Network Perception in Network Games*

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October 23, 2023

Abstract

In many situations, people make decisions based on the actions of others, but have incomplete information about the social structure they form. This paper presents a novel approach to the analysis of network games based on group theory. I propose a model of Bayesian updating in which players have incomplete information about the network they are part of, and form beliefs about it based on a set of signals before playing the game. I characterize posteriors for a variety of information setups, and provide conditions under which equilibrium actions depend monotonically on certain aspects of players' network position (e.g., on their number of friends). The conditions relate network information to the distribution of players in the set of feasible networks, and allow the design of information structures compatible with monotone equilibria. Moreover, they show that, regardless of the structure of the network that players are part of, equilibrium outcomes tend to be more influenced by the topology of networks that are more asymmetric, among all those compatible with their information.

Keywords: Network Games, Graphical Games, Network Cognition, Incomplete Information, Structural Equivalence, Networks, Symmetry.

JEL: D85, C72, L14, Z13.

1 Introduction

People's behavior often depends on that of their peers. When making a decision such as getting vaccinated (Brunson, 2013), voting (Harmon et al., 2019), adopting a technology (Ferrali et al., 2020), or doing someone else a favor (Jackson et al., 2012), individuals are often influenced by the decisions of their contacts. There is a huge literature on this phenomenon, which has motivated the development of models in which the influence that people exert on each other is represented as an

*I am especially grateful to the board of the Association of Southern European Economic Theorists (ASSET) for the Louis-André Gérard-Varet prize awarded to this paper.

I also thank Eric Bahel, Michel Grabisch, Emiliya Lazarova, Ahmed Saber Mahmud, Souvik Roy, Agnieszka Rusinowska and the entire GamesNet team for the best paper prize awarded to this work in the GamesNet Young Scholars' Competition.

I greatly appreciate comments from Coralio Ballester, Renaud Bourlès, Yann Bramoullé, Sergio Cappellini, Frèdèric Deröian, María Paz Espinosa, Penélope Hernández, Jaromír Kovářík, Norma Olaizola, Arnold Polanski and Fernando Vega-Redondo.

Financial support from *Ministerio de Ciencia e Innovación* (PID2019-108718GB-I00, PID2019-106146GB-I00) and *Ministerio de Educación* (FPU15/01715) is gratefully acknowledged. Declarations of interest: none.

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interaction matrix or network (see, for example, Ballester et al., 2006; Jackson and Yariv, 2007; Bramoullé and Kranton, 2007; Bramoullé et al., 2014; Bourlès et al., 2017; 2021; Bloch et al., 2021, López-Pintado and Meléndez-Jiménez, 2021). The main goal of this literature is to characterize the impact of network geometry on behavior.

Most papers on network games assume that players have complete information about the network to which they belong (see, for example, Goyal and Moraga-González, 2001; Ballester et al., 2006; Bramoullé and Kranton, 2007; Bramoullé et al., 2014; Bourlès et al., 2017; 2021). This is a strong assumption that may not be true in reality; even when complete information is available, people show cognitive limitations in their ability to encode and recall the network (Brashears et al., 2015; Dessi et al., 2016). On the other hand, a wide range of equilibrium outcomes is possible when players have complete information about the network, making it difficult to draw general conclusions about the effects of network structure on behavior. A smaller body of work assumes that players' information is limited to a particular aspect of the network architecture. Galeotti et al. (2010), for example, consider a setup in which players know only their number of friends (their degree) and their beliefs about the network are summarized by a probability degree distribution. An important contribution of this work is the identification of conditions under which equilibrium actions depend monotonically on players' degrees.[1]

The information setup of Galeotti et al. (2010) applies to many real-life situations. A person may decide to learn a language, start a business, get vaccinated, etc., based solely on her expected number of future interactions, without necessarily knowing the identity of her future contacts. In such a situation, the only network characteristic that influences her decision may be the number of people she expects to interact with from a relatively unbounded population (e.g., a country). A natural way to model this type of situation (in which the network is typically very large and decisions are made before links are formed) is to identify network beliefs with a probability degree distribution, setting aside beliefs about other network aspects.

Notwithstanding this, in many circumstances people have incomplete network information, but they do know who they interact with, how popular their opponents might be, whether they know each other, etc. (Killduf et al., 2008). Such more detailed information in turn allows network members to deduce other properties of the network to which they belong. Indeed, research in social psychology has shown that people who join a social group tend to create a cognitive map of the existing network–a mental picture of the connections that captures who is connected to whom in the group (Krackhardt,

[1] They show that when agents have degrees with either independent probabilities or probabilities that are positively (negatively) correlated, any symmetric equilibrium is monotonically non-decreasing (non-increasing) in players' degrees under strategic complements (substitutes).

1987). For example, a newcomer to a company may discover who is popular, who shares an office with whom, who holds what position in the company, and form mental representations of the company network. These mental representations significantly influence one's behavior. In contexts where such mental representations of networks emerge (e.g., a company, a class, or a faculty), beliefs about the network extend beyond a probability degree distribution, leading to novel research questions:

- Can we identify regularities in network beliefs in these situations with intermediate network information? If so, can we leverage them to address the equilibrium selection problem?
- Following Galeotti et al. (2010), can we identify conditions on players' (posterior) beliefs under which equilibrium actions depend monotonically on certain aspects of their network position, like the number of friends, clustering, centrality, etc.?

This paper aims to answer questions. I propose a model of Bayesian updating in which agents have incomplete information about the network they are embedded in. Agents have an arbitrary prior over the set of networks that could be the one they are part of. They then receive some signals that are used to update their priors before playing the game.

I have three main results. First, the probability assigned to each feasible network geometry increases monotonically with its degree of asymmetry, measured by the size of its automorphism group. This result holds if the prior probability assigned to each feasible network is non-decreasing in its degree of asymmetry and the received signals do not contradict these priors. However, it may also hold if these conditions are not satisfied, as illustrated below. A direct implication of this finding is that, for a variety of information structures and priors, equilibrium behavior is more influenced by the topology of the networks that are more asymmetric, among all those that are compatible with agents' information.

Intuitively, the number of distinct networks that can be formed with a given geometry z is constrained by the extent to which z is symmetric. Consider for example a population consisting of three individuals i, j, and k. They can form a triangle only if i is connected to j, j is connected to k, and k is connected to i. However, there are three different ways they can be arranged to form a v-shaped structure. These are three distinct networks that differ only in the identity of the agent with two links (i, j, or k). Thus, if both geometries (the triangle and the v-shaped structure) are feasible given players' information and players have uniform priors over all feasible networks, then they will assign a higher probability to the v-shaped geometry because there are more networks with this structure. The same may occur even if agents assign a lower probability to the networks that are more asymmetric.[2]

[2]Suppose that each network with the v-shaped (triangle) geometry has probability $\frac{37}{150}$ ($\frac{13}{50}$) according to players'

Second, I demonstrate that the degree of symmetry of a network, as captured by the size of its automorphism group, increases with the number of *automorphically equivalent nodes* (agents that occupy the same network position) and *structurally equivalent nodes* (agents that have the same neighbors). While these concepts are well-established in social network analysis (see, for example, Burt, 1987; Faust, 1988; Borgatti and Everett, 1992; Michaelson and Contractor, 1992: Freeman, 2017) they have not been related, to the best of my knowledge, to network perception.

Finally, I provide conditions on players' (posterior) beliefs for the existence of Bayes-Nash equilibria in which players' actions can be ordered as a function of particular aspects of their network position, which are jointly interpreted as their type. The conditions are general enough to encompass various information setups and payoff structures. They also facilitate the identification of scenarios in which equilibria will be reached through computational methods[3] and the design of information structures compatible with the existence of specific equilibria.

The main innovation of this work is the application of group theory to the study of network games. The paper uncovers the potential of this machinery to mitigate the equilibrium selection problem. Indeed, a fundamental criticism of the analysis of games under incomplete information is that the equilibrium achieved strongly depends on the specific assumptions that are made on beliefs (Weinstein and Yildiz, 2007). The wide variety of assumptions that can be made regarding players' beliefs leads to a similarly wide variety of possible equilibrium outcomes, making it difficult to draw a general conclusion about the effects of networks on behavior. The current approach is a first step towards mitigating this problem, which is accomplished by: (i) making a general prediction for the impact of networks on behavior under all information setups covered by the model (namely, that asymmetric feasible geometries tend to have a greater impact on behavior), and (ii) identifying conditions under which equilibrium actions can be ordered as a function of specific aspects of the agents' network position.

As for the literature, the proposed framework provides a first step in bridging the two extreme assumptions regarding network knowledge: extremely incomplete network information (Galeotti et al., 2010; Jackson and Yariv, 2007; Ruiz-Palazuelos, 2021) and complete information (Goyal and Moraga-González, 2001; Ballester et al., 2006; Bramoullé and Kranton, 2007; Bramoullé et al., 2014). Each of these approaches leads to radically different results. The current model also adds to the extensive literature on network cognition in social psychology and sociology (Krackhardt, 1987; Carley, 1987; Michaelson and Contractor, 1992; Freeman, 1992; Kumbasar et al., 1994; Casciaro,

priors. If no signal contradicts these priors, the probability assigned to the v-shaped geometry is $3 * \frac{37}{150} = \frac{37}{50}$, while the probability assigned to the triangle is $\frac{13}{50}$.

[3]There is a literature in computer science on games played on networks (see, for example, Kearns et al. (2001)), which focuses on finding algorithms to compute equilibria.

1998; Janicik and Larrick, 2005) and more recently in economics (Dessi et al. 2016; Jackson, 2019) by characterizing network beliefs under incomplete information. There is also literature in computer science on network games, for which the results in this paper might be particularly useful.[4]

The paper is organized as follows. Section 2 presents some background definitions. Section 3 provides some simple examples that illustrate the main insights of the paper. Sections 4 presents the model and Section 5 the results in network games. Section 6 concludes.

2 Background Definitions

Let $g = (N, L)$ be an undirected network composed of a set of agents $N = \{i, j, ..., z\}$ and a set of links L among them. Each agent is represented by a node, and there are $n = |N|$ nodes in the network. Let $g_{ij} = 1$ if i and j are linked in g and $g_{ij} = 0$ otherwise. The set of i's neighbors are the agents directly linked to i, i.e., $N_i(g) = \{j \in N : g_{ij} = 1\}$. The degree of node i, denoted by $k_i(g) = |N_i(g)|$, is the number of i's neighbors. The (frequency) degree distribution, denoted by $\mathcal{F}_g(k)$, specifies, for all $k \in \{0, 1, ..., n-1\}$, the proportion of individuals that have degree k in the network, $\mathcal{F}_g(k) = \frac{1}{n}|\{i \in N : k_i(g) = k\}|$. Network g is fully described by a $n \times n$ adjacency matrix $A(g) = (g_{ij})_{i,j \in N}$, where $g_{ii} = 0$. The set of all possible networks is $\mathcal{G}$ ($|\mathcal{G}| = \infty$).

Remark 1. *Two networks g and g' are different if and only if $A(g) \neq A(g')$.*

For example, networks g and g_1 in Figure 1 are distinct, since

$$A(g) = \begin{pmatrix} g_{ii} & g_{is} & g_{im} \\ g_{si} & g_{ss} & g_{sm} \\ g_{mi} & g_{ms} & g_{mm} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \neq A(g_1) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

On the contrary, g_2 and $g_2' = g_2$ in Figure 2 represent the same network, since $A(g_2) = A(g_2')$.

