The Matching Function:
A Unified Look into the Black Box

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Abstract

The matching function, the central building block of models with search frictions, remains largely a "black box." In this paper, we use tools from network theory to unpack it showing how the structure of the underlying connections between applicants and firms determines the emergent matching function's properties. Our overarching message is that *structure counts*. We show that for complex structures, captured by non-random graphs, the matching function depends on whole sets of connections rather than just the sizes of the two sides of the market. For simpler, random graph structures, the matching function depends only on the sizes of the two sides and a few structural parameters, as typically assumed in the literature. Structures characterized by greater asymmetries in the connections of applicants reduce the matching function's overall match efficacy, while more connections across applicants can have ambiguous effects on it. In the special case when the underlying connections are given by an Erdös-Rényi network, we illustrate that the way applicants' links vary with the sizes of the two sides of the market plays a critical role for the matching function to exhibit constant returns to scale, or even to be of specific functional forms, like Cobb-Douglas or CES.

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1 Introduction

The matching function is the central building block of most models that depart from the Walrasian equilibrium to capture search frictions in the market. It is considered largely consistent with empirical findings, and the number of contexts in which it has been used highlights its success: most notably the labor market, where unemployed and vacancies co-exist in equilibrium, but also credit markets, goods markets, assets trading over the counter, the new monetarist literature aiming to explain the emergence of money, international trade.

The matching function, however, has remained a "black box" for nearly forty years. It is a reduced-form object that economists use for its tractability but no systematic analysis has been done of the frictions implicitly assumed to underlie it, such as information limitations and coordination failures. More specifically, little is known of how the *structure* of the underlying frictions affects the matching function's properties.

Notable contributions that derive a matching function endogenously are Burdett, Shi, Wright (2001) and Albrecht, Gautier and Vroman (2006) in the directed search literature, Calvó-Armengol and Zenou (2005) in the social networks literature, and Stevens (2007) who uses a specific type of queuing system. Each of these contributions derive important implications, but in terms of the matching function, as we will show, each has focused on a particular, highly symmetric structure of the underlying frictions.

In this paper we use tools from the theory of networks to derive the matching function under different structures of the underlying connections that link applicants to firms. As we explain shortly, these connections can capture explicitly the frictions that are typically implicitly assumed to underlie the matching function. We thus provide the first systematic study of how different structures of the underlying frictions affect the emergent matching function's properties. The overarching message of our findings is that *structure counts.*

We recover existing matching functions as special cases and trace the determinants of "match efficacy" – the analog of TFP in the matching function typically taken to be exogenous – to the structure of underlying connections between applicants and firms. We obtain two main insights regarding match efficacy: Structures of greater asymmetry reduce it, while denser structures can have ambiguous effects on it. Our analysis thus provides a theory of the determinants of match efficacy without needing to impose the presence of a matching function at any level of (dis-)aggregation as has been done in the literature (e.g. Barnichon and Figura, 2015; Hall and Schulhofer-Wohl, 2018; Mukoyama, Patterson and Şahin, 2018).

Network theory is particularly appropriate to handle situations of granular information of the type "who knows of which job" which is typically the type of friction assumed to underlie the matching function. Importantly, network theory also provides the tools to both describe and handle asymmetries, allowing us to overcome pre-existing technical difficulties. Our treatment of the problem is at significantly greater generality than what has been done before: we do not need to impose a specific interpretation to a link; applicants can differ in the number of applications they send, and our analysis applies equally well to arbitrarily small or large economies.

A bipartite graph connects unemployed (applicants) and vacancies. The links of the graph can correspond to social ties – as in the social networks in the labor market literature (e.g. Calvó-Armengol, 2004), or skills required to apply for that job, or geographic restrictions the applicant has on where to work. In other words the graph can represent *any* relevant factors restricting the jobs an applicant can apply to, and we don't need to take a stance on it for our analysis. The network structure, that is the presence or not of a link between any applicant-vacancy pair, is thus making explicit precisely the frictions the search and matching literature has been assuming to implicitly underlie the matching function.

We adopt a simple applications-and-hiring protocol as to what happens over the network. Each unemployed person applies to all vacancies they are connected to; each vacancy makes one offer at random among all applications received (if any); each unemployed chooses one offer at random among the offers received (if any). Formally, when referring to "matching" we mean the expected number of total matches over all possible scenario realizations.[1]

[1]A scenario example: applicants 1 and 2 both apply to firm 1. Firm 1 makes an offer to applicant 1, who also receives an offer from firm 2. Applicant 1 chooses (and gets hired by) firm 2. This outcome occurs with some probability in our model, and we take the expectation over all such possible outcomes.

We start by analyzing structures given by any non-stochastic network between applicants and firms. We derive a generalized matching function that depends on whole *sets* of unemployed and vacancies, rather than just their *sizes* as typically assumed. Yet, the matching function is given by a compact expression. In the special case of the complete network, we recover the matching function of the classic balls-in-bins model derived early in the literature (e.g. Butters, 1977) and later as an equilibrium outcome by Burdett, Shi, and Wright (2001).

In the second part of our analysis we analyze structures given by random networks between applicants and firms. The degrees of the applicants are drawn iid from a distribution. Links then fall at random on vacancies on the other side of the market. The resulting structures are fundamentally less complex than the ones studied in the first part of our analysis in the sense that they are characterized by a few parameters only – the parameters of the applicant-degree distribution. In these types of structures there is “anonymity,” that is both applicants and firms are ex-ante identical and thus interchangeable within their set. However, there is still a well-defined notion of asymmetries as mean-preserving spreads in the applicants-degree distribution.

We derive the matching function for any random network thus generated. It is again given by a compact expression, but it depends only on the *sizes* of the two sides of the market and the parameters of the applicant-degree distribution. The fact that in this case the matching function does not depend on whole sets is the outcome of the anonymity of nodes embedded in random graphs. In the special case when the applicant-degree distribution is degenerate, i.e. all applicants send the same number of applications, and we take the large-economy limit of our model, we recover the matching function of Albrecht, Gautier and Vroman (2006).

Structures of greater asymmetry correspond to redistribution of links across applicants, while holding the firms side fixed, in a way that “the rich get richer.” Such types of link swaps increase the inequality in access to jobs: they take links from applicants who rely more on them – because the probability that the applicant receives no job from the rest of their connections is relatively high – to give them to applicants who need them less. In other words, such swaps hurt the losing applicant of the swap more than they benefit the winner, thus having a negative overall effect.

Structures of higher density correspond to cases where a greater fraction of all possible links exists. These are cases of higher search intensity, or equivalently cases of reduced information frictions broadly defined. The limit case when information frictions vanish is captured by the complete graph in our setup. Standard intuition might suggest that such reductions in information frictions would always improve the matching rate. We find this to be the case for one type of network only, the Erdös-Rényi. There are structures for which match efficacy is maximized when all applicants have a unique connection, and others where the matching function has an inverse-U shape as a function of applicant degree.

The reason for the ambiguous effect of search intensity on match efficacy is that whenever an applicant applies to a new firm, they benefit individually but they impose a negative externality on all other applicants linking to that firm. Additional links to firms that have received no application are necessarily beneficial, thus a "sufficient" degree of connectivity improves the efficacy of the matching process. But as applicants keep sending more applications, beyond a point, congestion effects of multiple firms making an offer to the same applicant dominate and higher search intensity becomes harmful for match efficacy.

The inversion of the effects of search intenstity is not part of the canonical matching functions, which assume higher search intensity is always beneficial for matching (e.g. Pissarides, 2000 chapter 5). Our result thus casts doubt on analyses which, based on such matching functions, argue that higher search intensity would unambiguously be shifting the Beveridge curve inwards (e.g. Elsby, Michaels and Ratner, 2015; Mukoyama, Patterson and Şahin, 2018). Our results on the effects of asymmetries suggests that such types of analyses should also be looking at higher moments of the link distributions beyond average search intensity.

When all applicants have the same degree, the matching function has an inverse-U shape that is shared with Calvó-Armengol and Zenou (2005), who derive a matching function in a similar setup. The key difference is that we directly consider the network between applicants and vacancies, while Calvó-Armengol and Zenou (2005) and the social networks literature more generally take as primitive the (social) network between applicants. We manage to draw closer connections to the standard matching function and to the search literature more broadly precisely because we work with networks of that type.

The result of structures of higher density possibly having a negative impact on match efficacy hinges on our application-and-hiring protocol having a single round of offer-acceptance between firms and applicants, while, as we show, the results of more asymmetric structures unambiguously having a negative effect do not. The single round assumption determines the amount of coordination frictions underlying the matching function. It is very common in the literature (e.g. Shimer, 2005), but it is also a reasonable assumption that can emerge endogenously in a richer environment. More concretely, in a dynamic setup where there is heterogeneity in the quality of the job-applicant matches (e.g. Martellini and Menzio, 2020), firms that are rejected by their top choice of applicant will likely want to wait to re-sample a candidate from next period's pool rather than locking in a less productive relationship now.

Finally, we find the "scaling" of the applicant degree distribution, that is how applicants' degrees vary with changes in the sizes of unemployed and vacancies, to play a critical role for the properties of the aggregate matching function. We characterize the conditions on applicants' mean degree (average search intensity) under which an Erdös-Rényi network of connections exhibits constant returns to scale or even be of specific forms, e.g. CES. These results echo and generalize Stevens (2007) who uses the "telephone line" queuing system, a similar setup to the Erdös-Rényi network. Search intensity and a matching technology of certain properties are thus not two separate things, where the former is super-imposed on the latter as in the textbook treatment. They are one and the same.

The rest of the paper is structured as follows. Section 2 lays out the economic setup and introduces notation. Section 3 derives the matching function for an arbitrary given network and goes over some useful special cases of networks. In section 4 we give the main result of the first part of our analysis regarding overall match efficacy. In section 5 we do the analysis for random networks. Section 6 presents the implications of scaling. Section 7 discusses the model's assumptions at length and especially the single-round offer-acceptance assumption of the application-and-hiring protocol. Section 8 concludes.

2 The setup

We start by introducing the economic environment, some terminology and notation.

Primitives: The economic environment consists of two sets of agents $\mathcal{U}$ and $\mathcal{V}$, of size $U, V \in \mathbb{N}$ respectively, and a (bipartite) graph G linking elements between the two sets.

We take the elements of $\mathcal{U}$ to correspond to *applicants*, i.e. workers searching for a job, and the elements of $\mathcal{V}$ to correspond to *jobs* offered by firms. In other words $\mathcal{U}$ contains the *unemployed*, while $\mathcal{V}$ contains *vacancies* in our setup.

As a convention, we will be indexing the elements of $\mathcal{U}$ by $i = 1, 2, ..., U$ and the elements of $\mathcal{V}$ by $j = 1, 2, ..., V$. Following the search and matching literature we will assume that each firm has a single vacancy to fill, thus we may interchangeably refer to firm j or vacancy j as the counterparty of an applicant i.

The graph G is represented by an adjacency matrix – denote $G = \big(g_{ij}\big)$, where $g_{ij} = 1$ if applicant i is connected to firm j, and $g_{ij} = 0$ otherwise.[2] $G + ij$ (resp. $G - ij$) denotes network G after adding (resp. deleting) a link between applicant i and vacancy j.

We will denote by $d_i = \sum_j g_{ij}$ an applicant's *degree*, that is the number of firms the applicant connects to. Similarly a firm's degree $d_j = \sum_i g_{ij}$ corresponds to the number of applicants the firm connects to. As a matter of accounting it has to hold that the total number of degrees on the two sides are equal, i.e. $\sum_i d_i = \sum_j d_j$.

Finally we refer to an applicant's *neighborhood* as the set of firms the applicant connects to. Specifically, for an applicant i, define $N_i = \{j \in \mathcal{V} : g_{ij} = 1\}$. Similarly we can define the neighborhood of a firm j. It follows that the size of a node's neighborhood equals their degree; for applicant i denote $|N_i| = d_i$.

As in what follows we will be making connections to the search and matching literature, let us also define the central quantity of that literature, *market tightness*, $\theta = \frac{V}{U}$.

An application and hiring protocol: Taking the network of links G as given, we assume

[2] Applicants correspond to rows and firms to columns.

applicants apply to all firms they connect to. A firm chooses one of the applicants uniformly at random to whom it makes an offer. Each applicant chooses to accept one offer uniformly at random among the offers they receive.

