioms)**.** *Reconsider the learning processes of example 3.3.*

1. $\bar{L}$ *violates A1 and satisfies A2-A4.*
2. $\hat{L}$ *violates A2 and satisfies A1 and A3-A4.*
3. $\tilde{L}$ *violates A3 and satisfies A1-A2 and A4.*
4. $\dot{L}$ *violates A4 and satisfies A1-A3.*
5. L^* *satisfies A1-A4.*

Example A.2 (WARP-EXT, WARNI-EXT, and Learning)**.** *Assume that* $X = \{x, y, z\}$ *and* $\mathcal{T} = \{0, \ldots, 3\}$*. Consider learning process* $L = (\prec_0^L, \ldots, \prec_3^L) \in \mathcal{L}(X)$ *defined by equation 1 and the shaded maximal chain of figure 1. The first four rows of table 1 display the DM's choices at a fixed learning stage, i.e., when* $f(A) = \bar{t}$ *for all* $A \in 2^X \backslash \emptyset$ *and some* $\bar{t} \in \mathcal{T}$*. The last row, instead, illustrates the DM's choices at different learning stages. The fourth column indicates whether WARP-EXT and WARNI-EXT are satisfied (green) or violated (red).*

Note that although the choices at learning stages 0 and 2 are generated by the maximisation of a partially ordered set, WARP-EXT is satisfied in both cases. The reason is that, while we restrict our attention to partially ordered sets by imposing anti-symmetry, a DM's choice behaviour satisfying WARP-EXT is equivalent to the maximisation of a weak order, that allows for indifference. The antichain $\preceq_0^L$ *can be interpreted as the DM being indifferent between all alternatives and poset* $\preceq_2^L$ *as a rational preference whereby the DM is indifferent between* x *and* z*.*[13]

[13]See Eliaz and Ok (2006) for a discussion on the distinction between indifference and indecisiveness.

t	$\preceq_t^L$	f	$c(A, f(A))$, where $A =$ $\{x,y\}$	$\{x,z\}$	$\{y,z\}$	$\{x,y,z\}$	Axioms
0	x y z	$f(A) = 0, \forall A$	x, y	x, z	y, z	x, y, z	WARP-EXT, WARNI-EXT
1	x | y z	$f(A) = 1, \forall A$	x	x, z	y, z	x, z	WARP-EXT, WARNI-EXT
2	x z \ / y	$f(A) = 2, \forall A$	x	x, z	z	x, z	WARP-EXT, WARNI-EXT
3	x | z | y	$f(A) = 3, \forall A$	x	x	z	x	WARP-EXT, WARNI-EXT
–	–	$f(\{x,y\}) = f(\{x,z\}) = 3$, $f(\{y,z\}) = f(\{x,y,z\}) = 2$	x	x	z	x, z	WARP-EXT, WARNI-EXT

Table 1: Examples of Choice Patterns Violating/Satisfying WARP-EXT and WARNI-EXT

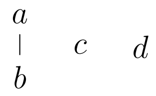
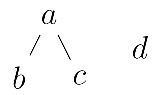
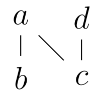
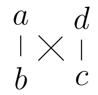
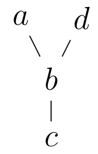
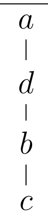
t	$\preceq_t^L$	$c(A,t)$, where $A =$ $\{a,b\}$	$\{a,c\}$	$\{a,d\}$	$\{b,c\}$	$\{b,d\}$	$\{c,d\}$	$\{a,b,c\}$	$\{a,b,d\}$	$\{a,c,d\}$	$\{b,c,d\}$	$\{a,b,c,d\}$
0	a b c d	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
1		a	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
2		a	a	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	a	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
3		a	a	$\emptyset$	$\emptyset$	$\emptyset$	d	a	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
4		a	a	$\emptyset$	$\emptyset$	d	d	a	$\emptyset$	$\emptyset$	d	$\emptyset$
5		a	a	$\emptyset$	b	d	d	a	$\emptyset$	$\emptyset$	d	$\emptyset$
6		a	a	a	b	d	d	a	a	a	d	a

Table 2: Maximal Dominant Choice Over the Learning Stages of the Learning Process of Figure 2

B Proofs

Proof of Theorem 3.1. Let $M = (E_0, E_1, \ldots, E_{n^*})$ denote a sequence of preference relations in the set of order extensions of the antichain $E_0 = (X, \preceq_0)$, where $n^* \leq T$.