The geometry of a network is the structure created by its edges. Two networks $g = (N, E)$ and $g' = (N', E')$ have the same geometry if and only if there exists a bijection (an isomorphism) $f : N \to N'$, such that $ij \in E$ if and only if $f(i)f(j) \in E'$ (see Borgatti and Everett, 1992). If an isomorphism exists between g and g', then the networks are called isomorphic and denoted as $g \simeq g'$. In Figure 1, $g \simeq g_1 \simeq g_2$.

An automorphism f is a bijection that preserves the adjacency matrix: $f : N \to N$, where $ij \in E$ if and only if $f(i)f(j) \in E$. The set of all automorphisms of g is the automorphism group

[4] See footnote 4.

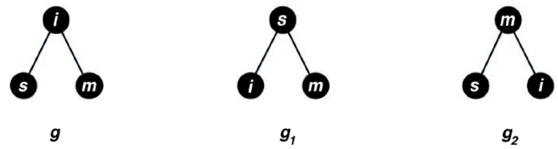

Figure 1: Three isomorphic networks

of g, denoted by $Aut(g)$. In Figure 2, $Aut(g_2) = \{f, f'\}$ where f and f' are the automorphisms of g_2 that result in g_2 and $g_2' = g_2$, respectively. In Figure 3, $Aut(g_3) = \{f, f', f'', f'''\}$ where f, f', f'' and f''' are the automorphisms of g_3 that result in g_3, $g_3' = g_3$, $g_3'' = g_3$ and $g_3''' = g_3$, respectively. The order of $Aut(g)$ is $|Aut(g)|$ and captures the degree of symmetry of g.

Remark 2. *The greater (lower) $|Aut(g)|$, the more (a)symmetric g is.*

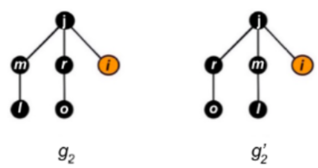

Figure 2: Automorphisms in $Aut(g_2)$

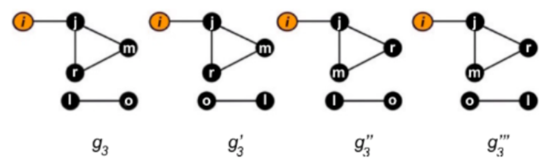

Figure 3: Automorphisms in $Aut(g_3)$

Two nodes i and j are automorphically equivalent if and only if they occupy the same network position. Formally, i and j are automorphically equivalent if and only if there exists an automorphism $f : N \in N$ such that $f(i) = l$. The notation $i \equiv l$ means that i and l are automorphically equivalent. In network g_2 of Figure 2, $m \equiv r$ and $l \equiv o$. Other pairs of agents are not automorphically equivalent.[5] The network position of i in network g is $o_i(g)$. The orbit of node i is the set composed of all nodes that occupy the same position as i, $Orv_i(g) = \{l \in N : o_i(g) = o_l(g)\}$.

Node i is structurally equivalent to l iff. $N_i(g) \setminus \{l\} = N_l(g) \setminus \{i\}$, and $i \equiv_s l$ means that i and l are structurally equivalent.[6] Structural equivalence is more demanding than automorphic equivalence: it requires not only that the nodes occupy indistinguishable structural positions in the network, but also that the identities of the agents connected to them are the same. Thus, structurally equivalent nodes must be automorphically equivalent, but the converse is not true. For example, in network g_2

[5]Automorphic equivalence has been used to identify *social roles* (Borgatti and Everett, 1992). For example, the social role of a CEO is her distinctive set of connections to he company's employees, directors, team leaders, administrative staff and so on. Similarly, the social role of an employee is defined by her connections to the company's team leaders, the CEO and others. Thus, two individuals occupy the same position if and only if they have the same social role.

[6]According to the standard definition, i and l are structurally equivalent iff. $N_i(g) = N_l(g)$ (Burt, 1976). Our more relaxed definition enlarges the set of structurally equivalent nodes. Thus, while the three nodes $\{i, j, k\}$ that make up a network triangle are not structurally equivalent according to the standard definition, they are according to our definition. Note that if $N_i(g) = \{l\}$ and $N_l(g) = \{i\}$, then i and l are structurally equivalent, since $N_i(g) \setminus \{l\} = N_l(g) \setminus \{i\} = \emptyset$. However, neither i nor l are structurally equivalent to an isolated node m, since $N_i(g) \setminus \{m\} = \{l\} \neq N_m(g) \setminus \{i\} = \emptyset$, and $N_l(g) \setminus \{m\} = \{i\} \neq N_m(g) \setminus \{l\} = \emptyset$.

in Figure 3, $r \equiv_s m$ and $l \equiv_s o$, and consequently $r \equiv m$ and $l \equiv o$. However, there is no pair of structurally equivalent nodes in network g_2 of Figure 2, although $m \equiv r$ and $l \equiv o$. The set of nodes that are structurally equivalent to i in network g is $E_i(g) = \{l \in N : N_l(g) \setminus \{i\} = N_i(g) \setminus \{l\}\}$.

3 The Effects of Network Information

Network games have been analyzed using two approaches: complete information and extremely incomplete information. In this section, I analyze a game with strategic substitutes (hereafter referred to as Game SS) to illustrate the equilibrium predictions based on each of the two information assumptions and the main insights of this paper. In the following, the network in which the players are embedded is referred to as network g.

Game SS. Every agent in g chooses an action in $X = \{0, 1\}$. Action 1 can be interpreted as contributing to a public good, and action 0 as not doing so. Let x_i be the action of player $i \in N$ and $x_{N_i} = \sum_{j \in N_i(g)} x_j$. The utility of every $i \in N$ is $u_i(x_i, x_{N_i})$ and takes the following values

$$u_i(x_i, x_{N_i}) = \begin{cases} 1 & \text{if} \quad x_i = 0 \quad \text{and} \quad x_{N_i} \geq 1 \\ 0 & \text{if} \quad x_i = 0 \quad \text{and} \quad x_{N_i} = 0 \\ 1 - c - \mu(x_{N_i}) & \text{if} \quad x_i = 1 \end{cases}$$

where $c \in (0, 1)$ is a cost and $\mu(x_{N_i})$ represents a player's regret if she incurs the cost of playing 1 when she could have free ridden: $\mu(x_{N_i}) = 0$ if $x_{N_i} = 0$ and $\mu(x_{N_i}) = \mu \in [0, 1 - c)$ otherwise.[7] ■

To start, complete information is assumed. In general, this means that a vast number of equilibria are possible.

Suppose that network g is as depicted in Figure 4(a). Each person in g is a player of Game SS and has complete information about g. Figure 4 presents the six Nash equilibria under this information scenario. Since all these equilibria exist for all parameter values (i.e., for all c and μ), they all are equally likely to emerge.

Consider now, instead, the setup of Galeotti et al. (2010). In their setup, players do not know the entire network, but only their own degree.[8] Players' (posterior) beliefs about the rest of the network are captured by a probability degree distribution, denoted $P_g(k)$, which specifies the probability that

[7] This is a "best-shot" game that applies to situations in which agents can free ride. For instance, Hendricks and Porter (1996) show that information spillovers reduce firms' exploratory drilling. Other examples, mentioned by Bilodeau and Slivinski (1996), include the cleaning of shared spaces or the chairing of an academic department. For a discussion of best-shot games, see Hirshleifer (1983).

[8] Agents do not even know the identity of their neighbors.

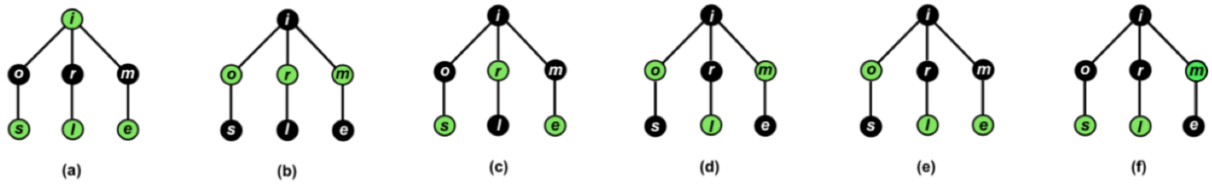

Figure 4: Equilibria under complete information. Black (green) nodes play 0 (1).

an individual has degree k for all $k \in \{0, 1, ..., n-1\}$. Galeotti et al. (2010) analyze the symmetric Bayes-Nash equilibria that arise under these conditions by identifying players' degrees with their types. A symmetric strategy is therefore a mapping σ that specifies a player's action as a function of her type.

The setup of Galeotti et al. (2010) fits well with large populations of agents making decisions before links are formed. They show that, under strategic complements (substitutes), any symmetric equilibrium is monotonically non-decreasing (non-increasing) in players' degrees when nodes have degrees with independent probabilities or probabilities that are positively (negatively) correlated.[9] Thus, in any network composed of players whose network beliefs conform to these patterns, players' equilibrium actions can be ordered as a function of their degrees.

To illustrate, consider Game SS. Suppose that players' beliefs about g are captured by $P_g(k)$ and, according to $P_g(k)$, each node in the network has degree $k \in \{0, 1, ..., n-1\}$ with probability p_k, independently on the degree of any other node. Suppose the agents playing 1 are those with a degree in set $D = \{k : \sigma(k) = 1\}$. When all agents follow the strategy σ, i's expected utility from playing 0, denoted by $E_{U_i}(0, \sigma)$, is the probability that at least one of her neighbors has a degree in D:

$$E_{U_i}\big(0, \sigma\big) = 1 - \Big[\sum_{k \notin D} p_k\Big]^{k_i(g)} \quad \forall i \in N.$$

Let $E_{U_i}\big(1, \sigma\big)$ be i's expected utility from playing 1. In equilibrium, each i plays 0 if and only if:

$$E_{U_i}\big(0, \sigma\big) \geq E_{U_i}\big(1, \sigma\big) = 1 - c - E_{U_i}\big(0, \sigma\big)\mu. \tag{1}$$

That is, iff.:

$$E_{U_i}\big(0, \sigma\big) \geq \frac{1-c}{1+\mu}, \tag{2}$$

[9]They focus on games of strategic complements (substitutes) in which an additional friend playing 0 does not affect the players' utility. Such a property is violated, for instance, if a player's utility depends on the average of her neighbors' actions.

Since $E_{U_i}\big(0,\sigma\big)$ is increasing in $k_i(g)$, players with more neighbors are more likely to have at least one neighbor who plays 1. Therefore, if an agent with k friends plays 0 in equilibrium, then a player with degree $k' > k$ must be best-responding by also playing 0. As a result, any equilibrium strategy is characterized by a degree threshold t: $\sigma(k) = 1$ for $k < t$, $\sigma(k) = 0$ for $k > t$ and $\sigma(k) \in \{0, 1\}$ for $k = t$.

Note that equilibrium behavior differs significantly depending on whether network information is complete or not, even if we focus on situations where players with the same degree choose the same actions. The equilibrium achieved depends on the specific assumptions on network beliefs. This paper aims to: (i) analyze equilibrium behavior in setups of intermediate network information, and (ii) provide a rationale for belief selection in network games under incomplete information in order to mitigate the equilibrium selection problem.