The key object of interest throughout our analysis is the *matching rate* defined as the expected number of matches given a particular network. We denote

$$m(G) = \mathbb{E}[\#\text{matches}|G]$$

An example: Let us consider the following instance[3]

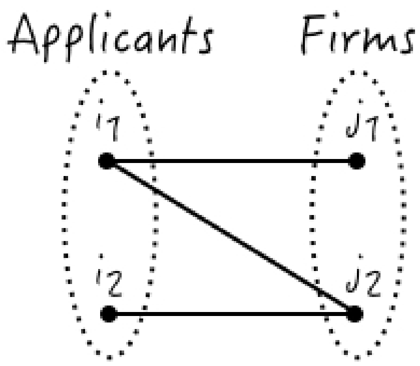


Figure 1: A 2-by-2 example.

Applicant i_1 applies both to firms j_1 and j_2, while applicant i_2 applies only to firm j_2.

Accordingly, firm j_1 makes an offer to applicant i_1, and firm j_2 chooses with probability 1/2 to make an offer to i_1 and with probability 1/2 to make an offer to i_2.

There are three possible outcomes:

(1) j_2 makes offer to i_2. i_1, and i_2 each accept their single offer.

(2) j_2 makes an offer to i_1. i_1 chooses j_2; i_2 has no offer.

[3]In matrix form the graph of this example is $G = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. For exposition, we also note that the degree of applicant i_1 is 2, and of applicant i_2 it is 1. Their corresponding neighborhoods are the sets $\{j_1, j_2\}$, and $\{j_2\}$ respectively.

(3) j_2 makes an offer to i_1. i_1 chooses j_1; i_2 has no offer.

Outcome 1 is the first best. Outcomes 2 and 3 are states of coexisting vacancy and unemployment, as a result of coordination failure, an issue we will illustrate further in the rest of the analysis.

According to our protocol, the first outcome occurs with probability 1/2; each of the other two outcomes occurs with probability 1/4. Thus, the expected number of matches is given by

$$\begin{aligned} m(G) &= \frac{1}{2} \cdot 2 + 2 \cdot \frac{1}{4} \cdot 1 \\ &= 3/2 \end{aligned}$$

A note on interpretation. The links of the graph can correspond to social ties – as in the social networks literature[4] (e.g. Calvó-Armengol, 2004), or skills required to apply for that job, or geographic restrictions the applicant has on where to work. In other words the graph can represent *any* relevant factors restricting the jobs an applicant can apply to, and we don't need to take a stance on it for our analysis.

We highlight that thus the network structure, that is the presence or not of a link between any applicant-vacancy pair, can be taken to make explicit precisely the frictions the search and matching literature has been assuming to implicitly underlie the matching function.[5] Overall the network can be thought to capture the underlying *information structure* in the economy.

[4]The seminal modern contribution here can be taken to be Calvó-Armengol (2004). A series of papers have followed including Calvó-Armengol and Jackson (2004), Calvó-Armengol and Zenou (2005), Ioannides and Soetevent (2006), Galenianos (2014, 2020), Galeoti and Merlino (2014), Espinosa, Kovářík and Ruiz-Palazuelos (2021). Montgomery (1991), even though earlier, can also be added as an important contribution to this literature.

[5]In the words of Barbara Petrongolo (VoxEU, 2010) "[Search] frictions derive from several sources, including imperfect information about trading partners, heterogeneous demand and supply, slow mobility, coordination failures and other similar factors."

3 Matching in an arbitrary graph

What we did for the small instance in the example above, we can do in the general case for an arbitrary graph G.

Proposition 1. *For any given arbitrary graph G, the matching rate defined as the expected number of total matches is given by*

$$m(G) = U - \sum_{i=1}^{U} \prod_{j \in N_i} \left(1 - \frac{1}{d_j}\right) \tag{1}$$

Proof. For any applicant i the probability to receive no offer is $\prod_{j \in N_i} \left(1 - \frac{1}{d_j}\right)$, and thus their probability of finding a job (= the probability of receiving at least one offer) is

$$\begin{aligned} f_i(G) &\equiv Pr\{i \text{ gets hired}|G\} \\ &= 1 - \prod_{j \in N_i} \left(1 - \frac{1}{d_j}\right) \end{aligned}$$

Now, for each applicant define the indicator r.v. showing if they find a job, where

$$Y_i = \begin{cases} 1, & \text{w.p. } f_i(G) \\ 0, & \text{w.p. } 1 - f_i(G) \end{cases}$$

Then the number of matches, taking the graph as given, which by definition is the number of applicants finding a job is also a r.v., and specifically $\#\text{matches}|G = \sum_i Y_i$.

The matching rate, i.e. the expected number of matches is then

$$\begin{aligned} m(G) &= \mathbb{E}[\#\text{matches}|G] \\ &= \mathbb{E}[\sum_{i=1}^{U} Y_i] \\ &= \sum_{i=1}^{U} \mathbb{E}[Y_i] \\ &= \sum_{i=1}^{U} f_i(G) \end{aligned}$$

$$= U - \sum_{i=1}^{U} \prod_{j \in N_i} \left(1 - \frac{1}{d_j}\right)$$

□

Let us pause and appreciate how compact an expression (1) is for how general it is: it gives us the expected number of matches for *any* possible graph G, for *any* two sets $\mathcal{U}, \mathcal{V}$.

We think it is useful to highlight that each applicant i finding a match is a Bernoulli trial with probability of success f_i. The Bernoulli trials are not independent, they all depend on G, but to compute $m(G)$ independence is not required; we only use the linearity of expectation. We also note that the derivation of f_i hinges on each firm deciding independently from all other firms which applicant to make an offer to.

Let us now see how (1) specializes in specific types of graphs, and specifically how it reduces in being a function only of the sizes U, V of the two sets as is usual in the search and matching literature.

Example 1: The *complete graph* (or family of graphs) is the case where all applicants connect to all firms. In this case $N_i = \mathcal{V},\ \forall i$, and $d_j = U,\ \forall j$, thus (1) becomes

$$\begin{aligned} m(G) &= U - \sum_{i=1}^{U} \left(1 - \frac{1}{U}\right)^{V} \\ &= U \left(1 - \left(1 - \frac{1}{U}\right)^{V}\right) \end{aligned}$$

It can be seen that the matching function in this case is increasing and concave in its two arguments.[6] We also note the complete graph case is precisely the classic balls-in-bins setup. Thus, it is no surprise the above is the same type of matching function derived early in the literature using the standard balls-in-bins model (e.g. Butters, 1977), and as an equilibrium object by Burdett, Shi and Wright (2001).[7]

The complete graph is an interesting special case as it corresponds to the case when information frictions are eliminated – all applicants know of and apply to all existing jobs, thus

[6] Shown in the appendix.

[7] Replace U with m, and V with n to get their eq. 18. See also Wright et al. (2021), eq. 46.

matching is only the outcome of coordination frictions.

It is broadly accepted in the literature (e.g. Petrongolo and Pissarides, 2001; Wright et al., 2021) that the empirical relevance of this functional form and its limiting version[8] is quite limited.

Example 2: A *double regular graph* (or family of graphs) is the case where every applicant is connected to d_U firms, and each firm is connected to d_V applicants.

This is a doubly-symmetric graph where all applicants and all firms search with the same "intensity," and we show it relates closely to the standard search-and-matching setup. We note that a double regular graph can be thought to correspond to a symmetric equilibrium.

For such graphs (1) gives us the matching function being

$$m(G) = U\left[1 - \left(1 - \frac{1}{d_V}\right)^{d_U}\right]$$

However, by accounting it holds that $Ud_U = Vd_V$, and utilizing this equation we can write

$$m(U, V) = U\left[1 - \left(1 - \frac{1}{d_U}\frac{V}{U}\right)^{d_U}\right]$$

where d_U is taken to be a parameter, and d_V is determined from $Ud_U = Vd_V$. All applicants have the same job-finding probability. Denoting $\theta = \frac{V}{U}$ this probability is[9]

$$f(\theta; d_U) = 1 - \left(1 - \frac{1}{d_U}\theta\right)^{d_U}$$

The matching function in this case can be shown to possess standard properties assumed in the literature: it is constant returns to scale, increasing, and concave in both U, and V. The

[8] Using the result $\lim_{n\to+\infty}\left(1 + \frac{x}{n}\right)^n = e^x$, the above asymptotically exhibits constant returns to scale, as for large U the matching function can be taken to be approximately $m(V, U) \approx U\left(1 - e^{-\frac{V}{U}}\right)$.

[9] Naturally, not *any* choice of a d_U will do; d_U has to be an integer and it has to be such that d_V is also an integer. This points to the limitation of this model (or special case) we are considering here. Also the standard "relaxation" of integer constraints applies when we are talking about thousands or millions of objects.

matching function can also be shown to be approximated at a first-order by a Cobb-Douglas function.[10] We show these results formally in the appendix.

We think the following result is interesting to compare with the more general results that follow on match efficacy.

Proposition 2. *For the double regular network the matching rate is maximized when* $d_U = 1$.

Proof. See appendix. □

4 Structure and overall match efficacy

The main result of this section is about aggregate match efficacy. We first provide a related comparative static for the individual level, which has natural empirical analogs and generalizes analogous results of the standard matching function.

Proposition 3. *In terms of their job-finding probability, an applicant (a) invariably benefits by connecting to a new firm, and (b) is hurt if another applicant links to a firm they are connected to.*

Proof. In notation the above comparative statics are respectively

$$f_i(G + ik) > f_i(G), \forall k \notin N_i$$
$$f_i(G + i'k) < f_i(G), \forall k \in N_i, i' \neq i$$

They follow directly from the expression $f_i(G) = 1 - \prod_{j \in N_i} \left(1 - \frac{1}{d_j}\right)$. □

Part (a) illustrates that an individual's job-finding probability is always improved from higher search intensity. Part (b) is the classic externality the higher search intensity of one applicant imposes on the job-finding probabilities of other applicants. Contrary to the standard matching function where the externality affects all other applicants, in our case it

[10] We note that the 1st-order approximation result is not specific to this family of graphs: *any* matching function that exhibits CRS to a 1st-order approximation is Cobb-Douglas.

is "local," affecting only the applicants connected to the firm to which the link is added; the rest of the applicants are unaffected.

In other words, an applicant receiving (or losing) a link creates winners and losers, and thus its effect on aggregate match efficacy is a priori ambiguous. The next example illustrates such a trade-off.

Example 3. Consider the following instance where we swap a link from applicant i to i'.

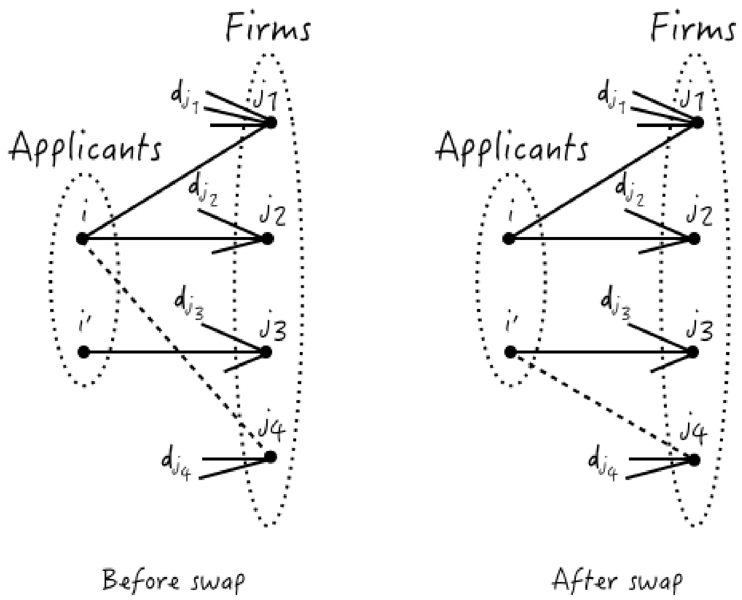


Figure 2: Aggregate efficacy is hurt from the swap iff $\left(1-\frac{1}{d_{j_1}}\right)\left(1-\frac{1}{d_{j_2}}\right) > \left(1-\frac{1}{d_{j_3}}\right)$. In this case applicant i' who is better off even without the extra link, gets a link at the expense of applicant i, raising inequality in access to jobs between them.

