Regional Equilibrium Unemployment with Agglomeration Effects[*]

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Abstract

Patterns of unemployment considerably vary along the geographical dimension. In this article, I study the role of agglomeration effects on the dynamics of local unemployment. An original model of the regional labor market with search and mobility frictions is built. The impact of place-based subsidies and unemployment benefits crucially depends on the sign and strength of the agglomeration forces. With agglomeration productivity gains, negative regional employment shocks are amplified because profit opportunities deteriorate, inducing higher mobility out of the region. The model is able to reproduce the strong persistence of the shock on the unemployment rate and the region's size.

JEL Code: J21, J61, L26, R13, R23

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Introduction

Unemployment rates vary widely across regions within countries. This is true in European countries, featuring low inter-regional mobility, as well as in the Unites States where workers are more mobile across states. Recent research suggests that these trends do not reflect compensating differentials but rather the persistence of the lack of jobs in some regions. Agglomeration effects are invoked to explain the emergence of long-run geographical disparities regarding region's size, industrial specialization, wages, or housing prices for instance. Regional unemployment, however, is missing in this list. I argue in this paper that agglomeration externalities can explain patterns of regional unemployment and constitute a plausible source of shock persistence.

Agglomeration forces can increase the persistence of shocks on the labor market. Imagine a firm closes for exogenous reasons. In absence of spillover effects, new firms and jobs would replace the bankrupt firm, attracted by profit opportunities from an excess labor supply. If firms are negatively impacted as the economic geography literature extensively documents instead, new firms may be more reluctant in entering the market because the excess labor supply may not counterbalance the loss in production efficiency. This mechanism is amplified if workers anticipate a slower replacement process of jobs. Workers may then leave the region, which reduces the excess labor supply and so the profit opportunities.

The main balancing mechanism following a shock on the labor market is migration. This paper thus focuses on the dynamics of a single region connected to the rest of the world only through worker and firm mobility. The persistence mechanism suggested above requires three main assumptions: i) productivity and local costs may depend intrinsically on the size of each group of agents, what I call agglomeration effects; ii) search frictions on the labor market prevent firms and workers to match instantaneously, resulting in unemployment; iii) inter-regional mobility is imperfect. In this context, I build a fully consistent model with rational forward-looking agents, deciding between migrating or living in the region. The labor market is based on the Diamond-Mortensen-Pissarides framework (see Pissarides (2000) for an introduction) in which firms are held by entrepreneurs.

My first contribution is to build a convenient workhorse model to think about the complex interaction between mobility, agglomeration forces and expectations. Industry specialization, labor force education, housing supply constitute many dimensions that result in different agglomeration forces across regions. The analysis is kept general as it does not rely on a particular mechanism for agglomeration effects. I analyze two types of policies: local subsidies/taxes to firms or workers, and place-based unemployed benefits. The effect of these policies on the region's size and local unemployment crucially depends on the sign and strength of agglomeration economies. In the standard search and matching framework, (non-financed) unemployment benefits improve the bargaining position of workers and thus reduce job creation. In presence of agglomeration effects like an aggregate demand effect, such a policy can attract more workers in the region to stimulate the production sector and to provide incentives for opening more jobs.

My second contribution is a quantitative analysis of the regional labor market dynamics from the model. In the simulations, I consider a regional economy with productivity gains from input sharing and congestion on the housing market. After a negative initial shock in employment, the unemployment rate monotonically returns to its steady state value whereas the response function of the size of the labor force is U-shaped. Workers leave the region in the first periods because of the lack of employment perspectives until a point when a sufficient number of firms entered the region. I find that agglomeration effects slow down the recovery of the local economy but do not increase significantly the magnitude of the shock on the region's size. The model is able to generate strong persistence in the region's size but relatively low magnitude.

The seminal work of Blanchard and Katz (1992) focuses on regional dynamics of the labor market in the United States. They find that shocks on relative unemployment rates last between 6 and 10 years for the U.S. states for the period 1972-1990. Although they argue that the persistence is low, it contradicts a more recent study by Amior and Manning (2015) showing that joblessness is persistent and hardly explained by compensating differentials. My paper explains the persistence observed by Amior and Manning (2015) with agglomeration effects, a mechanism which is absent in Blanchard and Katz (1992)'s article. In the spirit of Marshall (1890), agglomeration effects are modeled as external economies of scale impacting directly living costs and productivity. I take an agnostic approach in the sense that I do not choose particular micro-foundations in order to derive general properties. The literature on urban and regional economics is abundant in micro-foundations for agglomeration effects. Duranton and Puga (2004) provide a comprehensive study of the variety of agglomerations effects. Helpman (1995) builds a model encompassing centripetal forces attracting firms and centrifugal forces dispersing workers in a spatial model to study regional population. He follows the idea of increasing returns to scale in production with monopolistic competition developed by Krugman (1991). My approach is different as I consider an open region in which workers and firms decide to migrate as long as the endogenous value from living in the region is higher than the exogenous value from living elsewhere, in the idea of Roback (1982).

My article considers unemployment resulting from search frictions on the labor market. Kline and Moretti (2013) reassess public intervention for local unemployment. They consider, however, a particular channel of agglomeration effects through housing costs, which ignores an important channel through productivity gains. My article goes further in the sense that the effects of public policy are obtained without priors for agglomeration economies. I also study a larger set of public interventions. Beaudry et al. (2014) build a tractable model to estimate on U.S. data. Though their approach is empirical, their structural model is very close as it accounts for agglomeration forces in a frictional environment. My paper differs in two dimensions. First, the dynamic analysis I conduct relies on the intrinsic dynamics of the frictional labor market, whereas these authors consider regional labor markets hit by exogenous shocks but remaining at steady state. In other words, they assume the dynamics of the labor market is negligible relative to the exogenous dynamics of shocks. Second, my focus is also on place-based public policies and how agglomeration forces modify their efficiency. In my model, workers must be physically present in the region to participate to the labor market. Lutgen and Van der Linden (2015) discuss this hypothesis and show that an inefficiency arises when the unemployed can search before migrate in a region, calling for public intervention. Lastly, my paper is related to the work of Lkhagvasuren (2012). He provides an explanation for the observed negative correlation between regional unemployment and gross mobility based on individual location-specific productivity. Both our papers reassess the dynamics of local labor markets with richer credible mechanisms.

How agglomeration forces are accounted in the model of this paper is similar to the approach of Mortensen (1999). In his model, the unemployed create an externality on production resulting in the same coordination problem and the same stability analysis. My model accounts for richer possibilities for externalities, not only on production, and incorporates inter-regional mobility of firms and workers. Nevertheless, the higher dimensionality prevents me from conducting an exhaustive study of the dynamics as Mortensen (1999) (or more recently Sniekers (2013)) does.

The structure of the paper is the following. In the first section, the theoretical framework is introduced and the equilibrium is characterized. I conduct a policy analysis in the second section. In a third section, I investigate the dynamic behaviors of the model through a linear approximation and I simulate it. In the fourth section, I explore the issues of steady-state stability and multiplicity. Lastly, I conclude in the last section.

1 A Frictional Labor Market with External Agglomeration Forces

In this section, I present a continuous-time model of a regional economy. Workers and entrepreneurs with perfect foresight choose to enter or leave the region. Search frictions result in unemployment, and agglomeration effects result in external returns to scale.

1.1 Framework

A region is populated by two types of agents at time t; the measure of workers is l_t, and the measure of entrepreneurs is n_t. Agents may die at rates ρ^U for workers and ρ^E for entrepreneurs. They are instantaneously replaced by newborn individuals. The perspective of dying introduces a discount factor in the evaluation of asset values. Any psychological discount rate could be added to the benchmark but it is not useful for the analysis. The cost of living in the region is h_t for workers and the local cost of business k_t for entrepreneurs. These two variables aggregate various effects: housing prices, congestion, local infrastructure, cost of inputs, cost of ideas, entrepreneurial culture. Firms, held by entrepreneurs, produce a homogeneous good sold on the local market. The production technology has constant returns to scale in labor, the sole input. Entrepreneurs are able to manage at most a measure α of jobs.[1] The activity of each job yields y_t. Entrepreneurs start their business without employees. Workers start as unemployed and enjoy a flow b (indifferently non-financed unemployment benefits, home production, leisure) until they get hired and obtain the negotiated wage w_t. I reasonably assume the marginal productivity of a worker y_t to be higher than b. The region is integrated in a global economy through migration so that the measures of individuals, l_t and n_t, are endogenously determined.

Agglomeration effects This paper do not provide microfoundations for agglomeration externalities. Those have been widely documented in the literature on urban economics and economic geography, in Duranton and Puga (2004) for example. Instead, the emphasis is on the consequences of agglomeration effects keeping a general setting that encompasses a variety of foundations. Let $\mathcal{Y}(.,.)$, $\mathcal{H}(.,.)$ and $\mathcal{K}(.,.)$ to be three differentiable functions. Agglomeration effects write as

$$\begin{cases} y_t = \mathcal{Y}(l_t, n_t) \\ h_t = \mathcal{H}(l_t, n_t) \\ k_t = \mathcal{K}(l_t, n_t) \end{cases} . \tag{1}$$

Define the semi-elasticities of the agglomeration functions at a point (l, n), $\mathcal{Y}_l = l\frac{\partial \mathcal{Y}}{\partial l}(l, n)$, $\mathcal{Y}_n = n\frac{\partial \mathcal{Y}}{\partial n}(l, n)$ and analogously $\mathcal{H}_l$, $\mathcal{H}_n$, $\mathcal{K}_l$, $\mathcal{K}_n$. Table 1 provides examples of micro-founded agglomeration forces compatible with my model. Most of the specifications presented by Duranton and Puga (2004) can be adapted, except when the labor market has features hardly compatible with a random search framework. In Kline and Moretti (2013)'s article, absentee landlords produce housing with a convex technology and individuals desire one unit of housing. The equilibrium price of housing is thus increasing in

[1] The results remain if one includes an endogenous α with a convex management cost $C(\alpha)$. The wage bargaining would be unchanged because the managerial cost does not depend on the decision for a worker to accept the job, which rules out intra-firm bargaining as in Cahuc et al. (2008).

the number of residents. Krugman (1991) exploits the monopolistic competition setting as microfoundations for external increasing returns to scale. Simply because residents value variety, workers can enjoy higher utility and firms can benefit from a higher demand in a large market economy. Ethier (1982) focuses on the gains for firms to set in the same region in order to share inputs and produce more efficiently. Romer (1986) insists on the diffusion of knowledge as a productivity-improving mechanism. Large markets produce more knowledge as externalties, benefiting to everybody.

Table 1 – Examples of agglomeration effects

Agglomeration Force	**Effect**	**Articles**
Housing prices	$\mathcal{H}_l > 0$	Kline and Moretti (2013)
Indivisible production of variety	$\mathcal{H}_n < 0$, $\mathcal{Y}_n > 0$	Krugman (1991)
Input sharing	$\mathcal{K}_n < 0$, $\mathcal{Y}_n > 0$	Ethier (1982)
Knowledge spillovers	$\mathcal{H}_l, \mathcal{H}_n < 0$, $\mathcal{Y}_l, \mathcal{Y}_n > 0$	Romer (1986)

Note: The list of articles with these agglomeration forces is not exhaustive. Most of the papers are referenced in the survey by Duranton and Puga (2004).