To illustrate the core idea, suppose g is the network in Figure 4(a). Imagine that in absence of any initial information about g, any network in $\mathcal{G}$ can be network g according to agents' beliefs. Agents have uniform priors over the networks in $\mathcal{G}$. Then, every $i \in N$ receives the following information about g:

$$I_i(g) = \Big\{\{i\}, N_i(g), k_i(g), [\mathcal{F}_g(1), \mathcal{F}_g(2), \mathcal{F}_g(3)], n, \alpha(g)\Big\} = \Big\{i, N_i(g), k_i(g), \Big\{[\frac{3}{7}, \frac{3}{7}, \frac{1}{7}], 7, 3\Big\}, \quad (3)$$

where $\alpha(g)$ is the number of distinct network positions. For example, a recently hired worker may know the people she interacts with ($N_i(g)$), the proportion of interactions of other company members ($\mathcal{F}_g(k)$), the company size ($n = 7$) and the number of different professions in the firm ($\alpha(g) = 3$).[10]

From (3), i can infer that g has one of the two geometries appearing in Figure 5. Depending on how agents are allocated in the network, there are different networks that could be network g given $I_i(g)$. In particular, there exist seven different networks that could be g according to the beliefs of an $i \in N$ with $k_i(g) = 3$: six networks with geometry 1 and one network with geometry 2. Table 1 contains the number of feasible networks for $k_i(g) \in \{1, 2\}$.[11] Note that, regardless of $k_i(g)$, there is a greater number of feasible networks with geometry 1. The reason is that geometry 1 is more asymmetric: $|Aut(g_1)| = 6 < |Aut(g_2)| = 36$, where g_z is a network with geometry $z \in \{1, 2\}$.

The larger (smaller) number of feasible networks with geometry 1 (2) results in a larger (smaller) probabilistic weight assigned to that geometry. Thus, every $i \in N$ believes that network g has

[10] See footnote 6.

[11] See Lemma B in the Appendix.

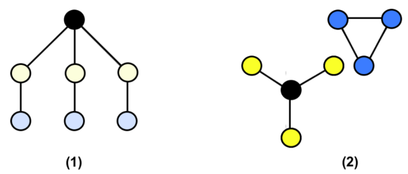

Figure 5: Two feasible geometries given (3). In each geometry, nodes in blue (yellow) occupy the same network position. Nodes in dark blue (yellow) are structurally equivalent.

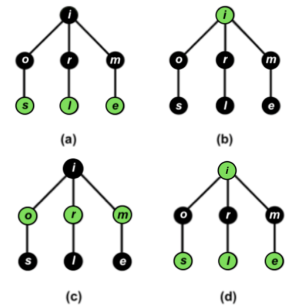

Figure 6: Equilibria of Game SS when $I_i(g)$ is (3). Black (green) nodes play 0 (1).

Feasible networks	Geometry 1	Geometry 2
$k_i(g) = 1$	24	4
$k_i(g) = 2$	60	10
$k_i(g) = 3$	6	1

Table 1: Feasible networks according to i's posterior beliefs.

Equilibria	σ_1	σ_2	σ_3	σ_4
$k_i(g) = 1$	1	0	0	1
$k_i(g) = 2$	0	0	1	0
$k_i(g) = 3$	0	1	0	1
$\frac{1-c}{1+\mu}$	$[0, \frac{1}{7}]$	$[0, \frac{1}{7}]$	$[\frac{1}{7}, \frac{6}{7}]$	$[\frac{1}{7}, \frac{6}{7}]$

Table 2: Equilibrium strategies when $I_i(g)$ is (3) $\forall i \in N$.

geometry 1 (2) with probability $\frac{6}{7}$ ($\frac{1}{7}$).[12] Players make their choices on the basis of this probability distribution. Since they do not observe the network, their actions depend not only on the topology of the network they are part of, but also on that of all the networks that are compatible with their information.

Suppose, for instance, that agents in g play Game SS. If all agents follow the strategy σ_1 in Table 2, then $E_{U_i}(0, \sigma_1) = \frac{6}{7}$ ($\frac{1}{7}$) for $k_i(g) = 2$ (3), since i has at least one neighbor playing 1 if and only if g has geometry 1 (2). Analogously, $E_{U_i}(0, \sigma_1) = 0$ for $k_i(g) = 1$, since it is not possible that i has a neighbor with degree one given $I_i(g)$. Then, σ_1 is an equilibrium strategy for $\frac{1-c}{1+\mu} \in [0, \frac{1}{7}]$. Table 2 provides the four equilibrium strategies that arise under this information setting.

Note that the greater probabilistic weight assigned by players to geometry 1 affects their behavior. Although there are four symmetric equilibrium strategies, the only ones that are sustainable for most parameter values are σ_3 and σ_4, which are the symmetric equilibrium strategies in the most asymmetric feasible networks under complete information (i.e., the networks with geometry 1). Similar results tend to hold for other information setups and priors over the feasible networks.[13]

In summary, the above examples illustrate that:

[12] Observe in Table 1 that $\frac{6}{7} = \frac{24}{24+4} = \frac{60}{60+10} = \frac{6}{6+1}$.

[13] Suppose that any network with geometry 1 (2) in Figure 5 in which $k_i(g) = 1$ has probability 0.035 (0.16) according to i's priors. If $I_i(g)$ is (3) and $k_i(g) = 1$, the posterior probability assigned by her to geometry 1 (2) is $0.035 * 24 = 0.84$ ($0.16 * 1 = 0.16$).

(i) Under complete information, multiple equilibria can exist and all them sustain for the same parameter values.

(ii) Under extreme incomplete information (see, for example, Galeotti et al. (2010)), equilibria depend on the assumed belief structure. Equilibrium actions depend monotonically on players' degrees under certain conditions of the probability degree distribution.

(iii) Under intermediate network information, multiple equilibria are possible. Nonetheless, the degree of symmetry of the feasible networks has a clear impact on behavior for a large variety of information environments. Equilibrium actions depend on the topology of all the networks that are compatible with players' posteriors, but tend to depend to a greater extent on the topology of those that are more asymmetric, since there are more networks with asymmetric topologies.

4 Network Perception

4.1 Priors

Initially, agents have no information about the network in which they are embedded, which we refer to as network g. They may have some prior beliefs about g, represented by a probability distribution over a set of feasible networks. The set of feasible networks according to i's priors is $B_i^0(g)$. The prior probability that i assigns to $g_z \in B_i^0(g)$ is $\mu_i^0(g_z) = p_i[g = g_z]$. To simplify the exposition, I assume that networks with the same geometry have identical probabilities according to players' priors beliefs (i.e., $\mu_i^0(g_z) = \mu_i^0(g_y)$ whenever $|Aut(g_z)| = |Aut(g_y)|$).[14]

Agents may have different priors about the set of feasible networks and their probabilities. However, since individuals have no initial information about g, it is reasonable to assume that $B_i^0(g) = \mathcal{G}\ \forall i \in N$ and the networks in $B_i^0(g)$ are uniformly distributed (i.e., $|B_i^0(g)| = \infty$ and $\mu_i^0(g_z) \simeq 0\ \forall g_z \in B_i^0(g)$ and $\forall i \in N$).

4.2 Information

Private information. Individuals are privately informed about a set of measurable features of their network position, which are jointly interpreted as their type. We write $t_i(g)$ to denote the type of agent i. Agents are informed about the same aspects of their network position (i.e., if $t_i(g) = k_i(g)$, then $t_j(g) = k_j(g)\ \forall j \in N$). Neither the identity of i nor the identity of any $j \neq i$ defines $t_i(g)$.

[14] This assumption allows a simplification of Proposition 1 and does not affect its main insight, as discussed below.

Agents can privately know the identity of their neighbors ($N_i(g)$). However, no i has information either about the neighbors of any $j \neq i$ or about the network position of any $j \neq i$. The number of individuals whose identity is known by i is n_{I_i}, and $n_{I_i} \in \{1, ..., k_i(g) + 1\}$.

Common knowledge. Agents know the size of the network (n) and may have information about the network structure as a whole (e.g., the degree distribution $F_g(k)$). They may also receive a signal θ about the process of network formation. For example, θ can represent an employee's knowledge about the number of interactions between the company's employees or about the type of networking activities that take place in the company.

Denote by $I_i(g)$ the information set of $i \in N$ about network g. Below are some examples:

- *Setting A*. For all $i \in N$, $I_i(g) = \Big\{\{i\}, t_i(g), \mathcal{F}_g(k), n\Big\} = \Big\{\{i\}, k_i(g), \mathcal{F}_g(k), n\Big\}$. For example, at the beginning of the school year, teachers may know the number of students in their class ($k_i(g)$), but not their identities. In addition, they may have information about $F_g(k)$: based on comments from previous teachers, they may know which proportion of children are more interactive, which proportion are more introverted, etc. Teachers may also know the school size (n).

- *Setting B*. For all $i \in N$, $I_i(g) = \Big\{\{i\}, N_i(g), t_i(g), \bar{k}(g), n\Big\} = \Big\{\{i\}, N_i(g), k_i(g), \bar{k}(g), n\Big\}$, where $\bar{k}(g) = \frac{1}{n}\sum_{i\in N} k_i(g)$. For example, a virtual platform user can monitor who visits her profile and how many visits other platform users have on average, in addition to the number of platform users.

- *Setting C*. For all $i \in N$, $I_i(g) = \Big\{\{i\}, t_i(g), n, \theta\Big\} = \Big\{\{i\}, \Big(k_i(g), \bar{k}_{N_i}, c_i(g)\Big), n, \theta\Big\}$, where $\bar{k}_{N_i} = \frac{1}{k_i(g)}\sum_{j\in N_i(g)} k_j(g)$ and $c_i(g)$ is the clustering coefficient of i; the proportion of i's neighbors that are connected.[15] This setup is similar to that in Ruiz-Palazuelos (2021). For example, a person invited to an event can anticipate the number of people she will interact with ($k_i(g)$), the average popularity of her contacts (k_{N_i}) and the probability that they are connected ($c_i(g)$). Similarly, she may have some information about the mechanisms that drive network formation, as captured by θ.

4.3 Posteriors

4.3.1 Feasible networks, geometries and positions

Agents update their priors according to Bayes' rule. The set of feasible networks according to i's posterior beliefs is $B_i(g) = \{g_z : p_i[g = g_z \mid I_i(g), \mu_i^0(g_z)]) > 0\}$, and $b_i(g) = |B_i(g)|$. The networks

[15] Formally, $c_i(g) = \frac{\sum_{j\neq k, j\neq i, k\neq i} g_{ik} g_{ij} g_{jk}}{\sum_{j\neq k, j\neq i, k\neq i} g_{ik} g_{ij}}$.

in $B_i(g)$ can have different geometries. We say that a certain geometry is feasible according to i's posterior beliefs if at least one network in $B_i(g)$ has such a geometry. The set of feasible geometries according to i's posteriors is $G_i(g) = \{1, 2, ..., h\}$. The subset of feasible networks with geometry z is $B_i^z(g) \subseteq B_i(g)$, and $b_i^z(g) = |B_i^z(g)|$. In what follows, g_z is a network with geometry z, $\forall z \in G_i(g)$.

The posterior probability that $i \in N$ assigns to network $g_z \in B_i^z(g)$ is $\mu_i(g_z) = p_i[g = g_z \mid I_i(g), \mu_i^0(g_z)] = \frac{1}{b_i(g)} + \kappa_i^z$, where κ_i^z is the probability premium that i assigns to each network with geometry z on the basis of her priors and $I_i(g)$.[16] If i has uniform priors over the networks and $\theta = \emptyset$, then $\kappa_i^z = 0$. The same is true if i has uniform priors and she knows from θ that links were formed randomly. As mentioned earlier, a uniform prior over all possible networks is a reasonable default when agents have no information about g until they receive the signals.