Their respective job-finding probabilities before the swap are

$$f_i^{\text{before}} = 1 - \left(1-\frac{1}{d_{j_1}}\right)\left(1-\frac{1}{d_{j_2}}\right)\left(1-\frac{1}{d_{j_4}}\right)$$
$$f_{i'}^{\text{before}} = 1 - \left(1-\frac{1}{d_{j_3}}\right)$$

After the swap i loses a link and will necessarily be worse-off, while i' gains a link and will be better-off. Thus a trade-off emerges:

$$f_i^{\text{after}} = 1 - \left(1 - \frac{1}{d_{j_1}}\right)\left(1 - \frac{1}{d_{j_2}}\right) < f_i^{\text{before}}$$
$$f_{i'}^{\text{after}} = 1 - \left(1 - \frac{1}{d_{j_3}}\right)\left(1 - \frac{1}{d_{j_4}}\right) > f_{i'}^{\text{before}}$$

All other applicants (not shown in the figure) remain unaffected. The outcome of the trade-off depends on how their job-finding probabilities compare *without* the concerned link. More concretely aggregate efficacy is reduced iff $f_i^{\text{before}} + f_{i'}^{\text{before}} < f_i^{\text{after}} + f_{i'}^{\text{after}}$ or

$$\underbrace{\left(1 - \frac{1}{d_{j_1}}\right)\left(1 - \frac{1}{d_{j_2}}\right)}_{\text{Prob\{i doesn't get job from connections } \textit{excluding } j_4\}} > \underbrace{\left(1 - \frac{1}{d_{j_3}}\right)}_{\text{Prob\{i' doesn't get job from connections } \textit{excluding } j_4\}}$$

This condition states that applicant i is relatively more reliant on the additional link to j_4 than i' to get a job. That is because the probability that i does *not* find a job relying on all their *other* connections, $\left(1 - \frac{1}{d_{j_1}}\right)\left(1 - \frac{1}{d_{j_2}}\right)$ is greater than the respective probability for i', $\left(1 - \frac{1}{d_{j_3}}\right)$. Thus making the swap hurts i more than it benefits i', hence the net outcome is negative.

Theorem 1. *Take an arbitrary network G.*

(A) Let $\hat{G}$ denote the network resulting from swapping a link $ij \in G$ with link $i'j \notin G$. Then $m(\hat{G}) < m(G)$, if and only if

$$1 - f_i(\hat{G}) > 1 - f_{i'}(G) \tag{*}$$

(B) Let $\hat{G}$ denote the network resulting from adding link ij, where $ij \notin G$. Then $m(\hat{G}) < m(G)$, if and only if

$$1 - \bar{f}_{N_j}(G) > 1 - f_i(G) \tag{*'}$$

where $1 - \bar{f}_{N_j}(G) \equiv \frac{1}{d_j} \sum_{k \in N_j} \prod_{j' \in N_k/\{j\}} \left(1 - \frac{1}{d_{j'}}\right)$.

Proof. See appendix. □

Part (A) of theorem 1 generalizes precisely the instance illustrated in the foregoing example: $1 - f_i(\hat{G})$ is the probability applicant i – the loser of the swap, does *not* receive an offer from all their other connections excluding j, and $1 - f_{i'}(G)$ the respective probability for applicant i' – the winner of the swap. The link swap is harmful iff the former is above the latter, and thus the applicant who is more reliant on the extra connection – i, loses it for the benefit of the applicant who needs it less – i'.

We note that a link swap between applicants is the only operation allowed when holding the firms' degrees fixed and is thus a well-defined comparative static. It is also empirically relevant as it corresponds to cases of *redistribution* of links among applicants. Specifically theorem 1 says that any swap – and thus any sequence thereof, that *increases inequality in access to jobs* among applicants will hurt overall match efficacy. In other words situations of *more asymmetric network structures*, or more figuratively when "the rich get richer," imply a lower overall match efficacy. Conversely, link swaps that equalize the probabilities of applicants to be hired increase the efficacy of the matching process.

Part (B) of theorem 1 states that an exactly analogous condition determines whether an additional link, that is higher search intensity or equivalently network structures of higher density, will improve or hurt overall match efficacy. The right-hand side of the condition is the probability applicant i – the winner of the addition, receives no offer from all their *other* connections, thus determines the reliance of i on the new link; the left-hand side gives the corresponding quantity for the "average" loser of the addition, that is the average probability an applicant connecting to firm j before the addition receives no offer from all their *other* connections excluding firm j.

Theorem 1 formalizes the two main themes of our analysis for overall match efficacy: structures of higher asymmetry unambiguously hurt overall match efficacy (part A), while structures of higher density can have ambiguous results (part B). We will see different variants of these two themes going forward.

A first variant of the effects of asymmetry expressed only in terms of applicants' degrees can be attained in the special case when all firms have the same degree. In this case: dispersion in applicant's degrees reduces aggregate efficacy.

Proposition 4. *Suppose* $d_j = d_V, \forall j$*. If* (d'_i) *is a mean-preserving spread of* (d_i)*, then* $m(G') < m(G)$*.*

Proof. We have

$$m(G') < m(G) \Leftrightarrow$$
$$U - \sum_{i=1}^{U} \left(1 - \frac{1}{d_V}\right)^{d'_i} < U - \sum_{i=1}^{U} \left(1 - \frac{1}{d_V}\right)^{d_i} \Leftrightarrow$$
$$\sum_{i=1}^{U} \left(1 - \frac{1}{d_V}\right)^{d'_i} > \sum_{i=1}^{U} \left(1 - \frac{1}{d_V}\right)^{d_i}$$

Since $\left(1 - \frac{1}{d_V}\right)^x$ is a convex function of x, the last inequality holds (shown in the appendix).[11]

□

Corollary 1. *Suppose* $d_j = d_V, \forall j$*, and let* G_R *denote the corresponding doubly regular graph, if that exists. Then for any graph* G*,* $m(G) \leq m(G_R)$*.*

Thus, when the situation is homogeneous on the firms' side, match efficacy increases with homogeneity on the applicants' side as well. Conversely, any increase in the spread in applicants' degrees will reduce match efficacy.

5 Matching in a random graph

Up to now, we have taken the underlying network – the bipartite graph – as given and analyzed the implications for matching. We showed that, in this case, matching generally depends on whole sets. In such a setup, therefore, to get closer to the standard matching function that only depends on sizes of the two sides, one needs to impose extreme symmetry as that in the double regular graph, thus losing all structural richness. We now introduce

[11] The appendix of this section provides some background material on mean-preserving spreads over arbitrary vectors which are not necessarily a probability distribution.

the family of structures described as (bipartite) random graphs which feature symmetry, as all applicants are ex-ante identical, yet retain structural richness.

Random graph characterization: Take applicant degrees to be i.i.d. draws from a given distribution $\vec{p} = (p_0, p_1, ...p_V)$, where $p_k \equiv Pr\{d_i = k\}$. For each applicant, conditional on a given draw from that distribution, the links are assumed to fall at random on an equal number of distinct firms among the V.[12]

The applicant-degree distribution $\vec{p}$ can be any arbitrary distribution over the non-negative integers. The way the random graph is characterized induces a distribution of degrees on the firm side, which we will show is a binomial distribution.

We highlight that even though all applicants are ex-ante identical in these structures, we can still meaningfully talk about changes in the asymmetry of the structure in the form of mean-preserving spreads in the underlying distribution $\vec{p}$. We also note these type of structures – random graphs, are fundamentally "less complex" than the ones studied in sections 3 and 4, in the sense that they are characterized by a few parameters only – the parameters of the applicant-degree distribution. The "complex" structures of our foregoing analysis in contrast require all $U \cdot V$ entries of the G matrix to be specified.

We will denote the firm-degree distribution by $\vec{z} = (z_0, z_1, ...z_U)$, where $z_k \equiv Pr\{d_j = k\}$. We will also denote the mean degree on the applicant and firm sides by $\bar{d}_U, \bar{d}_V$ respectively, i.e. $\bar{d}_U = \sum_k kp_k$ and $\bar{d}_V = \sum_k kz_k$.

Lemma 1. *Conditional on an applicant-degree distribution $\vec{p}$, the degrees on the firm side follow a binomial distribution, denote $d_j \sim Bin(\lambda, U)$, where $\lambda = \frac{\bar{d}_U}{V}$.*

Proof. We have

$$z_k = Pr\left\{\sum_{i=1}^{U} X_{ij} = k\right\}$$

where X_{ij} is an indicator, being 1 if applicant i links to (has applied to) firm j.

[12]More formally that is random sampling of d_i elements from a population of V without replacement.

Since all i are ex-ante i.i.d, X_{ij} are also i.i.d with probability[13]

$$
\begin{aligned}
Pr\{X_{ij}=1\} &= \sum_{k=1}^{V} Pr\{X_{ij}=1|d_i=k\}p_k \\
&= \sum_{k=1}^{V} \frac{\binom{V-1}{k-1}}{\binom{V}{k}} p_k
\end{aligned}
$$

Define $\lambda \equiv \sum_{k=1}^{V} \frac{\binom{V-1}{k-1}}{\binom{V}{k}} p_k$. Now, by noticing that $\frac{\binom{V-1}{k-1}}{\binom{V}{k}} = \frac{k}{V}$, it follows that

$$\lambda = \frac{\bar{d}_U}{V}$$

Then X_{ij} are Bernoulli with probability of success λ, and thus $d_j \sim Bin(\lambda, U)$. □

We note that λ is a function of $\vec{p}, V$ but for notational simplicity we are not denoting this explicitly.

Remark: Since $d_j \sim Bin(\lambda, U)$, it follows that $\bar{d}_V = \lambda U$, and thus[14]

$$\frac{\bar{d}_V}{U} = \frac{\bar{d}_U}{V}$$

This is a useful relationship we will invoke again in our analysis later.

Corollary 2. *A mean-preserving spread in the distribution of d_i's leaves the distribution of d_j's unchanged.*

Proof. This follows from d_j's following a binomial distribution, and its parameter λ depending only on $\bar{d}_U, V$, which stay constant with a mean-preserving spread. □

We will return to this result when we do comparative statics.

[13] The second line follows from a standard combinatorial argument: we want to find how many choices include a particular element i, among all the $\binom{V}{k}$ possible choices. We fix element i, and are free to choose the remaining $k-1$ elements from the remaining $V-1$ elements of the pool: these are precisely $\binom{V-1}{k-1}$.

[14] In fact it can be shown this is an accounting identity that has to hold for any bipartite random graph.

5.1 Moving to matching

We now derive the matching rate in the stochastic network case. The matching rate now is a *double* expectation, over who makes an offer to whom (as before), but also over the realized network G. In other words, the matching rate now is

$$m = \mathbb{E}_G[m(G)]$$

We first prove a lemma we will need regarding the *excess degree* of a firm an applicant connects to. The excess degree – denote by $\tilde{d}$, refers to the number of edges leaving the firm *other than* the edge of the said applicant.[15]

Lemma 2. *The excess degrees of all firms an applicant connects to (a) are i.i.d, and (b) it holds that* $Pr\{\tilde{d} = k\} = \frac{(1+k)z_{1+k}}{\bar{d}_v}$.