Necessity. Let $\mathcal{M}$ denote the set of maximal chains in the set of order extensions and let $M = \{E_0, \ldots, E_{n^*}\} \in \mathcal{M}$. Construct a sequence $M' = (P_0^{M'}, \ldots, P_T^{M'})$ of lenght $T + 1$ as follows. Given $M \in \mathcal{M}$, let $P_t^{M'} = E_t$ for all $t = 0, \ldots, n^*$ and $P_t^{M'} = E_{n^*}$ for all $t > n^*$. We note that M is a subsequence of M' and we now show that M' satisfies axioms $[A1] - [A4]$ for every $M \in \mathcal{M}$. By definition of maximal chain in the set of order extensions, each element of M' is a poset (satisfying A1). Moreover, the minimal element of M' is an antichain (satisfying A2). Given $E, F \in M'$, F is a cover of E if and only if F is an order extension of E (satisfying A3) and has an extra comparability pair relative to E (satisfying A4). Therefore, the sequence $(E_0, E_1, \ldots, E_{n^*})$ constitutes the first $n^* + 1$ terms of a learning process M' in the behavioural learning set $\mathcal{L}(X)$.

Sufficiency. Assume that the sequence $(E_0, E_1, \ldots, E_{n^*})$ constitutes the first $n^* + 1$ terms of a learning process $L = (\preceq_0^L, \ldots, \preceq_{n^*}^L)$ in the behavioural learning set $\mathcal{L}(X)$. We show that the first $n^* + 1$ elements of the learning process L constitute a maximal chain in the set of order extensions. We argue by induction on the learning stages in the set $\{0, \ldots, n^*\}$.

Learning stage 0. By $[A2]$, $|\text{comp}(X, \preceq_0^L)| = 0$ for all $L \in \mathcal{L}(X)$, so that each $\preceq_0^L = \preceq_0$ for all $L \in \mathcal{L}(X)$ and $\preceq_0$ is an n-element antichain. Since every poset is an order extension of itself, it follows that $\preceq_0$ is an element of the set of order extensions of itself. Moreover, $\preceq_0$ is the unique minimal element of the set of order extensions of the antichain.

Learning stage t for some $t \in \{1, \ldots, n^*\}$. By the inductive hypothesis, $\preceq_{t-1}^L$ is the $t-1$th order extension of the antichain for some $t \in \{1, \ldots, n^*\}$. Therefore, $\preceq_{t-1}^L$ is a poset with $|\text{comp}(X, \preceq_{t-1}^L)| = t - 1$. We now show that $\preceq_t^L$ is a tth order extension of the antichain. From axiom $[A1]$, it follows that $(X, \preceq_t^L)$ is a partially ordered set. By $[A4]$, $|\text{comp}(X, \preceq_t^L)| = t$ and, by $[A3]$, if $x \prec_{t-1}^L y$, then $x \prec_t^L y$, implying that $\preceq_t^L$ is an extension of $\preceq_{t-1}^L$. Therefore, $\preceq_t^L$ is the tth order extension of the antichain. Hence, $\preceq_{n^*}^L$ is a linear order and the first $n^* + 1$ elements of L constitute a maximal chain in the set of order extensions. □

Proof of Proposition 3.1. By remark 3.2, $\text{ext}(X)$ is ranked by $\rho(\cdot)$. Hence, every maximal chain has the same length, which is given by $n^* + 1$, i.e., the height of ext(X). Hence, by Dilworth (1950), there exists an antichain partition of ext(X) consisting of exactly $n^* + 1$ antichains. The rank sets of ext(X) provide one such antichain partition. □

Proof of Proposition 3.2. From Theorem 3.1, M is a maximal chain of the set of order extensions if and only if the sequence $(E_0, E_1, \ldots, E_{n^*})$ constitutes the first $n^* + 1$ terms of a learning process L in the behavioural learning set $\mathcal{L}(X)$, and any linear order relation $E_L = (X, \preceq_{n^*})$ has height n.