Search frictions Vacancies posted by local entrepreneurs and local unemployed workers meet on a market with search frictions, in the spirit of the standard Diamond-Mortensen-Pissarides framework.[2] The number of contacts between job seekers and open vacancies is given each period by a matching technology with constant returns to scale. The meeting rate then only depends on the market tightness θ_t, i.e. the vacancy-unemployed ratio. At each instant, an entrepreneur meets an unemployed with a probability $q(\theta_t)$ for each open vacancy and, conversely, an unemployed receives a job proposal with a probability $\theta_t q(\theta_t)$. $q(.)$ is decreasing function from $(0, +\infty)$ onto $(0, +\infty)$ whose elasticity lies between -1 and 0. Consequently, $\theta q(\theta)$ is increasing for θ from $(0, +\infty)$ onto $(0, +\infty)$ with an elasticity between 0 and 1. After a meeting, the worker and the employer decide to match and share the surplus through a Nash bargain. The bargaining power of workers is denoted β in $(0, 1)$. The wage is renegotiated at each period. The job terminates either when the employer or the employee die, or for other exogenous reasons at rate δ.

Constant returns to scale in the matching function eliminates a possible source of agglomeration externalities. First, this hypothesis is standard in search models. Surveying the literature on empirical estimates of the matching function, Petrongolo and Pissarides (2001) admit that constant returns is a good benchmark. Mortensen (1999), whose analysis of dynamic unemployment is similar, also assumes constant returns in matching. Second, though it simplifies the characterization of equilibria, one can follow the same

[2]See Pissarides (2000) for a detailed introduction to this theory.

methodology for increasing or decreasing returns. This hypothesis will be repeated if it is crucial for some results.

Mobility constraints Mobility is the main innovation of the model. Contrary to the benchmark model of the labor market, not only firms but also workers face an entry condition. Modeling mobility decisions with consistent discounting of the value from living in the region in a tractable framework is a theoretical challenge. Enabling agents to move freely should affect the wage bargaining and the value of jobs because of endogenous job separation. To avoid these difficulties, mobility is constrained in the sense that agents can move in or out the region only at the beginning of their life. The potential movers make their decisions by comparing the present discounted value from living in the region at the beginning of the life and the present discounted value from leaving outside. I denote the difference between the two as U_t for workers and E_t for entrepreneurs. In absence of mobility frictions, the measure of agents l_t and n_t would be jump variables, instantaneously adjusting to maintain $U_t = 0$ and $E_t = 0$ at a non-degenerated equilibrium. Mobility frictions is captured by the positive and increasing functions Φ^U and Φ^E, with $\Phi^U(0) = \Phi^E(0) = 1$. For an infinitesimal period Δt, the region looses a measure $\rho l_t \Delta t$ of workers and $\rho n_t \Delta t$ of entrepreneurs. A measure $\Phi^U(U_t)\rho l_t \Delta t$ of workers and $\Phi^E(E_t)\rho n_t \Delta t$ of entrepreneurs replace them. There is net inter-regional emigration of workers when $U_t < 0$. A consistent story is that a share $\Phi^U(U_t)$ of newborn workers decide to stay while nobody immigrates. When $U_t > 0$, the size of the immigrant population is assumed to be proportional to the number of newborn agents. One can choose other stories for individual mobility decisions as long as these net mobility patterns are preserved. Denote $\dot{X}_t$ the time derivative of any variable X_t. The dynamics of the number of workers write

$$\dot{l}_t = (\Phi^U(U_t) - 1)\rho^U l_t. \tag{2}$$

Local entrepreneurs follow the same mobility condition,

$$\dot{n}_t = (\Phi^E(E_t) - 1)\rho^E n_t. \tag{3}$$

Agents are more mobile at the aggregate level as functions Φ^U and Φ^E are steeper. Φ^U and Φ^E can be different to account for difference in response to local shocks. For instance, firms can be faster than workers in reallocating elsewhere. Individuals are mobile enough so that a small deviation in the gain U, positive or negative, leads to a first-order response in mobility, $\Phi^{U'}(0) = \phi^U > 0$. The parameters ϕ^U and ϕ^E encompass the effect of mobility constraints close to a steady state. Importantly, differentiability in zero is equivalent to assume that net mobility responds symmetrically to a small surplus gain and to a small surplus loss. This assumption is required to linearize the model around a steady state.

Living elsewhere yields constant flow returns until death, μ for workers, and ϵ for entrepreneurs. The region is small enough so that these variables can be considered as exogenous.

1.2 Equilibrium characterization

Expected payoffs A job has a present-discounted value V_t when vacant, and J_t when filled. An employer finds a worker to hire at a rate $q(\theta)$, leading to a capital gain $J_t - V_t$. An occupied job yields a flow profit $y_t - w_t$ and turns vacant at a rate $\rho + \delta^U$ because of exogenous reasons or the employee's death. The asset values satisfy the Bellman equations

$$\rho^E J_t = y_t - w_t + (\rho^U + \delta)[V_t - J_t] + \dot{J}_t, \tag{4}$$

$$\rho^E V_t = q(\theta_t)[J_t - V_t] + \dot{V}_t. \tag{5}$$

These formula slightly differ from the standard framework because of the discounting due to death. A derivation of these formula as a limit of the discrete-time case is given in appendix. Setting up a business in the region, compared to live elsewhere, yields a value L_t, which is the discounted sum of the local costs k_t and the outside amenity ϵ,

$$\rho^E L_t = -k_t - \epsilon + \dot{L}_t. \tag{6}$$

I assume that entrepreneurs take the labor market tightness as given, although they open several job positions at the same time. An entrepreneur who has already γ employees out of α enjoys the surplus $\gamma J_t + (\alpha - \gamma)V_t + L_t$ from being in the region. As the entrepreneurs create a business without any employee, the surplus from staying in the region writes

$$\rho^E E_t = \alpha V_t + L_t \tag{7}$$

Turn to the worker's values. I define W_t as the capital gains from being employed in the region instead of living elsewhere. Whether she is employed or unemployed, a worker incurs the cost of living h_t and renounces to the amenity μ from living in the region. An employed worker earns the wage w_t, she joins the pool of unemployed at a rate $\rho^E + \delta$ if her employer dies or if the match exogenously breaks. An unemployed enjoys a flow b and switches to employment at rate $\theta_t q(\theta_t)$. The corresponding Bellman equations write

$$\rho^U W_t = w_t - h_t - \mu + (\rho^E + \delta)[U_t - W_t] + \dot{W}_t, \tag{8}$$

$$\rho^U U_t = b - h_t - \mu + \theta_t q(\theta_t)[W_t - U_t] + \dot{U}_t. \tag{9}$$

The dynamics of the capital gains from filling a vacancy and from switching to employment are obtained from equations (4) and (5) on one side, and (8) and (9) on the other side:

$$(\rho^U + \rho^E + \delta + q(\theta_t))[J_t - V_t] = y_t - w_t + [\dot{J}_t - \dot{V}_t], \tag{10}$$

$$(\rho^U + \rho^E + \delta + \theta_t q(\theta_t))[W_t - U_t] = w_t - b + [\dot{W}_t - \dot{U}_t]. \tag{11}$$

Notice that the amenities from living outside the region, ϵ and μ, do not account in the gains from matching for the firm and the employee.

Surplus sharing The wage is negotiated each period between the firm and the employee through a Nash-bargaining problem,

$$\underset{w_t}{\text{argmax}}\ (W_t - U_t)^\beta (J_t - V_t)^{1-\beta} \text{ s.t. (4) and (8),}$$

When the worker and the firm matches, they decide on a salary taking as given the outside options from not matching, U_t and V_t. These values are not impacted by the local agreement between the employer and the employee. Define the surplus of a job $S_t = J_t - V_t + W_t - U_t$. The first-order condition of the problem leads to the traditional surplus-sharing rule that splits the surplus of a job according the bargaining power of each party,

$$W_t - U_t = \beta S_t \quad \text{and} \quad J_t - V_t = (1-\beta) S_t. \tag{12}$$

The asset values from being unemployed and being an employer, net of the values from leaving elsewhere can be expressed in terms of the job surplus,

$$\rho^U U_t = b - h_t - \mu + \beta \theta_t q(\theta_t) S_t + \dot{U}_t, \tag{13}$$

$$\rho^E E_t = -k_t - \epsilon + \alpha(1-\beta) q(\theta_t) S_t + \dot{E}_t. \tag{14}$$

The recursive definition of the job surplus derives from the Bellman equations (10) and (11),

$$[\rho^U + \rho^E + \delta + \beta \theta_t q(\theta_t) + (1-\beta) q(\theta_t)] S_t = y_t - b + \dot{S}_t. \tag{15}$$

Equilibrium flow conditions Denote m_t the measure of matches at time t, the market tightness is defined as the vacancy-unemployment ratio,

$$\theta_t = \frac{\alpha n_t - m_t}{l_t - m_t}. \tag{16}$$

The stock of matches satisfies the following law of motion:

$$\dot{m}_t = \theta_t q(\theta_t)(l_t - m_t) - (\rho^U + \rho^E + \delta) m_t. \tag{17}$$

The first term in the right-hand side of the equation is the inflow of new jobs and the second term is the outflow of broken matches.

Proposition 1 *An equilibrium of the model $\mathcal{P}$ is characterized by a state $\mathcal{P}_t = (l_t, n_t, \theta_t)$ for any time $t \geq 0$ such that*

- *the measures of residents, l_t and n_t, follow their law of motion in* (2) *and* (3)*;*
- *the labor market tightness, θ_t, and the measure of jobs, m_t, fulfill* (16) *and* (17)*;*
- *the present-discounted values U_t, E_t and S_t are recursively defined by* (13)*,* (14) *and* (15)*;*
- *productivity, y_t, and local costs, h_t and k_t, satisfy the agglomeration effects specification* (1)*.*

Note that S_t, U_t and E_t are forward-looking whereas the remaining variables are predetermined. The model can be refined as a stochastic equilibrium by considering μ and ϵ as exogenous processes.

1.3 Steady state

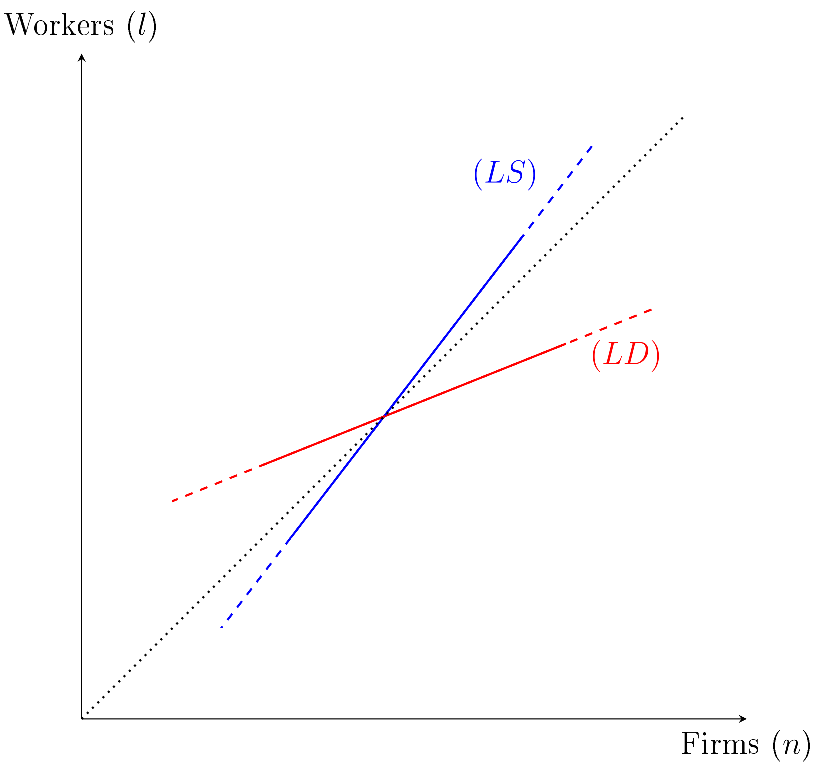

Figure 1 – Illustration of a steady state

Note: The intersection of the labor supply (LS) and the labor demand (LD) curves characterize a steady state equilibrium. The curves are not necessarily straight lines.