Proof. The degrees of firms are i.i.d following $Bin(\lambda, U)$. Thus the excess degrees of a firm an applicant connects to are also i.i.d and $\tilde{d} \sim Bin(\lambda, U-1)$, since $U-1$ only of the firm's degree Bernoulli trials remain to be determined. It follows that

$$\begin{aligned} Pr\{\tilde{d} = k\} &= \binom{U-1}{k} \lambda^k (1-\lambda)^{U-1-k} \\ &= \frac{(U-1)!}{k!(U-1-k)!} \lambda^k (1-\lambda)^{U-1-k} \\ &= \frac{1+k}{\lambda U} \frac{U!}{(1+k)!(U-1-k)!} \lambda^{1+k} (1-\lambda)^{U-1-k} \\ &= \frac{(1+k)z_{1+k}}{\bar{d}_V} \end{aligned}$$

□

Theorem 2. *The matching rate in our stochastic network model, defined as* $m = \mathbb{E}_G[m(G)]$, *is given by*

$$m = U\left(1 - \sum_{d_U=0}^{V} p_{d_U} (1-\phi)^{d_U}\right) \tag{2}$$

[15]We note that the result $Pr\{\tilde{d} = k\} = \frac{(1+k)z_{1+k}}{\bar{d}_v}$ is a special case of a more general result known for the configuration model (e.g. Newman (2003), Jackson (2010)). The result is exact in our case, while in the configuration model it is approximate and holds asymptotically for a large number of nodes.

where $\phi = \frac{1-z_0}{\bar{d}_V}$. $z_0 = (1-\lambda)^U$ *is the probability a firm receives no applications.*

Proof. We have that

$$Pr\{i \text{ finds job} \mid N_i, d_j \forall j \in N_i\} = 1 - \prod_{j \in N_i} \left(1 - \frac{1}{d_j}\right)$$

Therefore the (ex-ante) probability an applicant finds a job is given by

$$\begin{aligned}
f &= 1 - \mathbb{E}_{N_i}\left\{\mathbb{E}_{(d_j)_{j\in N_i}}\left\{\prod_{j\in N_i}\left(1-\frac{1}{d_j}\right)\middle| N_i\right\}\right\} \\
&= 1 - \sum_{N_i} p_{N_i} \mathbb{E}_{(d_j)_{j\in N_i}}\left\{\prod_{j\in N_i}\left(1-\frac{1}{d_j}\right)\middle| N_i\right\} \\
&= 1 - \sum_{N_i} p_{N_i} \prod_{j\in N_i}\left(1-\mathbb{E}_{d_j}\left\{\frac{1}{d_j}\middle| N_i\right\}\right) \\
&= 1 - \sum_{N_i} p_{N_i} \left(1-\mathbb{E}_{\tilde{d}}\left\{\frac{1}{1+\tilde{d}}\right\}\right)^{|N_i|} \\
&= 1 - \sum_{d_U} \sum_{N_i:|N_i|=d_U} p_{d_U} \frac{1}{\binom{V}{d_U}} \left(1-\mathbb{E}_{\tilde{d}}\left\{\frac{1}{1+\tilde{d}}\right\}\right)^{d_U} \\
&= 1 - \sum_{d_U} p_{d_U} \frac{1}{\binom{V}{d_U}} \left(1-\mathbb{E}_{\tilde{d}}\left\{\frac{1}{1+\tilde{d}}\right\}\right)^{d_U} \sum_{N_i:|N_i|=d_U} 1 \\
&= 1 - \sum_{d_U} p_{d_U} \left(1-\mathbb{E}_{\tilde{d}}\left\{\frac{1}{1+\tilde{d}}\right\}\right)^{d_U} \\
&= 1 - \sum_{d_U} p_{d_U} \left(1-\sum_{k=0}^{U-1}\frac{1}{1+k}\frac{(1+k)z_{1+k}}{\bar{d}_V}\right)^{d_U} \\
&= 1 - \sum_{d_U} p_{d_U} \left(1-\frac{1}{\bar{d}_V}\sum_{k=1}^{U} z_k\right)^{d_U} \\
&= 1 - \sum_{d_U} p_{d_U} \left(1-\frac{1-z_0}{\bar{d}_V}\right)^{d_U}
\end{aligned}$$

Since this is the probability of each applicant finding a job, the expected number of matches is given by $m = \sum_i f = Uf$.[16] $\square$

In the derivation of f, to go from the 2nd to the 3rd line we used the fact that d_j's are independently distributed within i's neighborhood (part (a) of lemma), and to go from the 3rd to the 4th line we used that d_j's are also identically distributed within any neighborhood (also part (a) of lemma). To go to the 5th line, we enumerate the neighborhoods by their size and use the fact that the probability to generate a particular neighborhood N_i of size d_U is to draw a degree of d_U and then choose the one among the $\binom{V}{d_U}$ neighborhoods of such size. To go to the 6th line we notice that nothing depends on the exact neighborhood N_i, only its size, thus we factor everything out of the second sum. To go to the 7th line we use the fact that the number of neighborhoods with d_U members is precisely $\binom{V}{d_U}$, hence this term cancels. To go to the 8th line we use part (b) of the lemma.

As the key object determining the matching rate is f, we will be working directly with it when convenient.

Remark: Suppose $\vec{p}$ is degenerate, say $\Pr\{d_i = d\} = 1$. Then all applicants have the same number of connections and thus send the same number of applications, and (2) becomes

$$\begin{aligned} m(U, V; \vec{p}) &= U \left(1 - \sum_{d_U=0}^{V} p_{d_U} (1-\phi)^{d_U} \right) \\ &= U \left(1 - \left(1 - \frac{1 - z_0}{\bar{d}_V} \right)^d \right) \end{aligned}$$

If also $U \to \infty$, then $z_0 \to e^{-\bar{d}_V}$. Using $Ud = V\bar{d}_V$,

$$m(U, V; d) = U \left(1 - \left(1 - \frac{1 - e^{-dU/V}}{dU/V} \right)^d \right)$$

We notice this is the matching function derived by Albrecht, Gautier and Vroman (2006).

[16] Note that $\phi < 1$: $\bar{d}_V = \sum_{k=1}^{U} k z_k = \sum_{k=1}^{U} z_k + \sum_{k=1}^{U} (k-1) z_k > \sum_{k=1}^{U} z_k = 1 - z_0$.

5.2 The special case of Erdös-Rényi

Given its prominence in the literature of random graphs, we derive the matching properties of the Erdös-Rényi model, which we show is a special case of our model.

Lemma 3. *When $d_i \sim Bin(\mu, V)$, our stochastic network model becomes the Erdös-Rényi model, that is the network that can be created drawing each link with the same probability μ.*

Proof. We have already shown the firm degree distribution is binomial with parameter $\lambda = \frac{\bar{d}_U}{V}$. But since $d_i \sim Bin(\mu, V)$, $\bar{d}_U = \mu V$. Thus $\lambda = \mu$, and $d_j \sim Bin(\mu, U)$.

It follows that in this case the model can be constructed drawing each link with probability μ as this process amounts to precisely V Bernoulli trials for each applicant, and U Bernoulli trials for each vacancy all with probability of success μ. □

Corollary 3. *In the case of the Erdös-Rényi model the matching function is given by*

$$m = U\left(1 - \left[1 - \frac{1-(1-\mu)^U}{U}\right]^V\right)$$

Proof. The matching function is $m = U \cdot f$. We will work with the job-finding probability f. Theorem 2 specializes in this case as

$$\begin{aligned}
f &= 1 - \sum_{d_U=0}^{V} p_{d_U}(1-\phi)^{d_U} \\
&= 1 - \sum_{d_U=0}^{V} \binom{V}{d_U} \mu^{d_U}(1-\mu)^{V-d_U}(1-\phi)^{d_U} \\
&= 1 - \sum_{d_U=0}^{V} \binom{V}{d_U} [\mu(1-\phi)]^{d_U}(1-\mu)^{V-d_U} \\
&= 1 - [1-\mu+\mu(1-\phi)]^V \\
&= 1 - [1-\mu\phi]^V \\
&= 1 - \left[1 - \mu\frac{1-z_0}{\mu U}\right]^V \\
&= 1 - \left(1 - \frac{1-(1-\mu)^U}{U}\right)^V
\end{aligned}$$

□

Two polar cases are readily verifiable: As we would expect, for $\mu = 0$, we have the empty graph, and $f = 0$; For $\mu = 1$, we have the complete graph, and $f = 1 - \left[1 - \frac{1}{U}\right]^V$.

Corollary 4. *The matching function in the Erdös-Rényi model is increasing in μ, and thus it is maximized when $\mu = 1$ (the complete graph).*

Proof. It follows directly from the expression for f. □

Contrasting this with the result on double regular graphs, it indicates that a higher search intensity has generally an ambiguous effect on match efficacy. We will see another variant of this finding in the more general analysis that follows.

5.3 The effects of asymmetry and density; again

The applicants' job-finding probability is given generally from the expression

$$f = 1 - \sum_{d_U=0}^{V} p_{d_U} (1 - \phi)^{d_U}$$

where $\phi = \frac{1-z_0}{\bar{d}_V}$.

We have already established that a mean-preserving spread in d_i's leaves the distribution of d_j's unchanged (corollary 3), and thus ϕ is left unchanged as well. We can further show the following proposition

Proposition 5. *A mean-preserving spread*[17] *in the distribution of d_i's reduces the applicants' job-finding probability f.*

[17] A mean-preserving spread is defined as a compound lottery, say $d'_U = d_U + Y$, where $\mathbb{E}[Y|d_U] = 0$ (Rothschild and Stiglitz, 1970). For example define $Y = 0$, if $d_U = 0$, and $Y = \begin{cases} +1, \text{w.p. } 1/2 \\ -1, \text{w.p. } 1/2 \end{cases}$ if $d_U \geq 1$.

Proof. Denote by $\vec{p}\,'$ a mean-preserving spread of $\vec{p}$. Equivalently, the two distributions have the same mean, and $\vec{p}$ second-order stochastically dominates (SOSD) $\vec{p}\,'$ (MCWG proposition 6.D.2). The definition of SOSD holds that for every non-decreasing concave functions $u(\cdot) : \mathbb{R}_+ \to \mathbb{R}$ it holds that

$$\sum_{d_U} p'_{d_U} u(d_U) \leq \sum_{d_U} p_{d_U} u(d_U)$$

Now, $-(1-\phi)^{d_U}$ is an increasing and (strictly) concave function, and then from the definition of SOSD we have

$$\sum_{d_U} p'_{d_U}[-(1-\phi)^{d_U}] \leq \sum_{d_U} p_{d_U}[-(1-\phi)^{d_U}] \Rightarrow$$
$$1 - \sum_{d_U} p'_{d_U}(1-\phi)^{d_U} \leq 1 - \sum_{d_U} p_{d_U}(1-\phi)^{d_U} \Rightarrow$$
$$f' \leq f$$

with equality holding iff $\phi = 0$ or $\phi = 1$. □

Corollary 5. *Given $\bar{d}_U$, the matching rate is maximized when everyone sends the same number of applications, $\bar{d}_U$.*

Proof. We have

$$1 - \sum_{d_U=0}^{V} p_{d_U}(1-\phi)^{d_U} < 1 - (1-\phi)^{\bar{d}_U}$$

following from Jensen's inequality. □

We note that proposition 5 and its corollary echo the results on the impact of heterogeneity on match efficacy of section 4.

We now move to study the effect of uniformly increasing search intensity across applicants. As illustrated in figure 3, when all applicants send the same number of applications, i.e. $\Pr\{d_i = d\} = 1$, the matching function exhibits an inverted-U shape as a function of d. Thus uniformly increasing search intensity has an ambiguous effect on match efficacy.

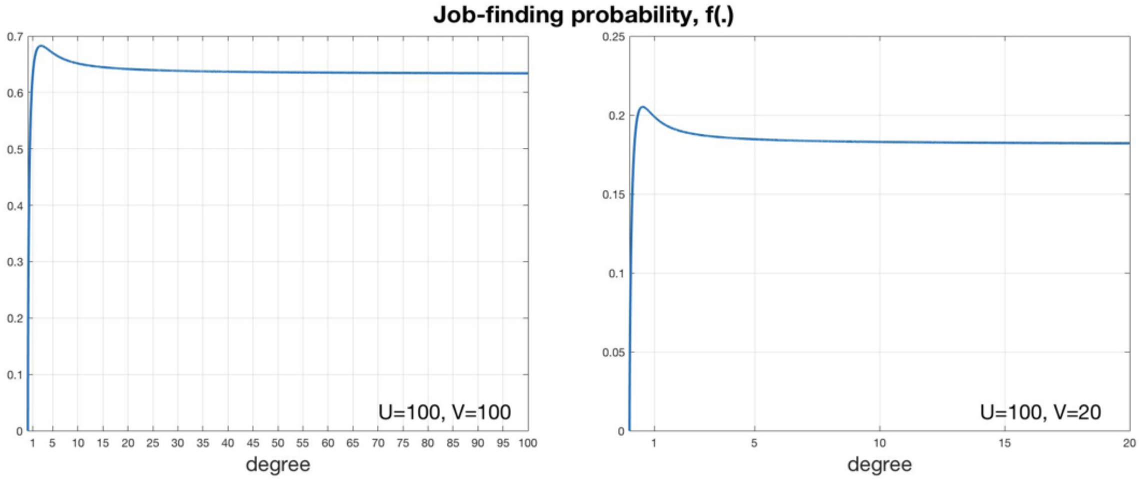


Figure 3: Job-finding probability $f(.)$ when all applicants have the same degree d as a function of $d \in [0, V]$, for given U, V. We notice a characteristically inverted-U shape of $f(.)$ and thus of $m(.)$ as a function of degree d.