A network position is feasible according to i's posterior beliefs if it is consistent with $I_i(g)$, i.e., if there is a positive probability that i occupies that position according to her beliefs. The set of feasible positions of i conditional on geometry $z \in G_i(g)$ is $O_i^z(g) = \{o_i(g_z) : g_z \in B_i^z(g)\}$. That is, $O_i^z(g)$ is the set of positions that i can occupy in the network if it has geometry z. The number of nodes that represent a feasible position of i conditional on g having geometry z is $n_i^z(g)$.[17] Thus, if $g_z = (N^z, E^z)$ is a network with geometry z, $n_i^z(g) = |\{i \in N^z : o_i(g_z) \in O_i^z(g)\}|$. The following example illustrates these concepts.

Example 1. Let $g = g_2$, where g_2 is shown in Figure 8. Assume $B_i^0(g) = \mathcal{G}\ \forall i \in N$, and agents have uniform priors over the networks in $\mathcal{G}$. Suppose $I_i(g) = \Big\{\{i\}, t_i(g), [\mathcal{F}_g(1), \mathcal{F}_g(2), \mathcal{F}_g(3)], n\Big\} = \Big\{\{i\}, k_i(g), [\frac{1}{2}, \frac{1}{3}, \frac{1}{6}], 6\Big\}\ \forall i \in N$. Figure 7 depicts $G_i(g) = \{1, 2, 3\}\ \forall i \in N$. Figure 8 represents a subset of feasible networks according to i's beliefs:[18] $\{g_1, g_4\} \subseteq B_i^1(g)$, $\{g_2, g_5\} \subseteq B_i^2(g)$ and $\{g_3, g_6\} \subseteq B_i^3(g)$. In this example, $O_i^1(g) = \{o_i(g_1), o_i(g_4)\}$, $O_i^2(g) = \{o_i(g_2), o_i(g_5)\}$ and $O_i^3(g) = \{o_i(g_3), o_i(g_6)\}$, and $n_i^z(g) = 3\ \forall z \in \{1, 2, 3\}$. ■

In Example 1, all agents have identical posteriors about the set of feasible geometries. This is a general property of agents' beliefs when the following conditions hold.

Claim 1. *If agents have identical priors about the feasible networks and their common knowledge includes the frequency distribution of types, then* $G_i(g) = G\ \forall i \in N$ *and* $n_i^z = n_i^y\ \forall z, y \in G$.

Note that, when the conditions in Claim 1 hold, the private information does not provide any

[16] Clearly, κ_i^z can be negative.
[17] Equivalently, $n_i^z(g)$ is the number of agents in a network with geometry z that occupy a feasible position of i.
[18] Note that $k_i(g) = 1$ in network g.

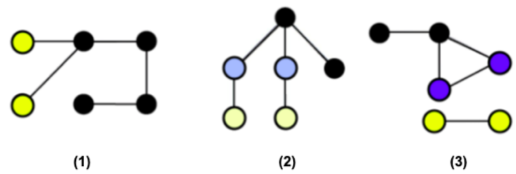

Figure 7: Feasible geometries in Example 1. Nodes in blue (yellow) are automorphically equivalent. Nodes in dark blue (yellow) are structurally equivalent.

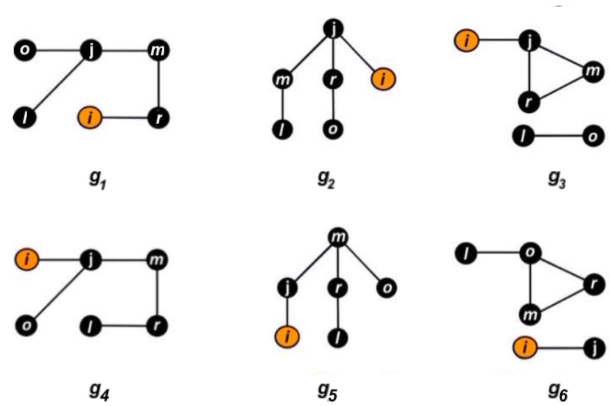

Figure 8: A subset of $B_i(g)$ in Example 1

information about the network structure beyond that conveyed by the common knowledge. Therefore, $G_i(g) = G \; \forall i \in N$. Example A in the Appendix shows that this symmetry in beliefs may not exist if the conditions in Claim 1 are not satisfied.

4.3.2 Probability distribution of feasible geometries

The posterior probability that i assigns to geometry $z \in G_i(g)$ is:

$$\rho_i^z(g) = \sum_{g_z \in B_i^z(g)} \mu_i(g_z) = b_i^z(g)\Big(\frac{1}{b_i(g)} + \kappa_i^z\Big). \tag{4}$$

Proposition 1 uncovers a negative relationship between the degree of symmetry of a geometry and the probability assigned to it according to agents' posterior beliefs.[19]

Proposition 1. *Every $i \in N$ believes that network g has geometry $z \in G_i(g)$ with probability:*

$$\rho_i^z = \frac{1}{1 + \sum\limits_{x \in G_i(g) \setminus \{z\}} \frac{n_i^x(g)|Aut(g_z)|}{n_i^z(g)|Aut(g_x)|}} + \underbrace{\frac{(n_{I_i} - 1)!(n - n_{I_i})! \; n_i^z(g)}{|Aut(g_z)|}}_{b_i^z(g)} \kappa_i^z \qquad \forall z \in G_i(g),$$

where $n_{I_i} \in \{1, ..., k_i(g) + 1\}$ is the number of nodes whose identity is known by i.

Proposition 1 states that the probability that i assigns to geometry z depends on three aspects:[20] (i) the number of nodes that represent a feasible position of i conditional on geometry z (i.e., $n_i^z(g)$),

[19] As mentioned earlier, the fact that $\mu_i^0(g_z) = \mu_i^0(g_y)$ holds whenever $|Aut(g_z)| = |Aut(g_y)|$ makes the expression in Proposition 1 simpler, since all networks with geometry z have the same probability according to i's beliefs. As explained in the proof of Proposition 1 (in the Appendix), this assumption does not affect the main insight of the proposition.

[20] ρ_i^z also depends on n_{I_i}, but this number does not vary with z.

(ii) z's degree of symmetry (captured by $|Aut(g_z)|$), and (iii) the probability premium that i assigns to z (which depends on her priors and θ). The larger $n_i^z(g)$ is, the larger the number of network locations that i can occupy in the network if it has geometry z, and the greater the likelihood that g has such a geometry. However, the larger $|Aut(g_z)|$ is, the smaller will be the array of feasible networks with geometry z, and the lower the likelihood that one of them is the network that i is part of. Recall that $|Aut(g_z)|$ captures the degree of the adjacency matrix's invariance under permutations of the network node labels. If $|Aut(g_z)|$ is large, the degree of invariance under permutations is high, and the number of different networks with geometry z that can be obtained by permuting the network node labels is small and vice versa.[21] Thus, if κ_i^z does not increase with $|Aut(g_z)|$ (which is the case if the prior probability assigned to networks with geometry z does not increase with $|Aut(g_z)|$ and θ does not reverse these priors), then ρ_i^z increases monotonically with the degree of asymmetry of z, *ceteris paribus*.

Note that since asymmetric networks are more numerous, agents can assign a larger probability to asymmetric geometries *even if the symmetric networks are more likely according to their priors.*[22] The inverse relationship between $|Aut(g_z)|$ and ρ_i^z is even more pronounced when the conditions in Claim 1 are satisfied, as the following corollary highlights.

Corollary 1. *If the conditions in Claim 1 are satisfied, then* $\rho_i^z = \rho^z = \frac{1}{1+\sum\limits_{x \in G \setminus \{z\}} \frac{|Aut(g_z)|}{|Aut(g_x)|}}$ $\forall z \in G$ *and* $\forall i \in N$.

Proposition 1 relates the degree of symmetry of a geometry to the probability that players assign to it. But, which network features determine a geometry's degree of symmetry? The following proposition relates the presence of automorphically equivalent and structurally equivalent agents in a network to the order of its automorphism group.

Proposition 2. *Let* $g = (N, E)$ *and* $g' = (N', E')$ *two networks such that* $N = N'$. *If* $|Orv_i(g)| \leq |Orv_i(g')|$ *and* $|E_i(g)| \leq E_i(g')|$ $\forall i \in N$ *and for some* $i \in N$ *either (i)* $|Orv_i(g)| < |Orv_i(g')|$ *or (ii)* $|E_i(g)| < E_i(g')|$ *or both, then* $|Aut(g)| < |Aut(g')|$.

Example 2. Consider the networks in Figure 8. Since $|Orv_i(g_2)| = |Orv_i(g_3)|$ $\forall i \in N$, $|E_r(g_2)| < |E_r(g_3)|$ for $r \in \{r, m, l, o\}$ and $|E_j(g_2)| = |E_j(g_3)|$ for $j \in \{j, i\}$, then $|Aut(g_3)| = 4 > |Aut(g_2)| =$

[21] Suppose for instance that g_z is a fully symmetric network, i.e., everyone is connected with everyone else. Since all nodes in the network occupy the same position, no permutation of the node labels modifies the network position of any agent. In that case, network g_z is the only one with geometry z. Imagine to the contrary that g_z is fully asymmetric. Given that all nodes occupy a distinct network position, each permutation of the node labels in g_z gives rise to a network that is distinct from g_z but has the same geometry as g_z.

[22] See footnote 14.

2[23]. Similarly, since $|Orv_i(g_3)| \geq |Orv_i(g_1)|$ and $|E_i(g_3)| \geq |E_i(g_1)|$ $\forall i \in N$ and $|E_m(g_3)| > |E_m(g_1)|$ for $m \in \{m, r\}$, then $|Aut(g_3)| = 4 > |Aut(g_1)| = 2$. ■

4.3.3 Inference about Network Features

Agents can infer the probability distribution of any network feature from their information. For example, they can learn the probability that their neighbors have certain types, even if they do not receive this information directly.

Define $\mathtt{t}_{\mathtt{N_i(g)}} = \big(t_1(g), ..., t_{k_i}(g)\big)$ as the vector of types of i's neighbors in network g, where $t_j(g)$ is the type of neighbor j ($j = 1, 2, ..., k_i(g)$). Let $\mathtt{t}_{\mathtt{k_i}} = (t_1, t_2, .., t_{k_i})$ be a feasible value of $\mathtt{t}_{\mathtt{N_i(g)}}$. Conditional on g having geometry z, the number of nodes representing a feasible position of i and having neighbors whose types are given by the vector $\mathtt{t}_{\mathtt{k_i}}$ is $n_i^z\big(g \,|\, \mathtt{t}_{\mathtt{k_i}}\big)$.[24] The following proposition defines the probability that i has particular types of neighbors according to her posteriors. Example D in the Appendix illustrates.