I investigate the features of a steady-state equilibrium. The time index is dropped in the variable notations for convenience. I exclude degenerated steady states, meaning when a type of agents has left the region, either $l = 0$ or $n = 0$. In the non-degenerated cases, the region's size remains constant over time when the net gains, U and E, are nil in (2) and (3). This writes in the two conditions:

$$F^U(\theta)(y - b) + b - h - \mu = 0, \tag{18}$$

$$\alpha F^E(\theta)(y - b) - k - \epsilon = 0, \tag{19}$$

with

$$F^U(\theta) = \frac{\beta\theta q(\theta)}{\rho^U + \rho^E + \delta + \beta\theta q(\theta) + (1-\beta)q(\theta)},$$

$$F^E(\theta) = \frac{(1-\beta)q(\theta)}{\rho^U + \rho^E + \delta + \beta\theta q(\theta) + (1-\beta)q(\theta)}.$$

Workers discount the future perspectives on the labor market such that it is equivalent to receive a flow utility, $F^U(\theta)(y - b)$, which is proportional to production net of unemployment benefits. Similarly, the possibility to fill a vacancy for an employer is equivalent to receive flow profits $F^E(\theta)(y - b)$. The fractions $F^U(\theta)$ and $F^E(\theta)$ are respectively increasing and decreasing in θ. A tighter labor market benefits to the unemployed workers as they find a job faster, it also improves their bargaining position. Consequently, the worker surplus increases with the market tightness. On the opposite side of the market, firms spend more time filling their vacancies and have to pay a higher wage bill as the labor market gets tighter. The number of employers derives from the definition of the market tightness (16) and the dynamics of the stock of jobs (17),

$$\theta = \Theta(l, n), \tag{20}$$

with $\Theta(l, n)$ solution of $\frac{\rho^U+\rho^E+\delta+q(\theta)}{\rho^U+\rho^E+\delta+\theta q(\theta)}\theta = \frac{\alpha n}{l}$. The labor market at the steady state is tighter as the number of entrepreneurs is higher or as the number of workers is lower.[3]

Proposition 2 *A non-degenerated steady-state equilibrium* $\mathcal{P}^* = (l^*, n^*, \theta^*)$ *is such that* (18), (19) *and* (20) *are satisfied given the specification of agglomeration forces* (1) *for* y, h *and* k.

By removing the market tightness θ, steady-state equilibrium can be characterized through two conditions defining labor supply and labor demand curves,

$$\mathcal{U}(l, n) = 0, \tag{LS}$$

$$\mathcal{E}(l, n) = 0, \tag{LD}$$

[3]The left-hand side of the equation is strictly continuously increasing in θ from 0 to $+\infty$ so the intermediate values theorem insures us the existence of Θ.

where $\mathcal{U}$ and $\mathcal{E}$ are the two functions defined by

$$\mathcal{U}(l,n) = F^U(\Theta(l,n))\,[\mathcal{Y}(l,n) - b] + b - \mu - \mathcal{H}(l,n),$$
$$\mathcal{E}(l,n) = \alpha F^E(\Theta(l,n))\,[\mathcal{Y}(l,n) - b] - \mathcal{K}(l,n) - \epsilon.$$

The first equation is the condition for workers to be indifferent between staying and living out of the region. The second one is the equivalent for the firms. The functions $\mathcal{U}$ and $\mathcal{E}$ define the equilibrium values of U and E depending on the size of the region. The two conditions represent two isoutility curves in a (n, l) plan as on a figure 1. Along the dotted line, the market tightness is constant. The market tightness increases with the angle of the dotted line, or equivalently the steady-state unemployment rate decreases. Without more assumptions on $\mathcal{Y}(.,.)$, $\mathcal{H}(.,.)$ and $\mathcal{K}(.,.)$, the model can generate multiple steady states. The model can be compared to the benchmark without agglomeration forces.

Corollary 1 *There are in general no non-degenerate equilibrium in an regional economy without agglomeration forces. For particular values of the parameters, an infinite number of non-degenerate equilibria exist and they are characterized by the same market tightness.*

In absence of agglomeration externalities, the productivity and local costs, y, h and k, are constant. The labor supply and demand curves are two straight lines passing through the origin (which is an equilibrium). In general the equilibrium market tightness cannot satisfy both equations (18) and (19) except for very particular values of the parameters. The parameters are such that the two equations are identical, the two curves thus superpose. The market tightness is the same for any equilibrium, and so is the ratio between the measure of workers to the measure of employers. There are no reasons for such conditions on the parameters to occur in practice.

2 Comparative Statics and Place-based Policies

This section focuses on the effects of different public policies. A normative analysis is irrelevant in such an open-region model because inter-regional migration will insure agents to receive constant incomes equivalent to μ for workers and ϵ for entrepreneurs in the long-run. Place-based public policies, however, can have other purposes than efficiency. Local political power may have incentives to improve the number of residents or to reduce the unemployment rate for electoral motives. The centralized authority may also have preferences for preserving regional culture and tradition, and thus for increasing the number of citizens. Consequently, I conduct a positive analysis of public policies. Another concern is the existence of multiple equilibria in general. A policy change may induce workers and firms to switch equilibrium. Here, I abstract from this scenario, which is postponed to the penultimate section. I also consider that public policies are small enough so that the linear approximation close to a steady state is valid.

Semi-elasticities of the flow returns Let $\mathcal{U}_l$ be the semi-elasticity of the flow returns from living in the region as a worker to the number of workers near a steady state, $\mathcal{U}_l = l^* \frac{\partial \mathcal{U}}{\partial l}(l^*, n^*)$. The terms $\mathcal{U}_n$, $\mathcal{E}_l$ and $\mathcal{E}_n$ are defined analogously. An increase of the number of workers by 1% implies an increase in the flow returns from living in the region by $\mathcal{U}_l/100$ units for workers and by $\mathcal{E}_l/100$ for entrepreneurs. The sign of these 4 parameters tells how each agent affects others' utility. For instance, $\mathcal{E}_n > 0$ means that firms benefit from the presence of other firms in the region.

The semi-elasticities depend explicitly on a limited number of variables that I define here. Denote η the elasticity of the matching function relative to the mass of employers at the steady state, $\eta = 1 + \frac{\theta^* q'(\theta^*)}{q(\theta^*)}$. The elasticities of the steady-state market tightness to the measure of workers and entrepreneurs are $\mathcal{T}_l = \frac{l^*}{\theta^*}\frac{\partial \Theta}{\partial l}(l^*, n^*)$ and $\mathcal{T}_n = \frac{n^*}{\theta^*}\frac{\partial \Theta}{\partial n}(l^*, n^*)$. The semi-elasticities of $\mathcal{U}$ and $\mathcal{E}$ are

$$\begin{aligned}
\mathcal{U}_l &= -\mathcal{H}_l + F^U(\theta^*)\mathcal{Y}_l + \left[\eta(\rho^U + \rho^E + \delta) + (1-\beta)q(\theta^*)\right] F^U(\theta^*) S^* \mathcal{T}_l, \\
\mathcal{U}_n &= -\mathcal{H}_n + F^U(\theta^*)\mathcal{Y}_n + \left[\eta(\rho^U + \rho^E + \delta) + (1-\beta)q(\theta^*)\right] F^U(\theta^*) S^* \mathcal{T}_n, \\
\mathcal{E}_l &= -\mathcal{K}_l + \alpha F^E(\theta^*)\mathcal{Y}_l - \alpha\left[(1-\eta)(\rho^U + \rho^E + \delta) + \beta\theta^* q(\theta^*)\right] F^E(\theta^*) S^* \mathcal{T}_l, \\
\mathcal{E}_n &= \underbrace{-\mathcal{K}_n}_{\text{local cost effect}} + \underbrace{\alpha F^E(\theta^*)\mathcal{Y}_n}_{\text{productivity effect}} - \underbrace{\alpha\left[(1-\eta)(\rho^U + \rho^E + \delta) + \beta\theta^* q(\theta^*)\right] F^E(\theta^*) S^* \mathcal{T}_n}_{\text{search externality}}.
\end{aligned}$$

Each semi-elasticity is the sum of three terms. The first component accounts for a change in the non-sharable local cost, h_t for workers and k_t for entrepreneurs. The second one represents the effect on job productivity y_t. The last one corresponds to a change in the labor market conditions. The market tightness responds symmetrically to a relative change in the number of firms and workers, $\mathcal{T}_n = -\mathcal{T}_l > 0$, because the matching technology has constant returns to scale. Denote $\mathbb{V}$ the matrix of interaction effects, $\mathbb{V} = \begin{pmatrix} \mathcal{U}_l & \mathcal{U}_n \\ \mathcal{E}_l & \mathcal{E}_n \end{pmatrix}$, and its determinant $v = \mathcal{U}_l\mathcal{E}_n - \mathcal{U}_n\mathcal{E}_l$. The determinant is informative of the configuration of a steady-state equilibrium by being closely related to the sign of the oriented angle between the two curves. How the two curves cross each other determines the effects of a change in parameter in a comparative statics analysis. For instance, the example on figure 1 is a case in which the isoutility curve for workers (LS) increases faster than the isoutility of entrepreneurs (LD) as the number of entrepreneur increases. It can correspond to a case in which $\mathcal{E}_l > 0 > \mathcal{U}_l$ and the determinant is positive.

In absence of agglomeration economies, the signs of the interaction matrix's elements are $\begin{pmatrix} (-) & (+) \\ (+) & (-) \end{pmatrix}$ and the determinant is nil. The sole interaction effects come from standard search externalities. One more job seeker reduces the job-finding rate of other workers but increases the job-filling rate of firms. This explains the signs in the first column of the matrix. The reverse applies for the entry of new job vacancies, which provides the signs of the second column. The determinant is nil because of the constant returns-to-scale assumption.

Local subsidies Imagine that the fiscal authority provides a financial aid (or a tax) for workers to live in an area, whether they are unemployed or not. Subsidizing local workers by an amount a is mathematically equivalent in the model to reducing the outside value from μ to $\mu - a$. Analogously, a subsidy to firms corresponds to a reduce in ϵ. The issue of financing the subsidy is ignored. One can assume that it is funded by taxes from the other regions in the country and can be considered as exogenous.

Proposition 3 *Assume the regional economy is at steady state. Consider subsidies to workers and firms that are small enough so that the linear approximation is still appropriate. In the long run, a subsidy to local workers*

- *increases the number of workers if and only if* $\frac{\mathcal{E}_n}{\mathcal{U}_l\mathcal{E}_n - \mathcal{U}_n\mathcal{E}_l} < 0$,
- *increases the number of employers if and only if* $\frac{\mathcal{E}_l}{\mathcal{U}_l\mathcal{E}_n - \mathcal{U}_n\mathcal{E}_l} > 0$.
- *decreases the unemployment rate (or increases the market tightness) if and only if* $\frac{\mathcal{E}_l + \mathcal{E}_n}{\mathcal{U}_l\mathcal{E}_n - \mathcal{U}_n\mathcal{E}_l} > 0$.