We notice the possibility that the peak of the inverted-U can be to the left of $d = 1$ in the case of a slack market of relatively low V (figure on the right, $U = 100, V = 20$). That means that in this case, having everyone send the same number of applications, maximum efficacy is achieved when everyone sends a single application. Furthermore, quantitatively, even when the peak is at a higher degree, it is early, or in other words the matching function has a "long" right tail. This means that even when there are benefits from multiple applications, congestion effects outweigh these benefits quite fast. The peak in the left figure, a perfectly balanced market with $U = V$ occurs at $d = 3$.

We think this is an interesting result as it echoes similar results in the social networks literature (Calvó-Armengol, 2004; Calvó-Armengol and Zenou, 2005), as well as Shimer (2004). Importantly, such a result is at odds with the assumption of the benchmark model (e.g. Pissarides, 2000 chapter 5) according to which an increase in search intensity across identical applicants unambiguously improves the number of matches (ceteris paribus), as well as the recently proposed matching function by Mukoyama, Patterson and Şahin (2018). It follows that such patterns of increase observed during recessions (e.g. the Great Recession) do *not necessarily* shift the Beveridge curve inwards, as recently argued in the literature

(Elsby, Michaels and Ratner, 2015; Mukoyama, Patterson and Şahin, 2018)

Finally, we compare the efficacy of three networks: (i) the double regular network, (ii) the Erdös-Rényi network, and (iii) the 1-side regular network where all applicants have the same degree (Albrecht, Gautier and Vroman, 2006). To do so, in the following figure we hold U, V fixed, and vary the applicants' degree d. In cases (i) and (iii) all applicants have exactly the same degree, d. In case (ii) there is (ex-post) heterogeneity in applicants' degrees, but they all have the same (ex-ante) expected degree, d; in other words for the three networks to be comparable, we vary μ in the Erdös-Rényi model by varying d, where $\mu = \frac{d}{V}$.

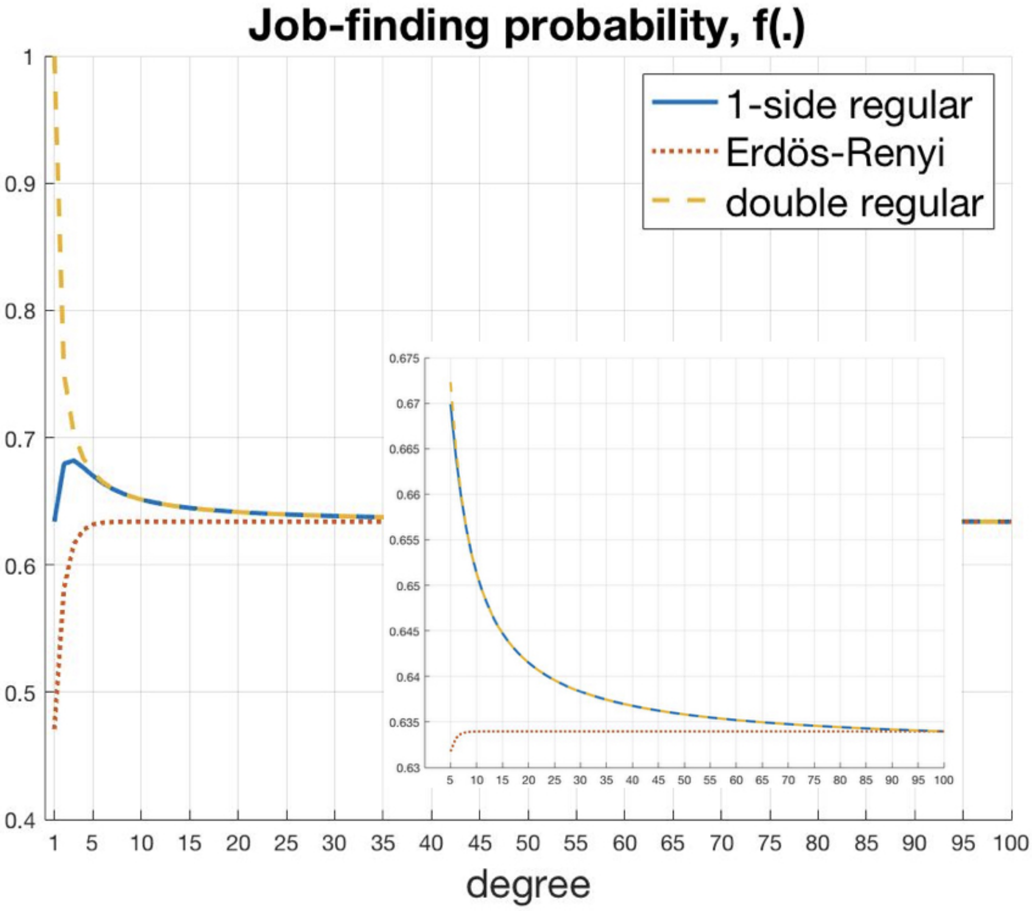


Figure 4: Comparing the efficacy of (i) the double regular network, (ii) the Erdös-Rényi network, and (iii) the 1-side regular network where all applicants have the same degree, holding fixed $U = V = 100$. The detail in the graph zooms in the range of $d = 5..100$ for visual clarity.

There are a few comments to make on the plot. First, we confirm that the job-finding probability is decreasing for the double regular graph, increasing for Erdös-Rényi, and of the inverted-U shape in the case of the 1-side regular network. We notice that for the double regular network the probability is exactly 1 for a degree of 1, as this is the case where every applicant links to a single firm and thus there are no search or coordination frictions.

Second, the fact that efficacy in Erdös-Rényi is lower everywhere reflects our results that applicant-side heterogeneity is harmful for efficacy: in Erdös-Rényi there is applicant-side heterogeneity in degrees, while in both other networks there isn't. The fact that efficacy is lower in the case of the 1-side regular network compared to the double regular network suggests that heterogeneity is harmful on the firm side as well: neither network has heterogeneity on the applicant side, but the double regular doesn't have heterogeneity on the firm side either, while the 1-side regular has the degrees on the firm side following a binomial distribution.

Lastly, we see the 1-side regular and double regular networks quickly converging to each other as degrees increase. This is because the matching functions of the two differ only by the probability of a firm to receive no applications – z_0, and this probability goes to 0 as d increases.[18] All three converge to the same limit at $d = V$, which corresponds to the matching rate of the complete network, i.e. the classic balls-in-bins case.

6 The importance of scaling

The complete graph is the special case when all applicants know of and apply to all vacancies. Now if the number of vacancies changes, for the graph to remain complete, each applicant has to scale scale up or down their degrees accordingly. In other words working with the complete graph structure implicitly supposes some type of *scaling* of the degree of all applicants as the size of the graph changes. The same is true for the double regular network.

[18] Our formulas for the two cases are $f(\theta; d) = 1 - \left(1 - \frac{1}{d}\theta\right)^d$ for the double regular network, and $f(\theta; d) = 1 - \left(1 - \frac{1-z_0}{d}\theta\right)^d$ for the 1-side regular network.

Upon reflection, the question of scaling applies to all graphs and relates to the question of how does behavior – applicant degree – change, when market conditions – the sizes of the two sides of the market – change. This question requires an economic model of network generation and thus lies outside the scope of this paper. We can, however, illustrate its significance following the classic analysis of the the Erdös-Rényi network, where the key parameter of the distribution is parameterized as a function of the network size, to study the change in properties of the graph (e.g. Jackson (2010), p. 89).

In the Erdös-Rényi network it holds that

$$f = 1 - \left[1 - \frac{1 - z_0}{U}\right]^V, \text{ where } z_0 = (1 - \mu)^U, \ \mu = \frac{\bar{d}_U}{V}$$

We take the limit where $U, V \to +\infty$ holding V/U fixed, to get

$$f \to 1 - e^{-(1-z_0)V/U}, \text{ where } z_0 \to e^{-U\bar{d}_U/V}$$

Now, scaling refers to how $\bar{d}_U$ changes when U, V change.[19] For example, if $\bar{d}_U$ is constant, the matching function will exhibit constant returns to scale, while, if $\bar{d}_U$ scales linearly in V (i.e. μ stays constant), the matching function will exhibit increasing returns to scale (figure 5). Finally, if $\bar{d}_U$ scales according to the following expression,

$$\bar{d}_U = -\frac{V}{U} \ln\left(1 + \frac{U}{V} \ln\left(1 - \frac{(U^{-\gamma} + V^{-\gamma})^{-\frac{1}{\gamma}}}{U}\right)\right), \ \gamma > 0$$

the matching function will be of the CES form, as shown in proposition 6 that follows. It can be seen that in this case $\bar{d}_U$ is homogeneous of degree 0 in U, V, and as illustrated in figure 6 it is increasing and concave in market tightness $\frac{V}{U}$.

[19] We notice this job-finding probability corresponds to a matching function that has been assumed in the literature (e.g. Petrongolo and Pissarides, 2001), where an exogenous fraction of applications z_0 is "lost." The fraction lost is endogenously derived in our case and corresponds to z_0. The question of scaling applies in such a setup as well though: if z_0 applications get "lost," does this fraction change when U, V change?