$(i \Longrightarrow ii)$. Assume that for all $t = 1, ..., n^*$ the preference relation $\preceq_t$ is a cover of $\preceq_{t-1}$ in the set of order extensions ext(X) and that $\preceq_t$ equals $\preceq_\lambda$ for all $t \geq n^*$. Because $\preceq_\lambda$ is a linear order, height$(X, \preceq_\lambda) = n$ and $h_{n^*} = n$. From corollary 2.2 in Brualdi, Jung and Trotter (1994), the sequence of poset heights $\{h_t\}$ is monotonically increasing. Clearly, the upper bound of this sequence is equal to n in the context of the n-element set X. Therefore, we have $h_t = n$ for all $t \geq n^*$.

$(ii \Longrightarrow iii)$. Assume that the sequence of ordered set heights $\{h_t\}$ associated with the learning process L is an increasing bounded sequence where $h_t = n$ for all $t \geq n^*$. Because $h_{n^*} = n$, the associated poset $E_{n*} = (X, \preceq_{n^*})$ is a linear order, and $w_{n^*} = 1$. From corollary 2.2 in Brualdi, Jung and Trotter (1994), the sequence of poset widths $\{w_t\}$ is monotonically decreasing. Clearly, the lower bound of this sequence is equal to 1. Therefore, we have $w_t = 1$ for all $t \geq n^*$, as required.

$(iii \Longrightarrow i)$. Assume that the sequence of ordered set widths $\{w_t\}$ associated with the learning process L is a decreasing bounded sequence, where $w_t = 1$ for all $t \geq n^*$. Then, from theorem 3.1 each poset $E_{t+1} = (X, \preceq_{t+1})$ is an order cover of $E_t = (X, \preceq_t)$, and for all $t \geq n^*$ the relation $\preceq_t$ must be a linear order. Since the only order extension of a linear order must be the same linear order, it follows therefore that $\preceq_t = \preceq_{n^*}$ for all $t \geq n^*$, as required. □

Proof of Proposition 4.1. Let E_j be an order extension of E_i. Then, by definition, $comp\,(E_i) \subseteq comp\,(E_j)$. Hence, it follows that $E_i \preceq_{d^{comp}} E_j$. Therefore, it is the case that $\preceq_{d^{ext}} \Longrightarrow \preceq_{d^{comp}}$.

We next prove that $E_i \preceq_{d^{comp}} E_j \Longrightarrow E_i \preceq_{d^{rank}} E_j$. Since the set of order extensions is ranked by the cardinality of comparability sets, $E_i \preceq_{d^{comp}} E_j \Longleftrightarrow comp\,(E_i) \subseteq comp\,(E_j) \Longrightarrow \rho\,(E_i) \leq \rho\,(E_j) \Longleftrightarrow E_i \preceq_{d^{rank}} E_j$. We therefore conclude that $E_i \preceq_{d^{ext}} E_j \Longrightarrow E_i \preceq_{d^{comp}} E_j \Longrightarrow E_i \preceq_{d^{rank}} E_j$ in the set of order extensions. $\square$

Proof of Proposition 4.2. Part (i): by proposition 3.2, learning process L settles on a linear order at learning stage n^*. By Arrow (1959), $c(A, f(A))$ satisfies WARP-EXT for any function $f \in \mathcal{F}$ and any $A \in 2^X \setminus \{\emptyset\}$ such that $f(A) \geq n^*$.

Part (ii): by part (i), $c(A, f(A))$ satisfies WARP-EXT for any function $f \in \mathcal{F}$ and any $A \in 2^X \setminus \{\emptyset\}$ such that $f(A) \geq n^*$. Since WARP-EXT implies WARNI-EXT, then WARNI-EXT holds too. Next, by axiom A1, each $(X, \preceq_t^L)$ is a partially ordered set. By the second part of theorem 2 in Eliaz and Ok (2006), $c(A, f(A))$ satisfies WARNI-EXT for any function $f \in \mathcal{F}$ and any $A \in 2^X \setminus \{\emptyset\}$ such that $f(A) = \bar{t}$ for some $\bar{t} \in \mathcal{T}$.

Part (iii): suppose first that there exists $A, B \in 2^X \setminus \emptyset$ such that $x, y \in A \cap B$ and $n^* > f(A) \neq f(B)$. Assume that $f(A) = f(B) - 1$ and suppose that $f(C) \in \{f(A), f(B)\}$ for all $C \in 2^X \setminus \{\emptyset, A, B\}$. Construct a learning process $L \in \mathcal{L}(X)$ with the following features: $x||_{f(A)}^L y$, x and y are $\preceq_{f(A)}^L$-maximal in X, and $y \prec_{f(B)}^L x$. Since $\preceq_{f(B)}^L$ is a cover of $\preceq_{f(A)}^L$ in the set of order extensions, then $\preceq_{f(B)}^L$ and $\preceq_{f(A)}^L$ differ only in the way the rank x and y and are otherwise identical. Then, $\{x, y\} \subseteq c(A, f(A))$. Similarly, $x \in c(B, f(B))$ and $y \notin c(B, f(B))$. We distinguish two cases.