In the long run, a subsidy to local companies

- *increases the number of workers if and only if* $\frac{\mathcal{U}_n}{\mathcal{U}_l\mathcal{E}_n - \mathcal{U}_n\mathcal{E}_l} > 0$,
- *increases the number of employers if and only if* $\frac{\mathcal{U}_l}{\mathcal{U}_l\mathcal{E}_n - \mathcal{U}_n\mathcal{E}_l} < 0$.
- *decreases the unemployment rate (or increases the market tightness) if and only if* $\frac{\mathcal{U}_l + \mathcal{U}_n}{\mathcal{U}_l\mathcal{E}_n - \mathcal{U}_n\mathcal{E}_l} < 0$.

Proof. Differentiating equations (LS) and (LD) with respect to l, n, μ and ϵ gives $\begin{pmatrix} d\mu \\ d\epsilon \end{pmatrix} = \mathbb{V} \begin{pmatrix} \frac{dl}{l^*} \\ \frac{dn}{n^*} \end{pmatrix}$. Inverse the system to obtain the effects on the number of workers and firms,

$$\begin{pmatrix} \frac{dl}{l^*} \\ \frac{dn}{n^*} \end{pmatrix} = \frac{1}{v} \begin{pmatrix} \mathcal{E}_n & -\mathcal{U}_n \\ -\mathcal{E}_l & \mathcal{U}_l \end{pmatrix} \begin{pmatrix} d\mu \\ d\epsilon \end{pmatrix}.$$

Now, we study the effect on the unemployment rate by showing that it is related to the market tightness. The steady-state unemployment rate is decreasing with the market tightness from its definition,

$$1 - \frac{m^*}{l^*} = \frac{\rho^U + \rho^E + \delta}{\rho^U + \rho^E + \delta + \theta^* q(\theta^*)}.$$

Log-differentiate $\theta = \Theta(l,n)$ to get

$$\gamma(\theta^*)\frac{d\theta}{\theta^*} = \frac{dn}{n^*} - \frac{dl}{l^*},$$

with $\gamma(\theta)$ as the log-derivative of expression $\frac{\rho^U+\rho^E+\delta+q(\theta)}{\rho^U+\rho^E+\delta+\theta q(\theta)}\theta$. After finding that $\gamma(\theta) > 0$, the growth rate of the market tightness, $\frac{d\theta}{\theta^*}$, is thus increasing in $\frac{dn}{n^*} - \frac{dl}{l^*}$. The sign of the effect on the unemployment derives from the effects on the number of firms and workers. ■

As the presence of a certain type of agents influence other agents' welfare through the interaction matrix, the effect of subsidizing workers or firms have ambiguous effect. Notice to implement any public policy listed, the government only needs to know two statistics to know the qualitative effects of a policy: a semi-elasticity depending on the type of policy and the determinant of the interaction matrix.

We can comment on the first bullet of this proposition. Consider a first case in which there are no crossed effects, $\mathcal{E}_l = \mathcal{U}_n = 0$; the matrix $\mathbb{V}$ is diagonal. If workers create negative externalities on their welfare, $\mathcal{U}_l < 0$, then a subsidy to workers would attract more workers so that the returns form living in the region equate μ. The firms' entry condition would not be affected by such a change in the labor force. In a second case, suppose the firms' effect on themselves is nil, $\mathcal{E}_n = 0$. A change in the size of the labor force cannot be compensated by a change in the number of entrepreneurs to maintain profits to their constant level. A subsidy to workers then has no impact on the size of the labor force. In the general case, the condition for workers to move in is equivalent to $\mathcal{U}_l < \frac{\mathcal{E}_l}{\mathcal{E}_n}\mathcal{U}_n$. The left-hand side is the direct effect of workers on their own welfare whereas the right-hand side corresponds to the indirect effect through the effect on firms' profits. Employers have an incentive to move in or out depending on the signs of the agglomeration forces, and thus impact workers' welfare. A similar interpretation can be given for a subsidy to employers.

The effect of a policy on the unemployment rate depends on the relative change of the number of employers compared to the change in the number of workers. Figure 2 summarizes the results for the unemployment rate. Any given interaction matrix $\mathbb{V}$ can represented as a dot on one of these 4 graphs. First, the sign of the crossed effects indicate the type of graphical representation among 4 cases. Then, the two diagonal elements of the matrix define coordinates $(\mathcal{U}_l, \mathcal{E}_n)$ on the plan. On the hyperbolas, the determinant of the matrix is nil. The horizontal line is defined by $\mathcal{E}_n = -\mathcal{E}_l$ and the vertical line by $\mathcal{U}_l = -\mathcal{U}_n$. In the degenerated case in which one of the crossed effect is nil, the hyperbolas become a vertical and a horizontal line.

It would be time-consuming to enumerate all the possibilities, and perhaps useless for cases that are unlikely to occur in practice. Instead, I look at departures from the

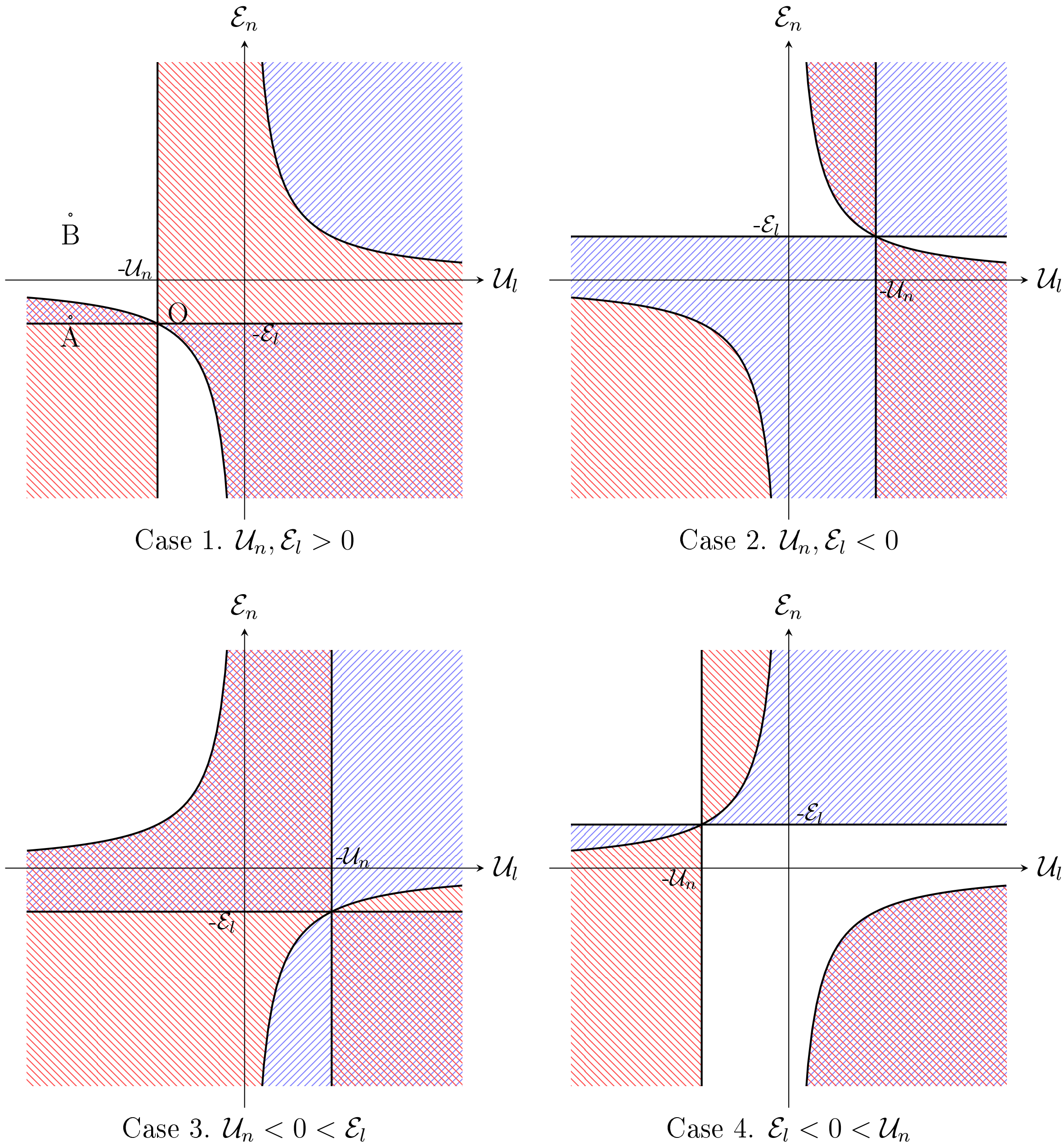

Figure 2 – Effects of policies on the unemployment rate depending on the interaction forces

Note. Any given interaction matrix $\mathbb{V}$ can represented as a dot on one of these 4 graphs (like point A for instance). When blue lines hatch the area from south-west to north-east, a subsidy to workers reduces unemployment. When red lines hatch the area from south-east to north-west, a subsidy to firms reduces unemployment. Both types of policy can also have the same effect, either increasing unemployment in the blank area or reducing unemployment in the double-hatched hatch area.

no agglomeration effects case. This situation corresponds to the first case illustrated in figure 2 at the coordinates $(\mathcal{U}_l, \mathcal{E}_n) = (-\mathcal{U}_n, -\mathcal{E}_l)$ (point O). Assume congestion or housing cost effects such that $\mathcal{H}_l > 0$ as in Kline and Moretti (2013) for instance and gains from input-sharing $\mathcal{K}_n < 0$ in the spirit of Ethier (1982). Points A and B are two cases for different negative values of $\mathcal{K}_n$. When subsidizing workers in case A, more workers enter the region. Firms have an incentive to open vacancies from a thick-market effect on the labor market. This new entry of firms has a multiplicative effect for firms due to gains from input sharing. Firms thus deteriorate their labor market conditions but produce at lower costs. The labor market eventually gets more favorable to workers, materialized by a decline in the unemployment rate. When the input-sharing gains are too high (point B), profits are exploding if firms decide to enter as the labor market can never be tight enough to maintain a constant level of profits. The steady state can only be achieved if firms decide to leave. Workers decide to leave and the labor market eventually deteriorates for workers as their utility is compensated by the reduce in housing prices.

Place-based Unemployment Benefits In the standard Diamond-Mortensen-Pissarides framework, local unemployment benefits are detrimental to firms' profit because workers can bargain a higher wage. Here, firms can enjoy agglomeration effects as workers are attracted in the region.

Proposition 4 *Assume the regional economy is at steady state. Consider subsidies to workers and firms that are small enough so that the linear approximation is still appropriate. In the long run, an increase in the local unemployment benefits*

- *increases the number of workers if and only if* $\frac{(1-F^U(\theta^*))\mathcal{E}_n - \alpha F^E(\theta^*)\mathcal{U}_n}{\mathcal{U}_l\mathcal{E}_n - \mathcal{U}_n\mathcal{E}_l} < 0$,
- *increases the number of employers if and only if* $\frac{(1-F^U(\theta^*))\mathcal{E}_l - \alpha F^E(\theta^*)\mathcal{U}_l}{\mathcal{U}_l\mathcal{E}_n - \mathcal{U}_n\mathcal{E}_l} > 0$.
- *decreases the unemployment rate (or increases the market tightness) if and only if* $\frac{(1-F^U(\theta^*))(\mathcal{E}_n+\mathcal{E}_l) - \alpha F^E(\theta^*)(\mathcal{U}_n+\mathcal{U}_l)}{\mathcal{U}_l\mathcal{E}_n - \mathcal{U}_n\mathcal{E}_l} > 0$.