Proposition 3. *The probability that i has neighbors of types $\mathtt{t}_{\mathtt{k_i}}$ is:*

$$p_i\big[\mathtt{t}_{\mathtt{N_i(g)}} = \mathtt{t}_{\mathtt{k_i}}\big] = \sum_{z \in G_i(g)} \frac{n_i^z\big(g \,|\, \mathtt{t}_{\mathtt{k_i}}\big)}{n_i^z(g)} \rho_i^z.$$

Proposition 3 characterizes the probability that i's neighbors have types given by vector $\mathtt{t}_{\mathtt{k_i}}$. By an analogous reasoning, the probability that i has a neighbor with a type set T is:

$$p_i[j \in N_i(g) : t_j(g) \in T] = \sum_{z \in G_i(g)} \frac{n_i^z\big(g \,|\, T\big)}{n_i^z(g)} \rho_i^z, \tag{5}$$

where $n_i^z\big(g \,|\, T\big)$ is the number of agents in a network with geometry z that occupy a feasible position of i and have some neighbor with a type in T.

5 Network Games

Following Galeotti et al. (2010) and Feri and Pin (2020), posteriors are taken as primitive to analyze the symmetric Bayes-Nash equilibria of the games.

[23] See Figure 2 and Figure 3.

[24] Formally, if $g_z = (N^z, E^z)$ is a network with geometry z, then $n_i^z(g) = |\{i \in N^z : o_i(g_z) \in O_i^z(g) \wedge \mathtt{t}_{\mathtt{N_i(g_z)}} = \mathtt{t}_{\mathtt{k_i}}\}|$.

5.1 Equilibrium Behavior

Let X be the set of actions. A strategy σ is symmetric if and only if $\sigma(t_i(g)) = \sigma(t_j(g))\ \forall i, j : t_i(g) = t_j(g)$, where $\sigma(t_i(g)) \in X$ is i's action, as specified by σ. The profile of actions of i's neighbors in network g induced by σ is $\boldsymbol{\sigma}(\mathtt{t_{N_i(g)}}) = \big(\sigma(t_1(g)), \sigma(t_2(g)), ..., \sigma(t_{k_i}(g))\big)$. When $\mathtt{t_{N_i}(g)} = \mathtt{t_{k_i}}$, this profile is $\boldsymbol{\sigma}(\mathtt{t_{k_i}})$. To simplify notation, i's action is sometimes denoted by x_i, where $x_i = \sigma\big(t_i(g)\big)$. The utility of i when all agents follow the strategy σ is denoted by $u_i(x_i, \boldsymbol{\sigma}(\mathtt{t_{N_i}}))$.

Depending on how the types are defined, it is natural to specify a particular order relationship $\succ$ over the type space.[25] Regardless of how the order $\succeq$ is defined, a symmetric strategy σ is monotonically non-decreasing if and only if $\sigma\big(t_i(g)\big) \geq \sigma\big(t_j(g)\big)$ when $t_i(g) \succeq t_j(g)$. Likewise, σ is monotonically non-increasing if and only if $\sigma\big(t_i(g)\big) \leq \sigma\big(t_j(g)\big)$ when $t_i(g) \succeq t_j(g)$.

Definition. *A symmetric strategy σ constitutes a Bayes-Nash equilibrium if and only if, for all x_i':*

$$E_{U_i}\big(x_i, \sigma\big) = \sum_{\forall \mathtt{t_{k_i}}} u_i\big(x_i, \boldsymbol{\sigma}(\mathtt{t_{k_i}})\big) * \left(\sum_{z \in G_i(g)} \frac{n_i^z\big(g \mid \mathtt{t_{k_i}}\big)}{n_i^z(g)} \rho_i^z \right) \geq E_{U_i}\big(x_i', \sigma\big).$$

The above condition shows that, for a wide range of priors and information environments, the topology of networks that are more asymmetric, among those that are compatible with the players' information, has a greater impact on equilibrium actions. For illustration, suppose that network g is the one depicted in Figure 4(a). Imagine that, for all $i \in N$, $B_i^0(g) = \mathcal{G}$, $I_i(g)$ is (3) and i has uniform priors over the networks in $\mathcal{G}$. As introduced in Section 3, the two geometries in Figure 5 are the feasible geometries given $I_i(g)$; each i assigns probability $\rho_i^1 = \frac{6}{7}$ to geometry 1 and probability $\rho_i^2 = \frac{1}{7}$ to geometry 2. Suppose that all agents in g play the game of strategic substitutes presented in Section 3 (Game SS). In this case, an agent wants to play 0 if and only if (2) holds. If all agents follow σ_1 in Table 2:

$$E_{U_i}\big(0, \sigma_1\big) = u_i\big(0, \boldsymbol{\sigma_1}((2,2))\big) * \left(\frac{n_i^1\big(g \mid (2,2)\big)}{n_i^1(g)} \rho_i^1 \right) + u_i\big(0, \boldsymbol{\sigma_1}((1,1))\big) * \left(\frac{n_i^2\big(g \mid (1,1)\big)}{n_i^2(g)} \rho_i^2 \right) = \frac{1}{7},$$

since $k_i(g) = 3$. Thus, since it is unlikely according to i's beliefs that her friends have degree 1, i only free rides if the cost of playing 1 or the regret of not playing 0 is high ($\frac{1-c}{1+\mu} \leq \frac{1}{7}$). By contrast,

[25] For example, if a player's type is jointly defined by her degree and the average degree of her neighbors it can be reasonable to say that $t_i(g) \succeq t_j(g)$ if and only if $k_i(g) \geq k_j(g)$ and $\bar{k}_{N_i} \geq \bar{k}_{N_j}$ in some contexts. In others, however, it may be more reasonable to establish that $t_i(g) \succeq t_j(g)$ if and only if $k_i(g) \geq k_j(g)$ and $\bar{k}_{N_i} \leq \bar{k}_{N_j}$.

if types playing 1 are those with degree of two (i.e., the types of i's neighbors in any network with geometry 1), then i's expected utility from playing 0 is $\frac{6}{7}$, and i best responds with action 0 for a broader range of parameter values ($\frac{1-c}{1+\mu} \leq \frac{6}{7}$). As can be seen in Table 2, the unique symmetric equilibrium strategies are σ_3 and σ_4 for most parameter values, which are the symmetric equilibrium strategies under complete information in the most asymmetric feasible networks. Such a result is a direct consequence of the following remark:

Remark 3. *The degree of substitutability between players' actions and the actions of their feasible neighbors increases with the degree of asymmetry of the networks to which such neighbors belong, ceteris paribus.*

Let $T_x(\sigma)$ be the set of types for which σ specifies action $x \in \{0, 1\}$. In Game SS, the equilibrium condition is simple; it only depends on the probability that at least one of i's neighbors is of a type in set $T_1(\sigma)$. The same applies in the following game of strategic complements (referred to as Game SC), as Remark 4 highlights.

Game SC. Every i chooses an action in $X = \{0, 1\}$. An example might be purchasing a software package. An individual may only decide to purchase the software if a neighbor of hers is already using it. Thus:

$$u_i(x_i, \bar{x}_{N_i}) = \begin{cases} 1 - c & \text{if} \quad x_i = 1 \quad \text{and} \quad x_{N_i} \geq 1 \\ -c & \text{if} \quad x_i = 1 \quad \text{and} \quad x_{N_i} = 0 \\ -\mu(\bar{x}_{N_i}) & \text{if} \quad x_i = 0 \end{cases}$$

where $\mu(x_{N_i}) = 0$ if $x_{N_i} = 0$ and $\mu(x_{N_i}) = \mu \in [0, c)$ otherwise. In this case, μ represents the regret of players when they chose action 0 and observe that they could have obtained a larger payoff by playing 1. ■

Remark 4. *For Game SS (SC), σ constitutes a Bayes-Nash equilibrium if and only if, for each player i of a type in $T_0(\sigma)$ ($T_1(\sigma)$):*

$$\sum_{z \in G_i(g)} \frac{n_i^z\big(g \,|\, T_1(\sigma)\big)}{n_i^z(g)} \rho_i^z \geq h$$

.

where $h = \frac{1-c}{1+\mu}$ in Game SS and $h = \frac{c}{1+\mu}$ in Game SC, while the reverse inequality holds for each player with a type in $T_1(\sigma)$ ($T_0(\sigma)$).

Monotonicity of equilibria. The above examples show that the belief structure that emerges under incomplete information does not generally lead to an equilibrium in which players' actions vary monotonically by type, even if network information is very limited. Galeotti et al. (2010) identify conditions on players' posterior beliefs under which players' actions vary monotonically with their degrees. Proposition 4 complements their results by identifying conditions under which equilibrium actions depend monotonically on alternative features of players' network position, about which agents are informed once the network is realized. The condition makes it possible to determine computationally under which information structures an equilibrium is reached, and to design information scenarios consistent with the existence of monotone equilibria.[26].

Proposition 4. *If for every symmetric equilibrium strategy σ and for all $x' \geq x$:*

$$\sum_{\forall \mathtt{t_{k_i}}} \Big[u_i\big(x, \boldsymbol{\sigma}(\mathtt{t_{k_i}})\big) - u_i\big(x', \boldsymbol{\sigma}(\mathtt{t_{k_i}})\big] \left(\sum_{z \in G_i(g)} \frac{n_i^z(g \mid \mathtt{t_{k_i}})}{n_i^z(g)} \rho_i^z \right)$$

is non-decreasing (non-increasing) in i's type for all $i \in N$, then every symmetric equilibrium strategy is monotonically non-increasing (non-decreasing).

Consider, for instance, Game SS (Game SC). The strategy σ is monotonically non-increasing if $\forall i, j:\ t_i(g) \succeq t_j(g)$:

$$E_{U_i}(0, \sigma) - E_{U_i}(1, \sigma) \geq E_{U_j}(0, \sigma) - E_{U_j}(1, \sigma),$$

and monotonically non-decreasing if the reverse inequality holds. In terms of our framework, this holds if the following conditions are satisfied.

Corollary 2. *Assume agents in g play Game SS (Game SC). The strategy σ is monotonically non-increasing (non-decreasing) if $\forall i, j : t_i(g) \succeq t_j(g)$:*

$$\sum_{z \in G_i(g)} \frac{n_i^z\big(g \,|\, T_1(\sigma)\big)}{n_i^z(g)} \rho_i^z \geq \sum_{z \in G_j(g)} \frac{n_j^z\big(g \,|\, T_1(\sigma)\big)}{n_j^z(g)} \rho_j^z, \tag{7}$$

and monotonically non-decreasing (non-increasing) if the reverse inequality holds.

Corollary 2 shows that the existence of a monotone non-decreasing (non-increasing) equilibrium depends solely on the distribution of types in feasible networks and on the probabilities of the feasible geometries. Equilibrium actions in games with strategic substitutes (strategic complements)

[26]See footnote 4

can increase or decrease with the types of the players depending on their network beliefs. This constitutes an important difference from Galeotti et al. (2010), where the existence of non-increasing (non-decreasing) equilibria is related to whether the actions of players reinforce or offset each other.

5.2 Equilibrium Welfare

In my framework, the set of symmetric equilibria does not (only) depend on the geometry of g, but on the geometry of all the feasible geometries. Given a belief structure, the set of symmetric equilibria is the same in all networks that have a feasible network geometry. However, the payoffs in these networks may differ if their geometries are different. Consider, for example, two networks g_1 and g_2 with geometries 1 and 2, respectively, as shown in Figure 5. The four strategies in Table 2 are equilibrium strategies when, for all $i \in N$, $B_i(g) = \mathcal{G}$, $I_i(g)$ is (3) and agents have uniform priors over the networks. However, when all agents play σ_3, the sum of players' payoffs is $7 - 3c$ in g_1, while it is $4 - 3c$ in g_2. Similarly, conditional on σ_4, the total welfare is $7 - 4c$ in g_1 and $4 - 4c$ in g_2. Since geometry 1 is more likely in players' beliefs, equilibrium welfare is higher in networks with geometry 1 for most parameter values.