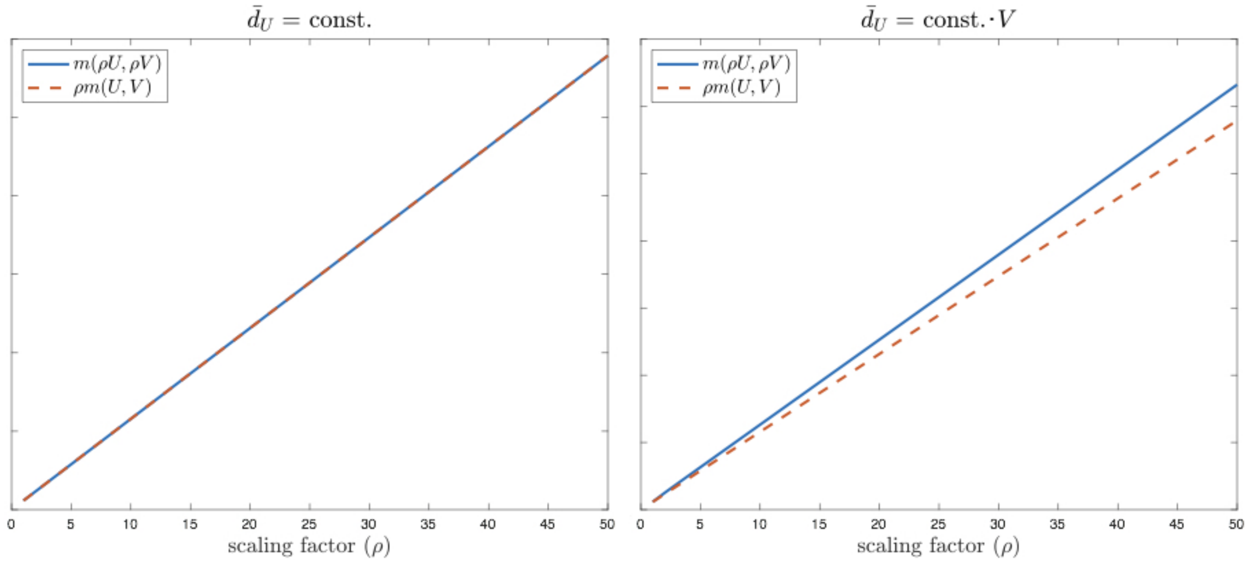


Figure 5: Returns to scale

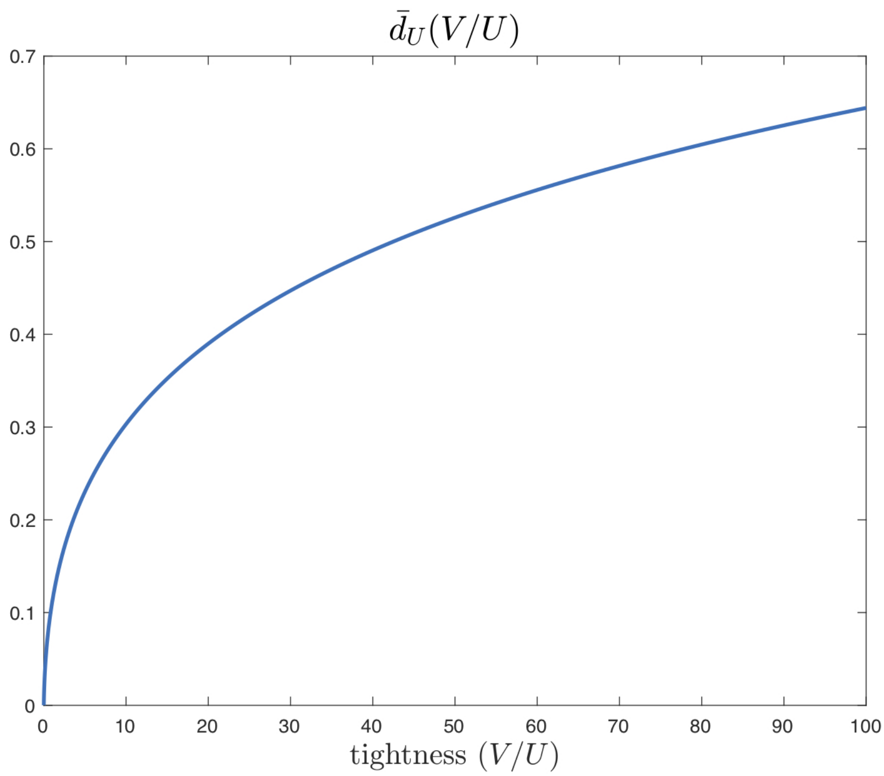


Figure 6: $\bar{d}_U$ yielding CES

Proposition 6. *Take a function $\tilde{m}(U,V)$ such that $\tilde{m}(U,V) < U, V$. Then, in the Erdös-Rényi network, if*

$$\bar{d}_U = -\frac{V}{U}\ln\left(1+\frac{U}{V}\ln\left(1-\frac{\tilde{m}(U,V)}{U}\right)\right) \tag{**}$$

it will hold that $f = \frac{\tilde{m}(U,V)}{U}$, and $\bar{d}_U > 0$.

Proof.

$$\begin{aligned}
\bar{d}_U &= -\frac{V}{U}\ln\left(1+\frac{U}{V}\ln\left(1-\frac{\tilde{m}(U,V)}{U}\right)\right) \Leftrightarrow \\
e^{-U\bar{d}_U/V} &= 1+\frac{U}{V}\ln\left(1-\frac{\tilde{m}(U,V)}{U}\right) \Leftrightarrow \\
1-e^{-U\bar{d}_U/V} &= -\frac{U}{V}\ln\left(1-\frac{\tilde{m}(U,V)}{U}\right) \Leftrightarrow \\
\frac{V}{U}(1-e^{-U\bar{d}_U/V}) &= -\ln\left(1-\frac{\tilde{m}(U,V)}{U}\right) \Leftrightarrow \\
\frac{V}{U}(1-z_0) &= -\ln\left(1-\frac{\tilde{m}(U,V)}{U}\right) \Leftrightarrow \\
f &= \frac{\tilde{m}(U,V)}{U}
\end{aligned}$$

Now, since $\bar{d}_U$ corresponds to a mean, it has to be that $\bar{d}_U > 0$; this is indeed the case. As long as $\tilde{m}(U,V) < U$ sufficiently, so that $\ln\left(1-\frac{\tilde{m}(U,V)}{U}\right) \approx -\frac{\tilde{m}(U,V)}{U}$ is a good approximation, from the 3rd line above we get $1-e^{-U\bar{d}_U/V} = \frac{\tilde{m}(U,V)}{V}$, and since $\frac{\tilde{m}(U,V)}{V} < 1$ from our assumption on $\tilde{m}(U,V)$, this equation pins down a unique $\bar{d}_U > 0$. □

It follows that if $\tilde{m}(U,V) = m_0\big(U^{-\gamma}+V^{-\gamma}\big)^{-\frac{1}{\gamma}}$, $\gamma > 0, m_0 \in (0,1]$,[20] the Erdös-Rényi network gives rise to the CES matching function. The Leontief can be derived as the limit case of the CES when $\gamma \to \infty$. The Cobb-Douglas can also be derived as a limit case of the more general CES function $m_0\big((1-\eta)U^{-\gamma}+\eta V^{-\gamma}\big)^{-\frac{1}{\gamma}}$, $\eta \in (0,1)$, when $\gamma \to 0$.[21]

Our result on being able to generate specific matching functions can be taken to illustrate "how much" or rather "what type" of a knife-edge case the Cobb-Douglas, the CES, or in

[20] For this functional form it can be checked that $\tilde{m}(U,V) < U, V$.

[21] For the Cobb-Douglas we have to restrict U, V in the regions where $m < U, V$. The fact that Cobb-Douglas is less tractable than the CES in the discrete case is known in the literature. Cobb-Douglas can also be derived as the 1st-order approximation to the CES as shown in the appendix of section 3.

fact any specification of the aggregate matching function are. Our analysis suggests that, assuming one of these specifications amounts to assuming a particular type of *scaling* of the applicant-degree distribution: For any pair of U, V, $\bar{d}_U$ has to scale "appropriately" – as given by (**), for the matching function to be of the respective functional form.

Seen differently, applicants' search effort, i.e. the applicant degrees, are elevated to first-tier citizens in our analysis: search intensity and a matching technology of certain properties are not two separate things, they are one and the same. More concretely, it is only under a specific distribution of applicant-degrees (search intensity) that certain functional forms emerge. The question as to *how* search intensity scales with the network size is ultimately an empirical one, thus our analysis suggests what type of data are needed to address it.

Connection to Stevens (2007). In the large-economy limit of the Erdös-Rényi network we are working with in this section, the applicant-degree distribution is Poisson with parameter $\bar{d}_U$. Stevens (2007) describes the underlying network of connections by a queuing system. Even though the queuing system cannot be mapped exactly to the Erdös-Rényi network because its characterization is inherently dynamic, we show the two are closely related.

Stevens (2007) describes the underlying network of connections and offer protocol as a ("telephone-line") queuing system (Cox and Miller, 1965, section 4.4; Ross, 2010, section 6.3). Applications arrive at a Poisson rate of $u\alpha$. These can equivalently be thought as independent arrivals from u applicants each at a Poisson rate α. This means that at any time interval τ the number of applications ("customer arrivals") from each applicant is a Poisson distributed random variable with parameter $\alpha\tau$. Thus on the applicant side the model is virtually identical to our Erdös-Rényi network. Things change slightly from our setup regarding how applications "fall" on vacancies, and how offers (matches) are made: the firm is modeled as a "server" processing applications at a Poisson rate of $v\gamma$. As soon as an application arrives, if it finds the server empty, it starts being processed; if the server is busy, the application is lost. Once processing finishes, an offer (match) is made and the server returns to being empty and ready to process another application. The (stationary) probability the server is empty is known to be $\frac{v\gamma}{v\gamma+u\alpha}$, and thus the matching rate is

$$u\alpha\frac{v\gamma}{v\gamma+u\alpha}$$

that is the rate of arrival times the fraction of time it finds the server empty.

This is a useful comparison for two reasons: (a) it links to an alternative characterization of the network of connections and offers protocol which are specified as arrival processes and thus for comparison we have to see them over some time interval τ; (b) it relates to scaling, as one of the main insights of Stevens (2007) is that the matching function is of the CES form when average application intensity – α, and recruitment intensity – γ, are endogenously chosen in a way that depends on market conditions, that is, when intensities "scale" appropriately (Stevens, 2007, proposition 3).

7 Discussion of assumptions

This section discusses some of our modeling assumptions with special emphasis on the single-round offer-acceptance (or rejection) assumption of the application-and-hiring protocol.

Single-round of offer-acceptance (or rejection). One critical assumption maintained throughout the paper is that firms make an offer to a candidate which the candidate accepts or rejects. The process ends there, without further rounds of offer-acceptance/rejection. This assumption determines the amount of coordination frictions in the matching function. It is commonly held among related papers that look at the allocation of jobs to applicants in the cross-section (e.g. Shimer, 2005; Albrecht, Gautier and Vroman, 2006; Galenianos and Kircher, 2009; as well as Calvó-Armengol, 2004; Calvó-Armengol and Zenou, 2005).

In this section we argue the assumption we make is a reasonable one which could endogenously emerge in a dynamic setup. In addition, to appreciate its critical role, we show how making the polar assumption of allowing for arbitrarily many rounds until all possible matches are exhausted affects our results. This type of analysis is closely related to Kircher (2009) and Gautier and Holzner (2017) who, however, contrary to us, impose some amount of coordination of how applicants are recalled over rounds.

In a dynamic setup, where there is heterogeneity in the quality of the job-applicant matches (e.g. Martellini and Menzio, 2020), a natural trade-off emerges: a firm that does not get its first-choice applicant in the first round of offer-acceptance/rejection process, will choose between hiring a lower productivity worker and producing at a reduced rate until the relationship ends, against keeping the vacancy open to get a chance for a new draw next period of a higher productivity applicant. Assuming relationships are expected to be sufficiently long-term, and the cost of maintaining a vacancy is sufficiently small, it seems reasonable that optimizing firms will be choosing the latter than the former. Thus the offer-acceptance/rejection process is likely to end in a single (or very few) rounds.[22]

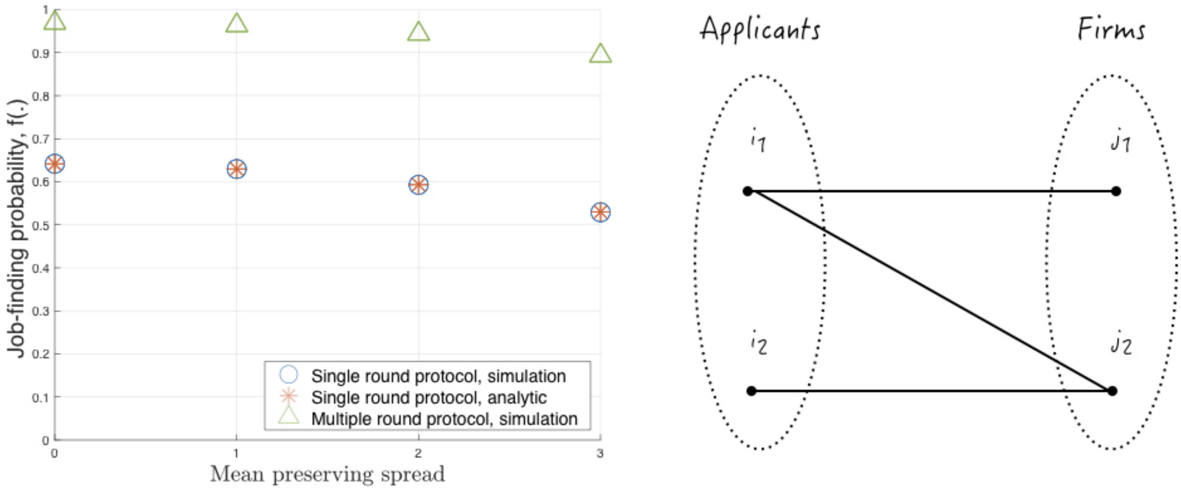
As already said, even though we think this is a reasonable assumption to maintain, let us see how relaxing it affects our results. Firstly, the result that increasing the number of links across applicants can have a negative impact on match efficacy due to congestion effects will go away. This can easily be seen in the limit: in the complete graph, that is when each applicant has the maximum number of links they can, allowing for multiple rounds necessarily means the matching function reaches its efficacy limit, that is $m(U, V) = \min\{U, V\}$, i.e. all possible matches will be formed.