Proof. Differentiating equations (LS) and (LD) with respect to l, n and b gives

$$-\begin{pmatrix} 1-F^U(\theta^*) \\ \alpha F^E(\theta^*) \end{pmatrix} db = \mathbb{V} \begin{pmatrix} \frac{dl}{l^*} \\ \frac{dn}{n^*} \end{pmatrix}.$$

Inverse the system to obtain the effects on the number of workers and firms,

$$\begin{pmatrix} \frac{dl}{l^*} \\ \frac{dn}{n^*} \end{pmatrix} = -\frac{1}{v} \begin{pmatrix} \mathcal{E}_n & -\mathcal{U}_n \\ -\mathcal{E}_l & \mathcal{U}_l \end{pmatrix} \begin{pmatrix} 1-F^U(\theta^*) \\ \alpha F^E(\theta^*) \end{pmatrix} db.$$

■

An increase in unemployment benefits is equivalent to a subsidy to workers and a tax to employers simultaneously. From the previous proposition, we can immediately deduce that the unemployment rate unambiguously reduces (increases) after a rise in unemployment benefits in the single-hatched areas in figure 2. Conclusions regarding the blank and double-hatched areas require further investigation.

3 Dynamics

3.1 Formal approximated dynamics

A contribution of the paper is the study of regional labor market dynamics. The model with its 3 forward-looking and 3 predetermined variables, however, is hardly tractable for an exhaustive analysis. Mortensen (1999) and Sniekers (2013) do it in 2-variable model of a frictional labor market. Instead, I consider the dynamics close to a steady state through a linear approximation.

I use the following notations for the variables' deviation to their steady-state values, $\tilde{a}_t = \frac{a_t - a^*}{a^*}$ for $a = l, n, m, \theta$ and $\tilde{A}_t = A_t - A^*$ for $A = S, U, E$. Define the column of predetermined variables $N_t = \begin{pmatrix} \tilde{l}_t \\ \tilde{n}_t \\ \tilde{\theta}_t \end{pmatrix}$, and the column of forward-looking variables $R_t = \begin{pmatrix} \tilde{U}_t \\ \tilde{E}_t \\ \tilde{S}_t \end{pmatrix}$. The linear model writes

$$\dot{N}_t = M_{NN}.N_t + M_{NR}.R_t, \tag{21}$$

$$\dot{R}_t = M_{RN}.N_t + M_{RR}.R_t, \tag{22}$$

with M_{NN}, M_{NR}, M_{RN} and M_{RR} are 3x3 matrices. I denote $\zeta^U = \rho^U + \rho^E + \delta + \theta^* q(\theta^*)$ and $\zeta^E = \rho^U + \rho^E + \delta + q(\theta^*)$. The market tightness is the only state variable whose dynamic movements depend on current position and current numbers of workers and entrepreneurs, hence the two first rows of M_{NN} are nil,

$$M_{NN} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{\zeta^U \zeta^E}{\rho^U + \rho^E + \delta} & \frac{\zeta^U \zeta^E}{\rho^U + \rho^E + \delta} & -[(1-\eta)\zeta^U + \eta\zeta^E] \end{pmatrix}.$$

This mechanism corresponds to dynamic adjustments of the labor market because jobs are destructed and created at each instant. If there is an excess of workers, the labor market will tend to be slacker over time as more job seekers are going to participate to the matching process. The first parameter in the third row of M_{NN} is thus negative. The reverse applies for an excess of employers. When the stock of jobs is in excess today, more matches are going to break mechanically so that the stock of jobs will reduce. This mechanism explains the negative sign of the third parameter in the third row of the matrix.

The dynamic behavior of the demographic variables depends also on the asset values R_t because of mobility decisions;

$$M_{NR} = \begin{pmatrix} \phi^U \rho^U & 0 & 0 \\ 0 & \phi^E \rho^E & 0 \\ -\frac{\zeta^U \phi^U \rho}{\rho^U + \rho^E + \delta} & \frac{\zeta^E \phi^E \rho}{\rho^U + \rho^E + \delta} & 0 \end{pmatrix}.$$

In absence of inter-regional migration, matrix M_{NR} would be filled by zeros. When the value of being unemployed in the region is too high, more workers migrate in the region so the sign of the first parameter in the first row is positive. This entry of workers crowds out the labor market so the the sign of the first parameter in the last row is negative. The two other non-zero parameters are the symmetric mechanisms for entrepreneurs.

The values R_t are recursively defined in the Bellman equation, the matrix M_{RR} captures the discounting effect of future values;

$$M_{RR} = \begin{pmatrix} \rho^U & 0 & -\beta\theta^* q(\theta^*) \\ 0 & \rho^E & -\alpha(1-\beta)q(\theta^*) \\ 0 & 0 & \beta\zeta^U + (1-\beta)\zeta^E \end{pmatrix}.$$

Lastly, the asset values are functions of the demographic configuration of the region through agglomeration effects and search externalities. They are encompassed in the matrix

$$M_{RN} = \begin{pmatrix} \mathcal{H}_l & \mathcal{H}_n & -\beta\eta\theta^* q(\theta^*)S^* \\ \mathcal{K}_l & \mathcal{K}_n & \alpha(1-\beta)(1-\eta)q(\theta^*)S^* \\ -\mathcal{Y}_l & -\mathcal{Y}_n & [\beta\eta\theta^* q(\theta^*) - (1-\beta)(1-\eta)q(\theta^*)]S^* \end{pmatrix}.$$

The two first column are the agglomeration externalities due to a change in the number of workers and firms respectively. The last column sums up the search externalities from the frictional labor market. Workers are better off when the market is tighter so the first parameter of the last column is positive. On the reverse, firms are worse off. The last parameter in the third column is the effect of excessive tightness on the value of jobs, which is ambiguous. The dynamic linear system is summarized in a block matrix $\mathcal{M} = \begin{pmatrix} M_{NN} & M_{NR} \\ M_{RN} & M_{RR} \end{pmatrix}$.

3.2 Simulations of a local labor market

The aim of this section is to investigate the ability of the model to replicate the dynamics observed in the data. The approach is to start with a parsimonious calibration of the labor market and to study the role of mobility frictions and the agglomeration effects. The dynamics can be simulated using the same solving strategy as any standard DSGE model. In the simulation, I only consider situations in which the Blanchard and Kahn (1980) conditions are satisfied meaning there is a unique trajectory converging to the steady state. The equivalent of the Blanchard-Kahn conditions in a continuous-time setting is formulated by Buiter (1984). Abstracting for singular cases, a departure from the steady state admits a unique dynamic equilibrium if and only if the matrix $\mathcal{M}$ has as many eigenvalues with positive real part as forward-looking variables, precisely 3. The eigenvectors associated to the eigenvalues with positive real parts define three linear static relationships between the variables. The three other eigenvectors correspond each to a non-explosive exponential trend. See for instance Lamo et al. (2011) for a simulation of a frictional labor market with a proof of local determination.

Parameter	**Target/Motive**	**Value**
Replacement rate, ρ^U, ρ^E	10% of the agents are replaced each year	0.00878
Exogenous job destruction rate, δ	10% of jobs break each quarter	0.0176
Steady-state job-filling rate, $q(\theta^*)$	Daily job-filling rate of 5%	1.56
Steady-state market tightness, θ^*	Unemployment rate at 6%	0.35
Elasticity of the matching function, η	Agnostic choice/Symmetry	0.5
Bargaining Power, β	Hosios-Pissarides condition, $\beta = 1 - \eta$	0.5
Number of jobs max per firm, α		10
Steady-state value of a job, S^*	Normalization $y - b = 100$	91.7

Table 2 – Parameters of the simulation

I normalize the time scale so that one period is a month. The values chosen for the simulation are provided in table 2. I arbitrarily assume that a worker or an entrepreneur has 10% chance of being replaced per year. It implies in the model that the region cannot loose more than 10% of residents per year. I use Shimer (2005)'s estimate to fix the exogenous job destruction rate ρ: 10% of jobs break each quarter because of death or exogenous reasons. Because the model is linearized close to a steady state, it is not necessary to specify each parameter of the model. In particular, we only require the steady-state values of three variables: the market tightness θ^*, the job-filling rate $q(\theta^*)$ and the value of jobs S^*. The job-filling rate is chosen to fit the average daily rate estimated by Davis et al. (2013) at 5%. The steady-state market tightness is then taken so that the unemployment rate is 6%. The elasticity of the matching function η is set to 0.5, in the range of estimates surveyed by Petrongolo and Pissarides (2001). The bargaining power of workers is such that the Hosios-Pissarides condition is satisfied. It implies that workers and firms receive a share of the match surplus corresponding to their public marginal contribution in creating jobs.[4] The maximum number of jobs per firm is fixed at 10. Lastly, the value of jobs is chosen so that net flow output of a match, $y - b$, is normalized to 100. The analysis is particularly sensitive to the mobility parameters and the agglomeration externalities. I will then use different set of values. Using these parameters, the interaction matrix $\mathbb{V}$ in absence of agglomeration externalities is equal to $\begin{pmatrix} -450 & 450 \\ 4683 & -4683 \end{pmatrix}$. As expected, the matrix has a negative first diagonal, a positive second diagonal and is of rank 1. The difference of magnitude between the first and the second row comes from the

[4] See Hosios (1990); Pissarides (2000).

number of jobs per firm α. One more firm in the region implies 10 more jobs to fill, the search externalities are thus stronger. The magnitude of the coefficients are relevant to fix different values of the agglomeration effects. Comparing the simulations using different values for the 6 agglomeration externalities parameters is untractable. Instead, I make the assumptions that some channels are shut down. I emphasize on two mechanisms. I assume first that $\mathcal{H}_l < 0$ because of congestion or convex cost on the housing technology for instance. Second, the production technology has external increasing returns to scale, $\mathcal{Y}_n < 0$ or $\mathcal{K}_n > 0$. Two reasons support this approach: these two mechanisms are the most emphasized in the urban economics literature and they are sufficient to reproduce plausible dynamics. The negative externality results in a centripetal force that prevents all workers from leaving the region. The positive externality creates persistence as firms gain less in setting a business in the region during a crisis.

My approach, at a preliminary stage, consists in fixing an arbitrary benchmark and to investigate various deviations. The agglomeration externalities parameters must be bounded to satisfy the Blanchard-Kahn conditions. When the conditions are almost binding, the dynamic responses are the strongest, which constitue a good benchmark. I thus consider $\mathcal{H}_l = 200$ and $\mathcal{K}_n = -1000$. A 1% increase in the number of workers induces an increase in the housing cost by 2 units and a 1% increase in the number of firms reduces the entrepreneurial cost by 10 units. These quantities can be compared in magnitude to the steady-state match output at $y - b = 100$. The relative mobility of entrepreneurs and workers is a dimension that affects dynamics. I then start with the symmetric case $\phi^E = \phi^U$. The mobility parameter is chosen so that the magnitude of the dynamics is the strongest, hence $\phi^E = \phi^U = 0.0001$. I simulate the dynamics of an economy following an unanticipated loss of 5% in employment, corresponding to the closure of 5% of the firms.