Case (1): x is the only $\preceq_{f(B)}^L$-maximal alternative in B. Hence, $\{x\} = c(B, f(B))$. Then, a violation of WARNI-EXT is immediately obtained, as y should be chosen from B as well.

Case (2): x is not the only $\preceq_{f(B)}^L$-maximal alternative in B. Let z be $\preceq_{f(B)}^L$-maximal alternative in B. Hence, $z \in c(B, f(B))$. Moreover, given that x is $\preceq_{f(A)}^L$-maximal in X, then so is z. This implies that $z||_{f(A)}^L y$ and $z||_{f(B)}^L y$, as $\preceq_{f(B)}^L$ and $\preceq_{f(A)}^L$ differ only in the way the rank x and y. Therefore, $c(\{z, y\}, f(A)) = c(\{z, y\}, f(B)) = \{z, y\}$. However, this implies that WARNI-EXT is violated, as y should be chosen from B as well, which is the desired result.

In the other direction, we prove the contrapositive. Suppose that there do not exist $A, B \in 2^X \setminus \emptyset$ such that $x, y \in A \cap B$ and $n^* > f(A) \neq f(B)$. We now show that it cannot be the case that $f(A) \neq f(B)$ for some distinct non-singleton menus $A, B \in 2^X \setminus \emptyset$. Assume, for instance, that $f(X) = \tilde{t}$ for some $\tilde{t} \in \mathcal{T}$. However, this leads immediately to a contradiction, as since $A, B \subseteq X$, then it must be that $f(A) = f(B) = f(X) = \tilde{t}$. Hence, $f(A) = \tilde{t}$

for all non-singleton menus $A \in 2^X \setminus \emptyset$ and some $\tilde{t} \in \mathcal{T}$. Since choices from singleton menus are the same across learning stages, then, by part (*ii*), $c(\cdot, f(\cdot))$ satisfies WARNI-EXT, as desired. $\square$

Proof of Proposition 4.3. Let $c(A, t)$ denote the choice correspondence (2) of the maximal dominant choice procedure.

(*i*). Suppose that $c(A, 0) \neq \emptyset$. Since $\preceq_0^L$ is an antichain, then $Best(\{x\}, \preceq_0^L) = \{x\}$ for any $x \in X$ and $Best(A, \preceq_0^L) = \emptyset$ for any $A \in 2^X \setminus \emptyset$ such that $|A| > 1$. Hence, $c(A, 0) \neq \emptyset$ implies $|A| = 1$. In the other direction, the result immediately follows from the fact that $Best(\{x\}, \preceq_0^L) = \{x\}$ for any $x \in X$.

(*ii*). Assume that $c(A, t) \neq \emptyset$. Note that, whenever $c(A, t) \neq \emptyset$, $|c(A, t)| = 1$, because each $(X, \preceq_t^L)$ is a poset. Hence, assume that $\{x\} = Best(A, \preceq_t^L)$. This implies that the restriction of $\preceq_t^L$ to A is a join semi-lattice with unique best element x. By axiom [A3], $\preceq_s^L$ with $s > t$ is an order extension of $\preceq_t^L$. Hence, the restriction of $\preceq_s^L$ to A is also a join semilattice with unique best element x and the added comparabilities rank alternatives that are dominated by x. Therefore, $\{x\} = Best(A, \preceq_s^L) \neq \emptyset$, as desired.

(*iii*). Assume that $c(A, t) = \emptyset$. Hence, the restriction of $\preceq_t^L$ to A is not a join semi-lattice. We now show that $s = n^*$ serves the purpose. By proposition 3.2, L settles on a linear order at learning stage n^*. Hence, the restriction of $\preceq_{n^*}^L$ to A is also a linear order and, in particular, a join semilattice. Hence, $Best(A, \preceq_{n^*}^L) \neq \emptyset$. This implies that $c(A, s) \neq \emptyset$ when $s = n^* > t$, as desired. $\square$