The following proposition reveals a relationship between the degree of asymmetry of a network and the welfare obtained by its members in equilibrium. Let $W_{t_i}(\sigma, g_z)$ be the average welfare of players of type t_i in g_z conditional on σ. Assume that the set of feasible geometries is $G_i(g) = G$ $\forall i \in N$. If conditional on g having a more asymmetric geometry in G the average welfare of each type in $T_x(\sigma)$ is at least as high as that of each type in $T_x(\sigma')$, then σ is an equilibrium strategy for at least the same parameter values as σ'.

Proposition 5. *Suppose that agents are playing Game SS or Game SC and the conditions in Claim 1 hold. Let σ and σ' be two symmetric equilibrium strategies. If $\forall i, j \in N : \sigma(t_i(g)) = \sigma'(t_j(g))$*

$$\sum_{z \in G(g):|Aut(g_z)| \leq q} W_{t_i}(\sigma, g_z) \geq \sum_{z \in G(g):|Aut(g_z)| \leq q} W_{t_j}(\sigma', g_z) \quad \forall q, \tag{8}$$

then σ is an equilibrium for at least the same range of parameter values as σ'. The strategy σ is an equilibrium for a broader range of parameter values than σ' if, additionally, (8) holds with strict inequality for some $i, j \in N : \sigma(t_i(g)) = \sigma'\big(t_j(g)\big)$ and some q.

Finally, the following proposition provides a sufficient condition for a network to be *efficient*. Network g is efficient if the aggregate welfare of its members in equilibrium is at least as high as that in any other network with a feasible geometry.

Proposition 6. *Suppose the conditions in Claim 1 hold, network g has geometry* $z \in G(g)$, *and* $\mu = 0$. *Then, network g is efficient if:*

1. $c < \frac{\rho^z}{n_i(g)}$ *in Game SS, or*

2. $1 - c < \frac{\rho^z}{n_i(g)}$ *in Game SC,*

where $\frac{\rho^z}{n_i(g)} \leq \frac{\rho^z}{n_j(g)}$ $\forall j \in N$.

6 Concluding Remarks

Empirical research demonstrates the importance of social networks in explaining behavior in strategic contexts. However, the analysis of network games raises a fundamental problem: even if one focuses on a particular network, multiple equilibria are possible, making it difficult to draw general conclusions about the effects of network structure on behavior. To the best of my knowledge, this paper is the first to apply group theory to the study of network games. The main contribution of this work is to show the potential of this approach for analyzing equilibrium behavior and mitigating the equilibrium selection problem.

The paper shows that, given a large set of priors and information structures, individuals assign a greater probability to the feasible geometries that are more asymmetric. This finding provides a microfoundation for belief selection in the analysis of network games under incomplete information, and can help to identify perception biases in the empirical study of an individual's social perception. Given the paucity of economic research on network perception, the results will hopefully stimulate further research on this subject.

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Appendix

A. Definitions

I first provide some definitions that are necessary for the proofs in Section C.

Stabilizer of a node. The stabilizer of a set of nodes $\bar{N}$ in g is the set of all automorphisms that map each node in $\bar{N}$ into itself, $Stab_{\bar{N}}(g) = \{f \in Aut(g) : f(i) = i \ \forall i \in Stab_{\bar{N}}(g)\}$. In network g_2 in Figure 2, $Aut(g_2) = Stab_i(g_2) = \{f, f'\}$, resulting in g_2 and g_2' in Figure 2, respectively. In network g_3 in Figure 3, $Aut(g_3) = Stab_i(g_3) = \{f, f', f'', f'''\}$, resulting in g_3, $g_3' = g_3$, $g_3'' = g_3$ and $g_3''' = g_3$ in the same figure, respectively.

The Orbit-Stabilizer Theorem. Let $Stab_i(g) = \{f \in Aut(g) : f(i) = i\}$ be the stabilizer of node $i \in N$. Then,

$$|Aut(g)| = |Orv_i(g)| * |Stab_i(g)| .$$

B. Isomorphisms of a Graph

Two networks are isomorphic if and only if they have the same geometry. Let $\bar{N}$ be a subset of nodes $\bar{N} \subseteq N$, with $\bar{n} = |\bar{N}|$. Lemma A calculates the number of distinct labelings of the nodes in $N \setminus \bar{N}$. That is, the number of distinct networks with the same geometry as g that can be obtained by permuting exclusively the labels of the nodes in $N \setminus \bar{N}$. This number is denoted $y(g \mid \bar{N})$

Labels of the nodes in $\bar{N}$ are not permuted, they are maintained fixed. Notice that in some cases we may permute the labels of some nodes in $N \setminus \bar{N}$ without any incidence in the adjacency matrix of the network. In other words, we may permute the labels of some nodes in $N \setminus \bar{N}$ and get a network $g' = g$. The set of different ways in which we can (exclusively) permute the labels of the nodes in $N \setminus \bar{N}$ without affecting the adjacency matrix of g is given by the stabilizer of $\bar{N}$, $Stab_{\bar{N}}(g)$.

Lemma A. *Let $g = (N, E)$. The total number of distinct isomorphic networks to g that can be obtained by exclusively permuting the labels of the nodes in $N \setminus \bar{N}$ is:*

$$y(g \mid \bar{N}) = \frac{(n - \bar{n})!}{|Stab_{\bar{N}}(g)|}.$$

Proof. There are $(n-\bar{n})!$ possible permutations of the labels of the nodes in $N \setminus \bar{N}$. For each of these $(n-\bar{n})!$ possible permutations, $|Stab_{\bar{N}}(g)| - 1$ of the others are identical: they produce the same network (network g). This is because for each of these permutations there are $|Stab_{\bar{N}}(g)|$ different ways in which we can permute the labels of the nodes in $N \setminus \bar{N}$ without altering the adjacency matrix of the network. Thereby, $y(g \mid \bar{N}) = \frac{(n-\bar{n})!}{|Stab_{\bar{N}}(g)|}$. Example B (in Section D of this Appendix) illustrates Lemma A. ■

C. Proofs

The proofs make use of the concepts defined in Section A in this Appendix. Lemma B calculates the total number of feasible networks according to i's posterior beliefs.

Lemma B. *Let* $g_z \in B_i^z(g)$ *and* n_{I_i} *the number of nodes whose identity is known by* i. *According to* i*'s posterior beliefs:*

$$b_i^z(g) = \frac{(n_{I_i}-1)!(n-n_{I_i})!\; n_i^z(g)}{|Aut(g_z)|} \quad \text{and} \quad b_i(g) = \sum_{z \in G_i(g)} b_i^z(g)$$

Proof. (i) Assume first that i has no information about the identity of her neighbors ($n_{I_i} = 1$). If the set of feasible positions of i conditional on geometry z is $O_i^z(g) = \big\{o_i(g_z), o_i(g_s), \ldots, o_i(g_l)\big\}$ (where $\{g_z, g_s, ..., g_l\} \subseteq B_i^z(g)$), then there exist at least one feasible network in which i occupies the position $o_i(g_z)$, at least one feasible network in which i occupies the position $o_i(g_s)$, and similarly for other positions in $O_i^z(g)$. By Lemma A, there are $y\big(g_z \mid \{i\}\big) = \frac{(n-1)!}{\big|Stab_i(g_z)\big|}$ networks in $B_i^z(g)$ in which i occupies the position $o_i(g_z)$; all these networks differ in how agents different from i are allocated. Similarly, there are $y\big(g_s \mid \{i\}\big) = \frac{(n-1)!}{\big|Stab_i(g_s)\big|}$ networks in $B_i^z(g)$ where i occupies the position $o_i(g_s)$, and analogously for other positions in $O_i^z(g)$. Hence, if $O_i^z(g) = \{o_i(g_z), o_i(g_s), ..., o_i(g_l)\}$:

$$b_i^z(g) = y\big(g_z \mid \{i\}\big) + y\big(g_s \mid \{i\}\big) + ... + y\big(g_l \mid \{i\}\big) = \frac{(n-1)!}{\big|Stab_i(g_z)\big|} + \frac{(n-1)!}{\big|Stab_i(g_s)\big|} + ... + \frac{(n-1)!}{\big|Stab_i(g_l)\big|}$$

$$= \frac{(n-1)!\big|Orv_i(g_z)\big|}{\big|Aut(g_z)\big|} + \frac{(n-1)!\big|Orv_i(g_s)\big|}{\big|Aut(g_s)\big|} + ... + \frac{(n-1)!\big|Orv_i(g_l)\big|}{\big|Aut(g_l)\big|} = \frac{(n-1)!\; n_i^z(g)}{|Aut(g_z)|}.$$

where the penultimate equality holds applying the Orbit-Stabilizer Theorem (in Section A of this Appendix) and noticing that $|Aut(g_z)| = |Aut(g_s)| = ... = |Aut(g_l)|$ and $\big|Orv_i(g_z)\big| + \big|Orv_i(g_s)\big| + ... + \big|Orv_i(g_l)\big| = n_i^z(g)$.

(ii) When i knows her identity but not that of her neighbors is $b_i^z(g) = \frac{(n-1)!\; n_i^z(g)}{|Aut(g_z)|}$. Note that, in this case, there are $\binom{n-1}{n_{I_i}-1}$ feasible identities for i's neighbors. If i knows $N_i(g)$, there is only one feasible identity for i's neighbors. Hence, if $N_i(g) \in I_i(g)$:

$$b_i^z(g) = \frac{1}{\binom{n-1}{n_{I_i}-1}} \frac{(n-1)!\, n_i^z(g)}{|Aut(g_z)|} = \frac{(n_{I_i}-1)!(n-n_{I_i})!\, n_i^z(g)}{|Aut(g_z)|}.$$

■

Proof of Proposition 1.

(a) Suppose that $\kappa_i^z = 0$. Considering Lemma B (above) and operating:

$$\rho_i^z = \frac{b_i^z(g)}{b_i(g)} = \frac{1}{1 + \sum\limits_{x \in G_i(g) \setminus \{z\}} \frac{n_i^x(g)|Aut(g_z)|}{n_i^z(g)|Aut(g_x)|}}. \tag{9}$$

(b) If $\kappa_i^z \neq 0$:

$$\rho_i^z = \frac{1}{1 + \sum\limits_{x \in G_i(g) \setminus \{z\}} \frac{n_i^x(g)|Aut(g_z)|}{n_i^z(g)|Aut(g_x)|}} + b_i^z(g)\kappa_z,$$

where $b_i^z(g)$ is calculated in Lemma B. Example C (in this Appendix) illustrates.

(c) In my analysis, I assume that $\mu_i^0(g_z) = \mu_i^0(g_y)$ whenever $|Aut(g_z)| = |Aut(g_y)|$. If this assumption is relaxed, the posterior probability assigned to g_z by i would be:

$$\mu_i(g_z) = \frac{1}{b_i(g)}\big(1 + \kappa_i(g_z)\big),$$

where $\kappa_i(g_z)$ is the probability premium that i assigns to g_z based on θ and her priors. Under this scenario:

$$\rho_i^z = \frac{b_i^z(g)}{b_i(g)} + \sum_{g_z \in B_i^z(g)} \kappa_i(g_z).$$

Note that the main insight of Proposition 1 maintains: if the prior probability assigned to the networks is non-decreasing in their degree of asymmetry and θ does not contradict these priors, then ρ_i^z is decreasing in $|Aut(g_z)|$. Example C (in Section D of this Appendix) illustrates Proposition 1.