The impulse response functions are plotted on figure 3 for the 6 variables of the linearized system and the unemployment rate. Right after the shock, the labor market is slack because some firms disappeared and more workers are looking for jobs. Firms are attracted by the opportunities on the labor market, whereas workers prefer to leave the region. The positive agglomeration on the cost of business are weaker and delay the entry of firms. The low labor demand is an incentive for workers to leave the region. During the first months after the shock, the number of firms increase while the size of the local labor force shrinks. The labor market becomes more favorable to workers over time. As workers move out of the region, the cost of living decreases until the point when workers are better off in the region. The deviation of the value of unemployment thus becomes positive and starts decreasing towards zero as workers immigrate. In the benchmark model, the labor market recovers within 5 to 10 years. The number of workers return

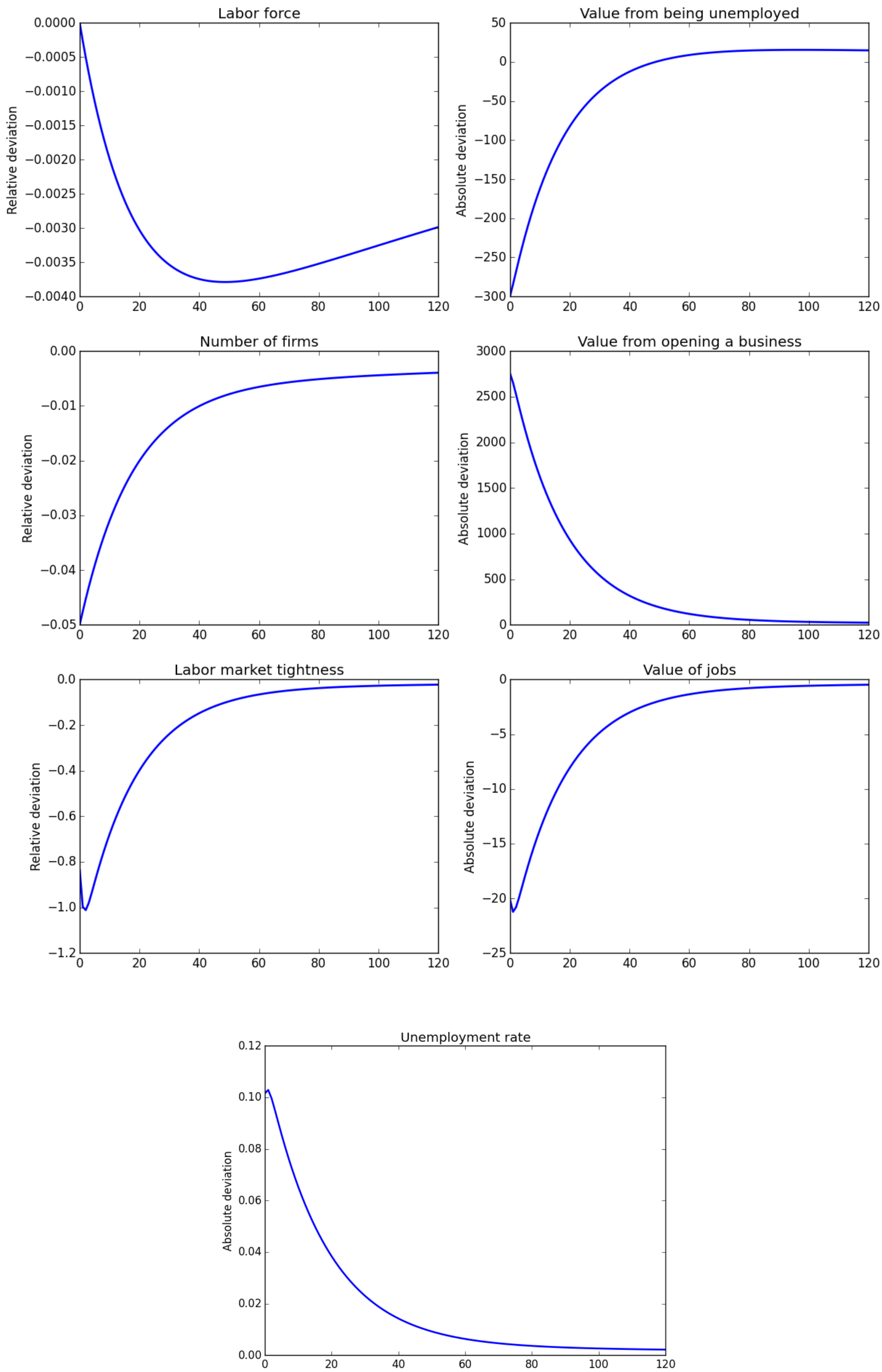

Figure 3 – Impulse response functions after a sudden destruction of firms and jobs in the benchmark model

Note. The X-axis is time in months. The economy starts with 5% less firms and 5% less employment compared to the steady state.

to the steady state more slowly. Its deviation reaches almost -0.4% four years after the shock, when the region has the lowest number of workers.

Model/ Deviation	Unemployment (p.p.) 1yr	5yrs	10yrs	Number of workers (%) 1yr	5yrs	10yrs	Spike
Benchmark	6.16	0.64	0.22	-0.21	-0.37	-0.3	-0.38 at t=48.4
Deviation 1: agglomeration effects							
twice weaker	5.87	0.44	0.11	-0.21	-0.37	-0.32	-0.37 at t=52.4
ten times weaker	5.67	0.3	0.03	-0.21	-0.39	-0.38	-0.39 at t=70.6
Deviation 2: firms involved							
twice less	3.05	0.32	0.11	-0.11	-0.19	-0.15	-0.19 at t=48.4
ten times less	0.56	0.06	0.02	-0.02	-0.04	-0.03	-0.04 at t=47.4
Deviation 3: absolute mobility							
ten times more mobile	1.68	0.1	0.03	-0.31	-0.14	-0.04	-0.32 at t=13.1
ten times less mobile	8.18	2.09	0.58	-0.15	-0.41	-0.44	-0.44 at t=215.8
Deviation 4: relative mobility							
firms twice more mobile	4.78	0.24	0.1	-0.14	-0.2	-0.15	-0.21 at t=38.3
firms ten times more mobile	1.68	0.03	0.02	-0.04	-0.04	-0.03	-0.05 at t=21.2
Deviation 5: sharable productivity gains							
half shared	6.06	0.57	0.18	-0.21	-0.37	-0.29	-0.37 at t=47.4
fully shared	5.97	0.51	0.14	-0.21	-0.36	-0.28	-0.37 at t=45.4

Table 3 – Moments from various simulations

Note. The three first columns concern the dynamics of the unemployment rate following the shock. The four last columns concern the size of the labor force. In type-1 deviation, the agglomeration effects are divided by 2 and 10. In type-2 deviation, the shock destroys the same number of jobs, but less firms close. In type-3 deviation, workers and firms are jointly more or less mobile. In type-4 deviation, workers are as mobile as in the benchmark but the mobility of firms differ. In type-5 deviation, the agglomeration gains in the production technology are shared with workers.

I conduct several simulations to compare to the benchmark. The results are exposed in table 3. As agglomeration forces gets weaker (precisely $\mathcal{H}_l$ and $\mathcal{K}_n$), the unemployment rate returns faster to the steady state while workers take more time to come back to the region. It is shown in type-1 deviation from the benchmark simulation. On one hand, firms are less affected in their production technology by the shock so new firms settler faster in the region. On the other hand, workers are more inclined to leave the region because the cost of living does not compensate the lack of job opportunities. Both mechanisms explain the faster recovery of the labor market. The negative spike in the level of number of workers is postponed as agglomeration effects disappear. The simulation thus replicates the Blanchard and Katz (1992)'s model at the limit: the employment level becomes permanently affected by a shock on the labor market. As the number of firms destroyed matter because of the agglomeration externalities, I consider the same shock on the jobs but more shared across firms so that less firms disappears. In type-2 deviation, I consider a loss of 5% in employment simultaneous to a closure of 2.5% and 0.5%. As expected, unemployment and the size of the labor force is much less impacted. The impulse response function follows the same dynamics but are weaker in magnitude. It shows that the dynamics produced in the benchmark comes mainly from the closure of firms rather than a shock on employment. In type-3 and type-4 deviation, the mobility coefficients, ϕ^E and ϕ^U, vary. The returns to the steady state is faster as both agents gets more mobile. More workers leave the region in the first periods following the shock, the spike happens sooner and is less strong. The unemployment rate thus deviates less from its steady-state value. When only firms becomes more mobile, new firms arrive sooner in the region after the shock, which reinforces the gains from staying in the region for workers. In the benchmark model, I consider $\mathcal{K}_n = -1000$ and $\mathcal{Y}_n = 0$, implying the productivity gains not to be shared with employees. In the last deviation exercise, I consider the case in which gains are half shared with workers, $\mathcal{K}_n = -500$ and $\alpha\mathcal{Y}_n = 500$, and the case in which they are fully shared, $\mathcal{K}_n = 0$ and $\alpha\mathcal{Y}_n = 1000$. The simulations results barely differ from the benchmark.

4 Multiplicity and Stability

The comparative statics analysis, as well as the simulations, ignore the existence of multiple equilibria. Firms and workers are supposed to coordinate to return to the original steady state, but they may also coordinate on an equilibrium that moves away from it. The concept of stability is required to characterize steady-state on which agents rationally coordinate. Due to the dimensionality of the problem, I only focus on the approximated linear model close to a steady state. The concepts are then true locally.

In the second part of this section, I investigate the link between the eigenvalues of matrix $\mathcal{M}$ and the agglomeration effects in the interaction matrix $\mathbb{V}$. Because explicit mathematical conditions are hardly tractable in the general case, I focus on a close variant of the original model in which jobs do not last.

4.1 Saddlepath stability

Before defining rigorously the concept of stability with the mathematics of the model, an intuitive idea is to consider departures from steady states. Consider a regional economy initially at a steady state. It is then hit by an exogenous shock in the number of residents, a sudden destruction of jobs, or a permanent change in the outside options ϵ and μ for instance. An economy at a stable steady state will be resilient to any departures, meaning the steady state will be recovered in the long run. On the contrary, unstable steady states will not be recovered in the long run; the economy will converge to other steady states or exhibit cyclical dynamics.

Definition 1 *A steady-state equilibrium $\mathcal{P}^*$ is locally stable if and only, for any equilibrium $\mathcal{P}$ such that $\mathcal{P}_t$ is in a neighborhood of $\mathcal{P}^*$ for some $t \geq 0$, the state $\mathcal{P}_t$ converges to the steady state $\mathcal{P}^*$.*

The following proposition is originated from Buiter (1984), who generalize the Blanchard and Kahn (1980) conditions to the continuous-time case. The stability of the linear system can be defined through the eigenvalues of matrix $\mathcal{M}$. We call the stable roots as the eigenvalues with non-positive real parts and unstable roots the eigenvalues with positive real parts.

Proposition 5 *A steady-state equilibrium $\mathcal{P}^*$ is (locally) stable if and only if matrix $\mathcal{M}$ has at most as many unstable roots as forward-looking variables, precisely 3 in this model. The equilibrium dynamics around a stable steady state is locally determinate if and only if there are exactly as many unstable roots as forward-looking variables. The steady-state is then called saddlepath-stable.*[5]

When the number of unstable roots exceed 3 in the model, the equilibrium dynamics is explosive. The linear approximation is insufficient to describe the dynamics in this case. Agents may coordinate on another steady state for instance. When the steady state is saddlepath-stable, the dynamics follows a unique trajectory converging to the steady state from any initial conditions. It can thus be simulated. When the steady state is stable but not not saddlepath-stable, agents can coordinate on different trajectories to the steady state.