However, the result that a mean-preserving spread in the applicant degree distribution hurts match efficacy is maintained, even though, naturally, the level of the job-finding probability changes. [23] In the left figure below, the y-axis plots the applicant job-finding probability and as we move to the right, the figure plots the outcome of four applicant-degree distributions each being a MPS of the one to its left. The reason the result carries through can be seen going back to our figure 1, reproduced below to the right: more well-connected applicants (i_1 in the figure) can receive and accept an offer (offer j_2 in the figure) that "block" less

[22]This should be especially the case when legal or other restrictions make firing less easy, or when the on-boarding process can be costly for the company and even more so when the quality signal is uncertain. Anecdotally, Facebook is known to try hard to avoid hiring an applicant who may not be a good fit, even though this means rejecting possibly good candidates as well.

[23]Regarding the size of improvement, in a richer model where a more well-connected applicant is also more skilled, and thus more likely to receive an offer first, the improvement in efficacy from allowing multiple rounds could be significantly smaller.

connected applicants from being able to receive an offer even if multiple rounds are allowed for.



(A) Protocol comparison

(B) A "blocking" pattern

Figure 7: Figure A plots the applicant job-finding probability and as we move to the right of the x-axis each applicant-degree distributions is a mean-preserving spread of the one to its left. More specifically, the sizes of the two sides of the market are held constant, $U = V = 100$; $\bar{d}_U = 20$; and for $x = \{0, 1, 2, 3\}$ half of the applicants are randomly assigned a high degree of $d^h = \bar{d}_U + x \cdot 5$, and the rest a low degree of $d^l = \bar{d}_U - x \cdot 5$. We run 3000 simulations of network creation and matching and take their average for each data point. For the single round protocol we have the analytic expression of the matching function as well and show it is identical to the expectation computed through simulation. Figure B illustrates that in the presence of such patterns in the network, when applicant i_1 receives and accepts an offer from firm j_2, applicant i_2 is cut out of the network even if we allow for multiple rounds thereafter.

Applicant links falling uniformly at random on vacancies. We note that essentially our random network model assumes firms have no limit on how many applications they want to screen; they accept as many applications as applicants send. One way of relaxing this assumption is by assuming that each firm has a randomly drawn degree, and "meeting" amounts to connecting the stubs of applicant and firm degrees. That model is the bipartite configuration model analyzed by Newman, Strogatz, and Watts (2001); asymptotically it will have the same behavior as ours, with the only difference being that z_k will not be necessarily binomial, but an arbitrary distribution, and a primitive. That setup will still have (asymptotically) a matching function given by (2) but z_0 will be a primitive.

A note on applicant-degree distributions. The fact that applicants with zero degree are included in the set we are referring to as "unemployed" may appear somewhat atypical. Most of the analysis will not be affected by limiting attention to distributions with $Pr\{d_i = 0\} = 0$, even though, naturally this precludes the Erdös-Rényi network which does have a positive fraction of applicants with 0 degree.

In other words, our random graph approach treats in a unified way the decision to enter the labor market (extensive margin of search – send 0 vs sending ≥ 1 applications) and the search effort an applicant puts (intensive margin of search – number of applications sent). People with 0 links (and hence 0 applications) will be the "voluntarily" unemployed, while people who send ≥ 1 applications and don't find a job, the "involuntarily" unemployed. The quotation marks are there to highlight that the "voluntarily" unemployed may just be shut out of the network of job search, not having for example the relevant skills at the moment, or the right connections.

Finally it is worth noting that some firms will also have zero degree, but this is perfectly normal: a firm can find no match either because nobody applied to its vacancy, or because it made an offer to someone who took another offer.

8 Concluding remarks

The key message of this paper is that *structure counts* for the properties of the emergent matching function, to the point we can claim that the underlying structure *is* the matching function. A natural next question is, of course, how do economic forces determine that structure and how that structure changes over time. To answer this question, one needs richer models endogenizing what we took as exogenous here to focus on our topic of interest: the matching function itself. The previous section has already discussed the importance of a dynamic extension for the assumption of single- (vs multiple-) round offers.

It is worth highlighting one sense of structure we have not covered in our analysis. For any plausible interpretation of a link – whether that reflects skills, or social ties between workers searching for jobs and friends employed at firms, or geographic restriction, one might expect correlation patterns to emerge. Then if applicant i connects to two jobs – say j, j', conditional of applicant i' connecting to job j, they may have a higher than average probability to also connect to job j'. This relates to the fundamental notion of *clustering* (or transitivity) in the networks literature, and it is absent from our model, for which the probabilities of each link are independent. We think this is an issue of first-order interest to be refined in future theoretical work and be tested empirically.

Apart from its realism, clustering is expected to matter quantitatively, as we would expect coordination failures be (potentially significantly) exacerbated in its presence: to put it simply, clustering implies that the same people compete for the same jobs. The presence of clustering is another notion of "structure" embedded in networks, and thus another avenue in which the network literature can shed light on the search literature. A description of the underlying network of connections featuring clustering can be seen as a relaxation of what is commonly done in empirical work of assuming fully segmented submarkets (e.g. Şahin, Topa and Violante, 2014; Barnichon and Figura, 2015), an approach that comes with the known empirical challenge of choosing the "boundaries" of these submarkets.

Networks can discipline further our collection of granular data-collection relevant for policy-making. For example, one can imagine compiling local measures of rationing unemployment (Michaillat, 2012): if 10 people compete for 3 jobs at a location, there is no way more than 3 of them get hired, so a policymaker would likely want to prescribe unemployment benefits, or retraining for the remaining 7 people. Furthermore, networks can guide refined welfare types of exercises,[24] where the planner takes into account the whole network of connections, thus finding the constrained optimum by eliminating only coordination frictions. The constrained optimum is a known problem in the networks literature, it is the max-flow problem.

[24]Such type of exercises have been the focus of large part of the literature, e.g. Moen, 1997; Shimer, 2005; Albrecht, Gautier and Vroman, 2006; Kircher, 2009; Galenianos and Kircher, 2012; Şahin, Topa and Violante, 2014

Starting from the job-finding probability for the random network, given in (2) and taking 1st-order Taylor approximation around $\bar{d}_u$, we get

$$f \overset{\text{1st}}{\approx} 1 - (1-\phi)^{\bar{d}_u}$$

which is the matching function of the 1-side regular network (Albrecht, Gautier and Vroman, 2006) giving that network an additional special role. Taking the 2nd-order Taylor approximation yields

$$f \overset{\text{2nd}}{\approx} 1 - (1-\phi)^{\bar{d}_u} \left[1 + \frac{1}{2}[\ln(1-\phi)]^2 Var(d_u)\right]$$

giving yet another variant of our theme that asymmetries, as these are captured here by the variance of the distribution of connections, hurt match efficacy.

In terms of empirical work, all parts of our analysis suggest that to further unpack the implications of the structure that underlies the emergent matching function, we need granular, applicant-firm-level data on links. And in terms of theoretical work what is needed is to further understand how these networks are formed: as the outcome of choices of people to acquire- and firms to advertise information of existing jobs, the choices of people to move to jobs of similar skills to theirs, to acquire new skills to expand the set of jobs they can apply to, or to relocate.

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Appendices

A Section 3, Matching in an arbitrary graph

Proposition. *The matching function in the case of the complete graph is increasing and concave in its two arguments*

Proof. The matching function in this case is

$$m(U,V) = U\left(1 - \left(1 - \frac{1}{U}\right)^V\right)$$

Its derivatives are of the respective signs:

$$\frac{\partial m}{\partial V} = -U\big(1 - \frac{1}{U}\big)^V \ln\big(1 - \frac{1}{U}\big) > 0, \quad \text{and} \quad \frac{\partial^2 m}{\partial V^2} = -U\big(1 - \frac{1}{U}\big)^V \big[\ln\big(1 - \frac{1}{U}\big)\big]^2 < 0$$
$$\frac{\partial m}{\partial U} = 1 - \big(1 - \frac{1}{U}\big)^{V-1}\big(\frac{U-(1+V)}{U}\big) > 0, \quad \text{and} \quad \frac{\partial^2 m}{\partial U^2} = -\frac{V^2}{U^3}\big(1 - \frac{1}{U}\big)^{V-2} < 0$$

□

Proposition. *The matching function m for a double regular graph exhibits constant returns to scale, and it is increasing and concave in each of its arguments.*

Proof. The matching function is

$$m(U,V;d_U) = U\left[1 - \left(1 - \frac{1}{d_U}\frac{V}{U}\right)^{d_U}\right]$$

Constant returns to scale follow from the definition, as $\forall\lambda > 0$

$$\begin{aligned} m(\lambda U,\lambda V;d_U) &= \lambda U\left[1 - \left(1 - \frac{1}{d_U}\frac{\lambda V}{\lambda U}\right)^{d_U}\right] \\ &= \lambda m(U,V;d_U) \end{aligned}$$

For the rest it helps to express m in terms of the job-finding probability $m(U,V) = Uf(\theta)$, where $f(\theta) = \left[1 - \left(1 - \frac{1}{d_U}\theta\right)^{d_U}\right]$, and we dropped the parameter d_U as an argument of the functions for notational convenience.

So for monotonicity and concavity we check the derivatives:

$$\frac{\partial m}{\partial V} = f'(\theta) \tag{i}$$
$$\frac{\partial^2 m}{\partial V^2} = f''(\theta)\frac{1}{U} \tag{ii}$$
$$\frac{\partial m}{\partial U} = f(\theta) - f'(\theta)\theta \tag{iii}$$
$$\frac{\partial^2 m}{\partial U^2} = f''(\theta)\theta^2 U^{-1} \tag{iv}$$

We first show that f is increasing and concave, signing conditions (i), (ii), (iv):

$$f'(\theta) = (1-\theta)^{d_U - 1} \geq 0$$
$$f''(\theta) = -(d_U - 1)(1-\theta)^{d_U - 2} \leq 0$$

We also note that $f(0) = 0$, and $f(1) = 1 - \left(1 - \frac{1}{d_U}\right)^{d_U} \leq 1$.

To get the sign of (iii) we show that $f(\theta) - f'(\theta)\theta \geq 0$: Define $Q(\theta) = f(\theta) - f'(\theta)\theta$. But $Q' = -\theta f'' \geq 0$. And since $Q(0) = 0$, $Q(\theta) \geq 0$. □

We note the elasticity of $m(\cdot)$ is not constant.[25] Specifically, denote $\eta(\theta) \equiv \frac{\partial m}{\partial V}\frac{V}{m}$, then

$$\eta(\theta) = \frac{f'(\theta)\theta}{f(\theta)}$$

Of course, from CRS we have that $\frac{\partial m}{\partial U}\frac{U}{m} = 1 - \eta(\theta)$. It follows from the concavity of m w.r.t U that $\eta(\theta) < 1$, as we showed above that $f'(\theta)\theta \leq f(\theta)$.

Proposition. *For tightness $\theta = \frac{V}{U}$ around 1, the matching function m for a double regular graph is equal to a Cobb-Douglas function up to 1st-order. Specifically, one can write*

$$m(U, V) \approx m_0 V^{\tilde{\eta}} U^{1-\tilde{\eta}},$$

where $m_0 = f(1) \leq 1$, and $\tilde{\eta} = \eta(1) < 1$.

[25] Constant elasticity is not considered one of the characteristic properties of the matching function. For example the Cobb-Douglas has constant elasticity, while the specification $m(V,U) = \left[V^{-\gamma} + U^{-\gamma}\right]^{-\frac{1}{\gamma}}, \gamma > 0$ does not.