■

Proof of Proposition 2. Since $|Orv_i(g)| \leq |Orv_i(g')| \, \forall i \in N$, for each automorphism $f : N \to N$ exists there is an identical automorphism $f : N' \to N'$. If $\exists i \in N : \big|Orv_i(g)\big| < \big|Orv_i(g')\big|$, then there exist at least one automorphism in g' that does not exist in g. Thereby, $\big|Aut(g)\big| < \big|Aut(g')\big|$.

I now prove that if $\left|Orv_i(g)\right| = \left|Orv_i(g')\right| \forall i \in N$, and $\exists m \in N' : \left|E_m(g)\right| < \left|E_m(g')\right|$, then $\left|Aut(g)\right| < \left|Aut(g')\right|$. Suppose $\exists m \in N' : |E_m(g')| > |E_m(g)|$. For each $r \in E_m(g') \setminus E_m(g)$, there exists an automorphism $f : N' \to N'$ such that:

$$f(w) = \begin{cases} r & if\ w = m \\ m & if\ w = r \\ w & \text{otherwise} \end{cases}$$

On the contrary, there does not exist such an automorphism between m and r in network g, since $r \notin E_m(g)$. As a result, $\left|Aut(g)\right| < \left|Aut(g')\right|$.

■

Proof of Proposition 3.

(i) Assume first that i does not know the identity of her neighbors ($n_{I_i} = 1$). Define $O_i^z(g \mid \mathtt{t_{k_i}})$ as the set of i's feasible positions such that, if i has occupies any of these positions, then: (i) i is part of a network with geometry z, and (ii) i she has neighbors with types given by the vector $\mathtt{t_{k_i}}$. That is:

$$O_i^z(g \mid \mathtt{t_{k_i}}) = \{o_i(g_z) \in O_i^z(g) : \mathtt{t_{N_i(g_z)}} = \mathtt{t_{k_i}}\}.$$

The probability that $i \in N$ has neighbors with types $\mathtt{t_{k_i}}$ is the probability that i occupies a position in $O_i^z(g \mid \mathtt{t_{k_i}}) = \big\{o_i(g_z), o_i(g_y), ..., o_i(g_r)\big\}$. According to Lemma A (in this Appendix), there exist $y\big(g_z \mid \{i\}\big)= \frac{(n-1)!}{\left|Stab_i(g_z)\right|}$ feasible networks according to i's beliefs in which i occupies the position $o_i(g_z)$. Analogously, there are $y\big(g_y \mid \{i\}\big) = \frac{(n-1)!}{\left|Stab_i(g_y)\right|}$ feasible networks in which i occupies the position $o_i(g_y)$, and similarly for other positions in $O_i^z(g \mid \mathtt{t_{k_i}})$. Then, conditional on network g having geometry z, the probability that i has neighbors with types $\mathtt{t_{k_i}}$ is:

$$\begin{aligned} p\Big[o_i(g) \in O_i^z(g \mid \mathtt{t_{k_i}}) \mid I_i(g)\Big] &= \Big[y(g_z \mid \{i\}) + y(g_y \mid \{i\}) + ... + y(g_r \mid \{i\})\Big]\Big(\frac{1}{b_i(g)} + \kappa_z\Big) \\ &= \left[\frac{(n-1)!}{|Stab_i(g_z)|} + \frac{(n-1)!}{|Stab_i(g_y)|} + ... + \frac{(n-1)!}{|Stab_i(g_r)|}\right]\Big(\frac{1}{b_i(g)} + \kappa_z\Big). \end{aligned} \tag{10}$$

Applying the Orbit-Stabilizer Theorem (in Section A) and noticing that $|Aut(g_z)| = |Aut(g_y)| = ... = |Aut(g_r)|$

$$p\Big[o_i(g) \in O_i^z(g \mid \mathtt{t_{k_i}}) \mid I_i(g)\Big]$$
$$= \left[\frac{(n-1)!|Orv_i(g_z)|}{|Aut(g_z)|} + \frac{(n-1)!|Orv_i(g_y)|}{|Aut(g_y)|} + ... + \frac{(n-1)!|Orv_i(g_r)|}{|Aut(g_r)|}\right]\Big(\frac{1}{b_i(g)} + \kappa_z\Big) \quad (11)$$
$$= \left[\frac{(n-1)!\, n_i^z(g \mid \mathtt{t_{k_i}})}{|Aut(g_z)|}\right]\Big(\frac{1}{b_i(g)} + \kappa_z\Big).$$

By Lemma B, $b_i(g) = \sum\limits_{z \in G_i(g)} b_i^z(g) = \sum\limits_{z \in G_i(g)} \frac{(n-1)!\, n_i^z(g)}{|Aut(g_z)|}$. Substituting $b_i(g)$ into (11) and operating:

$$p\Big[o_i(g) \in O_i^z(g \mid \mathtt{t_{k_i}}) \mid I_i(g)\Big] = \frac{n_i^z(g \mid \mathtt{t_{k_i}})}{n_i^z(g)}\left[\frac{1}{1 + \sum\limits_{x \in G_i(g) \setminus \{z\}} \frac{n_i^x(g)|Aut(g_z)|}{n_i^z(g)|Aut(g_x)|}} + \frac{(n-1)!\, n_i^z(g)}{|Aut(g_z)|}\kappa_z\right]. \quad (12)$$

Then, the probability that i has neighbors with types $\mathtt{t_{k_i}}$ given $I_i(g)$ is:

$$p\big[\mathtt{t_{N_i(g)}} = \mathtt{t_{k_i}} \mid I_i(g)\big] = \sum_{z \in G_i(g)} \frac{n_i^z(g \mid \mathtt{t_{k_i}})}{n_i^z(g)}\rho_i^z.$$

(i) If i has information about the identity of her neighbors (i.e., $n_{I_i} > 1$), the equivalent expression to (12) is:

$$p\Big[o_i(g) \in O_i^z(g \mid \mathtt{t_{k_i}}) \mid I_i(g)\Big] = \frac{n_i^z(g \mid \mathtt{t_{k_i}})}{n_i^z(g)}\left[\frac{1}{1 + \sum\limits_{x \in G_i(g) \setminus \{z\}} \frac{n_i^x(g)|Aut(g_z)|}{n_i^z(g)|Aut(g_x)|}} + \frac{(n_{I_i}-1)!(n-n_{I_i})!\, n_i^z(g)}{|Aut(g_z)|}\kappa_z\right]$$

and the result follows.

■

Proposition 5.

(i) *Game SS*. Recall that $T_1(\sigma)$ is the set of types that play 1 according to σ. Conditional on g having geometry z, the proportion of type t_i agents occupying a feasible position of i and having a neighbor with a type in $T_1(\sigma)$ is: $\frac{n_i^z(g \mid T_1(\sigma))}{n_i^z(g)}$. Observe that:

$$W_{t_i}(\sigma, g_z) = \begin{cases} \frac{n_i^z(g \mid T_1(\sigma))}{n_i^z(g)} & \text{if} \quad \sigma(t_i(g)) = 0 \\ (1-c) - \frac{n_i^z(g \mid T_1(\sigma))}{n_i^z(g)}\mu & \text{if} \quad \sigma(t_i(g)) = 1. \end{cases}$$

(a) Suppose $\sigma(t_i(g)) = 0$. The expected utility of i of playing $\sigma(t_i(g)) = 0$ conditional on σ is:

$$E_{U_i}(0, \sigma) = \sum_{z \in G} \frac{n_i^z(g \mid T_1(\sigma))}{n_i^z(g)}\rho^z = \sum_{z \in G} W_{t_i}(\sigma, g_z)\ \rho_z.$$

Action $\sigma(t_i(g)) = 0$ is a best response if and only if (2) holds. That is, iff.:

$$\frac{1-c}{1+\mu} \leq t = \sum_{z \in G} W_{t_i}(\sigma, g_z)\ \rho^z. \tag{13}$$

Likewise, $\sigma'(t_j(g)) = 0$ is a best response iff.:

$$\frac{1-c}{1+\mu} \leq t' = \sum_{z \in G} W_{t_j}(\sigma', g_z)\ \rho^z. \tag{14}$$

Recall that ρ^z increases the degree of asymmetry of z (see Corollary 1). Therefore, if

$$\sum_{z \in G:|Aut(g_z)| \leq q} W_{t_i}(\sigma, g_z) \geq \sum_{z \in G:|Aut(g_z)| \leq q} W_{t_j}(\sigma', g_z) \quad \forall q \tag{15}$$

is satisfied, then $t \geq t'$, since the weighted sum in the right size of (13) is greater than that in (14). As a result, $\sigma(t_i(g)) = 0$ is a best response for at least the same range of parameter values as $\sigma'(t_j(g)) = 0$.

(b) Suppose now that $\sigma(t_i(g)) = 1$ and $\sigma'(t_j(g)) = 1$. In this case, $\sigma(t_i(g)) = 1$ is a best response iff.:

$$\frac{1-c}{1+\mu} \geq t = \sum_{z \in G} \frac{n_i^z(g \mid T_1(\sigma))}{n_i^z(g)}\ \rho^z, \tag{16}$$

while $\sigma'(t_j(g)) = 1$ is a best response iff.:

$$\frac{1-c}{1+\mu} \geq t' = \sum_{z \in G} \frac{n_j^z(g \mid T_1(\sigma'))}{n_j^z(g)}\ \rho^z. \tag{17}$$

Note that, if $\sigma(t_i(g)) = \sigma'(t_j(g)) = 1$, then (15) is satisfied if and only if:

$$\sum_{z \in G:|Aut(g_z)| \leq q} \frac{n_i^z(g \mid T_1(\sigma))}{n_i^z(g)} \leq \sum_{z \in G:|Aut(g_z)| \leq q} \frac{n_j^z(g \mid T_1(\sigma'))}{n_j^z(g)} \quad \forall q. \tag{18}$$

If (18) hold, then $t \leq t'$, and $\sigma(t_i(g)) = 1$ is an equilibrium strategy for at least the same range of parameter values as $\sigma'(t_j(g)) = 1$.