A research literature investigated properties between the eigenvalues of the linearized model with the original model (see Grandmont et al. (1998) among others). Mortensen (1999) and Sniekers (2013) study economic cycles on the labor market by focusing on an economy close to a bifurcation (meaning when the stability properties change) with the bargaining power as the fundamental parameter. A similar study of the non-linearized dynamics of the model would be impossible in the context of this model. The higher dimensionality of the problem (3 predetermined and 3 forward-looking variables instead of 1-1 in Mortensen's model) is not the main reason. A general analysis would require the

[5]This proposition is true almost everywhere in the language of measure theory. In other words, I abstract for any particular degenerated cases.

full knowledge of the agglomeration forces in equations (1), which is beyond the scope of this paper. Other local properties like supercriticality of bifurcations also involve higher-order conditions on the agglomeration effects.

4.2 A variant of the model with non-lasting jobs

In this subsection only, the notations will slightly differ to describe the variant of the model. I abusively decided to keep the same notations to refer to the same concepts. The first departure to the original model is that jobs do not last, equivalently δ tends to infinity. Second, when a firm and a worker matches, the worker produces once for all a stock y_t net of unemployment benefits b, she shares it with the employer and they continue searching for another match. Thus, y_t, w_t and b become stocks in this variant instead of flows in the original model. If one does not make this assumption, the gains from matching would be nil. A last assumption is that entrepreneurs and workers have the same death rate, $\rho^U = \rho^E = \rho$

A rigorous derivation of the Bellman equations as a limit of the discrete-time model is given in appendix C. Equations (10) and (11) defining the capital gains from matching now write:

$$J_t - V_t = y_t - w_t, \tag{10'}$$

$$W_t - U_t = w_t - b. \tag{11'}$$

The surplus from matching is accordingly $S_t = y_t - b$. As everybody keeps looking for a partner, the labor market tightness is simply the vacancies-workers ratio, $\theta_t = \frac{\alpha n_t}{l_t}$.

I now define the linear system characterizing the equilibrium dynamics. Assuming that jobs do not last eliminate two variables from the analysis. The market tightness and the match surplus do not follow a dynamic differential equation anymore, so they can be substituted in the linear system:

$$\begin{pmatrix} \dot{\tilde{l}}_t \\ \dot{\tilde{n}}_t \\ \dot{\tilde{S}}_t^U \\ \dot{\tilde{S}}_t^E \end{pmatrix} = \mathcal{M} \begin{pmatrix} \tilde{l}_t \\ \tilde{n}_t \\ \tilde{U}_t \\ \tilde{E}_t \end{pmatrix} \tag{23}$$

with a new definition of matrix $\mathcal{M}$

$$\mathcal{M} = \begin{pmatrix} 0 & 0 & \phi^U \rho & 0 \\ 0 & 0 & 0 & \phi^E \rho \\ -\mathcal{U}_l & -\mathcal{U}_n & \rho & 0 \\ -\mathcal{E}_l & -\mathcal{E}_n & 0 & \rho \end{pmatrix}.$$

The elements of matrix $\mathbb{V}$ also have a slightly modified definition:

$$\begin{aligned}
\mathcal{U}_l &= -\mathcal{H}_l + \beta\theta^* q(\theta^*)\mathcal{Y}_l - \eta\beta\theta^* q(\theta^*)S^*,\\
\mathcal{U}_n &= -\mathcal{H}_n + \beta\theta^* q(\theta^*)\mathcal{Y}_n + \eta\beta\theta^* q(\theta^*)S^*,\\
\mathcal{E}_l &= -\mathcal{K}_l + \alpha(1-\beta)q(\theta^*)\mathcal{Y}_l + \alpha(1-\beta)(1-\eta)q(\theta^*)S^*,\\
\mathcal{E}_n &= \underbrace{-\mathcal{K}_n}_{\text{local cost effect}} + \underbrace{\alpha(1-\beta)q(\theta^*)\mathcal{Y}_n}_{\text{productivity effect}} - \underbrace{\alpha(1-\beta)(1-\eta)q(\theta^*)S^*}_{\text{search externality}}.
\end{aligned}$$

The equivalent of Proposition 5 is that a steady state is stable if $\mathcal{M}$'s eigenvalues has at most two unstable roots. Thanks to the simple bloc structure of matrix $\mathcal{M}$, the Blanchard-Kahn conditions of the system can be expressed as a function of $\mathbb{V}$. Solving for the characteristics polynomial, $Det(\mathcal{M} - xI_4)$ is equivalent to solve for the determinant of a smaller matrix, $Det\left(\mathcal{W}\mathbb{V} - \frac{x(\rho-x)}{\rho\sqrt{\phi^U\phi^E}} I_2\right)$, with I_i as the identity matrix of dimension i and $\mathcal{W} = \begin{pmatrix} \sqrt{\frac{\phi^U}{\phi^E}} & 0 \\ 0 & \sqrt{\frac{\phi^E}{\phi^U}} \end{pmatrix}$. The conditions for stability reduce to conditions on the matrix $\mathbb{V}$, weighted by the coefficient in matrix $\mathcal{W}$.

Proposition 6 *In the non-lasting-job model, a steady state is saddlepath-stable if and only if matrix*

$$\mathcal{W}\mathbb{V} = \begin{pmatrix} \sqrt{\frac{\phi^U}{\phi^E}} & 0 \\ 0 & \sqrt{\frac{\phi^E}{\phi^U}} \end{pmatrix} \begin{pmatrix} \mathcal{U}_l & \mathcal{U}_n \\ \mathcal{E}_l & \mathcal{E}_n \end{pmatrix}$$

satisfy one of the following properties:

- *the two eigenvalues are real and non-positive,*
- *the two eigenvalues are complex conjugate with a non-positive real part,*
- *the two eigenvalues are complex conjugate with a positive real part low enough so that* $v_{im}^2 > \frac{\rho}{\sqrt{\phi^U\phi^E}} v_{re}$. v_{re} *and* v_{im} *denote the real and imaginary parts of the eigenvalues.*

A proof of the proposition is given in the appendix. In the first case when the two eigenvalues are negative and real, the four roots of the dynamic system are real. The dynamics after a departure from a steady state are then aperiodic. On the contrary, dynamics are quasi-periodic in the two other cases.

A new interpretation of the stability conditions is suggested by this Proposition. A steady-state is saddlepath stable if the centripetal forces that attracts agents are stronger than the centrifugal forces. Focus on the case in which agents do not create externalities on the other types, $\mathcal{U}_n = \mathcal{E}_l = 0$ and $\mathcal{W}\mathbb{V}$ is diagonal. The steady state is saddlepath stable when both semi-elasticities, $\mathcal{U}_l$ and $\mathcal{E}_n$, are negative. Imagine an unanticipated exogenous increase in the number of workers at $t = 0$, $\tilde{l}_0 > 0$. Workers' welfare incurs a loss, which forces some workers to move out of the region, whereas firms are not affected. As workers leave the region, the equilibrium goes back to the initial steady state. A similar story applies for firms after an exogenous change in the number of entrepreneurs. Effects are more complex in presence of crossed effects in the interaction matrix $\mathcal{W}\mathbb{V}$. As the real parts of $\mathcal{W}\mathbb{V}$'s eigenvalues increase, the system becomes less stable. Because of mobility frictions preventing agents from leaving instantaneously the region, ϕ^U and ϕ^E nonzero, a steady state may be saddlepath stable even if $\mathcal{W}\mathbb{V}$'s eigenvalues have a positive real part.

As agglomeration forces disappear and only search externalities remain, one of $\mathcal{W}\mathbb{V}$'s eigenvalues tend to zero while the other is non-positive. Search externalities contribute to the centripetal forces, and thus enhance equilibrium stability. Consider an example in which housing costs are increasing in the number of workers (because of absentee landlords producing housing at a convex cost) and business costs are decreasing in the number of firms (from input-sharing benefits): $\mathcal{H}_l > 0 > \mathcal{K}_n$ and $\mathcal{H}_n = \mathcal{K}_l = \mathcal{Y}_l = \mathcal{Y}_n = 0$. The steady state is stable if search frictions are strong enough. Such a stationary equilibrium would have been unstable in absence of search frictions. In other words, search frictions constitute a plausible mechanism to improve stability.

When workers and firms are equally mobile, the weight matrix $\mathcal{W}$ is the identity matrix, and stability conditions in Proposition 6 only relies on matrix $\mathbb{V}$'s eigenvalues. When firms are more mobile than workers for instance, the stability conditions are distorted through the weight matrix $\mathcal{W}$. We then have an interesting property that a change in the relative mobility of employers comparing to workers do not affect the steady state contrary to Blanchard and Katz (1992)'s model. Instead, such a change modifies the stability property of steady states.

5 Conclusion

This article investigates the consequences of agglomeration forces on the spatial distribution of workers, firms and jobs. Search frictions provide a reasonable mechanism to explore the dynamics of an open region in which workers and firms can freely enter. New results are obtained about the impact of local public intervention on the labor market, accommodating for any strength and sign of agglomeration forces. The simulations produce high persistence of employment shocks but low magnitude regarding the region's size. A relevant comparison can be made with the estimated cycles of Amior and Manning (2015). For an initial negative shock of 10% in employment, the population is still

6.6% lower than the steady state ten years after the shock.

I also conduct a stability analysis. Though I do not discuss unstable steady states, I do think their analysis can be relevant. First, how agglomeration economies have emerged is a complex dynamic process that is beyond the scope of the paper. Technological change or globalization for instance are deep permanent shocks that have possibly modified agglomeration forces irreversibly, turning a stable steady state to an unstable one. Second, departures from unstable steady states can explain non-stationary behaviors of time series on regional unemployment rates. Such dynamics echo the current discussion on the future of macroeconomics between Blanchard and P. Romer among others. Some researchers advocate a return to non-linear dynamics, chaotic behaviors and hysteresis phenomena.

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A Limit of the discrete-time model

Consider the economy moving from time t to time $t+\Delta t$, with Δt close to 0. With probability $1-\rho\Delta t$, a worker survives between the two periods. The measure of surviving workers is $(1-\rho\Delta t)l_t$. The measure of (net) newcomers is $\Phi^U(U_t)\rho l_t\Delta t$. The measure of workers at $t+\Delta t$ then writes $l_{t+\Delta t} = (1-\rho\Delta t)l_t + \Phi^U(U_t)\rho l_t\Delta t$. At the limit, this equation tends to the continuous-time equivalent equation (2). Equation (3) is defined analogously.