Proof. We take logs of the matching function

$$\ln(m) = \ln(U) + \ln\left(1 - \left[1 - \frac{1}{d_U} e^{\ln\left(\frac{V}{U}\right)}\right]^{d_U}\right)$$

Define $L(x) \equiv \ln\left(1 - \left[1 - \frac{1}{d_U} e^x\right]^{d_U}\right)$, where $x \equiv \ln\left(\frac{V}{U}\right)$. We can take the Taylor expansion of $L(x)$ around any $x_0 \in (-\infty, \ln(d_U))$; we choose to do so around $x_0 = 0$:

$$L(x) = \sum_{n=0}^{\infty} \frac{L^{(n)}(0)}{n!} x^n$$

The 1st-order approximation yields

$$L(x) \approx L(0) + L'(0)x$$

And hence the matching function is (approximately) of the Cobb-Douglas form:

$$\ln(m) \approx L(0) + (1 - L'(0)) \ln(U) + L'(0) \ln(V)$$

where $L(0) = \ln\left(1 - \left(1 - \frac{1}{d_U}\right)^{d_U}\right)$, $L'(0) = \frac{\left(1-\frac{1}{d_U}\right)^{d_U-1}}{1-\left(1-\frac{1}{d_U}\right)^{d_U}}$. □

We also notice that $L'(0) = \frac{f'(1)\cdot 1}{f(1)} = \eta(1)$, and $L(0) = \ln(f(1))$. Thus we have shown that at a 1st-order, for cases when $V \approx U$, and all applicants and firms are symmetric we can write

$$m(U, V) \approx m_0 V^{\tilde{\eta}} U^{1-\tilde{\eta}},$$

where $m_0 = f(1) \leq 1$, and $\tilde{\eta} = \eta(1) < 1$.

Proposition. *For the double regular network the matching rate is maximized when $d_U = 1$.*

Proof. The job-finding probability in the double regular graph case is

$$f = 1 - \left(1 - \frac{1}{d_U}\theta\right)^{d_U}$$

To study its monotonicity holding θ fixed and varying d_U, let us define and study the monotonicity of the auxiliary function

$$h(d_U) = \left(1 - \frac{1}{d_U}\theta\right)^{d_U}$$

where naturally $d_U \geq \theta$ for the function to be well-defined.[26] From now on we drop the subscript U to simplify notation.

Define

$$\begin{aligned}\tilde{h}(d) &= \ln(h(d)) \\ &= d\ln\left(1 - \frac{1}{d}\theta\right)\end{aligned}$$

Now,

$$\begin{aligned}\tilde{h}'(d) &= \ln\left(1 - \frac{1}{d}\theta\right) + \frac{d\frac{1}{d^2}\theta}{1 - \frac{1}{d}\theta} \\ &= \ln\left(1 - \frac{1}{d}\theta\right) + \frac{\frac{1}{d}\theta}{1 - \frac{1}{d}\theta}\end{aligned}$$

We can show this is always > 0. Exponentiate both sides to get

$$e^{\tilde{h}'} = \left(1 - \frac{1}{d}\theta\right) e^{\frac{\frac{1}{d}\theta}{1-\frac{1}{d}\theta}}$$

But we know $e^x \geq 1 + x, \forall x \geq 0$, thus

$$\begin{aligned}e^{\frac{\frac{1}{d}\theta}{1-\frac{1}{d}\theta}} &\geq 1 + \frac{\frac{1}{d}\theta}{1 - \frac{1}{d}\theta} \Rightarrow \\ e^{\frac{\frac{1}{d}\theta}{1-\frac{1}{d}\theta}} &\geq \frac{1}{1 - \frac{1}{d}\theta} \Rightarrow \\ \left(1 - \frac{1}{d}\theta\right) e^{\frac{\frac{1}{d}\theta}{1-\frac{1}{d}\theta}} &\geq 1\end{aligned}$$

Since $e^{\tilde{h}'} \geq 1$, it follows that $\tilde{h}' \geq 0$, thus $\tilde{h}$ is increasing in d, thus h is increasing in d, and hence f is decreasing in d. $\square$

[26] This constraint is imposed in our model from $Ud_U = Vd_V$, and $d_V \geq 1$.

B Section 4, Structure and overall match efficacy

B.1 Proof of theorem 1

Theorem. *Take an arbitrary network G.*

(A) Let $\hat{G}$ denote the network resulting from swapping a link $ij \in G$ with link $i'j \notin G$. Then $m(\hat{G}) < m(G)$, if and only if

$$1 - f_i(\hat{G}) > 1 - f_{i'}(G)$$

(B) Let $\hat{G}$ denote the network resulting from adding link ij, where $ij \notin G$. Then $m(\hat{G}) < m(G)$, if and only if

$$1 - \bar{f}_{N_j}(G) > 1 - f_i(G)$$

where $1 - \bar{f}_{N_j}(G) \equiv \frac{1}{d_j} \sum_{k \in N_j} \prod_{j' \in N_k/\{j\}} \left(1 - \frac{1}{d_{j'}}\right)$.

Proof. Part (A): Since nothing changes for any other applicant other than i, i', only these two matter, thus

$$m(\hat{G}) < m(G) \Leftrightarrow$$

$$f_i(\hat{G}) + f_{i'}(\hat{G}) < f_i(G) + f_{i'}(G) \Leftrightarrow$$

$$-\prod_{k \in N_i/\{j\}} \left(1 - \frac{1}{d_k}\right) - \prod_{k \in N_{i'} \bigcup\{j\}} \left(1 - \frac{1}{d_k}\right) < -\prod_{k \in N_i} \left(1 - \frac{1}{d_k}\right) - \prod_{k \in N_{i'}} \left(1 - \frac{1}{d_k}\right) \Leftrightarrow$$

$$-\prod_{k \in N_i/\{j\}} \left(1 - \frac{1}{d_k}\right) + \prod_{k \in N_i} \left(1 - \frac{1}{d_k}\right) < -\prod_{k \in N_{i'}} \left(1 - \frac{1}{d_k}\right) + \prod_{k \in N_{i'} \bigcup\{j\}} \left(1 - \frac{1}{d_k}\right) \Leftrightarrow$$

$$-\prod_{k \in N_i/\{j\}} \left(1 - \frac{1}{d_k}\right) + \prod_{k \in N_i/\{j\}} \left(1 - \frac{1}{d_k}\right)\left(1 - \frac{1}{d_j}\right) < -\prod_{k \in N_{i'}} \left(1 - \frac{1}{d_k}\right) + \prod_{k \in N_{i'}} \left(1 - \frac{1}{d_k}\right)\left(1 - \frac{1}{d_j}\right) \Leftrightarrow$$

$$-\prod_{k \in N_i/\{j\}} \left(1 - \frac{1}{d_k}\right)\left[1 - \left(1 - \frac{1}{d_j}\right)\right] < -\prod_{k \in N_{i'}} \left(1 - \frac{1}{d_k}\right)\left[1 - \left(1 - \frac{1}{d_j}\right)\right] \Leftrightarrow$$

$$\prod_{k \in N_i/\{j\}} \left(1 - \frac{1}{d_k}\right) > \prod_{k \in N_{i'}} \left(1 - \frac{1}{d_k}\right) \Leftrightarrow$$

$$1 - f_i(\hat{G}) > 1 - f_{i'}(G)$$

Part (B): In this case only i and the applicants in the neighborhood of firm j are affected. Specifically,

$$
\begin{aligned}
m(\hat{G}) < m(G) &\Leftrightarrow \\
f_i(\hat{G}) + \sum_{k\in N_j} f_k(\hat{G}) < f_i(G) + \sum_{k\in N_j} f_k(G) &\Leftrightarrow \\
-\left(1-\frac{1}{1+d_j}\right)\left\{\prod_{j'\in N_i}\left(1-\frac{1}{d_{j'}}\right) + \sum_{k\in N_j}\prod_{j'\in N_k/\{j\}}\left(1-\frac{1}{d_{j'}}\right)\right\} < -\prod_{j'\in N_i}\left(1-\frac{1}{d_{j'}}\right) - \left(1-\frac{1}{d_j}\right)\sum_{k\in N_j}\prod_{j'\in N_k/\{j\}}\left(1-\frac{1}{d_{j'}}\right) &\Leftrightarrow \\
\frac{1}{1+d_j}\prod_{j'\in N_i}\left(1-\frac{1}{d_{j'}}\right) < \left(\frac{1}{d_j}-\frac{1}{1+d_j}\right)\sum_{k\in N_j}\prod_{j'\in N_k/\{j\}}\left(1-\frac{1}{d_{j'}}\right) &\Leftrightarrow \\
\frac{1}{1+d_j}\prod_{j'\in N_i}\left(1-\frac{1}{d_{j'}}\right) < \frac{1}{d_j(1+d_j)}\sum_{k\in N_j}\prod_{j'\in N_k/\{j\}}\left(1-\frac{1}{d_{j'}}\right) &\Leftrightarrow \\
\prod_{j'\in N_i}\left(1-\frac{1}{d_{j'}}\right) < \frac{1}{d_j}\sum_{k\in N_j}\prod_{j'\in N_k/\{j\}}\left(1-\frac{1}{d_{j'}}\right) &\Leftrightarrow \\
1-f_i(G) < \underbrace{\frac{1}{d_j}\sum_{k\in N_j}\prod_{j'\in N_k/\{j\}}\left(1-\frac{1}{d_{j'}}\right)}_{\equiv 1-\bar{f}_{N_j}(G)} &
\end{aligned}
$$

To go from line 2 to line 3 we used the expression for an applicant's job-finding probability, cancelled the 1's from all terms, and factored out $\left(1-\frac{1}{1+d_j}\right)$ on the left hand side for compactness. To go from line 3 to line 4 we collected terms on the two sides. □

B.2 Background material on mean-preserving spreads

In this section we provide some background material on mean-preserving spreads over arbitrary vectors which are not necessarily a probability distribution.

Definition *A vector x' is* ***a mean preserving spread*** *(MPS) of vector x if they have the same mean $\sum_i x_i = \sum_i x'_i$ and if x can be obtained from x' by a series of* ***Pigou-Dalton transfers****, ignoring the identities of the agents.*

Definition *A transfer $t > 0$ from one agent to another when the two agents are endowed with x_1, x_2 respectively of some quantity, is* ***a Pigou-Dalton transfer*** *if $x_1 > x_2$ AND $x_1 - t \geq x_2 + t$.*

That is a Pigou-Dalton transfer between two agents is one such that an amount is transferred from the richer to the poorer agent preserving their relative positions. The quantity under consideration can be anything, e.g. wealth, number of friends etc.

We make the following observations following straight from the definitions.

Remark 1: Any sequence of Pigou-Dalton transfers is mean-preserving.

Remark 2: It can be helpful to think of a mean-preserving spread (MPS) x' of a vector x, as created from x doing "inverse" Pigou-Dalton transfers. "Inverse" Pigou-Dalton transfers are transfers where the rich become richer and the poor poorer.

Remark 3: A mean-preserving spread (MPS) increases inequality in the outcomes, while a Pigou-Dalton transfer decreases it.

For a MPS we can show the following basic result:
Proposition. *For a strictly convex function $h(\cdot)$, if x' is a MPS of x, then it holds that*

$$\sum_i h(x'_i) > \sum_i h(x_i)$$

Proof. It suffices to show the inequality holds for a single Pigou-Dalton transfer (up to relabeling). Then by applying it repetitively, we can show it holds for any sequence of such transfers. Let us assume a transfer occurs between agents 1 and 2.

Assume $x'_2 = x_2 + t, x'_1 = x_1 - t$, and $x_2 \geq x_1$, where $t > 0$. Then

$$\begin{aligned}
\sum_i h(x'_i) > \sum_i h(x_i) \Leftrightarrow \\
h(x_1 - t) + h(x_2 + t) > h(x_1) + h(x_2) \Leftrightarrow \\
h(x_2 + t) - h(x_2) > h(x_1) - h(x_1 - t) \Leftrightarrow \\
\frac{h(x_2 + t) - h(x_2)}{t} > \frac{h(x_1) - h(x_1 - t)}{t}
\end{aligned}$$

But we know from the mean value theorem there exist $\tilde{c} \in (x_2, x_2 + t)$ and $\tilde{\tilde{c}} \in (x_1 - t, x_1)$ s.t.

$$h'(\tilde{c}) = \frac{h(x_2 + t) - h(x_2)}{t}$$

$$h'(\tilde{\tilde{c}}) = \frac{h(x_1) - h(x_1 - t)}{t}$$

We also know that since $h(\cdot)$ is convex, $h'(\cdot)$ is increasing thus $h'(\tilde{c}) > h'(\tilde{\tilde{c}})$, completing the proof. □

Note: Even though typically the outcome vectors x, x' are taken to be positive in applications (e.g. income redistribution), this is not a requirement. The result holds equally well for positive and negative outcome vectors.