(ii) As for Game SC,

$$W_{t_i}(\sigma, g_z) = \begin{cases} \frac{n_i^z(g \mid T_1(\sigma))}{n_i^z(g)} - c & \text{if} \quad \sigma(t_i(g)) = 1 \\ -\frac{n_i^z(g \mid T_1(\sigma))}{n_i^z(g)}\mu & \text{if} \quad \sigma(t_i(g)) = 0. \end{cases}$$

Following a similar reasoning as for Game SS, the result follows. ■

Proof of Proposition 6. Suppose g is a network with geometry z, $g = g_z$. If the conditions of Claim 1 are satisfied, then $n_i^z(g) = n_i(g)\ \forall z \in G$.

i) *Game SS*. Suppose that the set of feasible geometries is $G = \{1, 2, ..., h\}$ and $\sigma\big(t_i(g)\big) = 0$. Let $\bar{W}_{t_i}(g_z, \sigma)$ be the aggregate welfare of type t_i players in g_z, (i.e., the number of type t_i agents in g_z with a neighbor playing 1). Recall that $W_{t_i}(g_z, \sigma) = \frac{\bar{W}_{t_i}(g_z,\sigma)}{n_i(g)}$ is the average welfare of type t_i players in g_z. Imagine that $\bar{W}_{t_i}(g_y, \sigma) > \bar{W}_{t_i}(g_z, \sigma)$, $y \in G$. That is, $\bar{W}_{t_i}(g_z, \sigma) = \bar{W}_{t_i}(g_y, \sigma) - \epsilon$, $\epsilon \geq 1$. Playing $\sigma\big(t_i(g)\big) = 0$ is a best response for i if and only if (2) holds. That is, if the probability that i has at least one neighbor playing 1 is greater than $1 - c$. Formally, if:

$$\begin{aligned} E_{U_i}\big(0, \sigma\big) &= \sum_{z\in G} \frac{n_i^z(g \mid T_1(\sigma))}{n_i^z(g)} \rho^z = \sum_{z\in G} \frac{n_i^z(g \mid T_1(\sigma))}{n_i(g)} \rho^z \\ &= W_{t_i}(\sigma, g_z)\ \rho^z + W_{t_i}(\sigma, g_y)\ \rho^y + \sum_{s\in G\setminus\{z,y\}} W_{t_i}(\sigma, g_s)\ \rho^s \\ &= \Big(\frac{\bar{W}_{t_i}(\sigma, g_y) - \epsilon}{n_i(g)}\Big)\ \rho^z + W_{t_i}(\sigma, g_y)\ \rho^y + \sum_{s\in G\setminus\{z,y\}} W_{t_i}(\sigma, g_s)\ \rho^s \geq 1 - c. \end{aligned}$$

Equivalently, $\sigma(t_i(g)) = 0$ is an equilibrium strategy iff.:

$$c \geq 1 - \Big[W_{t_i}(\sigma, g_y)(\rho^z + \rho^y) + \sum_{s\in G\setminus\{z,y\}} W_{t_i}(\sigma, g_s)\ \rho^s\Big] + \frac{\rho^z}{n_i(g)}\ \epsilon. \tag{19}$$

Since $1 - \Big[W_{t_i}(\sigma, g_y)(\rho^z + \rho^y) + \sum_{s\in G\setminus\{z,y\}} W_{t_i}(\sigma, g_s)\ \rho^s\Big]$ is always positive and $\epsilon \geq 1$, (19) is not satisfied if $\frac{\rho^z}{n_i(g)} > c$. Then, $\sigma(t_i(g)) = 0$ is not a best response if $\frac{\rho^z}{n_i(g)} > c$. Given that $\frac{\rho^z}{n_i(g)} \leq \frac{\rho^z}{n_j(g)}$ $\forall j \in N$, then (19) is not satisfied for any $i \in N$ when $\frac{\rho^z}{n_i(g)} > c$. Then, there is not a symmetric equilibrium strategy σ: $\sigma\big(t_i(g)\big)= 0$ and $\bar{W}_{t_i}(g_y, \sigma) > \bar{W}_{t_i}(g_z, \sigma)$ for any $y \in G$.

When $\sigma\big(t_i(g)\big)= 1$, $\bar{W}_{t_i}(g_z, \sigma) = n_i(g)(1 - c) = \bar{W}_{t_i}(g_y, \sigma)\ \forall g_z, g_y : z, y \in G$. Then, there is not an equilibrium strategy σ such that $\sigma\big(t_i(g)\big)= 1$ and $\bar{W}_{t_i}(g_y, \sigma) > \bar{W}_{t_i}(g_z, \sigma)$. As a result, $\bar{W}(\sigma, g_z) = \sum_{t_i} \bar{W}_{t_i}(\sigma, g_z) \geq \bar{W}(\sigma, g_y) = \sum_{t_i} \bar{W}_{t_i}(\sigma, g_y)\ \forall y \in G$ whenever $\frac{\rho^z}{n_i(g)} \geq c\ \forall j \in N$.

ii) *Game SC*. Reasoning is analogous under strategic complements. Suppose $g = g_z$, $\sigma\big(t_i(g)\big) = 1$ and $\exists g_y : W_{t_i}(\sigma, g_y) > W_{t_i}(\sigma, g_z)$, $y \in G$. In equilibrium, i plays 1 if:

$$\begin{aligned} E_{U_i}\big(1, \sigma\big) =& W_{t_i}(\sigma, g_z)\ \rho^z + W_{t_i}(\sigma, g_y)\ \rho^y + \sum_{s\in G\setminus\{z,y\}} W_{t_i}(\sigma, g_s)\ \rho^s \\ &= \Big(\frac{\bar{W}_{t_i}(\sigma, g_y) - \epsilon}{n_i(g)}\Big)\ \rho^z + W_{t_i}(\sigma, g_y)\ \rho^y + \sum_{s\in G\setminus\{z,y\}} W_{t_i}(\sigma, g_s)\ \rho^s \geq c. \end{aligned} \tag{20}$$

That is, if:

$$1 - c \geq 1 - \Big[W_{t_i}(\sigma, g_y)(\rho^z + \rho^y) + \sum_{s \in G \setminus \{z,y\}} W_{t_i}(\sigma, g_s)\ \rho^s\Big] + \frac{\rho^z}{n_i(g)}\ \epsilon \tag{21}$$

If $\frac{\rho^z}{n_i(g)} > 1-c$, (21) does not hold and $\sigma\big(t_i(g)\big) = 1$ is not a best response for i. Since $\frac{\rho^z}{n_i(g)} \leq \frac{\rho^z}{n_j(g)}$ $\forall j \in N$, (21) is not satisfied for any $j \in N$ whenever $\frac{\rho^z}{n_i(g)} > 1-c$. Then, applying the same reasoning as for Game SS, the results follows. ■

D. Examples

Example A (*asymmetric beliefs about the geometry*). Suppose that g is g_3 in Figure 8, $B_i^0(g) = \mathcal{G}$ $\forall i \in N$ and agents have uniform priors over the networks. Let $I_i(g) = \Big\{\{i\}, t_i(g), [\mathcal{F}_g(1), \mathcal{F}_g(2), \mathcal{F}_g(3)], n\Big\}$ $= \Big\{\{i\}, \Big(k_i(g), \bar{k}_{N_i}\Big), [\frac{1}{2}, \frac{1}{3}, \frac{1}{6}], 6\Big\}\Big\}$ $\forall i \in N$, where $\bar{k}_{N_i} = \frac{1}{k_i(g)} \sum_{j \in N_i(g)} k_j(g)$. From $I_m(g)$, individual m knows that $\bar{k}_{N_m} = 2.5$. Hence, $G_m(g) = \{1, 3\}$, as depicted in Figure 7. Conditional on $I_l(g)$, the only feasible geometry is, in contrast, geometry 3 in Figure 7, since it is the only one with the degree distribution of g in which $\bar{k}_{N_l} = 1$. By similar arguments, $G_j(g) = \{2, 3\}$. ■

Example B (*Lemma A*). Assume $\bar{N} = \{i\}$.

(a) Consider network g_4 in Figure 8. Since there is no pair of automorphically equivalent nodes in $N \setminus \{i\}$, each permutation of the labels of the nodes in $N \setminus \{i\}$ gives rise to a different network. Therefore, $Stab_i(g_4) = \{f\}$, where $f(i) = i$ $\forall i$, and $|Stab_i(g_4)| = 1$. Hence, $y(g_4 \mid \{i\}) = \frac{(n-1)!}{1} = 120$. Similarly, $y(g_5 \mid \{i\}) = 120$.

(b) Consider now g_2 in Figure 8. Since $m \equiv r$ and $l \equiv o$, $|Stab_i(g_2)| = 2$, as shown in Figure 2. Therefore, $y(g_2 \mid \{i\}) = \frac{(n-1)!}{2} = 60$. Analogously, $y(g_1 \mid \{i\}) = y(g_6 \mid \{i\}) = 60$.

(c) As for network g_3 in Figure 8, $Stab_i(g_3) = \{f, f', f'', f'''\}$, resulting in g_3, g_3', g_3'' and g_3''' in Figure 3, respectively. Since $|Stab_i(g_3)| = 4$, $y(g_3 \mid \{i\}) = \frac{n-1!}{2} = 30$. ■

Example C (*Proposition 1*) Consider the information structure in Example 1, with $k_i(g) = 1$. By Lemma A (in Section A of this Appendix), there are $y\big(g_1 \mid \{i\}\big)$ feasible networks according to i's beliefs in which i occupies position $o_i(g_1)$. Thus:

$$b_i^1(g) = y\big(g_1 \mid \{i\}\big) + y\big(g_4 \mid \{i\}\big) = 60 + 120 = \frac{(n-1)!\; n_i^1(g)}{|Aut(g_1)|} = \frac{5!\;3}{2} = 180.$$

$$b_i^2(g) = y\big(g_2 \mid \{i\}\big) + y\big(g_5 \mid \{i\}\big) = 60 + 120 = \frac{(n-1)!\; n_i^2(g)}{|Aut(g_2)|} = \frac{5!\;3}{2} = 180.$$

$$b_i^3(g) = y\big(g_3 \mid \{i\}\big) + y\big(g_6 \mid \{i\}\big) = 30 + 60 = \frac{(n-1)!\; n_i^3(g)}{|Aut(g_3)|} = \frac{5!\;3}{4} = 90.$$

Since $|Aut(g_1)| = |Aut(g_2)| = 2$ and $|Aut(g_3)| = 4$, $\rho_i^1 = \frac{1}{1+\frac{|Aut(g_1)|}{|Aut(g_2)|}+\frac{|Aut(g_1)|}{|Aut(g_3)|}} = \frac{b_i^1(g)}{b_i(g)} = \frac{180}{180+180+90} = 0.4 = \rho_i^2$ and $\rho_i^3 = 0.2$. ■

Example D (*Proposition 3*). Consider the belief structure in Example 1, where $G_i(g) = \{1, 2, 3\}$ is represented in Figure 7 and g_z has geometry $z \in \{1, 2, 3\}$. According to Proposition 1, $\rho_i^1 = \rho_i^2 = \frac{2}{5}$ and $\rho_i^3 = \frac{1}{5}$, since $|Aut(g_1)| = |Aut(g_2)| = 2$ and $|Aut(g_3)| = 4$. Consider agent i (with $k_i(g) = 1$). Conditional on geometry 1, there are two nodes that represent a feasible position of i and are linked to a degree-three node (depicted in yellow in Figure 7 (1)). Therefore, $n_i^1\big(g \,|\, \mathtt{t}_{\mathtt{k_i}}\big) = n_i^1\big(g \,|\, (3)\big) = 2$. Similarly, $n_i^2\big(g \,|\, (3)\big) = n_i^3\big(g \,|\, (3)\big) = 1$. Then, the probability that i has a degree-three neighbor is:

$$p_i\big[\mathtt{t}_{\mathtt{N_i(g)}} = (3)\big] = \frac{n_i^1\big(g \,|\, (3)\big)}{n_i^1(g)}\rho_i^2 + \frac{n_i^2\big(g \,|\, (3)\big)}{n_i^2(g)}\rho_i^2 + \frac{n_i^3\big(g \,|\, (3)\big)}{n_i^3(g)}\rho_i^2 = \frac{2}{3} * \frac{2}{5} + \frac{1}{3} * \frac{2}{5} + \frac{1}{3} * \frac{1}{5} = \frac{7}{15}.$$

■