When a firm employs a worker, it receives the flow profits $(y_t - w_t)\Delta t$. If the employer survives with probability $(1-\rho^E\Delta t)$, the firm receives the asset value $J_{t+\Delta t}$ or $V_{t+\Delta t}$ depending on the survival of the job. Conditional on the survival of the employer, the job remains if it does not break for exogenous reasons and if the employee survives, with probability $(1-\rho^U\Delta t)(1-\delta\Delta t)$. The discrete-time equivalent of equation (4) is

$$\begin{aligned} J_t = (y_t - w_t)\Delta t + (1-\rho^E\Delta t)\left\{\left[(1-\rho^U\Delta t)(1-\delta\Delta t)\right] J_{t+\Delta t}\right. \\ \left. + \left[1-(1-\rho^U\Delta t)(1-\delta\Delta t)\right] V_{t+\Delta t}\right\} \end{aligned} \tag{24}$$

The Bellman equations for the firm's side write accordingly:

$$V_t = (1-\rho^E\Delta t)\left\{q(\theta_t)\Delta t J_{t+\Delta t} + (1-q(\theta_t)\Delta t)V_{t+\Delta t}\right\}, \tag{25}$$

$$L_t = -k_t\Delta t - \epsilon\Delta t + (1-\rho^E\Delta t)L_{t+\Delta t}. \tag{26}$$

Consider an unemployed worker. She receives the net amenities and benefits $b\Delta t - h_t\Delta t - \mu\Delta t$. She survives with probability $1-\rho^U\Delta t$ and then receives the asset value of being employed or unemployed. The employment event occurs at a probability $q(\theta_t)\Delta t$. The Bellman equations for the worker's side are:

$$U_t = b\Delta t - h_t\Delta t - \mu\Delta t + (1-\rho^U\Delta t)\left\{q(\theta_t)\Delta t W_{t+\Delta t} + (1-q(\theta_t)\Delta t)U_{t+\Delta t}\right\}, \tag{27}$$

$$\begin{aligned} W_t = w_t\Delta t - h_t\Delta t - \mu\Delta t + (1-\rho^U\Delta t)\left\{\left[(1-\rho^E\Delta t)(1-\delta\Delta t)\right] W_{t+\Delta t}\right. \\ \left. + \left[1-(1-\rho^E\Delta t)(1-\delta\Delta t)\right] U_{t+\Delta t}\right\}. \end{aligned} \tag{28}$$

Lastly, the stock of jobs evolve according to this dynamics:

$$m_{t+\Delta t} = m_t + \theta_t q(\theta_t)\Delta t(l_t - m_t) - \left(1 - (1-\rho^U\Delta t)(1-\rho^E\Delta t)(1-\delta\Delta t)\right) m_t. \tag{29}$$

The second term in the right-hand side is the measure of workers who become employed between t and $t+\Delta t$, and the third term is the measure of broken matches during the period.

B Linearized model around the steady state

I derive the linearized equations of the model. The dynamics of the measure of residents from (2) and (3) give

$$\dot{\tilde{l}}_t = \phi^U \rho \tilde{U}_t, \tag{30}$$

$$\dot{\tilde{n}}_t = \phi^E \rho \tilde{E}_t. \tag{31}$$

From the dynamics of the stock of jobs (16) and (17), one can obtain the two expressions:

$$\dot{\tilde{m}}_t = \eta(\rho^U + \rho^E + \delta)\tilde{\theta}_t + (\rho^U + \rho^E + \delta + \theta^* q(\theta^*))(\tilde{l}_t - \tilde{m}_t),$$
$$\dot{\tilde{m}}_t = (\eta - 1)(\rho^U + \rho^E + \delta)\tilde{\theta}_t + (\rho^U + \rho^E + \delta + q(\theta^*))(\tilde{n}_t - \tilde{m}_t),$$

which give

$$\dot{\tilde{m}}_t = (1-\eta)(\rho^U + \rho^E + \delta + \theta^* q(\theta^*))(\tilde{l}_t - \tilde{m}_t) + \eta(\rho^U + \rho^E + \delta + q(\theta^*))(\tilde{n}_t - \tilde{m}_t),$$
$$(\rho^U + \rho^E + \delta)\tilde{\theta}_t = (\rho^U + \rho^E + \delta + q(\theta^*))(\tilde{n}_t - \tilde{m}_t) - (\rho^U + \rho^E + \delta + \theta^* q(\theta^*))(\tilde{l}_t - \tilde{m}_t)$$

Define $\zeta^U = \rho^U + \rho^E + \delta + \theta^* q(\theta^*)$, $\zeta^E = \rho^U + \rho^E + \delta + q(\theta^*)$ and $u = (1-\eta)\zeta^U + \eta\zeta^E$. From the second equation above,

$$(\rho^U + \rho^E + \delta)(\dot{\tilde{\theta}}_t + u\tilde{\theta}_t) = \zeta^E(\dot{\tilde{n}}_t + u\tilde{n}_t) - \zeta^U(\dot{\tilde{l}}_t + u\tilde{l}_t) - (\zeta^E - \zeta^U)(\dot{\tilde{m}}_t + u\tilde{m}_t)$$

then

$$(\rho^U + \rho^E + \delta)(\dot{\tilde{\theta}}_t + u\tilde{\theta}_t) = \zeta^E(\dot{\tilde{n}}_t + u\tilde{n}_t) - \zeta^U(\dot{\tilde{l}}_t + u\tilde{l}_t) - (\zeta^E - \zeta^U)[(1-\eta)\zeta^U\tilde{l}_t + \eta\zeta^E\tilde{n}_t]$$

Finally,

$$(\rho^U + \rho^E + \delta)(\dot{\tilde{\theta}}_t + u\tilde{\theta}_t) = -\zeta^U\zeta^E\tilde{l}_t + \zeta^U\zeta^E\tilde{n}_t - \zeta^U\phi\rho\tilde{U}_t + \zeta^E\phi\rho\tilde{E}_t \tag{32}$$

Equation (21) is

$$\begin{pmatrix} \dot{\tilde{l}}_t \\ \dot{\tilde{n}}_t \\ \dot{\tilde{\theta}}_t \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{\zeta^U\zeta^E}{\rho^U+\rho^E+\delta} & \frac{\zeta^U\zeta^E}{\rho^U+\rho^E+\delta} & -u \end{pmatrix} \begin{pmatrix} \tilde{l}_t \\ \tilde{n}_t \\ \tilde{\theta}_t \end{pmatrix}$$
$$+ \begin{pmatrix} \phi^U\rho & 0 & 0 \\ 0 & \phi^E\rho & 0 \\ -\phi^U\rho\frac{\zeta^U}{\rho^U+\rho^E+\delta} & \phi^E\rho\frac{\zeta^E}{\rho^U+\rho^E+\delta} & 0 \end{pmatrix} \begin{pmatrix} \tilde{S}_t^U \\ \tilde{S}_t^E \\ \tilde{S}_t^J \end{pmatrix}.$$

Now, differentiate the surpluses in equations (13), (14) and (15),

$$\rho\tilde{S}_t^U = -\mathcal{H}_l\tilde{l}_t - \mathcal{H}_n\tilde{n}_t + \beta\theta^* q(\theta^*)\tilde{S}_t^J + \beta\eta\theta^* q(\theta^*)S^*\tilde{\theta}_t + \dot{\tilde{S}}_t^U, \quad (33)$$

$$\rho\tilde{S}_t^E = -\mathcal{K}_l\tilde{l}_t - \mathcal{K}_n\tilde{n}_t + \alpha(1-\beta)q(\theta^*)\tilde{S}_t^J + \alpha(1-\beta)(\eta-1)q(\theta^*)S^*\tilde{\theta}_t + \dot{\tilde{S}}_t^U, \quad (34)$$

$$[\rho^U + \rho^E + \delta + \beta\theta^* q(\theta^*) + (1-\beta)q(\theta^*)]\tilde{S}_t^J + [\beta\eta\theta^* q(\theta^*) + (1-\beta)(\eta-1)q(\theta^*)]S^*\tilde{\theta}_t \quad (35)$$

$$= \mathcal{Y}_l\tilde{l}_t + \mathcal{Y}_n\tilde{n}_t + \dot{\tilde{S}}_t^J. \quad (36)$$

Define $a_0 = \rho^U + \rho^E + \delta + \beta\theta^* q(\theta^*) + (1-\beta)q(\theta^*)$. Equation (22) is the matrix writing of the previous equations:

$$\begin{pmatrix} \dot{\tilde{S}}_t^U \\ \dot{\tilde{S}}_t^E \\ \dot{\tilde{S}}_t^J \end{pmatrix} = \begin{pmatrix} \rho & 0 & -\beta\theta^* q(\theta^*) \\ 0 & \rho & -\alpha(1-\beta)q(\theta^*) \\ 0 & 0 & a_0 \end{pmatrix} \begin{pmatrix} \tilde{U}_t \\ \tilde{E}_t \\ \tilde{S}_t \end{pmatrix} + \begin{pmatrix} \mathcal{H}_l & \mathcal{H}_n & -\beta\eta\theta^* q(\theta^*)S^* \\ \mathcal{K}_l & \mathcal{K}_n & -\alpha(1-\beta)(\eta-1)q(\theta^*)S^* \\ -\mathcal{Y}_l & -\mathcal{Y}_n & [\beta\eta\theta^* q(\theta^*) + (1-\beta)(\eta-1)q(\theta^*)]S^* \end{pmatrix} \begin{pmatrix} \tilde{l}_t \\ \tilde{n}_t \\ \tilde{\theta}_t \end{pmatrix}.$$

C Variant of the model

Among the Bellman equations defined in appendix A, only the equations regarding J_t and W_t have to be redefined:

$$J_t = y_t - w_t + (1-\rho\Delta t)V_{t+\Delta t} \quad (37)$$

$$W_t = w_t - b - h_t\Delta t + (1-\rho\Delta t)U_{t+\Delta t} \quad (38)$$

The stock of jobs follow the new dynamics

$$m_{t+\Delta t} = \theta_t q(\theta_t)\Delta t(l_t - m_t), \quad (39)$$

which implies that $m_t = 0$.

Proof of Proposition 6 Denote v_1 and v_2 the eigenvalues of $\mathcal{W}\mathbb{V}$. The eigenvalues of $\mathcal{M}$ solve

$$\frac{x(\rho - x)}{\rho\sqrt{\phi^U\phi^E}} = v_i \quad (40)$$

for $i = 1$ or 2. Denote $im^2 = -1$. $x = x_1 + x_2.im$ is a solution if and only if

$$\frac{x_1(\rho - x_1) + x_2^2}{\rho\sqrt{\phi^U\phi^E}} + \frac{x_2(\rho - 2x_1)}{\rho\sqrt{\phi^U\phi^E}}.im = v_i \quad (41)$$

Given that $\mathcal{W}\mathbb{V}$ is a real matrix, either v_1 and v_2 are real, or they are complex conjugate.

Consider the case when v_1 and v_2 are real (case 1). If $v_i > 0$ then any x solution of (40) has a positive real part. If v_i is negative, one solution of (40) is positive (real) and another is negative. Necessarily, v_1 and v_2 must be negative so that 2 eigenvalues are positive and 2 are negative.

Consider the case when v_1 and v_2 are complex conjugate. Denote $v_1 = v_{re} + v_{im}.im$ and $v_2 = v_{re} - v_{im}.im$. By substituting x_2 in the real part of (41)'s left-hand side,

$$\rho\sqrt{\phi^U\phi^E}v_{im}^2 = \left(v_{re} - \frac{x_1(\rho - x_1)}{\rho\sqrt{\phi^U\phi^E}}\right)(\rho - 2x_1)^2 \tag{42}$$

After an analysis of the 4th order polynomial in the right hand side, we find that there can be only two possible values of x_1. Either these two possible values are positive if $\rho\sqrt{\phi^U\phi^E}v_{im}^2 > \rho^2 v_{re}$, or one is positive and one is negative in the other case. The only possibility to obtain a saddlepath stable equilibrium is that this condition is not satisfied so that the system has 2 stable and 2 unstable roots.